Tunnel effect for Krammers-Fokker-Planck operators

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Models

Kinetic inhomogeneous equations with a potential.
The typical equation has the following form

$$\partial_t f + v\partial_x f - \partial_x V \partial_v f = Q(f) \quad \text{or} \quad Q(f,f),$$

where Q is the collision kernel and acts only in velocity v.
$f(t,x,v)$ the the density of probability of presence of particles with position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geq 0$.
Here we study the (Krammers) Fokker-Planck equation

$$\partial_t f + v\partial_x f - \partial_x V \partial_v f = \gamma\partial_v(\partial_v + v)f,$$

where the collision term is made of a friction term and a diffusion term.

Other collision kernels

transport equation in a force field

$$Q = 0$$

(Krammers-)Fokker-Planck equation

$$Q_{\mathrm{FP}} = \gamma \partial_v (\partial_v + v)$$

linear relaxation equation

$$Q_{\mathrm{LR}}(f) = \gamma(\rho\mu - f) \quad \text{where } \rho(t,x) = \int f(t,x,v)dv \ \text{ and } \ \mu(v) = \frac{e^{-v^2/2}}{(2\pi)^{d/2}}$$

Boltzmann equation

$$Q_{\mathrm{B}}(f,f) = \iint_{\mathbb{R}^d, S^{d-1}} \mathbf{B}(|v - v^*|, \cos(\theta))\,(f'_* f' - f_* f)\,dv_* d\sigma.$$

Landau equation

$$Q_{\mathrm{L}}(f,f) = \partial_v . \int_{\mathbb{R}^d} \mathbf{A}(|v - v^*|)\,(f_* \partial_v f - f \partial_v f_*)\,dv_*$$

which potentials ?

Confining potentials

$$e^{-V} \in L^1(dx) \Longrightarrow \mathcal{M} = C^{-1} e^{-(v^2/2+V(x))} \in L^1(dxdv)$$

non-confining potentials

$$e^{-V} \notin L^1(dx) \Longrightarrow \mathcal{M} = e^{-(v^2/2+V(x))} \notin L^1(dxdv)$$

Mean field potentials (Vlasov-Poisson)

$$V = V_{\mathrm{nl}} \text{ where } -\Delta V_{\mathrm{nl}} = \kappa \int f(t,x,v)dv$$

Torus case

$$V = 0$$

boundary conditions : (specular, bounce back ...)

Admissible potentials

quadratic growth : $\mathcal{C}^\infty$ with

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{2-\min(|\alpha|,2)} \quad \text{and} \quad |\partial V(x)| \geq C \ \text{ for large } x.$$

Potentiels with polynomial growth : $\mathcal{C}^\infty$ with

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{2n-\min(|\alpha|,2)} \quad \text{and} \quad |\partial V(x)| \geq C \langle x \rangle^{2n-1} \ \text{ for } x \text{ large.}$$

Morse hypothesis : admissible with a finite number of non-degenerate critical points.

Potentials with spectral gap

$$\min\left(\sigma(-\partial_x(\partial_x + \partial_x V)) \cap \mathbb{R}^{+,*}\right) > 0$$

as an operator on $L^2(e^V dx)$ (Witten Laplacian).

Thermodynamic quantities

Mass

$$\int f(t,x,v)dxdv = 1$$

Energy

$$\int (v^2/2+V(x))f(t,x,v)dxdv = \begin{cases} \text{conserved (Boltzmann, Landau)} \\ \text{not conserved (Fokker-Planck, rel. lin.)} \end{cases}$$

Entropy

$$H(f) = \int f(t,x,v)\ln(f(t,x,v))dxdv$$

Relative Entropy

$$0 \leq H(f|\mathcal{M}) = \iint f \ln\left(\frac{f}{\mathcal{M}}\right) dxdv \quad \text{(confining case)}$$

$$0 \leq H(f|\mathcal{M}) = \iint \left(\left(1+\frac{f}{\mathcal{M}}\right)\ln\left(1+\frac{f}{\mathcal{M}}\right) - \frac{f}{\mathcal{M}}\right)\mathcal{M}dxdv \quad \text{(other cases)}$$

H Theorem of Boltzmann

entropy dissipating functional

$$D(f|\mathcal{M}) = -\iint Q(f)\ln(f/\mathcal{M})dxdv \quad \text{ou} \quad -\iint Q(f)\ln(1+f/\mathcal{M})dxdv$$

H Theorem of Boltzmann :

$$\frac{d}{dt}H(f|\mathcal{M}) = -D(f|\mathcal{M}) \leq 0$$

explicit constant problematic :

Explicit hypocoercivity

Explicit estimate (w.r.t physical constants, in a constructive way) of the time decay (polynomial, exponential) of the (relative) entropy.

Fokker-Planck equation

$$\partial_t f + v\partial_x f - \frac{1}{m}\partial_x V(x)\partial_v f - \gamma\partial_v\left(\frac{1}{m\beta}\partial_v + v\right) f = 0$$

here $\beta = 1/kT$, T temperature, k Boltzmann constant, m total mass, γ friction coefficient.

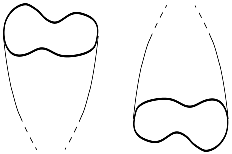
Theorem (HN04)

Let V be an admissible potential (polynomial growth). Then for all probability density f_0 such that $(f_0 - \mathcal{M})/\mathcal{M}^{1/2}$ (confining case) or $f_0/\mathcal{M}^{1/2}$ (non confining case) is a bounded measure, and for all $t > 0$, the solution $f(t)$ satisfies

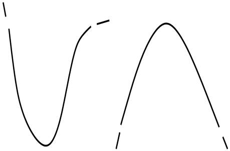
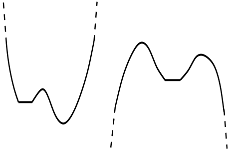
$$H(f|\mathcal{M})(t) \leq CC_{f_0} e^{-\tau t},$$

where τ and C are explicit and C_{f_0} is the measure of $\mathbb{R}^{2d}$ w.r.t. the mesures $(f_0 - \mathcal{M})/\mathcal{M}^{1/2}$ or $f_0/\mathcal{M}^{1/2}$ (depending on the cases).

High temperature

$\beta \preceq 1$ $(n \geq 1)$	strict. -convex (concave)	
$\sqrt{m}\gamma\beta^{\frac{n-1}{2n}} \preceq 1$	$\gamma\beta^{\frac{n-1}{n}} \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma\beta^{\frac{3n-1}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$	$\gamma \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma\beta^{\frac{3n-1}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$
$1 \preceq \sqrt{m}\gamma\beta^{\frac{n-1}{2n}}$	$\frac{1}{m\gamma} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma\beta^{\frac{-1-n}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$	$\frac{1}{m\gamma\beta^{\frac{n-1}{n}}} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma\beta^{\frac{-1-n}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$

Low temperature

$\beta \succeq 1$ $(n \geq 1)$	 Morse function with 1 critical point	 other cases
$\sqrt{m}\gamma \preceq \beta^{\frac{n-1}{2n}}$	$\frac{\gamma}{\beta^{\frac{2(n-1)}{n}}} \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma\beta^{\frac{-n-1}{n}}\right)}{\sqrt{m}}$	$\tau \preceq \frac{e^{-\frac{\beta}{3\kappa_V}}}{\sqrt{m}} \log\left(\sqrt{m}\gamma + \frac{1}{\sqrt{m}\gamma}\right)$
$\beta^{\frac{n-1}{2n}} \preceq \sqrt{m}\gamma$	$\frac{1}{m\gamma\beta^{\frac{n-1}{n}}} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma_0\beta^{\frac{3n-1}{n}}\right)}{\sqrt{m}}$	$\tau \preceq \frac{e^{-\frac{\beta}{3\kappa_V}}}{\sqrt{m}} \log\left(\sqrt{m}\gamma\right)$

Low temperature II

We can precise the results in the low temperature case.
a semi-classical study is possible.
Pose $T = h/2$ (other constants equal to 1 except γ).

$$h\partial_t f + vh\partial_x f - \partial_x V(x) h\partial_v f - \frac{\gamma}{2} h\partial_v \left(h\partial_v + 2v\right) f = 0$$

The maxwellian is given by

$$\mathcal{M}_h(x,v) = e^{-2(v^2/2+V(x))/h} \Big/ \iint e^{-2(v^2/2+V(x))/h} dxdv$$

In the case when V is a Morse function, the semiclassical study allows in the one hand to justify the quadratic approximation, and the other hand to study the tunnel effect.

Single critical point

Theorem (HSS06)

Suppose V is admissible and a Morse function with a single critical point. Then for all density f_0 such that $(f_0 - \mathcal{M}_h)/\mathcal{M}_h^{1/2} \in L^2$ (confining case) or $f_0/\mathcal{M}_h^{1/2} \in L^2$ (other cases), and for all $t > 0$, the solution $f(t)$ satisfies

$$H(f|\mathcal{M}_h)(t) \leq CC_{f_0,h}e^{-\tau t/h},$$

where τ and C are explicit and $C_{f_0,h}$ is the L^2 norm of $(f_0 - \mathcal{M}_h)/\mathcal{M}_h^{1/2}$ (resp. of $f_0/\mathcal{M}_h^{1/2}$) . besides we have

$$\tau = h\mu + o(h)$$

where μ is independant of h and given by a simple explicit formula.

Tunnel effect

In the case when the critical points are non-degenerate (Morse), we have a precise description of the return to equilibrium in some "bad cases".

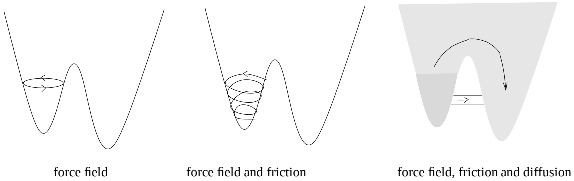
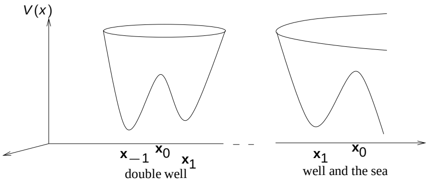


FIG.: tunnel effect for Krammers-Fokker-Planck

Two examples of tunnel effect

Let us see the "double well" and "the well and the sea" case ((HHS07a) et (HHS07b))



i) If V has precisely 3 critical points, 2 minima $\mathbf{x}_{\pm 1}$ and a saddle point $\mathbf{x}_0$ of index 1 then

$$\tau = h\left(a_1(h)e^{-2(V(\mathbf{x}_0)-V(\mathbf{x}_1))/h} + a_{-1}(h)e^{-2(V(\mathbf{x}_0)-V(\mathbf{x}_{-1}))/h}\right)$$

ii) If V has precisely 2 critical points, one local minimum $\mathbf{x}_1$ and a saddle point $\mathbf{x}_0$ of index 1 then

$$\tau = h\left(a_1(h)e^{-2(V(\mathbf{x}_0)-V(\mathbf{x}_1))/h}\right)$$

Linearization

In the Fokker-Planck case it seems to be transparent (linear equation). Anyway the procedure is rather deep

- confining case

$$f = \mathcal{M} + \mathcal{M}^{1/2}u \quad \text{et} \quad \int \mathcal{M}^{1/2}u dxdv = 0$$

  Natural space : $\{u \in L^2; \;\; u \perp \mathcal{M}^{1/2}\}$.

- non confining case

$$f = \mathcal{M}^{1/2}u$$

  Natural space : whole L^2.

In all cases : $H(f|\mathcal{M}) \sim \frac{1}{2}\iint u^2 dxdv$. In fact we have

$$0 \leq H(f|\mathcal{M}) \leq \iint u^2 dxdv .$$

common properties of linearized equations

We pose $X_0 = v\partial_x - \partial_x V(x)\partial_v$. The general form of linearized equations is

$$\partial_t u + Pu = 0, \qquad P = X_0 - L \quad \text{ou } P = X_0 - R^2 L$$

- $-L$ is the linearized collision kernel
- $R(x) = e^{-V(x)/2}/\sqrt{\int e^{-V} dx}$ (confining case)
- L acts only in velocity, and we have

$$L\mathcal{M}^{1/2} = 0, \qquad \text{et} \qquad L \leq 0 \quad \text{in} \quad L^2(dv)$$

Some linearized models

Fokker-Planck equation

$$L_{\mathrm{FP}} = -b^* b \text{ avec } b = \gamma^{1/2} (\partial_v + v/2)$$

linear relaxation equation

$$L_{\mathrm{LR}} u = \left(\int u_* M_* dv_* \right) M - u, \text{ avec } M(v) \stackrel{\mathrm{def}}{=} e^{-v^2/4}/(2\pi)^{d/4}$$

Boltzmann equation (confining case)

$$L_{\mathrm{B}} = L_{\mathrm{B}}^{+} - L_{\mathrm{B}}^{*} - \nu(v) \quad \text{with} \quad \nu(v) = (\Phi * M^2)(v)$$

Landau equation (confining case)

$$L_{\mathrm{L}} = L_{\mathrm{L}}^{*} - b^*.\mathbf{A}_\nu(v).b \quad \text{with} \quad \mathbf{A}_\nu(v) = (\mathbf{A} * M^2)(v)$$

semiclassical KFP-type equations

Equation

$$h\partial_t u + P_h u = 0, \qquad T = h/2$$

semiclassical Fokker-Planck equation

$$P_h = vh\partial_x - \partial_x V h\partial_v + \frac{\gamma}{2}(-h\partial_v + v)(h\partial_v + v)$$

chains of coupled oscillators

$$\begin{aligned} P_h &= vh\partial_x - (\partial_x V - z)h\partial_v \\ &+ \frac{\gamma}{2}(-h\partial_{z_1} + (z_1 - x_1))(h\partial_{z_1} + (z_1 - x_1)) \\ &+ \frac{\gamma}{2}(-h\partial_{z_2} + (z_2 - x_2))(h\partial_{z_2} + (z_2 - x_2)) \end{aligned}$$

Chains of coupled oscillators

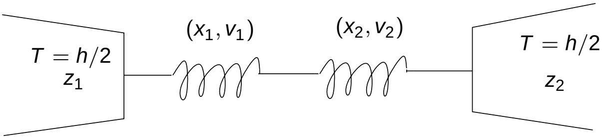


FIG.: Chaîn of 2 oscillators coupled with two heat baths

Potential : $V(x_1, x_2) = V_1(x_1) + V_2(x_2) + V_c(x_2 - x_1)$

Which microlocal analysis ?

- Hypoellipticity (à la Kohn, maximal, estimations pseudospectral estimates of cusp type, short time regularizing effect, ...)
  $\longrightarrow$ Fokker-Planck, Landau,
  Helffer, Nier, Villani, Sjöstrand, Kagei, Dencker, Hairer, Rey-Bellet, Eckmann, Bouchut ...
- Hypocoercivité (Hilbertian, entropy dissipation, à la Kohn, FIO methods, ...)
  $\longrightarrow$ KFP, linear relaxation, confining Boltzmann and Landau equations,
  Nier, Desvillettes, Mouhot, Dolbeault, Guo, ...
- semi-classical analysis (pseudospectrum, Grushin, spectrum, eigenfunctions, ...)
  $\longrightarrow$ KFP, Witten, chaines of oscillators,
  Sjöstrand, Hitrik, Helffer, Nier ...
- Supersymmetry (Witten, Hodge theory, tunnel effet, ...)
  $\longrightarrow$ KFP, Witten, linear relaxation, chains of oscillateurs,
  Tailleur, Tanase, Nicola, Kurchan, Lebeau, Bismut, Sjöstrand, Hitrik, ...

Hilbertian Hypocoercivity

recent concept.

Definition

Let $\mathcal{H}$ be an Hilbert space and P an unbounded maximal accretive operator on $\mathcal{H}$ with domain $D(P)$. Let $\mathcal{K} \subset \mathcal{H}$ an other Hilbert space with a norm $\|.\|_{\mathcal{K}}$ such that the restriction of $\|.\|_{\mathcal{H}}$ to $\mathcal{K}$ is equivalent to $\|.\|_{\mathcal{K}}$. P is said to be (Hilbert) hypocoercitive on $\mathcal{K}$ is the two following conditions occurs

- *Stability : The restriction of P to $\mathcal{K}$ is a maximal accretive operator with domain $\mathcal{K} \cap D(P)$ with value in $\mathcal{K}$.*
- *Coercivity : there exists a constant $\lambda > 0$ such that for all $u \in \mathcal{K} \cap D(P)$, we have*

$$\mathrm{Re}\,(Pu, u)_{\mathcal{K}} \geq \lambda \|u\|_{\mathcal{K}}^2$$

Some examples

Confining Fokker-Planck and linear relaxation equations

$$\mathcal{H} = L^2(dxdv), \quad \mathcal{K} = \left\{\mathcal{M}^{1/2}\right\}^{\perp}$$

non-confining Fokker-Planck and linear relaxation

$$\mathcal{H} = L^2(dxdv), \quad \mathcal{K} = \mathcal{H}$$

Confining Fokker-Planck and linear relaxation equations

$$\mathcal{H} = \mathbb{H}^1, \quad \mathcal{K} = \left\{\mathcal{M}^{1/2}\right\}^{\perp_{\mathbb{H}^1}}$$

Boltzmann and Landau on the Torus

$$\mathcal{H} = L^2(dxdv), \quad \mathcal{K} = \left\{\mathcal{M}^{1/2}, v_j\mathcal{M}^{1/2}, v^2\mathcal{M}^{1/2}\right\}^{\perp}$$

Let in general Π be a spectral projector for P, we take

$$\mathcal{H} = L^2(dxdv), \quad \mathcal{K} = (1 - \Pi)\mathcal{H}$$

Purpose/difficulty

The utility of this concept comes from the easy following corollary

Corollary

Let P an hypocoercive operator on $\mathcal{K} \subset \mathcal{H}$. then for all $u \in \mathcal{K}$ we have

$$\left\|e^{-tP}u\right\|_{\mathcal{H}} \leq C^2 e^{-\lambda t} \left\|u\right\|_{\mathcal{H}},$$

where C is the equivalence constant between the norm on $\mathcal{K}$ and the one on $\mathcal{H}$ restricted to $\mathcal{K}$.

In particular it implies the decay result on entropy mentioned in the beginning of this talk.

The main difficulty is to find a norm for which the operator will be coercive.

Mention also that it allows a perturbative study near steady states (in the confining case).

KFP and linear relaxation

We give now 2 examples of hypocoercivity, based on the use of the following cross operator

$$K = \Lambda^{-2} a^* b,$$

where $a = (\partial_x + \partial_x V(x)/2)$, $b = (\partial_v + v/2)$ and $\Lambda^2 = 1 + a^* a + b^* b$. We consider the Witten Laplacian $A_0 = a^* a$ and we suppose that

$$0 < \inf \left(\sigma(A_0) \cap \mathbb{R}^{+,*} \right) \stackrel{\text{def}}{=} \omega(V) \tag{1}$$

Theorem (HN05), (H06)

Let V admissible, confining and such that (1) happens. Let $\mathcal{H} = L^2(dxdv)$ and $\mathcal{K} = (\mathcal{M}^{1/2})^{\perp}$. Then there exists $\delta > 0$ explicit such that P_{FP} et P_{LR} are hypocoercive with the following norm on $\mathcal{K}$:

$$\|u\|_{\mathcal{K}}^2 \stackrel{\text{def}}{=} \|u\|_{\mathcal{H}}^2 + \delta \mathrm{Re}\, (u, Ku)_{\mathcal{H}}$$

for $u \in \mathcal{K}$, with λ explicit w.r.t. γ, δ, and a finite number of derivatives of V and $\omega(V)$.

semi-classical KFP

In (HSS05) and (HHS07a) we showed that if V is both admissible and Morse, then for $B > 0$ sufficiently large, the spectrum of P in $D(0, Bh)$ is discrete. for such a B, we denote by Π_B the spectral projector associated to P in $D(0, Bh)$.

Theorem (HHS07b)

Consider the semiclassical Fokker-Planck operator P_h. Suppose that V is Morse and admissible and that B is a large constant. Then there exists h_0 such that for all $0 < h \leq h_0$ we have the following :
There exists a Fourier integral operator with complex phase explicit and invertible A_δ uniformly bounded (and the same for its inverse) w.r.t. h and a constant $C > 0$ such that P_h is hypocoercif on $\mathcal{H} = L^2(dxdv)$ with $\mathcal{K} = (1 - \Pi_B)\mathcal{H}$ and

$$\|u\|_{\mathcal{K}} = \left\|A_\delta^{-1} u\right\|_{\mathcal{H}}^2$$

for $u \in \mathcal{K}$, with hypocoercive constant equal to Bh/C.

KFP semi-classique II

Hypocoercivity for P_h reads

$$\mathrm{Re}\,(A_\delta^{-1}P_h u, A_\delta^{-1}u) \geq \frac{Bh}{C}\left\|A_\delta^{-1}u\right\|^2$$

for $u \in \mathcal{K} \cap \mathcal{S}$, where we used the standard scalar product on $\mathcal{H} = L^2$. With $v = A_\delta^{-1}u$, this is equivalent to proving that

$$\mathrm{Re}\,(A_\delta^{-1}PA_\delta v, v) \geq \frac{Bh}{C}\|v\|^2 ,$$

i.e. that $P^\delta = A_\delta^{-1}P_h A_\delta$ is coercive on

$$A^\delta(1-\Pi_B)L^2(dxdv) = (1-\Pi_B^\delta)L^2(dxdv).$$

Let us just mention that the conjugated operator P^δ is very close to $P + \delta[X_0, K]$ where K is the cross operator.

Supersymmetry

We introduce two objects related to $\mathbb{R}^d$

- A real invertible $d \times d$ matrix $A = B + C$, B sym., C skewsym.
- A Morse function ϕ with $\partial^\alpha \phi$ and $\partial^\alpha \langle B\nabla\phi, \nabla\phi\rangle = \mathcal{O}(1)$.

We associated the following operator (on k forms) with symbol

$$p(x,\xi) = \sum_{j,k} A_{j,k}(-i\xi_k + \partial_{x_k}\phi)(i\xi_j + \partial_{x_j}\phi) \\ - h\,\mathrm{tr}(A\phi'') + 2h\sum_j (\phi'' \circ {}^tA)(dx_j)^\wedge \partial_{x_j}^\rfloor .$$

Principal symbole : $p = \langle B\xi, \xi\rangle + 2i\langle C\nabla\phi, \xi\rangle + \langle B\nabla\phi, \nabla\phi\rangle$.
of twisted Witten Laplacien form : $-\Delta_A = d_\phi^{A,*} d_\phi + d_\phi d_\phi^{A,*}$.

Examples

Witten Laplacien :

$$A = \frac{1}{2}\gamma Id \quad \phi(x) = V(x)$$

Fokker-Planck :

$$A = \frac{1}{2}\begin{bmatrix} 0 & Id \\ -Id & \gamma \end{bmatrix}, \quad \phi(x,v) = v^2/2 + V(x).$$

Chains of oscillators :

$$A = \frac{1}{2}\begin{bmatrix} 0 & Id & 0 \\ -Id & 0 & 0 \\ 0 & 0 & \gamma Id \end{bmatrix}$$

and $\quad \phi(x,v,z) = v^2/2 + V(x) + z^2/2 - zx.$

Exponentially small eigenvalues

The tunnel effect is a consequence of the existence of exponentially small eigenvalues. Supersymmetry allows to reduce the problem to a finite dimension problem, to compute the eigenvalues and to estimate the norm of the spectral projectors, in particular for $-\Delta_A^{(0)}$ et $-\Delta_A^{(1)}$ (Helffer-Sjöstrand 80').

In the double well case we have 2 exp. small eigenvalues for $-\Delta_A^{(0)}$: $\lambda_1 = 0$ and λ_2, and the eigenfunctions are build from quasimodes

$$\psi_j^{(0)}(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x) - \phi(x_j))}.$$

and 1 exp. small eigenvalue for $-\Delta_A^{(1)}$: $\lambda_1^{(1)} = \lambda_2$.

fundamental property

$$-\Delta_A^{(1)} d_\phi = d_\phi(-\Delta_A^{(0)}), \quad -\Delta_A^{(0)} d_\phi^{A,*} = -d_\phi^{A,*} \Delta_A^{(1)},$$

Return to equilibrium

- Thanks to hypocoercivity : exponential time decay for the "high spectrum" $(1-\Pi_B)u_0$.
- Thanks to resolvent estimates and Cauchy formula : exponential time decay for the "intermediate spectrum"
- thanks to supersymmetry : exponential time decay for the "exp. low spectrum"

$$e^{-tP/h} = e^{-tP/h}(\Pi_1 + \Pi_2 + \Pi_{(3-B)} + (1-\Pi_B)).$$

$\longrightarrow$ exp. return to equilibrium with rate

$$\tau = h\left(a_1(h)e^{-2(V(\mathbf{x}_0)-V(\mathbf{x}_1))/h} + a_{-1}(h)e^{-2(V(\mathbf{x}_0)-V(\mathbf{x}_{-1}))/h}\right)$$

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