ELLIPTIC COHOMOLOGY AND MODULAR FORMS

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§1. Introduction. The homology and cohomology theories of the title, which have been found in joint work with Doug Ravenel and Bob Stong [14], are periodic complex-oriented multiplicative theories, with the cohomology of a point naturally interpreted as a ring of modular functions. The formal groups that occur for these theories are obtained from the formal group of the Jacobi quartic

(1) $$Y^2 = 1 - 2\delta X^2 + \varepsilon X^4$$

over the ring $\mathbf{Z}[\frac{1}{2}]\ [\delta,\varepsilon]$ by passing to suitable localizations of this ring, where δ and ε are viewed as indeterminates of degrees 4 and 8. We view these theories as belonging to a tower:

bordism and cobordism (MU, MSpin, MSO)

↓

elliptic cohomology (Ell)

↓

K-theory (KU, KO)

↓

ordinary cohomology (H)

In the first part of this report, I want to provide an assurance that such theories exist. In the second part, I shall explore the connections with modular forms.

There are several prominent open questions in this subject, the main one being to give a geometric definition of the elliptic cohomology theories. A number of these problems will be collected at the end.

It is a pleasure to thank the many people with whom I have discussed these topics; by now the list is extremely long. Thanks are also due to the National

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§2. Elliptic genera. By a genus in the sense of Hirzebruch [7], one means a ring homomorphism

$$\varphi : \Omega_*^{SO} \to \Lambda$$

from the oriented bordism ring to a commutative $\mathbb{Q}$ - algebra with unit $(\varphi(1) = 1)$. Each such genus has a logarithm

$$g(x) = \int_0^x \sum_{n\geq 0} \varphi(\mathbb{C}P^{2n})\, t^{2n}\, dt ,$$

and a characteristic power series

$$u/g^{-1}(u) .$$

Following S. Ochanine [18], we call φ an elliptic genus if

$$g(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}}\, dt \tag{2}$$

with elements $\delta,\ \varepsilon \in \Lambda$. In this case, the corresponding formal group

$$F(x,y) = g^{-1}(g(x) + g(y))$$

has the following form found by Euler:

$$F(x,y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \varepsilon x^2 y^2} \tag{3}$$

where

$$R(x) = 1 - 2\delta x^2 + \varepsilon x^4 . \tag{4}$$

The signature (L-genus) and $\hat{A}$-genus are special cases. Namely, if $\delta = \varepsilon = 1$ then one has

$$g(x) = \int_0^x \frac{1}{1-t^2}\, dt = \tanh^{-1}(x)$$

and so obtains the characteristic series $u/\tanh\ u$ of the L-genus of Hirzebruch. And if $\delta = -1/8$, $\varepsilon = 0$ then one finds

$$g(x) = \int_0^x (1 + \tfrac{1}{4}t^2)^{-\frac{1}{2}}\, dt$$

and the characteristic series $u/2\ \sinh\ (u/2)$ of the $\hat{A}$-genus.

We remark that for any elliptic genus φ one has

$$\delta = \varphi(\mathbb{C}P^2) ,\quad \varepsilon = \varphi(\mathbb{H}P^2) . \tag{5}$$

Recalling that the Legendre polynomials $P_n(x)$ are defined by ([10])

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n\geq 0} P_n(x)\, t^n ,$$

one sees easily that

(6) $$\varphi(\mathbb{C}P^{2n}) = P_n(\delta/\sqrt{\varepsilon})\, \varepsilon^{n/2} =: P_n(\delta,\varepsilon) .$$

It is a pleasant surprise that on quaternionic projective spaces one has ([5])

(7) $$\varphi(\mathbb{H}P^n) = \begin{cases} \varepsilon^{n/2} , & n \text{ even} \\ 0 , & n \text{ odd} . \end{cases}$$

In view of (3) and the binomial expansion, it is immediate that all coefficients of $F(x,y)$ are in $\mathbf{Z}[\frac{1}{2}]\,[\delta,\varepsilon]$, so by Quillen's theorem φ maps Ω_*^{SO} into the subring $\mathbf{Z}[\frac{1}{2}]\,[\delta,\varepsilon]$ of Λ (see §5 for more precise results).

For a Jacobi quartic (1) or the corresponding elliptic genus, we introduce the discriminant

(8) $$\Delta = \varepsilon(\delta^2 - \varepsilon)^2 .$$

If δ , $\varepsilon \in \mathbb{C}$ and $\Delta \neq 0$, then $g^{-1}(u)$ is the expansion at the origin of an elliptic function $s(u)$, which is odd and of order 2 (a Jacobi sine; see [18] and §4). Note that the L-genus and $\hat{A}$-genus are "degenerate," i.e. $\Delta = 0$; the function $s(u)$ becomes singly periodic in these cases.

§3. <u>Elliptic homology and cohomology</u>. Continuing with the notation of the previous section, take δ and ε to be algebraically independent over $\mathbb{Q}$, and put

$$M_* = \mathbf{Z}[\tfrac{1}{2}]\,[\delta,\varepsilon] .$$

Then consider the rings:

$$\begin{array}{ccccc} & \nearrow & M_*[\Delta^{-1}] & \nwarrow & \\ M_*[\varepsilon^{-1}] & & & & M_*[(\delta^2-\varepsilon)^{-1}] \\ & \nwarrow & M_* & \nearrow & \end{array}$$

<u>Theorem 1</u> ([14]). <u>There are homology theories with each of these rings as homology of a point</u>. <u>These are multiplicative theories</u>, <u>the corresponding cohomology theories being complex-oriented</u>. <u>The formal group of each of these theories has</u>

logarithm given by (2), and the explicit form of (3) with $R(x)$ as in (4).

Note: I am viewing homology and cohomology as two sides of the same coin. We write $Ell_*(X)$ and $Ell^*(X)$ for such theories.

First proof. We produce a connective homology theory with $Ell_*(pt) \cong M_*$, by using bordism with singularities (the Sullivan-Baas construction [1]). Thus start with oriented bordism theory with 2 inverted, $\Omega_*^{SO}(X)[\frac{1}{2}]$, a module over

$$\Omega_*^{SO}[\tfrac{1}{2}] = \mathbf{Z}[\tfrac{1}{2}][x_4, x_8, x_{12}, \ldots] .$$

We can take

$$x_4 = [\mathbb{C}P^2] , \quad x_8 = [\mathbb{H}P^2]$$

and choose $x_{4n} = [M^{4n}]$ $(n \geq 3)$ so that the ideal

$$(x_{12}, x_{16}, \ldots)$$

consists of all bordism classes killed by elliptic genera; for the latter we follow Ochanine [18], generators for the ideal having the form $[\mathbb{C}P(\xi^{2m})]$ with ξ an even-dimensional complex vector bundle over a closed oriented manifold.

Now the Sullivan-Baas construction produces from the singularity set

$$\Sigma = \{x_{12}, x_{16}, \ldots\}$$

a theory $\Omega_*^{SO,\Sigma}[\frac{1}{2}](X)$ with $\Omega_*^{SO,\Sigma}[\frac{1}{2}](pt) \cong \mathbf{Z}[\frac{1}{2}][x_4, x_8] \xrightarrow{\sim} M_*$. Since 2 is inverted, one obtains a multiplicative homology theory, the obstructions to the existence of a good product all being 2-primary [16].

One can next simply invert Δ or its factors to obtain three further *periodic* theories.

Second proof. We shall follow a more insightful route, which yields the three periodic homology theories as a consequence of the exact functor theorem [12]. To explain the latter, let R be a commutative ring, and suppose given a formal group over R, i.e. a homomorphism from the complex bordism ring Ω_*^U to R (the formal group over Ω_*^U is universal). View R as a module over Ω_*^U. For each prime p and $n \geq 1$ define an element

$$u_n \in \Omega^U_{2(p^n-1)}$$

as the coefficient of z^{p^n} in the multiplication - by - p series

$$[p](z) = pz + \dots + u_1 z^p + \dots + u_2 z^{p^2} + \dots$$

for the universal formal group over Ω_*^U .

Exact Functor Theorem ([12]). In order that

$$X \to \Omega_*^U(X) \otimes_{\Omega_*^U} R$$

be a homology theory, it suffices that for each prime p

$$p, u_1, u_2, \dots, u_n, \dots$$

be a regular sequence on the Ω_*^U - module R . (I.e., multiplication by p on R and by each u_n on $R/(pR + \dots + u_{n-1}R)$ must be injective.)

Since we are inverting 2, it is the same to deal with

$$\Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}} R$$

and apply the criterion for all odd primes. Here we take, say,

$$R = M_*[\Delta^{-1}] = \mathbf{Z}[\tfrac{1}{2}][\delta, \varepsilon, \Delta^{-1}] .$$

With p an odd prime, multiplication by p on R is injective, and we pass to $\mathbb{F}_p[\delta, \varepsilon][\Delta^{-1}]$. In terms of the homogeneous Legendre polynomials of formula (6), one sees easily ([13]) that

(9) $$u_1 \equiv P_{(p-1)/2}(\delta, \varepsilon) \mod p .$$

That $u_1 \not\equiv 0 \mod p$ follows from the fact that $P_n(1) = 1$ for all n .

We are next obligated to examine $u_2 \mod (p, u_1)$, and here the principal facts are that

(10) $$\begin{cases} u_2 \equiv (-1)^{(p-1)/2} \varepsilon^{(p^2-1)/4} , \\ (\delta^2 - \varepsilon)^{(p^2-1)/4} \equiv \varepsilon^{(p^2-1)/4} \end{cases}$$

mod (p, u_1) in the ring $\mathbf{Z}[\tfrac{1}{2}][\delta, \varepsilon]$. The point is that mod (p, u_1) , inverting $\delta^2 - \varepsilon$ is equivalent to inverting ε , and so also to inverting Δ ; and that u_2 then becomes a unit. This ends the argument for $R = M_*[\Delta^{-1}]$, and also in case just one factor of Δ is inverted. □

Note. The congruences (10) can be better appreciated if $\varepsilon = 1$, and then read

$$(11)\qquad \begin{cases} u_2 \equiv (-1)^{(p-1)/2} \\ (\delta^2 - 1)^{(p^2-1)/4} \equiv 1 \end{cases}$$

mod $(p\ ,\ u_1)$ in the ring $\mathbf{Z}[\frac{1}{2}]\ [\delta]$. In this form, they were first pointed out by David and Gregory Chudnovsky [4]; the most direct proof is based on two papers of Igusa [8, 9]. Details will appear in [13], which also includes the following related result of Dick Gross.

Theorem ([6]). *Let* E *be a supersingular elliptic curve given by a Weierstrass equation over a field of characteristic* $p \geq 5$, *so that for its formal group one has* $[p]\ (z) = u_2\ z^{p^2} + \ldots$ *with* $u_2 \neq 0$, *where* $z = -x/y$ *is the standard uniformizing parameter. Then*

$$u_2 = (-1)^{(p-1)/2}\ .\ \Delta^{(p^2-1)/12}\ ,$$

where Δ *is the discriminant (expressed in terms of the coefficients of the Weierstrass equation).*

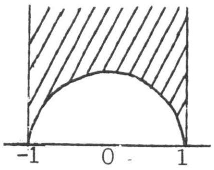
§4. $\mathbf{Z}[\frac{1}{2}]\ [\delta\ ,\ \varepsilon]$ as a ring of modular forms. Let Γ denote the modular group $SL_2(\mathbf{Z})/\{\pm 1\}$, acting as usual on the upper half-plane H . Whereas Γ is generated by

$$\tau \mapsto \tau + 1\quad ,\quad \tau \mapsto -\tau^{-1}\ ,$$

it will be convenient here to deal with the subgroup Γ_Θ generated by

$$\tau \mapsto \tau + 2\ ,\ \tau \mapsto -\tau^{-1}\ .$$

Γ_Θ , the "theta group," has index 3 in Γ , and the standard fundamental domain with two cusps:



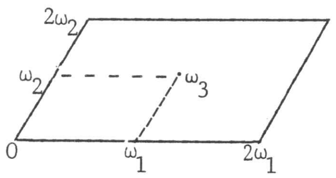
Moreover, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\Theta$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ mod 2 .

The appearance of level 2 modular forms was first noticed by David and Gregory Chudnovsky [3].

In [13] I have found the following classical picture helpful. Assume δ, $\varepsilon \in \mathbb{C}$ and $\Delta \neq 0$, so

$$Y^2 = 1 - 2\delta X^2 + \varepsilon X^4$$

is an elliptic curve over $\mathbb{C}$. This curve is uniformized by an elliptic function $s(u)$, odd and of order 2, with period lattice generated by $2\omega_1$ and $2\omega_2$.



The function $s(u)$ has poles at ω_1 and ω_2, and zeros at 0 and $\omega_3 = \omega_1 + \omega_2$. Hence one of the half-periods ω_3 is distinguished; assume $\tau = \omega_2/\omega_1 \in H$. The lattice has the usual Weierstrass function $\wp(u)$ and invariants g_2, g_3, as well as the half-period values $e_i = \wp(\omega_i)$. The following formulas are now easily obtained:

$$s(u) = -2\,(\wp(u) - e_3)/\wp'(u)$$

$$\begin{cases} g_2 = (\delta^2 + 3\varepsilon)/3 \\ g_3 = \delta(\delta^2 - 9\varepsilon)/27 \end{cases}$$

$$\begin{cases} \delta = 3e_3 \\ \varepsilon = (e_1 - e_2)^2 \\ \delta^2 - \varepsilon = 4(e_1 - e_3)(e_2 - e_3) \end{cases}$$

Moreover, one can as a further exercise (see [2]) express all these quantities in terms of τ via theta functions. Taking $\omega_1 = \pi$ to remove powers of π/ω_1 from the expressions, one has

$$\begin{cases} \delta = \theta_1^4 - \theta_2^4 \\ \varepsilon = (\theta_1^4 + \theta_2^4)^2 = \theta_3^8 . \end{cases}$$

Here the "theta-constants" are given by $(q = e^{\pi i \tau})$

$$\theta_1(\tau) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)}$$

$$\theta_2(\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}$$

$$\theta_3(\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} .$$

One then finds easily that δ and ε are modular forms of weights 2 and 4 for Γ_θ , and indeed that every modular form for Γ_θ is a polynomial in δ and ε . Recall that a modular form of weight k for Γ_θ is a holomorphic function $f(\tau)$ on H so that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$, and so that

$$f(\tau) = \sum_{n \geq 0} a_n q^n \qquad (q = e^{\pi i \tau})$$

with a similar holomorphicity condition at the other cusp. We see that

$$\delta(i\infty) = -1 , \quad \varepsilon(i\infty) = 1 ;$$

for the cusp at $\tau = 1$, we use $\tau \mapsto 1 - 1/\tau$ sending ∞ to 1 and find that

$$\delta(1) = 2 , \quad \varepsilon(1) = 0 .$$

We conclude that, up to inessential multiples, one finds the L-genus at $\tau = i\infty$ and the $\hat{A}$-genus (better, the A-genus) at $\tau = 1$.

Furthermore, in addition to

$$M_*(\Gamma_\theta) = \mathbb{C}[\delta , \varepsilon]$$

we can identify $M_* = \mathbf{Z}[\frac{1}{2}] [\delta , \varepsilon]$ with the ring of modular forms for Γ_θ with q-expansion coefficients in $\mathbf{Z}[\frac{1}{2}]$. In addition, we go on to identify $M_*[\Delta^{-1}]$ with those modular functions for Γ_θ which are holomorphic on H (poles at cusps only) and have q-expansion coefficients in $\mathbf{Z}[\frac{1}{2}]$. The rings $M_*[\varepsilon^{-1}]$ and $M_*[(\delta^2 - \varepsilon)^{-1}]$ are given similar interpretations, allowing a pole at one or the other cusp.

Thus we are to view δ and ε as modular forms, and so we should regard the elliptic genus

$$\varphi: \ \Omega_*^{SO} \to \mathbf{Z}[\tfrac{1}{2}]\ [\delta\ ,\ \varepsilon] = M_*$$

as assigning a modular form

$$\varphi(M^{4n}) = P_M(\delta\ ,\ \varepsilon)$$

to each oriented (or Spin) manifold. I leave it to Ed Witten [19] to explain a geometric procedure to produce such modular forms for Spin manifolds; we shall examine a formula for the resulting modular form in §6, given in terms of familiar constructions on vector bundles. In the next section, we answer the question: Which modular forms arise from oriented or from Spin manifolds?

§5. <u>Integrality and divisibility of elliptic genera</u>. The results stated here are taken from [5]. Let $\varphi: \ \Omega_*^{SO} \to \Lambda$ be an elliptic genus, with parameters $\delta\ ,\ \varepsilon \in \Lambda$.

<u>Theorem 2</u> ([5]). <u>For an elliptic genus, one has</u>

$$\varphi\Omega_*^{SO} = \mathbf{Z}[\delta,\ 2\gamma,\ 2\gamma^2,\ldots,\ 2\gamma^{2^s},\ldots]$$

<u>with</u> $\gamma = (\delta^2 - \varepsilon)/4$, <u>and</u>

$$\varphi\Omega_*^{Spin} = \mathbf{Z}[16\delta,\ (8\delta)^2,\ \varepsilon]\ .$$

<u>Corollary 1</u>. <u>For an elliptic genus</u> $\varphi: \ \Omega_*^{SO} \to \mathbb{Q}$ <u>one has</u>

$$\varphi\Omega_*^{SO} = \mathbf{Z}[\delta\ ,\ \gamma]\ ,$$

$$\varphi\Omega_*^{Spin} = \mathbf{Z}[8\delta\ ,\ \varepsilon]\ .$$

<u>Corollary 2</u> (Ochanine [17]). <u>The signature of a Spin manifold</u> M^{8k+4} <u>is divisible by 16</u>.

For the second corollary, take $\delta = \varepsilon = 1$ and note the degrees of the generators of $\varphi\Omega_*^{Spin}$.

We shall sketch the proof of the theorem for Ω_*^{Spin} .

Proof for Ω_*^{Spin} .

i) $\varphi(\mathbb{H}P^2) = \varepsilon$.

ii) There is a Spin manifold V^4 with signature 16, i.e. $\varphi(V^4) = 16\delta$.

iii) Kervaire and Milnor [11] constructed an almost parallelizable W^8 with $\hat{A}(W^8) = 1$; if

$$\varphi(W^8) = a\delta^2 + b\varepsilon$$

then $a = 64$; by i), we have $\varphi(M^8) = (8\delta)^2$ for a suitable Spin manifold.

iv) Hence $\mathbf{Z}[16\delta, (8\delta)^2, \varepsilon] \subset \varphi(\Omega_*^{Spin})$.

v) In [15] we constructed a sequence ρ_k , $k \geq 0$, of KO-theory characteristic classes of oriented bundles such that

$$\rho_k[M^{4n}] = \hat{A}(M)\ ch(\rho_k TM)\ [M^{4n}]$$

has the properties

a) $\rho_0[M^{4n}] = \hat{A}(M^{4n})$

b) $\rho_1[M^{4n}] = \hat{A}(M)\ ch(TM - 4n\mathbb{R})\ [M^{4n}]$

c) $\rho_k[M^{4n}]$ is integral on Spin manifolds, and is even when n is odd .

d) ρ_t: $\Omega_*^{SO} \to \mathbb{Q}[[t]]$ given by $\rho_t(M) = \sum_{k\geq 0} \rho_k[M]\ t^k$ is an elliptic genus,

for which

e) $\delta(t) \equiv -\frac{1}{8} + 3t \mod t^2\mathbf{Z}[[t]]$

f) $\varepsilon(t) \equiv -t \mod t^2\mathbf{Z}[[t]]$

(for the integrality in e) and f), see [3], [20] and the next section).

vi) Returning to the argument, if M^{8k} is a Spin manifold and

$$\varphi(M^{8k}) = a_0(8\delta)^{2k} + a_1(8\delta)^{2k-2}\varepsilon + \ldots + a_k\ \varepsilon^k$$

for an arbitrary elliptic genus φ , take $\varphi = \rho_t$ so that

$$\rho_t(M^{8k}) \in \mathbf{Z}[[t]]$$

to conclude easily that each $a_i \in \mathbf{Z}$. And if M^{8k+4} is a Spin manifold, write

$$\varphi(M^{8k+4}) = 16\delta \sum_{j=0}^{k} b_j (8\delta)^{2k-2j} \varepsilon^j$$

and argue that each $b_j \in \mathbf{Z}$. As an illustration of the simple method, if

$$\varphi(M^k) = 16\delta\ (b_0(8\delta)^2 + b_1\varepsilon)$$

then

$$16(-\frac{1}{8} + 3t + \ldots)\ [b_0(1 - 24t + \ldots)^2 + b_1(- t + \ldots)]$$

lies in $2\mathbf{Z}[[t]]$, so

$$b_0(1 - 24t + \ldots)^2 + b_1(1 - 24t + \ldots)\ (- t + \ldots)$$

lies in $\mathbf{Z}[[t]]$, whence

$$b_0 \in \mathbf{Z} \quad \text{(constant term)} ,$$

$$b_1 \in \mathbf{Z} \quad \text{(coefficient of } t) . \qquad \square$$

§6. Witten's formula for the elliptic genus. We refer to [19] for the geometry underlying the following considerations. For a real or complex vector bundle E , put

$$\lambda_t(E) = \sum_{k\geq 0} \lambda^k(E)\ t^k \ , \ S_t(E) = \sum_{k\geq 0} S^k(E)\ t^k$$

where $\lambda^k(E)$ and $S^k(E)$ denote the exterior and symmetric powers of E . In addition, put

$$\Theta_t(E) = \bigotimes_{n=1}^{\infty} [\lambda_{t^{2n-1}}(E) \otimes S_{t^{2n}}(E)] .$$

Evidently, $\Theta_t\ (E \oplus F) = \Theta_t(E) \cdot \Theta_t(F)$. For a complex line bundle L we have

$$\Theta_t(L) = \prod_{n=1}^{\infty} \frac{1 + t^{2n-1}L}{1 - t^{2n}L} ,$$

in particular

$$\Theta_t(1) = \prod_{n=1}^{\infty} \frac{1 + t^{2n-1}}{1 - t^{2n}} .$$

One computes:

$$\Theta_t(E) = 1 + tE + t^2(\lambda^2 E + E) + t^3(\lambda^3 E + E \otimes E + E) + \ldots$$

and immediately sees a connection with results of [15], indeed a step toward an explanation of these results.

Using this characteristic class, we obtain a genus

$$\Theta_t : \Omega_*^{SO} \to \mathbb{Q}[[t]]$$

by putting

$$\Theta_t(M^{4n}) = \hat{A}(M)\ \mathrm{ch}\ \frac{\Theta_t(TM)}{\Theta_t(1)^{4n}}\ [M^{4n}]$$

<u>Theorem 3</u>. <u>The genus</u> Θ_t <u>is an elliptic genus</u>, <u>coinciding with the natural choice for the elliptic genus</u> ρ_t .

After some clarifying remarks, one will see that this is really an easy observation. The analysis of elliptic genera ρ_t (see §5) in [15, 3] led to a formula for the characteristic power series

$$\frac{x/2}{\sinh(x/2)}\ f_t(e^x + e^{-x} - 2)\ .$$

Here

$$t \equiv -q \mod q^2\mathbf{Z}[[q]]\ ,$$

the natural parameter being q and the most natural choice of t being simply $t = -q$. In [3] (see also [20]) it is shown that, in terms of q ,

$$f_t(y) = \prod_{n=1}^{\infty} \frac{1 - y\,q^{2n-1}/(1 - q^{2n-1})^2}{1 - y\,q^{2n}/(1 - q^{2n})^2}\ .$$

Now the genus Θ_t is given in a form which permits one to easily find its characteristic power series. Indeed, with $t = -q$ as suggested above, one finds that for a complex line bundle L with $c_1(L) = x$ one has

$$\mathrm{ch}\,\Theta_t(L \oplus \bar{L} - 2)$$

$$= \prod_{n=1}^{\infty} \frac{(1 + t^{2n-1}e^x)(1 + t^{2n-1}e^{-x})/(1 + t^{2n-1})^2}{(1 - t^{2n}e^x)(1 - t^{2n}e^{-x})/(1 - t^{2n})^2}$$

$$= \prod_{n=1}^{\infty} \left[(1 - q^n e^x)(1 - q^n e^{-x})/(1 - q^n)^2\right]^{(-1)^{n-1}}$$

$$= \prod_{n=1}^{\infty} \left[1 - y\, q^n/(1 - q^n)^2\right]^{(-1)^{n-1}} ,$$

with $y = e^x + e^{-x} - 2$. We have therefore verified that

$$\mathrm{ch}\, \Theta_t(L \oplus \bar{L} - 2) = f_t(y) ,$$

whence the genera Θ_t and ρ_t (with $t = -q$) have identical characteristic series, the latter being an elliptic genus. The theorem is proved. □

Note. We refer to [20] for the interpretation of δ, ε, and so the elliptic genus of any oriented manifold, as modular forms for the group $\Gamma_0(2)$. This group is a conjugate of Γ_θ in Γ , consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with c even.

7. Some open problems. As of October 1986, here are some rather naive questions that deserve study.

A) Give an intrinsic geometric construction for an elliptic cohomology theory.

B) Construct such a theory in which it is not necessary to invert 2.

C) Find appropriate versions of representation theory and index theory fitting with elliptic cohomology.

D) Since we are producing periodic cohomology theories, one might seek a fundamental result analogous to Bott periodicity, in an appropriate setting.

E) Spin bundles are orientable for elliptic cohomology. Construct such an orientation, compatibly with the KO-theory orientation constructed by means of Clifford algebras.

F) Seek related invariants in dimensions not divisible by 4. This may call for modular forms of half-integral weight, or mod p modular forms (especially with $p = 2$) .

G) Develop a variant in which modular forms of level 1 occur.

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