DIFFERENTIAL FORMS AND 0-DIMENSIONAL SUPERSYMMETRIC FIELD THEORIES

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Abstract. We show that closed differential forms on a smooth manifold $X$ can be interpreted as topological (respectively Euclidean) supersymmetric field theories of dimension $0|1$ over $X$. As a consequence, concordance classes of such field theories are shown to represent de Rham cohomology. The main contribution of this paper is to make all new mathematical notions regarding supersymmetric field theories precise.

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1. Introduction

Two of us [ST] spent the last years to find a precise notion of supersymmetric Euclidean field theories of (super) dimension $d|\delta$ and relate it to certain multiplicative cohomology theories. We showed that in dimension $1|1$ the relevant cohomology theory is K-theory, see also [HST] for a much more current account. We also conjectured that for dimension $2|1$ one gets elliptic cohomology, or more precisely, the cohomology theory TMF of topological modular forms. In this paper we fill the gap in dimension $0|1$ by showing that de Rham cohomology arises in this easiest case. It is a very interesting question whether cohomology theories arise from other values of $d|\delta$.

The geometric cocycles we actually get from EFTs (which is short for Euclidean field theories) of dimension $0|1$ are closed differential forms, just like vector bundles with connection can be used to get Euclidian field theories of dimension $1|1$, see [D]. Our goal remains to show that EFTs of dimension $2|1$ are cocycles for TMF.

Our results are consistent with the formal group point of view towards (complex oriented) cohomology theories, where the additive formal group gives ordinary rational cohomology, the multiplicative group gives K-theory and the formal groups associated to elliptic curves lead to elliptic cohomology.

The precise definition of EFTs is pretty involved, so we won’t repeat it here but refer instead to our survey [ST2]. We will summarize in Section 4 the necessary information for dimension $0|1$ where the fancy definitions reduce to well known notions. This was actually
one of the design criteria for our definitions: everything has to work in arbitrary dimension but reduce to differential forms in dimension 0

In our definition, an EFT has a degree \( n \in \mathbb{Z} \) which is related to the central charge as well as to the degree of a cohomology class. If \( X \) is a smooth manifold, we also define EFTs over \( X \), which can be thought of as families of EFTs parametrized by \( X \). In this case, the degree \( n \) can be generalized to a twist over \( X \) which relates very well to twisted cohomology but will not be discussed in this paper. An EFT over \( X \) should be thought of as a geometric object over \( X \). This is best explained by our main result below, Theorem 1, which says that a closed differential form over \( X \) can be interpreted as a 0-dimensional EFT over \( X \) and vice versa.

Like differential forms or vector bundles with connection, EFTs over \( X \) of the same dimension \( d \| \delta \) can be added and multiplied. Addition preserves the degree \( n \), whereas multiplication adds degrees as expected. Moreover, \( d \| \delta \)-dimensional EFTs over a manifold \( X \) of degree \( n \) form a category \( d \| \delta \)-EFT\(^n\)(\( X \)) (in fact, a \( d \)-category, an issue we’ll ignore in this paper) and can be pulled back via smooth maps: a smooth map \( f: Y \to X \) determines a functor

\[
f^*: d \| \delta \text{-EFT}\(^n\)(X) \to d \| \delta \text{-EFT}\(^n\)(Y)
\]

and these functors compose strictly. We call two EFTs \( E_0, E_1 \in d \| \delta \text{-EFT}\(^n\)(X) \) concordant if there exists a field theory \( E' \in d \| \delta \text{-EFT}\(^n\)(X \times \mathbb{R}) \) and \( \epsilon > 0 \) such that \( E' \cong p_1^*(E_0) \) on \( X \times (-\infty, \epsilon) \) and \( E' \cong p_2^*(E_1) \) on \( X \times (1 - \epsilon, \infty) \).

We observe that concordance give an equivalence relation which can be defined for geometric objects over manifolds for which pullbacks and isomorphisms make sense. By Stokes’ Theorem two closed \( n \)-forms on \( X \) are concordant if and only if they represent the same de Rham cohomology class; two vector bundles with connections are concordant if and only if they are isomorphic as vector bundles (disregarding the connections). Passing from an EFT over \( X \) to its concordance class forgets the geometric information while retaining the topological information. We will write \( d \| \delta \text{-EFT}\(^n\)[X] \) for the set of concordance classes of \( d \| \delta \)-dimensional supersymmetric EFTs of degree \( n \) over \( X \).

**Theorem 1.** For smooth manifolds \( X \), there are natural group isomorphisms

\[
0 \| 1\text{-EFT}\(^n\)(X) \cong \begin{cases} 
\Omega_{cl}^{ev}(X) & n \text{ even} \\
\Omega_{cl}^{odd}(X) & n \text{ odd}
\end{cases}
\]

where \( \Omega_{cl}^{ev}(X) \), respectively \( \Omega_{cl}^{odd}(X) \), stands for the even, respectively odd, closed differential forms on \( X \). These isomorphisms take the tensor product of EFTs to the wedge product of differential forms.

It follows that on concordance classes we get isomorphisms

\[
0 \| 1\text{-EFT}\(^n\)[X] \cong \begin{cases} 
H_{dR}^{ev}(X) & n \text{ even} \\
H_{dR}^{odd}(X) & n \text{ odd}
\end{cases}
\]

where \( H_{dR}^{ev}(X) \), respectively \( H_{dR}^{odd}(X) \), stands for the direct sum of the even, respectively odd, de Rham cohomology groups of \( X \).

There is a beautiful interpretation of the Chern character form of a vector bundle with connection in terms of the map from 1\|1-dimensional to 0\|1-dimensional EFTs over \( X \), given by crossing with the standard circle, see [Ha]. It is hence essential that the result above yields differential forms of varying degrees. However, differential forms of a specific degree \( n \) arise by forgetting the Euclidean geometry (on superpoints) and working with TFTs (topological field theories) instead. Again, there are categories \( d \| \delta \text{-TFT}\(^n\)(X) \) of \( d \| \delta \)-dimensional
TFTs over a manifold $X$ of degree $n$ as well as their concordance classes $d|\delta\cdot\text{TFT}^n[X]$. In fact, the following result is true for any supermanifold $X$, whereas one would have to use pseudodifferential forms on $X$ to make Theorem 1 hold, see Sections refsec:pseudo and 5.2.

**Theorem 2.** For supermanifolds $X$, there are natural group isomorphisms

$$0|1\cdot\text{TFT}^n(X) \cong \Omega^n_{cl}(X)$$

compatible with multiplication (tensor product of TFTs versus wedge product of differential forms). Moreover, concordance classes lead to isomorphisms

$$0|1\cdot\text{TFT}^n[X] \cong H^n_{dR}(X)$$

It is well known that the canonical map induces an isomorphism

$$H^n_{dR}(X) \cong H^n_{dR}(X_{\text{red}}),$$

so we recover information only about the underlying reduced manifold. We will show in a forthcoming paper with Chris Schommer-Pries that Theorem 2 carries over to the case of twisted topological field theories which relate to differential forms, twisted by a flat vector bundle, and the resulting twisted de Rham cohomology.

From the above theorem, it is easy to recover the entire structure of de Rham cohomology from TFTs. What’s missing is the boundary map in Mayer-Vietories exact sequences for a covering of $X$ by open sets. Equivalently, we need to express the suspension isomorphisms

$$H^n(X) \cong H^{n+1}_{\text{cs}}(X \times \mathbb{R})$$

in terms of TFTs. Here the subscript $\text{cs}$ means compact vertical support (in the $\mathbb{R}$-direction). This isomorphism is given by taking the product with a particular class $u \in H^1_{\text{cs}}(\mathbb{R})$, the Thom class for the trivial line bundle over $X = \text{pt}$. Therefore, it suffices to express the condition of compact vertical support in terms of TFTs. However, this is easy since the first part of Theorem 2 describes the cocycles for de Rham cohomology in terms of TFTs and compactly supported cohomology is given by concordance classes of compactly supported cocycles. The second, cohomological, part of Theorem 2 alone would not be sufficient for this argument!

Similarly, it is the description of de Rham cocycles that enables us to use TFTs for building Eilenberg-MacLane spaces $K(\mathbb{R}, n)$: Consider extended standard k-simplices

$$\Delta^k_e := \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^{k} t_i = 1\}$$

which are smooth manifolds (without boundary or corners). The usual face and degeneracy maps are defined on these extended simplices and hence there are simplicial sets $K_\bullet(n)$ with k-simplices $0|1\cdot\text{TFT}^n(\Delta^k_e)$.

**Corollary 3.** The geometric realization $|K_\bullet(n)|$ is an Eilenberg-MacLane space of type $K(\mathbb{R}, n)$, where $\mathbb{R}$ has the discrete topology.

**Proof.** This result is well known for any ordinary cohomology theory (with arbitrary coefficients), where one replaces TFTs by the relevant cocycles for the theory. The easiest way for us to prove the result is to state the following result from [MW, Appendix]: Given any sheaf (on the big site of smooth manifolds) $F : \text{Man}^{op} \rightarrow \text{Set}$, there are natural bijections for manifolds $X$ as follows:

$$F[X] \cong [X, |F|]$$
The notation $SP_{\mathbb{R}}$ local our definition to be $R$. Here the translational part of the supermanifold of superpoints: $X$.

As a consequence, differential forms on an ordinary manifold is a point in $X$ are representable by the supermanifold $\pi_{SP\mathbb{R}}X$.$\n$ of the superpoint $R$.

Forms because the de Rham operator $d$ is defined locally.

The very last part of the above proof is our motivation for requiring the field theories in our definition to be local. Currently, we express this by saying that a $d|\partial$-dimensional field theory is a (symmetric monoidal) $d$-functor from a bordism $d$-category to a target $d$-category. The precise details of this definition for $d = 2$ are far from obvious.

This paper is organized as follows. After briefly introducing the category $SM$ of supermanifolds, we give a detailed proof that for any supermanifold $X$, the odd tangent bundle $ITX$ represents the inner Hom $SM(\mathbb{R}^{0|1}, X)$ of maps from the odd line $\mathbb{R}^{0|1}$ to $X$. We actually think of the odd line as the easiest way of thickening a point to a supermanifold and, as a consequence, view this inner Hom as the supermanifold of superpoints in $X$. We abbreviate this generalized supermanifold by $SP_{\mathbb{R}}X := SM(\mathbb{R}^{0|1}, X)$.

It is characterized by its $T$-points $SP_{\mathbb{R}}X(T) = SM(T \times \mathbb{R}^{0|1}, X)$ for any supermanifold $T$. The notation $SP_{\mathbb{R}}X$ for the superpoints in $X$ is similar to writing $LX = Man(S^1, X)$ for the generalized manifold of loops in $X$. Unlike this infinite dimensional case, the superpoints in $X$ are representable by the supermanifold $ITX$ which means roughly that a map $\mathbb{R}^{0|1} \rightarrow X$ is a point in $X$ together with an odd tangent vector, see Proposition 6 for a precise statement. As a consequence, differential forms on an ordinary manifold $X$ are the functions on this supermanifold of superpoints:

$$\Omega^*X \cong C^\infty(ITX) \cong C^\infty(SP_{\mathbb{R}}X)$$

This isomorphism explains both structures, the $\mathbb{Z}$-grading and de Rham $d$, on differential forms via symmetries of the superpoint as follows. The (inner) diffeomorphism supergroup of the superpoint $\mathbb{R}^{0|1}$ is

$$Diff(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$$

Here the translational part $\mathbb{R}^{0|1}$ induces an action on functions which turns out to be infinitesimally generated by $d$, see Lemma 8. The relation $d^2 = 0$ follows because translations commute with each other. Moreover, the $\mathbb{Z}$-grading of $\Omega^*(X)$ comes from the dilation action of $\mathbb{R}^\times$ on $\mathbb{R}^{0|1}$, see Corollary 11. Finally, the usual relations between dilations and translations show that $d$ must have degree one. We claim no originality for these results since they seem to be well known to several authors. However, we exhibit detailed arguments (including the case where $X$ is a supermanifold) in this paper because we couldn’t find a reference that contained a proof of this result, rather than just stating it.

In Sections 4 and 5 we will review the notions of field theories as functors, pioneered by Atiyah, Kontsevich in the topological case and Segal in the conformal case. We will give precise meaning for such functors to be smooth by introducing family versions of the relevant bordism categories. Once this is done, it will be easy to generalize this notion to super symmetric field theories which we then continue to study in the simplest case, that of dimension $0|1$. We’ll show in Proposition 30 that such field theories are functions on the quotient $SP_{\mathbb{R}}X/G$ on the site $SM$ of supermanifolds. Here $G$ is a subgroup of the (inner)
diffeomorphism group $\text{Diff}(\mathbb{R}^{0|1})$ defining the geometry on the superpoint $\mathbb{R}^{0|1}$, in the spirit of Felix Klein. For a topological field theory, $G = \text{Diff}(\mathbb{R}^{0|1})$ consists of all diffeomorphisms and hence

$$0|1\text{-TFT}^0(X) \cong C^\infty(\text{SP}X)^{\text{Diff}(\mathbb{R}^{0|1})} \cong \Omega^*X^{\text{Diff}(\mathbb{R}^{0|1})} \cong \Omega^0_{cl}(X)$$

which is the case of Theorem 2 for degree 0. A Euclidean field theory is defined by setting $G = \text{Iso}(\mathbb{R}^{0|1}) := \mathbb{R}^{0|1} \rtimes \{\pm 1\}$, allowing only translations and reflections as isometries of $\mathbb{R}^{0|1}$, but not all dilations. It follows that

$$0|1\text{-EFT}^0(X) \cong C^\infty(\text{SP}X)^{\text{Iso}(\mathbb{R}^{0|1})} \cong \Omega^*X^{\text{Iso}(\mathbb{R}^{0|1})} \cong \Omega^0_{ev}(X)$$

which is the case of Theorem 1 for degree 0. In both Theorems, the degree $n$ case is obtained by defining twisted field theories in a way that functions on the quotient $\Pi T X/G$ are replaced by sections of a line bundle given by the twist. This is explained in Section 6 for the easiest possible twists, giving the degree $n$ field theories.

These functions on the moduli spaces $\text{SP}X/G$ should be thought of as the “partition functions” of our field theories. In dimension 0|1 they obviously contain the entire information. It is also very natural that in the twisted case these are not functions but sections of certain line bundles on the moduli spaces, just like the (integral) modular forms arising as the partition functions of 2|1-dimensional EFTs in [ST2].

Our appendix is again expository, we’ll survey the notion of Grothendieck fibrations $V \to S$. In the case where $S$ is the category of manifolds respectively supermanifolds, this notion will be later used to define smooth respectively supersymmetric field theories. Here $V$ will be family versions of various bordism categories, respectively versions of the target categories like $\text{Pic}$ for the field theory. The only original result of the appendix is Proposition 53.

2. Quick Survey of Supermanifolds

Following a suggestion of the referee, we have removed our original survey from this paper, it now appears online in the Manifold Atlas (coordinated by Kreck and Crowley at the Hausdorff Institute for Mathematics in Bonn). The reader is also invited to have a look at the beautiful survey article on supermanifolds by Deligne and Morgan [DM] or at one of the standard references by Leites [L], Bernstein [B], Manin [M], or Voronov [V]. For the simplest super manifold $\mathbb{R}^{0|n}$ and its diffeomorphisms we recommend [KS].

We shall just summarize the absolutely basic notions, working with the ground field $\mathbb{R}$. A superalgebra is a monoidal object in the category of supervector spaces and is hence the same thing as a $\mathbb{Z}/2$-graded algebra. However, the interesting symmetry operators on this monoidal category implies that a superalgebra is commutative if for all homogenous $a, b \in A$ we have

$$ab = (-1)^{[a][b]}ba,$$

a very different notion than a commutative $\mathbb{Z}/2$-graded algebra. The derivations of such a commutative superalgebra $A$ are endomorphisms $D \in \text{End}(A)$ satisfying the Leibniz rule: \footnote{Whenever we write formulas involving the degree $|.|$ of certain elements, we implicitly assume that these elements are homogenous.}

$$D(a \cdot b) = Da \cdot b + (-1)^{|D|[a]}a \cdot Db.$$ Der $A$ is a super Lie algebra with respect to the bracket operation

$$[D, E] := DE - (-1)^{|D||E|}ED$$
A supermanifold $M$ of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative superalgebras $\mathcal{O}_M$ that is locally isomorphic to $\mathbb{R}^{p|q}$. The latter is the space $\mathbb{R}^p$ equipped with the sheaf $\mathcal{O}_{\mathbb{R}^{p|q}}$ of commutative superalgebras $U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$.

The category $\text{SM}$ of supermanifolds is defined by using morphisms of sheaves. There is a functor $\text{SM} \to \text{Man}$ that associates to a supermanifold $M$ its reduced manifold

$$M^{\text{red}} := (|M|, \mathcal{O}_M / \text{Nil})$$

obtained by dividing out the ideal of nilpotent functions. By construction, this quotient sheaf gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of supermanifolds $M^{\text{red}} \hookrightarrow M$. Note that the sheaf of ideals $\text{Nil} \subset \mathcal{O}_M$ is generated by the odd functions.

**Example 4.** Let $E$ be a real vector bundle of fiber dimension $q$ over the ordinary manifold $X^p$ and $\Lambda^*(E^*)$ the associated algebra bundle of alternating multilinear forms on $E$. Then its sheaf of sections gives a supermanifold of dimension $p|q$, denoted by $\Pi E$.

The following proposition gives two extremely useful ways of looking at morphisms between supermanifolds. We shall use the notation $C^\infty(M) := \mathcal{O}_M(M)$ for the algebra of (global) functions on a supermanifold $M$.

**Proposition 5.** For $S, M \in \text{SM}$, the functor $C^\infty$ induces natural bijections

$$\text{SM}(S, M) \cong \text{Alg}(C^\infty(M), C^\infty(S))$$

If $M \subset \mathbb{R}^{p|q}$ is an open supersubmanifold (a domain), $\text{SM}(S, M)$ is in bijective correspondence with those $(f_1, \ldots, f_p, \eta_1, \ldots, \eta_q)$ in $(C^\infty(S)^{\text{ev}})^p \times (C^\infty(S)^{\text{odd}})^q$ that satisfy

$$(|f_1|(s), \ldots, |f_p|(s)) \in |M| \subset \mathbb{R}^p$$

for all $s \in |S|$. The $f_i, \eta_j$ are called the coordinates of $\phi \in \text{SM}(S, M)$ and are defined by

$$f_i = \phi^*(x_i) \quad \text{and} \quad \eta_j = \phi^*(\theta_j),$$

where $x_1, \ldots, x_p, \theta_1, \ldots, \theta_p$ are coordinates on $M \subset \mathbb{R}^{p|q}$. Moreover, by the first part we see that $f_i \in C^\infty(S)^{\text{ev}} = \text{SM}(S, \mathbb{R})$ and hence $|f_i| \in \text{Man}(|S|, \mathbb{R})$.

The proof of the first part is based on the existence of partitions of unity for supermanifolds, so it is false in analytic settings. The second part always holds and is proved in [L].

Since sheaves are generally difficult to work with, one often thinks of super manifolds in terms of their $S$-points, i.e. instead of $M$ itself one considers the morphism sets $\text{SM}(S, M)$ where $S$ varies over all super manifolds. More formally, one embeds the category $\text{SM}$ of super manifolds in the category of contravariant functors from $\text{SM}$ to $\text{Set}$ by

$$Y : \text{SM} \longrightarrow \text{Fun}(\text{SM}^\text{op}, \text{Set}), \quad Y(M) = (S \mapsto \text{SM}(S, M)).$$

This Yoneda embedding is fully faithful and identifies $\text{SM}$ with the the category of representable functors, defined to be those in the image of $Y$. Following A.S. Schwarz, we will sometimes refer to an arbitrary functor $F : \text{SM}^\text{op} \to \text{Set}$ as a generalized super manifold.

Note that Proposition 5 makes it easy to describe the morphism sets $\text{SM}(S, M)$. We’d also like to point out that this functor of points approach is closely related to computations involving additional odd quantities (the odd coordinates of $S$ as opposed to those of $M$) in many physics papers.
3. The Superpoints in a Supermanifold

For a supermanifold $X$, we would like to talk about the supermanifold $\text{SP}X$ of superpoints in $X$. By definition, this is the inner Hom from the superpoint $\mathbb{R}^{0|1}$ to $X$ in the category $\text{SM}$ of supermanifolds, usually denoted by $\text{SP}X$, compare Remark 45.

More generally, for any supermanifold $M \in \text{SM}$, we can consider the inner Hom $\text{SM}(M, X)$ as a generalized supermanifold given by

$$\text{SM}(M, X)(S) := \text{SM}(S \times M, X) \quad \text{for } S \in \text{SM}$$

It is clear that if the dimensions of $M_{\text{red}}$ and $X$ are nonzero, this functor is not representable, at least not by finite dimensional supermanifold that we are studying here. However, it turns out that for $M = \mathbb{R}^{0|n}$ it actually is in the image of the Yoneda embedding

$$Y : \text{SM} \rightarrow \text{Fun}(\text{SM}^{\text{op}}, \text{Set}), \quad Y(M) = (S \mapsto \text{SM}(S, M)).$$

The following proposition will prove the case $n = 1$, the other cases follow by induction.

**Proposition 6.** For any supermanifold $X$, the odd tangent bundle $\Pi TX$ represents the inner Hom $\text{SM}(\mathbb{R}^{0|1}, X) = \text{SP}X$. More precisely, there is an isomorphism of generalized supermanifolds

$$(T \mapsto \text{SP}X(T)) \cong (T \mapsto \Pi TX(T))$$

In the following, we will think of the superpoints in $X$ as a supermanifold, i.e. $\text{SP}X \in \text{SM}$ and will ignore the fact that it is actually only a generalized supermanifold.

**Proof.** We split the proof of the desired bijection into the following natural correspondences, where in (3) $\text{Der}_f$ denotes derivations $C^\infty(X) \rightarrow C^\infty(S)$ with respect to $f$, in the sense that $C^\infty(S)$ is a $C^\infty(X) - C^\infty(X)$-bimodule using the algebra homomorphism $f$.

$$\text{SM}(S \times \mathbb{R}^{0|1}, X) \leftrightarrow_{(1)} \text{Alg}(C^\infty(X), C^\infty(S \times \mathbb{R}^{0|1}))$$

$$\leftrightarrow_{(2)} \text{Alg}(C^\infty(X), C^\infty(S) \otimes \Lambda^*(\mathbb{R}))$$

$$\leftrightarrow_{(3)} \{(f, g) \mid f \in \text{Alg}(C^\infty(X), C^\infty(S)), g \in \text{Der}_f^{\text{odd}}\}$$

$$\leftrightarrow_{(4)} \text{SM}(S, \Pi TX)$$

In (3) $\text{Der}_f$ denotes derivations $g : C^\infty(X) \rightarrow C^\infty(S)$ with respect to $f$ in the sense that

$$g(ab) = g(a)f(b) + (-1)^{|g||a|}f(a)g(b)$$

In other words, these are odd sections of the pulled back tangent bundle along $f$.

(1) follows directly from proposition 5 and (2) just uses the definition of products of supermanifolds together with $C^\infty(\mathbb{R}^{0|1}) = \Lambda^*(\mathbb{R})$. To see (3), decompose $\varphi : C^\infty(X) \rightarrow C^\infty(S) \otimes \Lambda^*(\mathbb{R}) = C^\infty(S)[\theta]$ as a sum

$$\varphi = f + \theta g,$$

with $f, g : C^\infty(X) \rightarrow C^\infty(S)$.

Here $\theta$ is the usual odd coordinate on $\mathbb{R}^{0|1}$. Note that $f$ preserves the grading, whereas $g$ reverses it. For $a, b \in C^\infty(X)$ we have $\varphi(ab) = f(ab) + \theta g(ab)$, and since $\varphi$ is an algebra homomorphism this is also equal to

$$\varphi(a)\varphi(b) = (f(a) + \theta g(a))(f(b) + \theta g(b))$$

$$= f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|}f(a)g(b)).$$
Comparing the coefficients we conclude that $f$ is an algebra homomorphism and that $g$ is an odd derivation with respect to $f$. Conversely, any such pair $(f, g)$ defines an algebra map $\varphi$. It is clear that the bijection is natural with respect to superalgebra maps $C^\infty(S) \to C^\infty(S')$.

Proposition 5 translates $f$ into a morphism $\hat{f} : S \to X$ in (4). Then $g$ is taken to a global section $\hat{g}$ of the sheaf $f^*(TX)^{\text{odd}}$. This pair $(\hat{f}, \hat{g})$ is an $S$-point of the supermanifold $\Pi TX$. The last statement holds more generally: Any vector bundle over $M$ (aka a locally free and finitely generated sheaf $\mathcal{E}$ of $O_M$-modules) has a total space $E \in \text{SM}$ that comes with a projection map $\pi : E \to M$. It can be most easily described in terms of its $S$-points

$$E(S) = \{(f, g) \mid f \in \text{SM}(S, M), g \in f^*(\mathcal{E}^\text{ev})\}$$

So $g$ is an even global section of the pullback bundle on $S$ and the projection $\pi$ comes from forgetting this datum. If we reverse the parity of $\mathcal{E}$ by tensoring fibrewise with $\mathbb{R}^{0,1}$, we obtain the sheaf $\Pi \mathcal{E}$ with total space $\Pi E$ determined by its $S$-points

$$\Pi E(S) = \{(f, g) \mid f \in \text{SM}(S, M), g \in f^*(\mathcal{E}^{\text{odd}})\}$$

This finishes the proof of the proposition. We would like to point out that Proposition 5 is not crucial for the proof. One can write down the equivalences in terms of maps of sheaves (instead of their restriction to global sections), the only thing that changes is that the notation becomes more complicated. 

Let us write down the above natural bijection more explicitly for superdomains $X = U \subseteq \mathbb{R}^{p|q}$. Let $y_1, \ldots, y_{p+q}$ be coordinate functions on $X$, where $y_1, \ldots, y_p$ are even and $y_{p+1}, \ldots, y_{p+q}$ are odd. Then a morphism $\varphi : S \times \mathbb{R}^{0,1} \to U$ is given by coordinates

$$\varphi(y_i) = (x_1 + \theta \hat{x}_1, \ldots, x_p + \theta \hat{x}_p, \eta_1 + \theta \hat{\eta}_1, \ldots, \eta_q + \theta \hat{\eta}_q) \in (O_S^{\text{ev}})^p \times (O_S^{\text{odd}})^q$$

In this case we can make the identification $\Pi TU \cong U \times \mathbb{R}^{q|p}$ with coordinates $(y_i, \hat{y}_i)$ where $\hat{y}_1, \ldots, \hat{y}_p$ are odd and $\hat{y}_{p+1}, \ldots, \hat{y}_{p+q}$ are even. Going through the above bijections one sees that the image $\hat{\varphi} : S \to U \times \mathbb{R}^{q|p}$ of the morphism $\varphi$ has coordinates

$$\hat{\varphi} \varphi(y_i, \hat{y}_i) = (x_1, \ldots, x_p, \eta_1, \ldots, \eta_q, \hat{x}_1, \ldots, \hat{x}_p) \in (O_S^{\text{ev}})^p \times (O_S^{\text{odd}})^q \times (O_S^{\text{ev}})^q \times (O_S^{\text{odd}})^p$$

3.1. The Translation Action of $\mathbb{R}^{0,1}$. Addition gives $\mathbb{R}^{0,1}$ a super Lie group structure which we denote by $m : \mathbb{R}^{0,1} \times \mathbb{R}^{0,1} \to \mathbb{R}^{0,1}$. We get a right action of $\mathbb{R}^{0,1}$ on itself and a left group action $\mu_0 : \mathbb{R}^{0,1} \times \text{SPX} \to \text{SPX}$ by pre-composing with right translations: To an $S$-point

$$(\eta, \varphi) \in \text{SM}(S, \mathbb{R}^{0,1}) \times \text{SM}(S \times \mathbb{R}^{0,1}, X) \cong \text{SM}(S, \mathbb{R}^{0,1} \times \text{SPX})$$

$\mu_0(S)$ associates the composition $\varphi_\eta \in \text{SM}(S \times \mathbb{R}^{0,1}, X) \cong \text{SM}(S, \text{SPX})$ given by

$$\varphi_\eta : S \times \mathbb{R}^{0,1} \xrightarrow{id \times m} (S \times \mathbb{R}^{0,1}) \times \mathbb{R}^{0,1} = S \times (\mathbb{R}^{0,1} \times \mathbb{R}^{0,1}) \xrightarrow{id \times m} S \times \mathbb{R}^{0,1} \xrightarrow{\varphi} X$$

The infinitesimal generator of this $\mathbb{R}^{0,1}$-action is a globally defined odd vector field $D$ on $\text{SPX} \cong \Pi TX$. Since $\mathbb{R}^{0,1}$ is commutative, we have $D^2 = \frac{1}{2}[D, D] = 0$ (which is not always true for odd vector fields). We next describe $D$ in local coordinates for superdomains $X = U \subseteq \mathbb{R}^{p|q}$.

It is easy to see that $\varphi_\eta$ is given by replacing $\theta$ by $\theta + \eta$ in the coordinate representation $(\varphi)$ above. Translating from $\text{SPX}$ to $\Pi TX$, this action becomes in coordinates for $\hat{\varphi}$:

$$(\eta, x_1, \ldots, \eta_q, \hat{x}_1, \ldots, \hat{x}_p) \mapsto (x_1 + \eta \hat{x}_1, \ldots, x_q + \eta \hat{\eta}_q, \hat{x}_1, \ldots, \hat{x}_p).$$

The fact that

$$x_i = \varphi^*(y_i), \ \hat{x}_i = \varphi^*(\hat{y}_i), \ i = 1, \ldots, p \quad \text{and} \quad \eta_i = \varphi^*(y_{p+i}), \ \hat{\eta}_i = \varphi^*(\hat{y}_{p+i}), \ i = 1, \ldots, q$$
together with formula (*) above implies that the action map $\mu_0 : \mathbb{R}^{0|1} \times \Pi TX \rightarrow \Pi TX$ pulls back these coordinate functions as follows:

$$\mu_0^*(y_i) = y_i + \eta \hat{y}_i \quad \text{and} \quad \mu_0^*(\hat{y}_i) = \hat{y}_i \quad \text{for} \quad i = 1, \ldots, p + q$$

Abusing notation, $\eta$ here denotes the standard odd coordinate function on $\mathbb{R}^{0|1}$. To get the infinitesimal generator $D$ for the action, we have to differentiate this formula with respect to $\eta$ and evaluate at $\eta = 0$. This gives $D(y_i) = \hat{y}_i$ and $D(\hat{y}_i) = 0$. Using $\partial_i := \frac{\partial}{\partial y_i}$, the local representation of our odd vector field $D$ is therefore given as the derivation

$$D = \sum_{i=1}^{p+q} \hat{y}_i \partial_i$$

3.2. The de Rham Complex for Supermanifolds. For a supermanifold $X$, the algebra of differential forms on $X$ has two gradings, the $\mathbb{Z}/2$-parity and the (cohomological) $\mathbb{Z}$-degree. There are two conventions how to deal with this situation, we will work with one that makes $\Omega^* X$ into a $\mathbb{Z}$-graded commutative superalgebra and leads to an odd de Rham differential $d$. This seems to be a natural choice, since we want to relate $d$ to the action of the odd vector field $D$ on the commutative superalgebra $C^\infty(\Pi TX)$.

Let $V$ be a module over the commutative superalgebra $A$. Following the convention of Bernstein-Leites we define the exterior algebra on $V$ to be

$$\Lambda_A^*(V) := \text{Sym}_A(\Pi V).$$

Here $\text{Sym}_A(W)$ is the quotient of the tensor algebra on $W$ by the ideal generated by all supercommutators $w_1 \otimes w_2 - (-1)^{\|w_1\|\|w_2\|} w_2 \otimes w_1$. The commutative superalgebra $\Lambda_A^*(V)$ has the universal property that giving a superalgebra map from $\Lambda_A^*(V)$ to any commutative $A$-superalgebra $B$ is the same as giving an $A$-module map $\Pi V \rightarrow B$.

Let $\Omega^1 X := \text{Hom}_{\mathcal{O}_X} (T X, \mathcal{O}_X)$ be the cotangent sheaf of $X$ and define

$$\Omega^* X := \Lambda_{\mathcal{O}_X}^*(\Omega^1 X).$$

Clearly, $\mathcal{O}_X$ and $\Pi \Omega^1 X$ are subsheaves of $\Lambda_{\mathcal{O}_X}^*(\Omega^1 X)$ in a natural way. The universal even differential $d_{ev} : \mathcal{O}_X \rightarrow \Omega^1 X$ is characterized by

$$d_{ev} f(\xi) = (-1)^{\|f\|\|\xi\|} \xi(f) \quad \text{for all} \quad \xi \in TX$$

Alternatively, we can think of this as an odd differential $d : \mathcal{O}_X \rightarrow \Pi \Omega^1 X$. The de Rham differential on $\Omega^* X$ is the extension of $d$ whose square is zero and which satisfies the Leibniz rule.

We will denote the global sections of the sheaf $\Omega^* X$ as usual by $\Omega^* X$, (non-bold).

3.3. Differential Forms as Functions. The next step is to interpret differential forms on $X$ as certain functions on $\Pi TX$. We thank the referee for pointing out that functions on $\Pi TX$ are also called pseudodifferential forms on $X$. This notation is motivated by the following well known result.

**Lemma 7.** There is an embedding of sheaves of $\mathcal{O}_X$-superalgebras

$$i : \Omega^* X \hookrightarrow \mathcal{O}_{\Pi TX}$$

that maps onto the functions that are polynomial on every fiber.
Proof. Let $x_1, \ldots, x_p, \eta_1, \ldots, \eta_q$ be local coordinates on $X$. We have canonically associated coordinates
\[(x_1, \ldots, x_p, \eta_1, \ldots, \eta_q, \hat{x}_1, \ldots, \hat{x}_p)\]
on $\Pi TX$. Recall that the $\hat{x}_i$'s are odd, whereas the $\hat{\eta}_j$'s are even. On the other hand, a local basis for the $\mathcal{O}_X$-module $\Omega^1 X$ is given by $dx_1, \ldots, d\eta_q$. According to the convention we picked for the definition of the de Rham complex, the $dx_i, d\eta_j \in \Pi \Omega^1 X$ have odd and even parity, respectively. Hence we can define a map of super $\mathcal{O}_X$-modules $\iota_0 : \Pi \Omega^1 X \to \mathcal{O}_{\Pi TX}$ by prescribing $dx_i \mapsto \hat{x}_i$ and $d\eta_j \mapsto \hat{\eta}_j$. It is not hard to check that this is independent of the coordinate system chosen.\footnote{In fact, one can see this using the (global!) vector field $D$ considered in Section 3.1: The map $\iota_0$ is equal to the composition $\Pi \Omega^1 X \xrightarrow{\Pi} \Omega^1 X \to \Omega^1 (\Pi TX) \xrightarrow{D} \mathcal{O}_{\Pi TX}$, where $D(\omega) = (-1)^{|\omega|} \omega(D)$.}

Hence we can define a map of super $\mathcal{O}_X$-modules $\iota : \Omega^* X \to \mathcal{O}_{\Pi TX}$. It is clear that $\iota$ is injective with image as stated above.

The map $\iota$ is surjective if and only if $X$ is an ordinary manifold. For example, if $X = \mathbb{R}^{0|q}$ then $\Omega^* X = \Lambda(\mathbb{R}^q)[x_1, \ldots, x_q]$, the polynomial ring on $q$ even generators $x_i$ over the ground ring $\Lambda(\mathbb{R}^q)$. It has to be completed in the $x_i$-directions to obtain
\[C^\infty(\Pi TX) = C^\infty(\mathbb{R}^{0|q}) = \Lambda(\mathbb{R}^q) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^q)\]

Since $D$ is an odd vector field on $\Pi TX$ we have the Leibniz rule
\[D(fg) = (Df)g + (-1)^{|f|} f(Dg)\]
for all functions $f, g$ on $\Pi TX$. Furthermore, we already know that $D^2 = 0$. Hence the restriction of $D$ to $\Omega^* X$ is the de Rham differential once we have shown

**Lemma 8.** The restriction of $D$ to $\mathcal{O}_X \subset \mathcal{O}_{\Pi TX}$ is the odd differential $d$. More precisely, we have
\[D = \iota d : \mathcal{O}_X \to \mathcal{O}_{\Pi TX}\]

Proof. It is clear from the local representation of $D$ that the image of $D$ is contained in $\iota(\Pi \Omega^1 X) \subset \mathcal{O}_{\Pi TX}$. The claim is equivalent to showing that the composition
\[\tilde{D} := \iota^{-1} D : \mathcal{O}_X \to \Omega^1 X\]
is equal to $d$, i.e. for all $f \in \mathcal{O}_X$ we have to check that
\[\tilde{D} f(\xi) = (-1)^{|f||\xi|} \xi(f)\]
for all vector fields $\xi \in TX$.

It suffices to prove this for (local) basis vector fields $\partial_j = \frac{\partial}{\partial y_j}$, $j = 1, \ldots, p + q$, where the $y_i$ are local coordinates on $X$. We first compute
\[\tilde{D} f = \iota^{-1} \left( \sum_{i=1}^{p+q} \hat{y}_i \partial_i f \right) = \iota^{-1} \left( \sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f) \hat{y}_i \right)\]

Since $\iota$ is even, we get
\[\tilde{D} f = \left( \sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f) dy_i \right) = \sum_{i=1}^{p+q} (-1)^{|y_i|(|f|+1)} (\partial_i f) dy_i\]

Therefore, $\tilde{D} f = D f$.
Applying this 1-form to \( \partial_j \) and using \((dy_i)\partial_j = (-1)^{|y_i||y_j|}\delta_{ij}\) yields
\[
\tilde{D}f(\partial_j) = (-1)^{|y_j|(|f|+1)+|y_j|} = (-1)^{|y_j||f|} \partial_i(f)
\]
as desired. \(\square\)

If \( X \) is purely even the cohomological degree of \( \alpha \in \Omega^*X \) is equal to the parity of its image in \( C^\infty \Pi TX \) modulo 2. Hence the Leibniz rule above is exactly the (graded) Leibniz rule for differential forms, and so \( D \) is equal to the usual de Rham differential on \( \Omega^*X \).

3.4. The Diffeomorphism Group of the Superpoint. We used the translation action of \( \mathbb{R}^{0|1} \) on itself to define an action of \( \mathbb{R}^{0|1} \) on the superpoints \( \Pi TX \cong \Pi TX \). In fact, the whole super Lie group \( \text{Diff}(\mathbb{R}^{0|1}) \) of diffeomorphisms of the supermanifold \( \mathbb{R}^{0|1} \) acts on \( \Pi TX \).

We briefly describe this action. By definition, \( \text{Diff}(\mathbb{R}^{0|1}) \) is the super Lie group representing the group-valued functor
\[
S \mapsto \text{Diff}_S(\mathbb{R}^{0|1} \times S, \mathbb{R}^{0|1} \times S).
\]

Here \( \text{Diff}_S(\mathbb{R}^{0|1} \times S, \mathbb{R}^{0|1} \times S) \) is the group of diffeomorphisms of \( \mathbb{R}^{0|1} \times S \) that are compatible with the projection to \( S \). The following result follows from a short computation together with Proposition 6 for \( X = \mathbb{R}^{0|1} \).

**Lemma 9.** There is an isomorphism of (generalized) super Lie groups
\[
\text{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}
\]
where the semi-direct product is defined by the right action of \( \mathbb{R}^\times \) on \( \mathbb{R}^{0|1} \), given by scalar multiplication.

The right action \( \mathbb{R}^{0|1} \times (\mathbb{R}^\times \ltimes \mathbb{R}^{0|1}) \rightarrow \mathbb{R}^{0|1} \) is on \( S \)-points given by
\[
(\theta, (a, \eta)) \mapsto \theta a + \eta \quad \text{where } a \in C^\infty(S)^{ev} \text{ and } \theta, \eta \in C^\infty(S)^{odd}.
\]

It follows that in our local coordinates \((y_i, \hat{y}_i)\) for \( \Pi TX \) from Section 3.1, the action of \( \lambda \in \mathbb{R}^\times \) is given by fixing the \( y_i \) and multiplying each \( \hat{y}_i \) by \( \lambda \).

**Remark 10.** In Lemma 9, we use the following convention for the \((S\)-points of the\) semi-direct product. If a group \( G \) acts on another group \( A \) on the right, written as \((a, g) \mapsto a^g\), elements of \( G \ltimes A \) are just pairs \((g, a)\) with (associative) multiplication
\[
(g_1, a_1) \cdot (g_2, a_2) := (g_1g_2, (a_1)^{g_2}a_2)
\]

In Remark 14 it will be useful to consider the case of a left action of \( G \) on \( A \), defined by \( g(a) := a^g^{-1} \). In particular, this will introduce an inverse for the scalar multiplication of \( \mathbb{R}^\times \).

Every function \( f \) on \( \Pi TX \) which is polynomial on fibers is locally a finite sum of functions of the form
\[
f = g\hat{y}_1^{i_1}...\hat{y}_{p+q}^{i_{p+q}}, \quad \text{where } g \in \mathcal{O}_X.
\]

It follows that the action of \( \lambda \in \mathbb{R}^\times \) on such an \( f \) is given by the formula
\[
(\lambda, f) \mapsto \lambda^{\sum_{k=1}^{p+q} i_k} f
\]
Conversely, if a function \( f \in \mathcal{O}_{\Pi TX} \) has degree \( n \) in the sense that
\[
(\lambda, f) \mapsto \lambda^n f \quad \forall \lambda \in \mathbb{R}^\times
\]
then \( f \in \Omega^n X \subset \mathcal{O}_{\Pi TX} \) must be homogenous of degree \( n \) along the fibres.
Corollary 11. The $\mathbb{R}^x$-action on $O_{\Pi TX}$ coming from dilations of the superpoint determines the $\mathbb{Z}$-degree operator and vice versa. More precisely, all $\lambda \in \mathbb{R}^x$ map $f \in O_{\Pi TX}$ to $\lambda^nf$ if and only if $f \in \Omega^nX$.

In the proof of Proposition 35 below we will need the following reformulations of the above computations.

Lemma 12. Let $\mu_0 : \mathbb{R}^{0|1} \times \text{SPX} \to \text{SPX}$ be the left action given by pre-composition with right translation as in Section 3.1. Then the induced action on functions is given by

$$\mu_0^*(f) = 1 \otimes f + \eta \otimes D(f) \in C^\infty(\mathbb{R}^{0|1} \times \text{SPX}) = \Lambda^*[\eta] \otimes C^\infty(\text{SPX})$$

for all $f \in C^\infty(\text{SPX})$.

Proof. Let $\eta$ be the standard coordinate on $\mathbb{R}^{0|1}$ and $D$ the vector field on $\Pi TX \cong \text{SPX}$ infinitesimally giving the action of the super Lie algebra of $\mathbb{R}^{0|1}$ on $C^\infty(\Pi TX)$. Using our explicit (local) coordinate representations of $\mu_0$ and $D$ in Section 3.1, the asserted equality is trivial to verify when $f$ is equal to the local coordinates $y_i$ and $\hat{y}_i$ of $\Pi TX$.

Furthermore, using $\eta^2 = 0$ and that $D$ is an odd derivation, it is easy to check that the right hand side of the asserted equality defines an algebra homomorphism from $C^\infty(\Pi TX)$ to $C^\infty(\mathbb{R}^{0|1} \times \Pi TX)$. Since the coordinates $y_i$ and $\hat{y}_i$ locally generate $O_{\Pi TX}$, it follows that $\mu_0^*$ and this algebra homomorphism are equal. Hence the asserted equality holds for all $f \in C^\infty(\Pi TX)$.

Proposition 13. Let $\mu : \text{Diff}(\mathbb{R}^{0|1}) \times \text{SPX} \to \text{SPX}$ be the left action of $\text{Diff}(\mathbb{R}^{0|1})$ on $\text{SPX}$ induced by the right action of $\text{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^x \times \mathbb{R}^{0|1}$ on $\mathbb{R}^{0|1}$. Then

$$\mu^*(f) = t^n \otimes 1 \otimes f + t^n \otimes \eta \otimes D(f) \in C^\infty(\mathbb{R}^x) \otimes \Lambda^*[\eta] \otimes C^\infty(\text{SPX})$$

for all $f \in \Omega^nX \subset C^\infty(\Pi TX) \cong C^\infty(\text{SPX})$. Here $t \in C^\infty(\mathbb{R}^x)$ is the standard coordinate coming from the inclusion $\mathbb{R}^x \subset \mathbb{R}$.

Proof. By Corollary 11, the dilation action

$$\nu : \mathbb{R}^x \times \text{SPX} \to \text{SPX}$$

induces on functions $f \in \Omega^n(X)$ the action $\nu^*(f) = t^n \otimes f$, where $t \in C^\infty(\mathbb{R}^x)$ is the standard coordinate. By Remark 14, our action map $\mu$ can be written as a composition

$$\mathbb{R}^x \times (\mathbb{R}^{0|1} \times \text{SPX}) \xrightarrow{\text{id} \times \mu_0} \mathbb{R}^x \times \text{SPX} \xrightarrow{\nu} \text{SPX}$$

It follows that for $f \in \Omega^n(X)$ one has

$$\mu^*(f) = (\nu \circ (\text{id} \times \mu_0))^*(f) = (\text{id} \times \mu_0)^*(t^n \otimes f) = t^n \otimes (1 \otimes f + \eta \otimes D(f))$$

which proves our claim.

Remark 14. In the above proof, we have used the following elementary fact about left actions of a semi-direct product $G \rtimes A$ on a set $Y$: A $G$-action and an $A$-action on $Y$ fit together to an action of $G \rtimes A$ if and only if the $A$-action map $A \times Y \to Y$ is $G$-equivariant. Here we assume that $G$ acts on the left on $A$ and on $Y$ and hence it acts on $A \times Y$ diagonally. This observation uses the conventions from Remark 10 for semi-direct products. In particular, the left action map

$$\delta : \mathbb{R}^x \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$$
is determined by $\delta^*(\eta) = t^{-1} \otimes \eta$ since it comes from the right action given by scalar multiplication, see Lemma 9. Then the above compatibility condition for the $\mathbb{R}^{0|1}$- and $\mathbb{R}^\times$-actions comes from the commutative diagram

$$
\begin{array}{c}
\mathbb{R}^\times \times (\mathbb{R}^{0|1} \times \text{SP}X) \\
\downarrow \text{id} \times \mu_0
\end{array} \xrightarrow{\Delta_{13}} \begin{array}{c}
(\mathbb{R}^\times \times \mathbb{R}^{0|1}) \times (\mathbb{R}^\times \times \text{SP}X) \\
\downarrow \mu_0
\end{array} \xrightarrow{\delta \times \nu} \mathbb{R}^{0|1} \times \text{SP}X
$$

which we leave for the reader to check.

4. **Topological Field Theories**

The usual definition of a $d$-dimensional TFT, going back to Atiyah and Segal, is in terms of a symmetric monoidal functor $E : d\mathcal{B} \to \text{Vect}$ or shorter

$$E \in \text{Fun}^\otimes(d\mathcal{B}, \text{Vect}).$$

The domain category $d\mathcal{B}$ is the bordism category whose objects are closed $(d-1)$-manifolds and whose morphisms are diffeomorphism classes of compact $d$-dimensional bordisms. The target of the functor is the category $\text{Vect}$ of finite dimensional vector spaces. The symmetric monoidal structures are given by disjoint union respectively tensor product. In [ST2] we explain a version of this definition using internal categories in which one can easily add several bells and whistles, for example geometry, supersymmetry and a notion of degree. We also describe what these definitions mean in dimension 0 and 0$|1$, which we shall summarize in the coming subsections.

One obvious simplification in these smallest possible dimensions is that the empty set is the only manifold of dimension $(-1|\delta)$. This implies that our language of internal categories in [ST2] can be reduced to ordinary categories which we shall stick to in this paper. Instead of working exclusively in dimension 0 and 0$|1$, however, we shall explain the part of our work that can be formulated in terms of categories alone. In any dimension, this is the part given by restricting attention to the empty set as the only relevant $(d-1|\delta)$-manifold.

**Definition 15.** Consider the categories $d\mathcal{B}_c$ (respectively $d\mathcal{B}_{cc}$) with objects closed (respectively closed, connected, non-empty) $d$-manifolds and morphisms being diffeomorphisms. The disjoint union operation makes $d\mathcal{B}_c$ into a symmetric monoidal category.

The subscripts ‘c’ respectively ‘cc’ stand for closed respectively closed connected manifolds but also for category. Notice that unlike for $d\mathcal{B}$, we are not considering diffeomorphism classes of manifolds but keep track of the diffeomorphisms as morphisms. This will be essential for supersymmetric and twisted field theories discussed below. We next point out a lemma that shows how one can simplify the discussions related to the symmetric monoidal structure. It follows from the fact that any compact manifold is the disjoint union of connected manifolds and that any diffeomorphism is uniquely determined by its restriction to connected components.

**Lemma 16.** For any symmetric monoidal category $\mathcal{C}$, there is an equivalence of functor categories

$$\text{Fun}^\otimes(d\mathcal{B}_c, \mathcal{C}) \simeq \text{Fun}(d\mathcal{B}_{cc}, \mathcal{C}).$$

In other words, $d\mathcal{B}_c$ is the free symmetric monoidal groupoid generated by its subcategory $d\mathcal{B}_{cc}$. Note that manifolds in $d\mathcal{B}_{cc}$ are assumed to be non-empty so that the monoidal unit does not lie in this subcategory.
For example, such a functor arises from a \(d\)-dimensional TFT by restricting it to the empty \((d - 1)\)-manifold (and hence to closed \(d\)-manifolds) and taking \(C := \mathbb{R}\), our chosen ground field, considered as a monoid via multiplication and as a discrete category, i.e. a category with identity morphisms only.

This is the well known observation that in the top dimension \(d\), TFTs give multiplicative diffeomorphism invariants of closed manifolds. Given \(E \in \text{Fun}(d-\mathcal{Bc}, \mathcal{C})\), it is the locality properties of \(E\) that tell whether it can be extended to a full fledged TFT. The Atiyah-Segal axioms address the codimension 1 gluing laws and higher codimensions can be handled by using \(d\)-categories, an aspect that’s very important but not relevant for the current paper.

4.1. 0-dimensional TFTs. Starting with the Atiyah-Segal definition and observing that there is only one \((-1)\)-dimensional manifold, \(\emptyset\), we have

\[
0\text{-TFT} := \text{Fun}^\otimes(0\mathcal{B}, \mathcal{Vect}) \cong \text{Maps}^\otimes(0\mathcal{B}(\emptyset, \emptyset), \mathbb{R})
\]

Here we have used that the monoidal unit \(\emptyset \in d\mathcal{B}\) has to be sent to the monoidal unit \(\mathbb{R} \in \mathcal{Vect}\), up to canonical isomorphism. By definition \(0\mathcal{B}(\emptyset, \emptyset)\) are the isomorphism classes of objects in \(0\mathcal{B}_c\) and since the category \(\mathbb{R}\) has only identity morphisms it follows that there are bijections

\[
0\text{-TFT} \cong \text{Maps}^\otimes(0\mathcal{B}(\emptyset, \emptyset), \mathbb{R}) \cong \text{Fun}^\otimes(0\mathcal{B}_c, \mathbb{R}) \cong \text{Fun}(0\mathcal{B}_{cc}, \mathbb{R})
\]

where the right most bijection follows from Lemma 16. Since the point has no non-trivial diffeomorphisms, it follows that \(0\mathcal{B}_{cc} = \{\text{pt}\}\) and hence we conclude

**Lemma 17.** There is a bijection

\[
0\text{-TFT} \cong \text{Fun}(0\mathcal{B}_{cc}, \mathbb{R}) \cong \text{Fun}(\text{pt}, \mathbb{R}) \cong \mathbb{R}
\]

sending a TFT \(E\) to the real number \(E(\text{pt})\).

Graeme Segal also introduced the notion of a field theory over a manifold \(X\) as a symmetric monoidal functor \(E : d\mathcal{B}(X) \to \mathcal{Vect}\) or

\[
E \in \text{Fun}^\otimes(d\mathcal{B}(X), \mathcal{Vect}).
\]

Here one replaces the domain category \(d\mathcal{B}\) by \(d\mathcal{B}(X)\) where both objects and bordisms are equipped with a smooth map to \(X\). Arguing exactly as above one sees that

\[
\text{Fun}^\otimes(0\mathcal{B}(X), \mathcal{Vect}) \cong \text{Fun}(0\mathcal{B}_{cc}(X), \mathbb{R}) \cong \text{Maps}(X, \mathbb{R})
\]

because \(0\mathcal{B}_{cc}(X) \cong X\) (as discrete categories). It follows that TFTs over \(X\) would be all real valued maps on \(X\). Note that there is no smoothness or continuity requirement on these functions! Our main contribution is to implement smoothness in such a way that it generalizes to

- higher dimension and to
- supersymmetry

For this purpose, we introduce in [ST2] family versions of all relevant categories and require that the functors extend in a natural way to these family versions. We’ll now explain these smooth families of manifolds in a way that can easily be extended to various other settings, in particular to super families of supermanifolds.
4.2. Smooth 0-dimensional TFTs over $X$.

**Definition 18.** The category $d\text{-}B_c^{\text{fam}}$ (respectively $d\text{-}B_{cc}^{\text{fam}}$) has objects smooth fibre bundles $Y \to S$ where the fibres are closed (respectively closed, connected, non-empty) $d$-manifolds. This is by definition a smooth $S$-family of such $d$-manifolds.

There are also categories $d\text{-}B_c(X)^{\text{fam}}$ (respectively $d\text{-}B_{cc}(X)^{\text{fam}}$) whose objects include in addition a smooth map $Y \to X$. In all cases, morphisms are smooth bundle maps

$$
\begin{array}{ccc}
Y' & \xrightarrow{\phi} & Y \\
\downarrow & & \downarrow \\
S' & \xrightarrow{f} & S
\end{array}
$$

that are fibrewise diffeomorphisms and commute with the map to $X$ (if present). Note that there are projection functors $d\text{-}B_c(X)^{\text{fam}} \to \text{Man}$ that take a bundle $Y \to S$ and sends it to its base (or parameter space) $S$.

Fortunately, there is already a very well developed language that deals with fibred categories such as $d\text{-}B_c(X)^{\text{fam}} \to \text{Man}$, going back to at least Grothendieck. So we borrow some language from algebraic geometry introduced for dealing with families of schemes and import them to manifolds and supermanifolds, see our Appendix for a quick survey.

Using Lemma 23 in the case $d = 0$ it follows that the following definition agrees with that given in [ST2] for arbitrary $d$. It is a much simplified version, for example the symmetric monoidal structure plays no role. In this paper we decided to give the simplest possible definitions and prove only later that they agree with the ones in arbitrary dimension.

**Definition 19.** A smooth 0-dimensional TFT over $X$ is a fibred functor into the representable fibred category $\mathbb{R}$, see Definition 43:

$$
\begin{array}{ccc}
0\text{-}B_{cc}(X)^{\text{fam}} & \xrightarrow{E} & \mathbb{R} \\
\downarrow & & \downarrow \\
\text{Man} & \xrightarrow{\text{Fun}_{\text{Man}}(0\text{-}B_{cc}(X)^{\text{fam}}, \mathbb{R})} & \\
& & \text{Fun}_{\text{Man}}(0\text{-}B_{cc}(X)^{\text{fam}}, \mathbb{R})
\end{array}
$$

In the notation introduced in the Appendix, we define actually the following category

$$0\text{-}\text{TFT}(X) := \text{Fun}_{\text{Man}}(0\text{-}B_{cc}(X)^{\text{fam}}, \mathbb{R})$$

However, by Lemma 42 this is indeed the set of 0-dimensional TFT’s over $X$.

There is an equivalence of fibred categories $0\text{-}B_{cc}(X)^{\text{fam}} \simeq X$ over $\text{Man}$ because any bundle $Y \to S$ with fibres a single point must be a diffeomorphism and only the map $S \to X$ remains as a datum. By the Yoneda lemma we end up with the desired result:

**Lemma 20.** There is a bijection between smooth 0-dimensional TFTs over $X$ and smooth functions:

$$0\text{-}\text{TFT}(X) \cong \text{Fun}_{\text{Man}}(X, \mathbb{R}) \cong \text{Man}(X, \mathbb{R}) \cong C^\infty(X; \mathbb{R})$$

Since any smooth function on $X \times \{0, 1\}$ can be extended to a smooth function on $X \times [0, 1]$ it follows that no cohomological information can be derived from 0-dimensional TFTs over $X$:

**Corollary 21.** There is a single concordance class of smooth 0-dimensional TFTs over $X$. In other words, $0\text{-}\text{TFT}[X] = 0$. 
Surprisingly, this changes as soon as we introduce one odd dimension which we shall do in the next subsection.

**Remark 22.** Restricting to closed, connected $d$-manifolds gives a functor

$$d\text{-TFT}(X) \to \text{Fun}_{\text{Man}}(d\text{-B}^{\text{cc}}(X)^{\text{fam}}, \mathbb{R})$$

where the left hand side is the category of smooth TFTs over $X$ in the sense of [ST2]. The same remark holds in the conformal setting and in fact the image of the analogous functor evaluated on the moduli stack of tori gives the partition function of the CFT.

For the careful reader we’d like to address the following subtlety. In Definition 19 we could have used disconnected 0-manifold fibres to obtain a symmetric monoidal family bordism category over Man which is in fact closer to the definition given in [ST2]. The following parametrized version of Lemma 16 shows that the outcome would not have been different because $\mathbb{R}$ with multiplication is a symmetric monoidal stack. Since this lemma only serves to justify Definition 19 above, we will not be overly careful in explaining the stacky notions used in the proof.

**Lemma 23.** If $C \to \text{Man}$ is a symmetric monoidal stack in the sense of Definition 61 then there is an equivalence of categories

$$\text{Fun}_{\text{Man}}(d\text{-B}^{\text{cc}}(X)^{\text{fam}}, C) \simeq \text{Fun}_{\text{Man}}^\otimes (d\text{-B}^{\text{cc}}(X)^{\text{fam}}, C)$$

**Proof.** We start with the observation that $d\text{-B}^{\text{cc}}(X)^{\text{fam}}$ is the stackification of a much simpler fibred category, namely $d\text{-B}^{\text{cc}}(X)^{\text{prfam}}$, whose objects consists of product families only. That is to say, all total spaces are of the form $Y = S \times F$ where $F$ is a closed, connected, non-empty $d$-manifold. Stackification is left adjoint to the forgetful functor (from stacks to prestacks), so since $C$ is a stack by assumption, there is an equivalence of categories

$$\text{Fun}_{\text{Man}}(d\text{-B}^{\text{cc}}(X)^{\text{fam}}, C) \simeq \text{Fun}_{\text{Man}}(d\text{-B}^{\text{cc}}(X)^{\text{prfam}}, C)$$

The symmetric monoidal structure on the fibred category $C \to \text{Man}$ gives a fibred functor

$$\otimes : C \times_{\text{Man}} C \to C$$

In the case of $d\text{-B}^{\text{cc}}(X)^{\text{fam}}$ (as well as $d\text{-B}^{\text{cc}}(X)^{\text{prfam}}$) it comes from the disjoint union of two total spaces $Y, Y'$ with a fixed base $S$. Note that this is not a symmetric monoidal structure on the category but rather of the fibred category $d\text{-B}^{\text{cc}}(X)^{\text{fam}} \to \text{Man}$.

The symmetric monoidal fibred category $d\text{-B}^{\text{cc}}(X)^{\text{prfam}}$ is freely generated by the fibred category $d\text{-B}^{\text{cc}}(X)^{\text{prfam}}$ and hence we obtain an equivalence as in Lemma 16

$$\text{Fun}_{\text{Man}}(d\text{-B}^{\text{cc}}(X)^{\text{prfam}}, C) \simeq \text{Fun}_{\text{Man}}^\otimes (d\text{-B}^{\text{cc}}(X)^{\text{prfam}}, C)$$

Finally, the symmetric monoidal stack $d\text{-B}^{\text{cc}}(X)^{\text{fam}}$ is the stackification of the symmetric monoidal fibred category $d\text{-B}^{\text{cc}}(X)^{\text{prfam}}$, leading to the final equivalence of categories by our assumption on $C$:

$$\text{Fun}_{\text{Man}}^\otimes (d\text{-B}^{\text{cc}}(X)^{\text{fam}}, C) \simeq \text{Fun}_{\text{Man}}^\otimes (d\text{-B}^{\text{cc}}(X)^{\text{fam}}, C)$$

Putting the 3 equivalences together finishes the proof of our lemma. □
5. Supersymmetric Field Theories

In this section we’ll be rewarded for expressing smooth functions on a manifold \(X\) in Lemma 20 in very fancy language as certain fibred functors over the site \(\text{Man}\). In fact, we can easily generalize all definitions to supermanifolds in the following straightforward way.

The naive extension of the Atiyah-Segal definition would say that a \(d|\delta\)-dimensional TFT associates a finite dimensional supervector space to any closed \((d - 1)|\delta\)-manifold and a linear map to a compact \(d|\delta\)-dimensional bordism, satisfying the usual gluing axioms. In the presence of a target \(X\), all supermanifolds would be in addition equipped with a smooth map to \(X\). Even for \(d = 0\) we then run into the question how to implement the smoothness of the functor. We find it very natural use the same formalism as in the previous subsection, except for using supermanifolds \(S\) as the parameter (or base) spaces for the family versions of our categories. For \(d > 0\), we also have to work with supermanifolds with boundary which were introduced in [VZ].

When trying to generalize, say, the fibred category \(d\mathcal{B}_{c}^{fam} \to \text{Man}\), we have to generalize the notions of a fibre bundle of supermanifolds. There is an obvious way of doing that, namely to start with trivial bundles and define general bundles via gluing data. In the language of fibred categories this procedure is exactly the stackification, already used in Lemma 23. Keeping with our spirit of giving the simplest possible definitions and using stacks only as a motivation, see Lemma 27, we proceed as follows.

**Definition 24.** A (supersymmetric) \(0|\delta\)-dimensional TFT over \(X\) is a fibred functor

\[
\begin{array}{ccc}
0|\delta\text{-}\mathcal{B}_{c}(X)^{prfam} & \overset{E}{\longrightarrow} & \mathbb{R} \\
\downarrow & & \downarrow \\
\text{SM} & \underset{\text{SM}}{\longrightarrow} & \text{SM}
\end{array}
\]

In other words, we set

\[
0|\delta\text{-TFT}(X) := \text{Fun}_{\text{SM}}(0|\delta\text{-}\mathcal{B}_{c}(X)^{prfam}, \mathbb{R})
\]

Since the dimension already signifies supermanifolds, the additional adjective ‘supersymmetric’ will be usually skipped.

The missing piece in this definition is that of the fibred category of *product families* of supermanifolds

\[
d|\delta\text{-}\mathcal{B}_{c}(X)^{prfam} \longrightarrow \text{SM} \text{ respectively } d|\delta\text{-}\mathcal{B}_{c}(X)^{prfam} \longrightarrow \text{SM}
\]

that we shall spell out for arbitrary \(d\). Recall that by definition, a supermanifold is closed (respectively connected) if and only if its underlying manifold is.

**Definition 25.** The category \(d|\delta\text{-}\mathcal{B}_{c}^{fam}\) has objects supermanifolds of the form \(S \times F\) where \(F\) is a closed \(d|\delta\)-manifold. For a fixed supermanifold \(X\), \(d|\delta\text{-}\mathcal{B}_{c}(X)^{prfam}\) is the category whose objects include in addition a smooth map \(S \times F \to X\). Morphisms are commutative diagrams in \(\text{SM}\)

\[
\begin{array}{ccc}
S' \times F' & \overset{\phi}{\longrightarrow} & S \times F \\
\downarrow & & \downarrow \\
S' & \underset{f}{\longrightarrow} & S
\end{array}
\]

(26)
that are fibrewise diffeomorphisms and commute with the map to \(X\) (if present). To explain the former, note that \(\phi = \psi \circ (f \times \text{id}_F)\) where \(\psi\) is a map over \(S'\):

\[
\begin{array}{ccc}
S' \times F' & \xrightarrow{\psi} & S' \times F \\
\downarrow & & \downarrow \\
S' & & 
\end{array}
\]

We say that \(\phi\) is a fibrewise diffeomorphism if \(\psi\) is a diffeomorphism. Alternatively, one can start with all commutative diagrams as in (26) and then restrict to the cartesian morphisms. It follows that there are fibrations

\[d|\delta\text{-}B_c(X)^{prfam} \longrightarrow SM\]

that take \(S \times F\) and send it to its base (or parameter space) \(S\). Finally, in the case of \(0|\delta\text{-}B_c(X)^{prfam}\) we only use fibres \(F := \mathbb{R}^{0|\delta}\).

Definition 24 is justified by the following result that is proven exactly as Lemma 23. The last description is the one used in [ST2]. Recall that the representable stack \(\mathbb{R} \to SM\) used for TFTs is symmetric monoidal with respect to multiplication.

**Lemma 27.** If \(C \to SM\) is a symmetric monoidal stack in the sense of Definition 61 then there are equivalences of categories

\[\text{Fun}_{SM}(d|\delta\text{-}B_c(X)^{prfam}, C) \simeq \text{Fun}_{SM}(d|\delta\text{-}B_c(X)^{fam}, C)\]

and

\[\text{Fun}_{SM}(d|\delta\text{-}B_c(X)^{fam}, C) \simeq \text{Fun}_{SM}(d|\delta\text{-}B_c(X)^{fam}, C)\]

where \(d|\delta\text{-}B_c(X)^{fam} \to SM\) is the stack of fibre bundles with closed \(d|\delta\)-dimensional fibres (obtained from stackifying \(d|\delta\text{-}B_c(X)^{prfam} \to SM\)).

5.1. 0|1-dimensional TFTs over \(X\). In the 0-dimensional case we used the equivalence of fibred categories

\[0\text{-}B_c(X)^{fam} \simeq X\]

to complete our computation of 0-TFT(\(X\)). In the 0|1-dimensional case, the corresponding result is more interesting due to the nontrivial diffeomorphisms of superpoints.

**Lemma 28.** For every supermanifold \(X\) and every \(\delta \in \mathbb{N}\) there is an equivalence of fibred categories over \(SM\)

\[0|\delta\text{-}B_c(X)^{prfam} \simeq SM(\mathbb{R}^{0|\delta}, X)/\text{Diff}(\mathbb{R}^{0|\delta})\]

The right hand side is the quotient construction explained in Definition 52 of the Appendix. Note that a priori, the two inner Homs are just presheaves on \(SM\) but that’s all one needs to form the fibred quotient category. It actually turns out that both presheaves are representable but we’ll only discuss this in the case \(\delta = 1\).

**Proof.** To simplify the discussion, fix a supermanifold \(S\) and only look at the fibre categories over \(S\) on both sides of the equation. The left hand side has objects

\[f \in SM(S \times \mathbb{R}^{0|\delta}, X) \cong SM(S; SM(\mathbb{R}^{0|\delta}, X)),\]

where the right hand side consists exactly of the objects in the quotient category. As for morphisms, Definition 25 explains why the left hand side has pairs \((\psi, f)\) where \(f : S' \to S\)
and \( \psi : S' \times \mathbb{R}^{0|\delta} \to S' \times \mathbb{R}^{0|\delta} \) is a diffeomorphism over \( S' \). Comparing this to the morphisms on the right hand side, we see that we just need to translate \( \psi \) into a map

\[
g : S' \to \text{Diff}(\mathbb{R}^{0|\delta})
\]

However, this translation is just the definition of the diffeomorphism group in terms of its \( S \)-points. It is not hard to see that these translations preserve the composition in the respective categories. Finally, we can use Lemma 40 or work things out by hand. \( \square \)

We now turn to the case \( \delta = 1 \) and abbreviate the superpoint \( \mathbb{R}^{0|1} \) as before. We conclude from the above Lemma that

\[0|1\text{-B}_{cc}(X)^{prfam} \simeq \text{SPX}/\text{Diff}(\mathbb{R}^{0|1})\]

where the supermanifold of superpoints in \( X \), \( \text{SPX} \), is represented by \( \Pi C^\infty(TX) \) by Proposition 6. From Lemma 9 we know that the diffeomorphism supergroup of \( \mathbb{R}^{0|1} \) is given by

\[\text{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^\times \rtimes \mathbb{R}^{0|1},\]

where \( \mathbb{R}^\times \) is the even dilational part and \( \mathbb{R}^{0|1} \) are the odd translations of \( \mathbb{R}^{0|1} \).

**Proposition 30.** For any supermanifold \( X \), there is a bijection between \( 0|1 \)-dimensional TFTs over \( X \) and closed \( 0|1 \)-forms on \( X \):

\[0|1\text{-TFT}(X) \cong \text{Fun}_{\text{SM}}(\text{SPX}/\text{Diff}(\mathbb{R}^{0|1}), \mathbb{R}) \cong \Omega^0_{cl}(X)\]

The right hand side equals \( \{ f \in C^\infty(X) \mid df = 0 \} = \{ f \in C^\infty(X_{\text{red}}) \mid df = 0 \} \).

**Proof.** By Corollary 57 we just need to determine those functions on the supermanifold \( \text{SPX} \) that are fixed by the supergroup \( \text{Diff}(\mathbb{R}^{0|1}) \). For the even action of \( \mathbb{R}^\times \) by dilations this is literally the fixed point set of the action on \( C^\infty(\text{SPX}) \). As explained in Section 3.4, the only functions on \( \text{SPX} \) that are fixed by all dilations are the functions on \( X \). For the odd part of the action, namely by translations of \( \mathbb{R}^{0|1} \), it is not hard to see that being ‘invariant’ in the sense of Corollary 57 is equivalent to being annihilated by the infinitesimal generators of translation. By Lemma 8, this infinitesimal generator of translation is just the de Rham \( d \) on \( C^\infty(X) \). \( \square \)

**Corollary 31.** Concordance classes of \( 0|1 \)-dimensional TFTs over \( X \) agree with degree 0 de Rham cohomology:

\[0|1\text{-TFT}[X] \cong H^0_{dR}(X) \cong H^0_{dR}(X_{\text{red}})\]

This finishes the proof of the degree 0 case of Theorem 2. Before going to non-trivial degrees, we shall prove the degree 0 case of Theorem 1.

5.2. \( 0|1 \)-dimensional EFTs over \( X \). In the spirit of Felix Klein’s Erlangen program, we introduce a **Euclidean geometry** on the superpoint by specifying its isometry group to be

\[\text{Iso}(\mathbb{R}^{0|1}) := \{ \pm 1 \} \rtimes \mathbb{R}^{0|1} \leq \mathbb{R}^\times \rtimes \mathbb{R}^{0|1} = \text{Diff}(\mathbb{R}^{0|1})\]

given by translations and reflections of the superpoint. This is analogous to the Euclidean group of \( \mathbb{R} \) inside all diffeomorphisms. It leads to a Euclidean bordism category \( 0|\delta\text{-EB}_{cc} \) and its family version and also to the notion of a **Euclidean** field theory by following the same steps as for the case of TFTs. So we define

\[0|1\text{-EFT}(X) := \text{Fun}_{\text{SM}}(0|\delta\text{-EB}_{cc}(X)^{prfam}, \mathbb{R})\]
and compute exactly as in Proposition 30 that it is isomorphic to
\[ \text{Fun}_{\text{SM}}(\text{SP}X/\text{Iso}(\mathbb{R}^{0|1}), \mathbb{R}) \cong \{ f \in C^\infty(\text{SP}X)^\text{ev} | D(f) = 0 \} \]

\( D \) is the infinitesimal generator of translations, acting on \( C^\infty(\text{SP}X) \) as explained in Section 3.1. One can thus think of such field theories as closed pseudodifferential forms on a supermanifold \( X \). For example, if \( X = \mathbb{R}^{0|1} \) then \( \text{SP}X = \mathbb{R}^{1|1} \) and
\[ 0|1\text{-EFT}(\mathbb{R}^{0|1}) \cong C^\infty(\mathbb{R}) \]

If \( X \) is an ordinary manifold then Proposition 6 and Lemma 8 imply
\[ C^\infty(\text{SP}X) \cong \Omega^*X \quad \text{and} \quad D = d \]

and hence the degree 0 case of Theorem 1 follows.

6. Twisted Field Theories

Recall from Definition 24 and Lemma 60 that we can express untwisted field theories as fibred natural transformations over \( \text{SM} \) as follows:
\[ 0|\delta\text{-TFT}(X) \cong \{ 0|\delta\text{-B}_{cc}(X)^\text{prfam} \} := \text{Nat}_{\text{SM}}(\mathbb{I}, \mathbb{I}) \]

where \( \text{Pic} \to \text{SM} \) is the symmetric monoidal stack of \( \mathbb{Z}/2 \)-graded real line bundles and \( \mathbb{I} : \text{SM} \to \text{Pic} \) is the monoidal unit, giving the trivial bundle for each \( S \in \text{SM} \). Hence the following definition is not surprising.

**Definition 32.** A twist for \( 0|\delta \)-dimensional TFTs over a supermanifold \( X \) is a fibred functor
\[ T \in \text{Fun}_{\text{SM}}(0|\delta\text{-B}_{cc}(X)^\text{prfam}, \text{Pic}) \]

Moreover, a \( T \)-twisted TFT over \( X \) is a fibred natural transformation
\[ E \in \text{Nat}_{\text{SM}}(\mathbb{I}, T) = \{ 0|\delta\text{-B}_{cc}(X)^\text{prfam} \} := \text{Nat}_{\text{SM}}(\mathbb{I}, T) \]

We write \( 0|\delta\text{-TFT}^T(X) \) for the set of \( T \)-twisted TFTs over \( X \).

In the case \( \delta = 1 \) we computed the bordism category in equation 29 to be equivalent to the quotient fibration
\[ QX := \text{SP}X/\text{Diff}(\mathbb{R}^{0|1}) \cong 0|1\text{-B}_{cc}(X)^\text{prfam} \]

and hence twists \( T \in \text{Fun}_{\text{SM}}(QX, \text{Pic}) \) are by definition line bundles over the quotient \( QX \), see Section 7.4 of our Appendix. Moreover, \( T \)-twisted field theories are sections of this line bundle:
\[ 0|1\text{-TFT}^T(X) \cong \Gamma(QX; T) := \text{Nat}_{\text{SM}}(\mathbb{I}, T) \]

We will compute all twists and their sections in the forthcoming paper with Chris Schommer-Pries, here we’ll finish by studying the simplest twists, namely those that don’t depend on \( X \) and give the notion of “degree”.

\textbf{Definition 34.} Consider the composition of supergroup homomorphisms

\[ \rho : \text{Diff}(\mathbb{R}^{0|\delta}) \to GL_\delta(\mathbb{R}) \to \mathbb{R}^\times \]

and recall from Section 7.4 that there is a corresponding even line bundle \( L_\rho \) on the quotient fibration

\[ \text{SM}(\mathbb{R}^{0|\delta}, X)/\text{Diff}(\mathbb{R}^{0|\delta}) \]

The \textit{degree 1 twist} is defined to be the odd partner of that line bundle, \( T_1 := \Pi L_\rho \). For each \( n \in \mathbb{Z} \) we define \( T_n \) to be the \( n \)-th power of \( T_1 \) with respect to the symmetric monoidal structure on \( \text{Pic} \). We denote by

\[ 0|\delta\text{-TFT}^n(X) := 0|\delta\text{-TFT}^{T_n}(X) \]

the set of \( 0|\delta \)-dimensional TFTs of \textit{degree} \( n \) over \( X \). In particular, since \( T_0 = 1 \) we see that degree 0 TFTs are by definition untwisted.

Even though this definition applies for all \( \delta \), we shall only continue the discussion for \( \delta = 1 \). By Section 7.4, the projection \( \rho : \text{Diff}(\mathbb{R}^{0|1}) \to \mathbb{R}^\times \) gives a canonical line bundle \( L \) on \( QX \):

\[ L : QX = SPX/\text{Diff}(\mathbb{R}^{0|1}) \to \text{pt} / \text{Diff}(\mathbb{R}^{0|1}) \xrightarrow{L} \text{pt} / \mathbb{R} \to \text{pt} / \mathbb{R} = \text{Pic} \]

Then \( T_1 := \Pi L \) is the odd partner of this line bundle and \( T_n = T_1^{\otimes n} \). Specializing equation 33 to \( T_n \), we see that the set 01-TFT\(^n\)(X) of 01-dimensional degree \( n \) TFTs over \( X \) can be identified with the set \( \Gamma(QX; T_n) \) of sections of the line bundle \( T_n \) over the quotient fibration \( QX \). Hence the following proposition implies Theorem 2.

\textbf{Proposition 35.} For any supermanifold \( X \), there is a canonical bijection between sections of the line bundle \( T_n \) on \( QX = SPX/\text{Diff}(\mathbb{R}^{0|1}) \) and closed differential forms of degree \( n \) on \( X \):

\[ \Gamma(QX; T_n) \cong \Omega^n_{cl}(X) \]

\textbf{Proof.} We apply Corollary 59 in the case where \( G := \text{Diff}(\mathbb{R}^{0|1}) \) acts on the superpoints \( SPX \) and \( \rho \) is the \( n \)-th power map on \( \mathbb{R}^\times \). We conclude

\[ \Gamma(SPX/G; T_n) \cong \{ f \in C^\infty(SPX) \mid \mu^*(f) = p_1^*(f) \cdot p_2^*(\rho) = f \otimes \rho \in C^\infty(SPX \times G) \} \]

Expanding \( C^\infty(SPX \times G) = C^\infty(SPX) \otimes \Lambda^*[\eta] \otimes C^\infty(\mathbb{R}^\times) \), the right hand side is given as \( p_1^*(f) \cdot p_2^*(\rho) = f \otimes \rho = f \otimes 1 \otimes t^n \), where \( t \in C^\infty(\mathbb{R}^\times) \) is the standard coordinate. By Corollary 11 this can only equal the left hand side if \( f \in \Omega^n X \subset C^\infty(SPX) \). Moreover, Proposition 13 says for any \( f \in \Omega^n X \)

\[ \mu^*(f) = f \otimes 1 \otimes t^n + D(f) \otimes \eta \otimes t^n \in C^\infty(SPX) \otimes \Lambda^*[\eta] \otimes C^\infty(\mathbb{R}^\times) \]

Comparing coefficients shows that \( D(f) = 0 \) and by Lemma 8 this is equivalent to \( d(f) = 0 \). \( \square \)
6.1. **Twisted Euclidean Field Theories.** Recall that the Euclidean structure on the superpoint $\mathbb{R}^{0|1}$ is defined by its isometry group

$$\text{Iso}(\mathbb{R}^{0|1}) := \{\pm 1\} \ltimes \mathbb{R}^{0|1} \leq \mathbb{R}^\times \ltimes \mathbb{R}^{0|1} = \text{Diff}(\mathbb{R}^{0|1})$$

given by translations and reflections. This leads to

$$0|1\text{-EFT}^n(X) := \text{Nat}_{\text{SM}}(\mathbb{1}, T_n) = \{0|1\text{-c.c. EB}(X)^{\text{mfam}} \downarrow \text{Pic} \}$$

which can be computed exactly as in Proposition 35 to be isomorphic to

$$\Gamma(\text{SP}_X/\text{Iso}(\mathbb{R}^{0|1}); T_n) \cong \{ f \in C^\infty(\text{SP}_X)^{\text{ev/odd}} \mid D(f) = 0 \}$$

where the functions are even respectively odd depending on the parity of $n$. Specializing to an ordinary manifold $X$, we get $\Omega^{\text{ev/odd}}_n(X)$ as before. This finishes the proof of Theorem 1.

7. **Appendix: Grothendieck Fibrations**

In this appendix we’ll give a survey of the language used in this paper to discuss smooth and supersymmetric families of categories. Excellent references on these (Grothendieck) fibrations are [C] and [Vi], here we only recall the main aspects for the convenience of the reader who does not want to look at other sources. We only claim originality for Proposition 53 which we generalize from groupoids to categories.

We will typeset categories in sans-serif $\mathcal{C}, \mathcal{S}, \mathcal{V}, \ldots$ and abbreviate $C \in \mathcal{C}$ to mean that $C$ is an object in $\mathcal{C}$. Similarly, $\phi \in \mathcal{C}(C', C)$ will denote a morphism in $\mathcal{C}$, $\phi : C' \to C$.

Consider a functor $p : \mathcal{V} \to \mathcal{S}$ where $\mathcal{S}$ will later be the category $\text{Man}$ of manifolds or $\text{SM}$ of supermanifolds. We also use the letter $\mathcal{S}$ to remind the reader that this is the (Grothendieck) site over which everything is happening. One motivating example to keep in mind is when $\mathcal{V}$ is the category of (super) vector bundles over (super) manifolds and $p$ is the map that takes a vector bundle to its base.

7.1. **Pullbacks and Categories of Fibred Functors.** In the following diagrams, an arrow going from an object $V$ of $\mathcal{V}$ to an object $S$ of $\mathcal{S}$, written as $V \hookrightarrow S$, will mean that $p(V) = S$ or in words, that “$V$ lies over $S$”. Furthermore, the commutativity of the diagram

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
S & \rightarrow & T \\
\downarrow & & \downarrow \\
V & \phi \rightarrow & W
\end{array}
$$

will mean that $p(\phi) = f$, or that “$\phi$ lies over $f$”.

**Definition 37.** A morphism $\phi \in \mathcal{V}(V, W)$ is cartesian if for any $\psi \in \mathcal{V}(U, W)$ and any $g \in \mathcal{S}(p(U), p(V))$ with $p(\phi) \circ g = p(\psi)$, there exists a unique $\theta \in \mathcal{V}(U, V)$ with $p(\theta) = g$ and $\phi \circ \theta = \psi$, as in the commutative diagram

$$
\begin{array}{ccc}
U & \rightarrow & V \\
\downarrow & \phi \rightarrow & \downarrow \\
R & \rightarrow & T
\end{array}
$$
If \( \phi \colon V \to W \) is cartesian, we say that the diagram (36) is a cartesian square. It is easy to see that cartesian morphisms are closed under composition. They should be thought of as the “fibrewise isomorphisms”. We refer to [C, 3.1.2] for a careful comparison of this notion and the one used by Grothendieck where \( g = \text{id}_S \) in the diagram above.

For example, any isomorphism \( \phi \) is clearly cartesian and in fact, this notion is meant to formalize that of ‘fibrewise isomorphisms’, compare also Definition 46.

**Definition 38.** A functor \( p \colon V \to S \) is a (Grothendieck) fibration if pullbacks exist: for every object \( W \in V \) and every morphism \( f \in S(S, p(W)) \), there is a cartesian square

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow & & \downarrow \\
S & \to & p(W)
\end{array}
\]

One can think of this property as existence of “categorical path-lifting”. We’ll define the fibres of such a fibration below and we shall see in Definition 41 that in the discrete case, path-lifting is unique, just like for covering maps (aka fibre bundles with discrete fibres).

A fibred category over \( S \) is a category \( V \) together with a functor \( p \colon V \to S \) which is a fibration. If \( p_V \colon V \to S \) and \( p_W \colon W \to S \) are fibered categories over \( S \), then a fibred functor \( F \colon V \to W \) is a functor with \( p_V \circ F = p_W \) that sends cartesian morphisms to cartesian morphisms, i.e. that preserves pullbacks.

For an object \( S \in S \) the fibre of the fibration \( p : V \to S \) over \( S \) is by definition the subcategory \( V_S \subseteq V \) whose objects are those \( V \in V \) lying over \( S \) and whose morphisms lie over \( \text{id}_S \). For example, \( \text{Vect} \to \text{Man} \) is a fibration with a fibre \( \text{Vect}_S \) being the category of vector bundles over the manifold \( S \).

After the choice of a cleavage, i.e. a certain collection of pullbacks, these fibres assemble into a pseudo-functor (or lax 2-functor)

\[
S^{\text{op}} \to \text{Cat}, \quad S \mapsto V_S
\]

which is a different way of looking at the fibration condition. This point of view leads naturally to the following

**Definition 39.** A fibred natural transformation \( \alpha \in \text{Nat}_S(F,G) \) between two fibred functors \( F, G : V \to W \) is a natural transformation \( \alpha : F \Rightarrow G \) such that for any object \( V \in V \), the morphism

\[
\alpha_V \in W(F(V), G(V))
\]

lies in \( W_S \), or equivalently \( \alpha_V \) lies over \( \text{id}_S \), where

\[
S := p_V(V) = p_W(F(V)) = p_W(G(V)) \in S.
\]

We shall write \( \text{Fun}_S(V,W) \) for the category of fibred functors and fibred natural transformations. The notion of an equivalence of fibred categories arises in the usual way from observing that fibred categories over \( S \) form a (strict) 2-category.

The following lemma is very useful, see [Vi, Prop.3.36] for a proof.

**Lemma 40.** A fibred functor \( F : V \to W \) is an equivalence of fibred categories if and only if the restrictions \( F_S : V_S \to W_S \) are equivalences for all objects \( S \in S \).
The next 3 definitions will give different ways of constructing fibred categories. An easy class of fibrations are those with discrete fibres, i.e. those where all fibres \(V_S\) have only identity morphisms. These are sometimes also referred to as categories fibred in sets. By [Vi, Prop.3.26], up to equivalence these always arise from a presheaf (aka functor) \(F : S^{op} \rightarrow \text{Set}\) as follows.

**Definition 41.** Define the objects of \(F\) to be pairs \((S,g)\) where \(S \in S\) and \(g \in F(S)\) and morphisms by

\[
F((S',g'),(S,g)) := \{f \in S(S',S) \mid F(f)(g) = g'\}.
\]

The forgetful map \(p : F \rightarrow S\) is easily seen to be a fibration, in fact, there are unique pullbacks in this case.

**Lemma 42.** The only fibred natural transformation between functors \(F,G \in \text{Fun}(S,F)\) is the identity. In particular, this category is discrete (in the sense that it has only identity morphisms).

The easiest examples of fibrations with discrete fibres come from representable presheaves, i.e. where the presheaf arises from a fixed object \(M \in S\) via \(F(S) := S(S,M)\).

**Definition 43.** For \(M \in S\), we write \(M \rightarrow S\) for the resulting representable fibration.

The 2-Yoneda’s lemma [Vi, 3.6.2] gives natural equivalences of categories

\[
F_{\text{un}}(M, W) \simeq W_M
\]

In the case \(W = N\) these reduce to natural bijections of sets

\[
F_{\text{un}}(M) \simeq N(M) = S(M,N).
\]

**Remark 45.** We have used the notation of ‘underlining’ in two different contexts: above we used it to get fibred categories over \(S\) from presheaves respectively objects of \(S\). In Section 3 we used it for distinguishing Hom from inner Hom in the category \(SM\). Inner Hom can also be discussed in our current general context, assuming that \(S\) has a monoidal structure \(\otimes\). For two fixed objects \(M,N \in S\) one can then consider the presheaf

\[
S^{op} \rightarrow \text{Set}, \quad S \mapsto S(S \otimes M,N)
\]

If this presheaf is representable then the representing object is determined uniquely up to canonical isomorphism by the usual Yoneda lemma. It is denoted by \(S(M,N) \in S\) and has the following \(S\)-points:

\[
S(S,S(M,N)) \cong S(S \otimes M,N)
\]

We hope that these two distinct underlining conventions will not confuse the reader, one makes fibred categories from objects, the other objects from morphism sets.

**Definition 46.** Given a functor \(G : S^{op} \rightarrow \text{Cat}\), there is a corresponding fibred category \(\hat{G} \rightarrow S\). The objects of \(\hat{G}\) are pairs \((S,g)\) where \(S \in S\) and \(g \in G(S)\). Moreover, define

\[
\hat{G}((S',g'),(S,g)) = \{(f,\phi) \in S(S',S) \times \text{mor} \ G(S') \mid s(\phi) = g', t(\phi) = G(f)(g)\}
\]

where \(s,t\) are the source respectively target maps of the category \(G(S')\). The composition law is not hard to guess but surprisingly can also be written down in the case \(G\) is just a pseudo-functor, see [Vi, 3.1.3]. In both cases, the reader is invited to check that a morphism \((f,\phi)\) is cartesian if and only if \(\phi\) is an isomorphism.
Remark 47. Every fibred category $p : \mathcal{V} \to S$ is canonically equivalent to one of the form $\hat{G}$, see [Vi, Thm.3.45]. Namely, one takes $G$ to be the functor that sends $S$ to $\text{Fun}_S(S, V)$.

7.2. Stacks. If $S$ is a (Grothendieck) site in the sense of [Vi, Def.2.24], i.e. it carries the notion of coverings of objects, one can ask for a generalization of the sheaf property of a presheaf $\mathcal{F}$ as above. It turns out that $\mathcal{F}$ is a sheaf if and only if the fibration $\mathcal{F}$ is a stack in the following sense.

Definition 48. A fibred category $\mathcal{V} \to S$ over a site $S$ is called a stack if for every covering $S_i \to S$ of an object $S \in S$, the natural functor

$$\mathcal{V}_S \to \mathcal{V}_{S_i \to S}$$

is an equivalence of categories. The right hand side is the descent category of the covering $S_i \to S$ whose objects are gluing data, see [Vi, 4.1.2].

Let us explain the descent category in the case of a fibred category $\hat{G}$ coming from a functor $G : S^{op} \to \text{Cat}$ and where the covering is given by a single morphism $U \to S$. By the properties of a site, the pullbacks

$$U^{[n+1]} = U \times_S \cdots \times_S U$$

exist for all $n \geq 0$ and they form a (Čech-like) simplicial object $U^\bullet$ in $S$, resolving the object $S \in S$, thought of as a constant simplicial object $S^\bullet$:

$$U^\bullet = (\cdots U \times_S U \times_S U \longrightarrow U \times_S U \longrightarrow U) \to S^\bullet$$

If we apply the functor $G$ to this simplicial map, we get a cosimplicial functor $G(S^\bullet) \to \hat{G}(U^\bullet)$. Using a version of Definition 54 below, we can form its homotopy limit

$$\hat{G}_S = G(S) = \text{hlim}^\times(G(S^\bullet)) \to \text{hlim}^\times(G(U^\bullet)) =: \hat{G}_{U \to S}$$

which is the functor in equation 49 that is required to be an equivalence for $\hat{G}$ to be a stack. Here the version of the homotopy limit $\text{hlim}^\times$ is defined exactly like the homotopy limit $\text{hlim}$ in Definition 54 below, except that $\phi \in C_1(s(C), t(C))$ is assumed to be an isomorphism. This is important for being able to glue objects together consistently. In particular, the right hand side $\hat{G}_{U \to S}$ is by definition the descent category of the covering $U \to S$.

We shall only use these descent conditions for motivating our definitions in the next sections, see Lemmas 23 and 27. The main tool we’ll need is the stackification functor from fibred categories to stacks over $S$, see [C, 4.2.2]. It is left adjoint to the forgetful functor in analogy to sheafification.

The example of vector bundles fibred over $S = \text{Man}$ is a stack. The descent conditions just formalize the fact that bundles can be constructed from their restriction to open subsets via gluing data. Here we use the usual notion of a covering of a manifold $S \in \text{Man}$, namely where $S_i \subseteq S$ are open subsets with union $S$. If we form $U := \coprod S_i$ then we get a single covering $U \to S$ in $\text{Man}$ and the reader is invited to check that the above definition of a descent category indeed equals the category formed by gluing data for vector bundles over $S$ with respect to the covering $S_i$. 
7.3. Internal Categories as Fibrations. An important special case in the construction of $\tilde{G}$ in Definition 46 is the case where the functor $G : S^{op} \to \text{Cat}$ takes values in small categories. Then $G$ is given by an internal category in presheaves on $S$. So

$$\text{obj } G(S) = F_0(S) \quad \text{mor } G(S) = F_1(S)$$

for presheaves $F_i : S^{op} \to \text{Set}$ and there are various structure maps defining the structure of the categories $G_i(S)$. These can be most easily summarized by the following diagram where $F_2 := F_1 \times_{F_0} F_1$ is the pullback in $\text{Presheaf}(S)$ formed using the maps $s, t$ below:

\[
\begin{array}{ccc}
F_2 & \xrightarrow{p_1} & F_1 \\
\downarrow c & & \downarrow s \\
F_0 & \xrightarrow{u} & F_0
\end{array}
\]

Here $p_i$ are the two projections, $c$ is the composition map, $s, t$ are source and target maps and $u$ is the unit (or identity) map. Some of the relations between these structure maps are elegantly expressed by the simplicial identities present in the above diagram, namely

$$t \circ u = \text{id} = s \circ u, \quad t \circ p_1 = t \circ c, \quad s \circ p_2 = s \circ c, \quad t \circ p_2 = s \circ p_1.$$  

However, to formulate the associativity of $c$, the above diagram is not sufficient and one needs to extend it to the nerve $\mathcal{F}_\bullet$ of the category by setting for $n \geq 1$

$$\mathcal{F}_n := F_1 \times_{F_0} \cdots \times_{F_0} F_1,$$

with the well known structure maps that make this into a simplicial presheaf. Diagram 50 above is its initial segment which we shall refer to as its 2-skeleton (even though we are missing two degeneracy maps $F_1 \to F_2$). It is an important elementary fact that all identities between the structure maps of our category are expressed in terms of simplicial identities (all appearing in the 3-skeleton). Vice versa, every simplicial presheaf satisfying equation 51 comes from an internal category. We will therefore use the notation $\mathcal{F}_\bullet$ for the internal category in presheaves on $S$.

As in Definition 46, the functor $G$ gives a fibration $\tilde{G} \to S$ which we shall abbreviate as $\mathcal{F}_0/\mathcal{F}_1$ in the case at hand. The common notation for the associated stack is $\mathcal{F}_0/\mathcal{F}_1$.

**Definition 52.** In our applications to field theories, the most important example of an internal category will arise from a monoid object $G \in S$ that acts (from the right) on another object $M \in S$. This action has an associated internal transport category in $S$ (and hence in presheaves on $S$)

$$M \times G \xrightarrow{\mu} M$$

and therefore the quotient fibration $M/G := M/(M \times G)$ is defined as above.

For example, for $S = \text{Man}$ and $G$ a Lie group acting on a manifold $M$, the quotient space is not a manifold in general and usually not even Hausdorff. It is therefore often wise to study instead the quotient fibration $M/G$. This is in general not a stack but its stackification, the quotient stack $M//G$, has objects $(P, m)$ where $P \to S$ is a $G$-principal bundle and $m \in S(P, M)$ is $G$-equivariant, see [C, 4.4.9]. The objects of $M/G$ correspond to trivial bundles and general bundles are obtained by the gluing construction.

More generally, if $pt$ is a terminal object in a site $S$ and $G \in S$ is a group object, one can define the category of $G$-principal bundles (sometimes also referred to as $G$-torsors) in $S$ to be the stackification of the fibration $pt/G$. Then the discussion in the previous paragraph applies to describe the quotient stack $M//G$ in this setting.
The following computation of the functor category between two fibred categories will be essential in our applications to field theories. We state a very general result also for the purpose of referring to it in future papers.

**Proposition 53.** For any fibration $\mathcal{W} \to S$ and any internal category $\mathcal{F}_\bullet$ in presheaves on $S$, there is an isomorphism of categories

$$\text{Fun}_S(\mathcal{F}_0/\mathcal{F}_1, \mathcal{W}) \simeq \text{hlim}(\text{Fun}_S(\mathcal{F}_\bullet, \mathcal{W}))$$

where the right hand side is the homotopy limit (defined below) of the cosimplicial category with $n$-simplices consisting of fibred functors $\mathcal{F}_n \to \mathcal{W}$.

Unfortunately, the left hand side above is computed wrongly in [C, Prop.3.7.5], where the homotopy pullback needs to be replaced by the homotopy limit defined as follows.

**Definition 54.** Let $C_\bullet$ be a cosimplicial category with 2-skeleton

$$\begin{array}{cccc}
C_2 & \xleftarrow{c} & C_1 & \xleftarrow{s} \xrightarrow{t} C_0 \\
\xrightarrow{p_2} & & \xleftarrow{p_1}
\end{array}$$

Then its homotopy limit $\text{hlim}(C_\bullet)$ is the category with objects $(C, \phi)$ where $C \in C_0$ and $\phi \in C_1(s(C), t(C))$ is a morphism such that

$$u(\phi) = \text{id}_C \quad \text{and} \quad p_1(\phi) \circ p_2(\phi) = c(\phi)$$

Note that the cosimplicial identity $p_1 \circ s = p_2 \circ t$ implies that the above composition exists in $C_2$ and the other two simplicial identities say that it has the same source and target as $c(\phi)$.

Morphisms from $(C', \phi')$ to $(C, \phi)$ are $f \in C_0(C', C)$ such that

$$t(f) \circ \phi' = \phi \circ s(f)$$

Note that the homotopy limit only depends on the 2-skeleton of $C_\bullet$. Moreover, if $C_1$ is discrete, then so is $\text{hlim}(C_\bullet)$. It is then just the equalizer of $s$ and $t$ on objects of $C_0$.

**Remark 55.** Given a cosimplicial set $C_\bullet$, we can form its limit in the category of sets. This is a certain subset of the direct product of all sets $C_n$. Using the cosimplicial identities, it turns out that this limit is canonically isomorphic to the limit (or equalizer) of the 1-skeleton

$$\begin{array}{ccc}
C_1 & \xleftarrow{s} & C_0 \\
\xrightarrow{t}
\end{array}$$

This is the analogue of the fact that the homotopy limit of a cosimplicial category in Definition 54 only depends on its 2-skeleton.

To make this analogy precise we would have to introduce an appropriate Quillen model structure on cosimplicial categories and discuss homotopy limits with respect to it. This was done for cosimplicial groupoids in [Ho] and in this setting Proposition 53 can be proven as follows:

- Define a Quillen model structure on categories fibred in groupoids.
- Show that in this model structure, $\mathcal{F}_0/\mathcal{F}_1$ is the homotopy limit of the diagram from equation 50.
- Show that the functor $\text{Fun}_S(-, \mathcal{W})$ takes homotopy limits to homotopy colimits (of cosimplicial groupoids as defined above).
This conceptional background might help some readers, even though we prefer the following direct argument. It also constructs an isomorphism of categories, rather than just an equivalence.

Proof of Proposition 53. Let $Z \in \text{Fun}_S(\mathcal{F}_0/\mathcal{F}_1, W)$. Define $C \in \text{Fun}_S(\mathcal{F}_0, W)$ as the composition

$$C : \mathcal{F}_0 \rightarrow \mathcal{F}_0/\mathcal{F}_1 \rightarrow W,$$

where the first arrow is the fibered functor including the subcategory $\mathcal{F}_0$ in $\mathcal{F}_0/\mathcal{F}_1$. Next, recall that a morphism in $\mathcal{F}_0/\mathcal{F}_1$ is given by a triple $(f, g, X)$, where $f : S' \rightarrow S$ is a morphism in $S$, $X \in \mathcal{F}_0(S)$, and $g \in \mathcal{F}_1(S')$ such that $t(g) = \mathcal{F}_1(f)(X)$. Now define an injection

$$\iota : \text{obj}(\mathcal{F}_1) \hookrightarrow \text{mor}(\mathcal{F}_0/\mathcal{F}_1), (S, g) \mapsto (\text{id}_S, g, t(g)).$$

Note that for elements $(\text{id}_S, g, t(g))$ in the image of $\iota$ the third entry in the triple is redundant. In order to avoid simplify notation, we will use the abbreviation $(\text{id}_S, g, \_)$ for such morphisms. The key observation we will use is that the morphisms in $\text{im}(\iota)$ and $\text{mor}(\mathcal{F}_0)$ generate all of $\text{mor}(\mathcal{F}_0/\mathcal{F}_1)$. Namely, any $(f, g, X) \in \text{mor}(\mathcal{F}_0/\mathcal{F}_1)$ can uniquely be written as

$$(f, g, X) = (f, \text{id}_{\mathcal{F}_1(f)(X)}, X) \circ (\text{id}_{S'}, g, \mathcal{F}_1(f)(X)) = (f, \text{id}, X) \circ (\text{id}, g, \_).$$

The asserted isomorphism of categories is given (on objects) by

$$\text{Fun}_S(\mathcal{F}_0/\mathcal{F}_1, W) \ni Z \mapsto (C, Z\iota) \in \text{hlim}(\text{Fun}_S(\mathcal{F}_0, W)).$$

Of course, we have to verify that the assignment $Z\iota : \text{obj}(\mathcal{F}_1) \rightarrow \text{mor}(W)$ indeed defines a natural transformation between the functors $C \circ s$ and $C \circ t : \mathcal{F}_1 \rightarrow W$ and hence an element in $\text{Fun}_S(\mathcal{F}_1, W)(s(C), t(C))$. To see this, let $f : S' \rightarrow S$, $X \in \mathcal{F}_0(S)$, and $g : X \rightarrow Y$ in $\mathcal{F}_1(S)$. Then $F = (f, \text{id}, X)$ and $G = (\text{id}, g, Y)$ are composable and

$$G \circ F = (f, \mathcal{F}_1(f)(g), Y) = (f, \text{id}, Y) \circ (\text{id}, \mathcal{F}_1(f)(g), \_) =: \tilde{F} \circ \tilde{G}.$$

Since $Z$ is a functor, we have $Z(G) \circ Z(F) = Z(G \circ F) = Z(\tilde{F}) \circ Z(\tilde{G})$. Equivalently, the relation

$$(Z\iota)(S, g) \circ C(f, \text{id}, X) = C(f, \text{id}, Y) \circ (Z\iota)(S', \mathcal{F}_1(f)(g))$$

holds. This equality, in turn, exactly expresses the naturality of the transformation $Z\iota$. To see this, note that the fibered functor $\mathcal{F}_1 \rightarrow S$ has discrete fibres. Hence any morphism in $\mathcal{F}_1$ is of the form $(f, \text{id}, g) : (S', \mathcal{F}_1(f)(g)) \rightarrow (S, g)$ with $f : S' \rightarrow S$ and $g \in \mathcal{F}_1(S)$. It is now easy to check that the naturality of $Z\iota$ for $(f, \text{id}, g)$ precisely amounts to the previous equation with $X = s(g)$, $Y = t(g)$.

Finally, since $Z$ is a functor, $Z\iota$ respects compositions and identities as required in the definition of the hlim. We give the argument for the composition and leave the easier identity $u(Z\iota) = \text{id}_C$ to the reader. The equality $p_1(Z\iota) \circ p_2(Z\iota) = c(Z\iota)$ amounts to showing that for all $(S, g_1), (S, g_2) \in \text{obj}(\mathcal{F}_1)$ that are composable in the sense that $t(S, g_1) = s(S, g_2)$ (such elements necessarily live over the same base $S$) we have

$$Z\iota p_1((S, g_2), (S, g_1)) \circ Z\iota p_2((S, g_2), (S, g_1)) = Z\iota c((S, g_2), (S, g_1))$$

or equivalently

$$Z(\text{id}, g_2, \_) \circ Z(\text{id}, g_1, \_) = Z(\text{id}, g_2g_1, \_) = Z(\text{id}, g_1, \_) \circ Z(\text{id}, g_2, \_),$$

which holds since $Z$ is a functor.

It is now easy to see how the inverse of the functor $Z \mapsto (C, Z\iota)$ is defined (on objects): its inverse takes $(C, \phi) \in \text{hlim}(\text{Fun}_S(\mathcal{F}_0, W))$ and builds a functor $Z$ as above. Using the unique
factorization of a morphism \((f, g, X) \in \text{mor}(\mathcal{F}_0/\mathcal{F}_1)\), we can extend the definition of \(Z\) to all of \(\text{mor}(\mathcal{F}_0/\mathcal{F}_1)\) by letting

\[
Z(f, g, X) := C(f, \text{id}, X) \circ \phi(S', g)
\]

where \(f : S' \to S\) and \(g \in \mathcal{F}_1(S')\). It remains to check that this yields a fibered functor \(Z\). Functoriality holds automatically on \(\text{im}(\iota)\) and \(\text{mor}(\mathcal{F}_0)\) and for a general morphism it comes down to checking

\[
\phi(S, g) \circ C(f, \text{id}, X) = C(f, \text{id}, X) \circ \phi(S', \mathcal{F}_1(f)(g))
\]

for \(f : S' \to S\) and \(g \in \mathcal{F}_1(S)\). However, as above, this is precisely the condition that \(\phi\) is a natural transformation.

It remains to prove that \(Z\) preserves cartesian morphisms. It follows readily from the definitions that a morphism \((f, g, X)\) is cartesian if and only if \(g\) is an isomorphism. Since the composition of two cartesian morphisms is cartesian, we only have to check that for such an \((f, g, X)\) the morphisms \(C(f, \text{id}, X)\) and \(\phi(S', g)\) are cartesian. The former holds since \(C\) is a fibered functor and \((f, \text{id}, X)\) is cartesian. The latter follows since we already know that \(Z\) is a functorial on \(\text{im}(\iota)\): this implies that \(\phi(S', g)\) is also invertible and thus cartesian.

The definition of the functor \(\text{Fun}_S(\mathcal{F}_0/\mathcal{F}_1, W) \to \text{hlim}(\text{Fun}_S(\mathcal{F}_*, W))\) on morphisms is easy. In both categories a morphism is a fibered natural transformation, given by an assignment

\[
N : \text{obj}(\mathcal{F}_0/\mathcal{F}_1) = \text{obj}(\mathcal{F}_0) \longrightarrow \text{mor}(W).
\]

Finally, we check that the naturality condition agrees in both cases. Using the same factorization as above, we see that \(N\) is natural when considered a transformation between functors \(Z_1\) and \(Z_2\) on \(\mathcal{F}_0/\mathcal{F}_1\) if and only if the diagram expressing naturality holds for all morphisms in \(\text{mor}(\mathcal{F}_0)\) and \(\text{im}(\iota)\). The condition for \(\text{mor}(\mathcal{F}_0)\) precisely expresses that \(N\) is a natural transformation between the functors \(C_1 = Z_1|_{\mathcal{F}_0}\) and \(C_2 = Z_2|_{\mathcal{F}_0}\). We claim that naturality for \((\text{id}, g, \_ ) \in \text{im}(\iota)\) is equivalent to the condition \(t(N) \circ \phi_2 = \phi_1 \circ s(N)\) in the definition of the \(\text{hlim}\), where \(\phi_i = Z_{\iota t}\). We leave this simple verification to the reader.  

Example 56. There are various important special cases of this result:

1. If \(W\) is discrete then so is \(\text{Fun}_S(V, W)\) by Lemma 42. Therefore, the homotopy limit is just an equalizer and we get a bijection of objects

\[
\text{Fun}_S(\mathcal{F}_0/\mathcal{F}_1, W) \cong \lim \text{Fun}_S(\mathcal{F}_0, W) \xrightarrow{s^*} \text{Fun}_S(\mathcal{F}_1, W)
\]

2. If the simplicial presheaf \(\mathcal{F}_\idot\) is represented by a simplicial object \(G_\idot\) in \(S\) then the 2-Yoneda lemma (44) implies \(\text{Fun}_S(\mathcal{F}_1, W) \cong W_{G_\idot}\) and hence

\[
\text{Fun}_S(G_0/G_1, W) \cong \text{hlim}(W_{G_\idot})
\]

3. If \(G_\idot\) is as in 2 and \(W\) is a presheaf on \(S\) with corresponding discrete fibration \(W_\idot\), then examples 1 and 2 above lead to a bijection

\[
\text{Fun}_S(G_0/G_1, W) \cong \{w \in W(G_0) \mid W(s)(w) = W(t)(w) \in W(G_1)\}
\]

4. If \(S\) has a terminal object \(\text{pt}\) and \(H \in S\) is a monoid in \(S\) with unit \(\text{pt} \to H\) then we can form the quotient fibration \(W = \text{pt} / H\) and get an equivalence of categories

\[
\text{Fun}_S(G_0/G_1, \text{pt} / H) \cong \text{hlim}(\text{pt} / H_{G_\idot})
\]
Since the categories \((pt/H)_S\) have only one object, the objects of the homotopy limit are morphisms \(\rho\) in the category \((pt/H)_{G_1}\) where the composition in
\[
\text{mor}((pt/H)_S) = S(S,H)
\]
is given by the monoid structure on \(H\). Moreover, the requirements on \(\rho : G_1 \to H\) from Definition 54 reduce to saying that \(\rho\) is an (internal) homomorphism.

If \(\rho', \rho \in S(G_1, H)\) are two such internal homomorphisms then the result above says that the morphisms in the category \(\text{Fun}_S(G_0/G_1, pt/H)\) from \(\rho'\) to \(\rho\) are in bijective correspondence with certain \(\alpha : G_0 \to H\):
\[
\text{Nat}_S(\rho', \rho) \cong \{ \alpha \in S(G_0, H) \mid \rho \cdot s^*(\alpha) = t^*(\alpha) \cdot \rho' \in S(G_1, H) \}
\]
where \(s^*(\alpha) = \alpha \circ s\) is multiplied ‘pointwise’ by \(\rho\).

7.4. Quotients of Supermanifolds and Line Bundles. Let’s apply the above results in some cases needed later. Consider a group action of \(G\) on \(M\) in the site \(S = SM\) of supermanifolds, for example \(G = \text{Diff}(\mathbb{R}^{0\mid 1})\) acting on the superpoints \(SPX\). Then example 56.3 above has the following corollary, using for \(\mathcal{W}\) the presheaf represented by \(\mathbb{R}\) respectively \(\mathbb{R}^{0\mid 1}\) and recalling that \(SM(M, \mathbb{R}) = C^\infty(M)^{ev}\) and \(SM(M, \mathbb{R}^{0\mid 1}) = C^\infty(M)^{odd}\).

Corollary 57. Functions on the quotient are invariants by the group:
\[
\text{Fun}_S(M/G, \mathbb{R}) \cong \{ f \in C^\infty(M)^{ev} \mid \mu^*(f) = p^*(f) \in C^\infty(M \times G)^{ev} \}
\]
To obtain odd functions, we need to replace \(\mathbb{R}\) by \(\mathbb{R}^{0\mid 1}\) in the above.

Remark 58. If \(g\) is the super Lie algebra of a super Lie group \(G\) with connected underlying manifold \(G_{\text{red}}\) then the infinitesimal action of \(g\) on the algebra \(C^\infty(M)\) of functions has the same fixed point set as described in Corollary 57 above (where we have to add the even and odd parts).

Let \(\text{Pic} \to SM\) be the symmetric monoidal stack of real line bundles, compare Definition 61. An object in \(\text{Pic}\) is a line bundle over a supermanifold and on each connected component, it can have superdimension 1\mid 0 respectively 0\mid 1. We refer to these cases as \(ev\) respectively \(od\) line bundles. Let \(1 : SM \to \text{Pic}\) be the monoidal unit, giving the trivial even line bundle for each \(S \in SM\). If \(M/G\) is a quotient fibration over \(SM\) then we define the category of line bundles over \(M/G\) to be
\[
\text{Fun}_{SM}(M/G, \text{Pic}),
\]
extending the case \(\text{Fun}_{SM}(M, \text{Pic}) \simeq \text{Pic}_M\). The stack \(\text{Pic}\) contains the fibred subcategory \(pt/\mathbb{R}\) of trivial line bundles, where \(\mathbb{R}\) is a monoid via multiplication. Since every line bundle is locally trivial, it follows that \(\text{Pic}\) is the stackification of \(pt/\mathbb{R}\). A functor \(M/G \to pt/\mathbb{R}\) can be given by a homomorphism \(\rho : G \to \mathbb{R}^\times\) because this induces a morphism of internal categories in \(SM\). We denote the corresponding even line bundle on \(M/G\) by \(L_\rho\). For example, the trivial bundle \(1\) comes from the trivial homomorphism. Again extending the non-equivariant case, we define the sections of \(L_\rho\) to be
\[
\Gamma(M/G; L_\rho) := \text{Nat}_{SM}(1, \rho) = \{ M/G \xrightarrow{1} E \xrightarrow{\rho} \text{Pic} \}
\]
We note that both functors \(1\) and \(\rho\) take values in \(pt/\mathbb{R} \subset \text{Pic}\) which, for each \(S \in SM\), is the full subcategory of trivial bundles over \(S\). Therefore, it doesn’t matter whether the
above natural transformations have target \( \mathbb{R} \) or \( \text{Pic} \) and example 56.4 above leads to the following computation.

**Corollary 59.** Consider a quotient fibration \( M/G \) over \( SM \) with an even line bundle \( L_\rho \) given by a homomorphism \( \rho : G \to \mathbb{R}^\times \). Then

\[
\Gamma(M/G; L_\rho) \cong \{ f \in C^\infty(M)^{ev} | \mu^*(f) = p_1^*(f) \cdot p_2^*(\rho) \in C^\infty(M \times G)^{ev} \}
\]

To obtain odd functions, we need to replace \( L_\rho \) by its odd partner \( \Pi L_\rho \).

### 7.5. Natural Transformations as Functors.

In order to motivate twisted field theories in Section 6, we will need the following yoga. Let \( w_0, w_1 \in \text{Fun}_S(S, W) \) be two sections of a fibred category \( W \to S \). Then one can define the presheaf

\[
\mathcal{F}_{w_0, w_1} : S^{op} \to \text{Set}, \quad S \mapsto W(w_0(S), w_1(S))
\]

using the existence and uniqueness properties of cartesian morphisms. The following result can be derived directly from the definitions.

**Lemma 60.** For any fibred category \( p : V \to S \), there is a natural bijection

\[
\text{Nat}_S(p \circ w_0, p \circ w_1) \cong \text{Fun}_S(V, \mathcal{F}_{w_0, w_1})
\]

We shall use this lemma in the case where \( w_0 = w_1 = \mathbb{I} \) is the monoidal unit in a *fibred monoidal category* \( W \). This is a category fibred over \( S \) together with fibred functors

\[
\otimes : W \times_S W \to W \quad \text{and} \quad \mathbb{I} : S \to W
\]

and fibred natural transformations (associator etc.) that satisfy the usual properties (pentagon etc.). In this case, the following picture describes Lemma 60 well if one remembers that everything is over \( S \):

\[
\begin{array}{c}
\{ V \xleftarrow{\mathbb{I}} \xrightarrow{E} W \} \cong \{ V \xrightarrow{E} \Omega_1 W \}
\end{array}
\]

Here the ‘based loops’ \( \Omega_1 W \) signify the presheaf \( S \mapsto W(\mathbb{I}(S), \mathbb{I}(S)) \). For example, if \( W = \text{Pic} \) is the category of real line bundles over \( SM \) then this is the presheaf represented by \( \mathbb{R} \) and we get the motivation for the definition of twisted field theories in Section 6.

The following definition will be used in Lemmas 23 and 27.

**Definition 61.** A *monoidal stack* is a fibred monoidal category \( W \to S \) that satisfies the descent conditions in the monoidal sense: Both sides of 49 are by assumption monoidal categories and we require that the natural functor is a monoidal equivalence. It is also clear how to define a *symmetric monoidal stack* over \( S \).

We note that none of this requires \( S \) to be monoidal, the definition only captures monoidal structures *along the fibres* of a fibration.

### References


