Invariance principles for non-isotropic long memory random fields

Frédéric Lavancier
LS-CREST, Paris
Laboratoire Paul Painlevé, UMR CNRS 8524
Bât. M2. Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq Cedex, France

Abstract

We prove that when a random field with bounded spectral density satisfies a Donsker type theorem, its dilated and properly normalised spectral field admits a weak limit. We apply this result to establish the convergence of partial sums for random fields obtained by filtering a white noise. In particular we prove the convergence of partial sums for strongly-dependent fields whose memory does not satisfy the regularity conditions previously met in the literature.

Keywords : Linear filtering, Non Gaussian limit, Partial sums, Donsker Theorem.

1 Introduction

A random field $X = (X_n)_{n\in\mathbb{Z}^d}$ is usually said to exhibit long memory, or strong dependence, when its covariance function $r(n)$, $n \in \mathbb{Z}^d$, is not absolutely summable : $\sum_{n\in\mathbb{Z}^d} |r(n)| = \infty$. An alternative definition relates on spectral properties : a random field is said to be strongly dependent if its spectral density is unbounded at certain frequencies. These two points of view are closely related but not equivalent.

Dobrushin and Major (1979) deals with partial sums of subordinated Gaussian fields. They assume that the Gaussian random field $X = (X_n)_{n\in\mathbb{Z}^d}$ admits a covariance function of the following form

$$r(h) \underset{h\to\infty}{\sim} |h|^{-\alpha} L(|h|)\, b\left(\frac{h}{|h|}\right), \qquad 0 < \alpha < d \tag{1.1}$$

where L is a slowly varying function at infinity and b is a continuous function on the unit sphere of $\mathbb{R}^d$, $|.|$ denotes the euclidean norm on $\mathbb{Z}^d$. They prove the convergence of the properly normalised partial sums $n^{k\alpha/2-d}L(n)^{-k/2}\sum_{i_1=0}^{[nt_1]}\cdots\sum_{i_d=0}^{[nt_d]} H(X_{i_1,\ldots,i_d})$, where k is the Hermite rank of H.

Condition (1.1) is also assumed in Doukhan et al. (2002) where the authors focus on the empirical process of a linear field having long memory. They prove that the properly normalised empirical process weakly converges to a degenerated process.

Condition (1.1) is therefore the standard hypothesis met in the literature on long memory random fields. It generalises the usual long memory hypothesis in dimension 1 : $r(h) = h^{-\alpha}L(h)$ where $0 < \alpha < 1$, $h \in \mathbb{Z}$ and L is a slowly varying function.

In this paper, we use a general spectral approach to investigate the asymptotic behaviour of the partial sums of random fields. We obtain some convergence results for a large class of random fields. In the case of weak dependence, our point of view is different from most of existing works (for instance the paper of Breuer and Major (1983) which deals with subordinated Gaussian processes), because these ones make assumptions on the covariance function structure whereas we choose a spectral approach. In case of strong dependence, we focus on a wider class of fields than in the standard setting (1.1). We are particularly interested in random fields which admit a non-isotropic strong dependence, in the sense that they don't satisfy the following definition.

Definition 1. A stationary random field exhibits isotropic long memory if it admits a spectral density which is continuous everywhere except at 0 where

$$f(x) \sim |x|^{\alpha-d}\, b\left(\frac{x}{|x|}\right) L\left(\frac{1}{|x|}\right), \quad 0 < \alpha < d, \tag{1.2}$$

where L is slowly varying at infinity and b is a continuous function on the unit sphere in $\mathbb{R}^d$.

Conditions (1.1) and (1.2) are related by a result of Wainger (1965) who proved that if a random field admits a covariance function of the form (1.1) and if its spectral density is continuous outside 0, then this random field exhibits isotropic long memory according to Definition 1.

One can easily construct non-isotropic long memory random fields. Given ξ, a white noise with variance $\sigma^2 \neq 0$, let X be defined by

$$X_{n_1,n_2} = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \hat{a}(k_1,k_2)\xi_{n_1-k_1,n_2-k_2},$$

where $\hat{a}$ is the Fourier transform of a function $a \in L^2([-\pi,\pi]^2)$. The spectral density of X is $f(x,y) = \sigma^2 a^2(x,y)$. Now, if $a(x,\ y) = |x+\theta y|^\alpha$, where $\theta \in \mathbb{R}$ and $-1/2 < \alpha < 0$, the field X is strongly dependent since its spectral density is unbounded at the origin, and this strong dependence is not isotropic. Moreover, Lemma 1 of Section 3 shows that the covariance function associated with this spectral density does not satisfy condition (1.1).

We follow the scheme of Lang and Soulier (2000) which relies on a spectral convergence theorem. In Section 2, we extend this convergence theorem to dimensions greater than 1. Let ξ be a d-dimensional random field having a bounded spectral density. We suppose that an invariance principle holds for ξ. Then we prove that its properly dilated spectral field weakly converges when the dilatation parameter tends to infinity. This theorem is directly applicable to obtain the asymptotic law of partial sums of fields obtained by filtering a white noise. These applications are presented in Sections 3 and 4. In Section 3, we obtain the limit in law of the partial sums of random processes and 2-dimensional random fields built from a filter which is either continuous and non-zero at $x = 0$, or equivalent at this point to an homogeneous function. In the first case the process exhibits weak dependence or isotropic long memory in the sense of Definition 1 and we merely find by a different method pre-existing results. In the second case there is non-isotropic long memory and the results we obtain for the partial sums are new. In dimension 1 we slightly extend the setting of Lang and Soulier (2000) as explained in Remark 2. In Section 4, we work with $d \geq 3$, situation in which it happens that the spectral theorem of Section 2 is less easily applicable. We restrict ourselves to some classes of random fields. Firstly we consider random fields for which the suitable normalisation of the partial sums is $n^{d/2}$. They are built from filters continuous and non zero at $x = 0$ and sufficiently smooth elsewhere. Secondly we present some non central limit theorems for partial sums with a normalisation stronger than $n^{d/2}$. Among them we consider non-isotropic long memory random fields having spectral singularities all over a linear subspace of $\mathbb{R}^d$.

Section 5 contains the proof of Theorem 1 and the appendix recalls some properties of approximations of unity.

2 The spectral convergence theorem

We show that when a random field satisfies a Donsker type theorem, the sequence of its renormalised spectral measure converges in a sense defined below.

Let $\xi := (\xi_k)_{k\in\mathbb{Z}^d}$ be a real stationary random field. We assume the following hypothesis about it :

H 1. *The stationary random field $(\xi_k)_{k\in\mathbb{Z}^d}$ is centered and has a spectral density f_ξ bounded by $M > 0$. Moreover, the sequence S_n^ξ of random functions defined on $]0,\infty[^d$ by :*

$$S_n^\xi(t_1,\dots,t_d) = n^{-d/2} \sum_{k_1=0}^{[nt_1]-1} \cdots \sum_{k_d=0}^{[nt_d]-1} \xi_{k_1,\dots,k_d} \tag{2.1}$$

converges in the finite dimensional distributions sense to a field B.

The sequence ξ_k admits the spectral representation :

$$\xi_k = \int_{[-\pi,\ \pi]^d} e^{i<k,x>} dW(x), \tag{2.2}$$

where the control measure of W has density f_ξ. For $n \geq 1$, we consider the dilated spectral measure W_n, the random measure on $[-n\pi;\ n\pi]^d$ defined by

$$W_n(A) = n^{d/2} W(n^{-1}A).$$

Theorem 1. *Under* **H 1***, there exists a linear application I from $L^2(\mathbb{R}^d)$ into $L^2(\Omega,\mathcal{A},P)$ which has the following properties :*

(i) $\forall \Phi \in L^2(\mathbb{R}^d) \quad E\left(I\left(\Phi\right)\right)^2 \leq (2\pi)^d M ||\Phi||_2^2$

(ii) $I\left(x \mapsto \prod_{j=1}^d \frac{e^{it_jx_j}-1}{ix_j}\right) = B(t_1,\dots,t_d)$

(iii) If Φ_n is a sequence of functions converging in $L^2(\mathbb{R}^d)$ to Φ, then $\int \Phi_n(x)dW_n(x)$ converges in law to $I\left(\Phi\right)$.

(iv) If ξ is a strong white noise, $\forall \Phi \in L^2(\mathbb{R}^d)$ $I(\Phi) = \int \Phi dW_0$, where W_0 is the Gaussian white noise measure.

The proof of this theorem is relegated to Section 5.

Remark 1. From (i), we see that I is not necessarily an isometry so that I cannot be viewed as a stochastic integral. The interpretation of I as a stochastic integral is allowed when ξ is a strong white noise. In this case, B is the Brownian sheet and property (ii) corresponds to its harmonisable representation :

$$B(t_1, \dots, t_d) = \int \prod_{j=1}^{d} \frac{e^{it_j x_j} - 1}{ix_j} dW_0(x_1, \dots, x_d).$$

Theorem 1 allows us to study the asymptotic of partial sums of random fields constructed by filtering the noise ξ. We now explain the way we use this theorem. The partial sums of X are :

$$S_n(t_1, \dots, t_d) = n^{-d/2} \sum_{k_1=0}^{[nt_1]-1} \dots \sum_{k_d=0}^{[nt_d]-1} X_{k_1,\dots,k_d}, \tag{2.3}$$

where

$$X_{n_1,\dots,n_d} = \sum \hat{a}(k_1, \dots, k_d)\xi_{n_1-k_1,\dots,n_d-k_d}, \tag{2.4}$$

where $\hat{a}(k_1, \dots, k_d)$ are, up to $(2\pi)^{d/2}$, the Fourier coefficients of the filter $a \in L^2([-\pi, \pi]^d)$ and verify

$$a(x) = \sum_{k \in \mathbb{Z}^d} \hat{a}(k_1, \dots, k_d) e^{-i<k,x>}.$$

Filter a is directly linked to the spectral density of X by the relation :

$$f_X(x) = f_\xi(x) a^2(x),$$

if f_ξ is the spectral density of ξ. In particular, when ξ is a white noise with variance σ^2, one has $f_X(x) = \frac{\sigma^2}{(2\pi)^d}|a^2(x)|$.

Then, using the spectral representation (2.2) and the definition of W_n :

$$S_n(t) = \int_{[-n\pi, n\pi]^d} a\left(\frac{x}{n}\right) \prod_{j=1}^{d} D_n(x_j, t_j) dW_n(x), \tag{2.5}$$

where

$$D_n(x_j, t_j) = \frac{e^{ix_j[t_j n]/n} - 1}{n(e^{ix_j/n} - 1)} \mathbb{1}_{[0,\ n\pi]}(x_j). \tag{2.6}$$

In order to investigate the convergence of partial sums $S_n(t)$, t being fixed, with the help of Theorem 1, one has only to study the $L^2(\mathbb{R}^d)$ convergence of $a(x/n) \prod_{j=1}^{d} D_n(x_j, t_j)$. The convergence of S_n will then follow according to (iii) of Theorem 1.

In the following, we use the notation

$$D(x_j, t_j) = \frac{e^{it_j x_j} - 1}{ix_j}. \tag{2.7}$$

We denote by $\xrightarrow{fidi}$ the convergence of the finite dimensional distributions.

3 Partial sums of random fields in dimension $d \leq 2$

The following theorem gives the asymptotic behaviour of the partial sums of a 2-dimensional random field constructed with a filter which is either continuous at the origin or equivalent at 0 to a homogeneous function. The notations are the same as in Section 2.

Theorem 2. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* **H 1**. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (2.4), constructed by filtering ξ through a and define $S_n(t)$ by (2.3) for all $t \in]0,\infty[^d$.*

(i) If the filter $a \in L^2([-\pi,\pi]^d)$ is continuous at the origin with $a(0) \neq 0$, then, for $d \leq 2$,

$$S_n(t) \xrightarrow[n\to\infty]{fidi} a(0)B(t), \tag{3.1}$$

where B is the limit of the partial sums of ξ introduced in hypothesis **H 1**.

(ii) If the function $a \in L^2([-\pi,\pi]^d)$ is equivalent at 0 to a homogeneous function $\tilde{a}$ with degree $\alpha \in]-1;0[$, i.e. $\forall\lambda\ \tilde{a}(\lambda x) = |\lambda|^\alpha\,\tilde{a}(x)$, then, for $d \leq 2$,

$$n^\alpha S_n(t) \xrightarrow[n\to\infty]{fidi} I\left(\tilde{a}(x)\prod_{i=1}^{d} D(x_i,t_i)\right), \tag{3.2}$$

where I is the linear application defined in Theorem 1.

Remark 2. When $d = 1$, the setting is the same as in Lang and Soulier (2000) ; however, we slightly extend the results of these authors since when they assume the filter to be continuous at the origin and bounded elsewhere, we only need the continuity at the origin. Similarly, in (ii), the filter does not need to be homogeneous on $[-\pi,\pi]$ but only at $x = 0$.

Remark 3. Filtering a white noise through a filter satisfying the hypothesis in (i) can produce a weakly dependent random field. It is the case for instance if a is continuous on $[-\pi,\pi]^d$. It can also produce long memory, for instance when a is unbounded at one ore several non-zero frequencies. In this situation, the memory is long as far as the covariance function is not absolutely summable. But, as expected, this memory, which involves only non-zero singularities of the spectral density, does not modify the limit obtained under weak dependence.

Remark 4. The form of the limit process in (3.2) is not very explicit in the general setting but remember that when ξ is a strong white noise, the limit process can be written as a stochastic integral with respect to a Gaussian white noise measure (cf Remark 1).

Remark 5. Condition (ii) of Theorem 2 can be satisfied with isotropic or non-isotropic long-memory. The long memory is non-isotropic for instance when the filter is a tensorial product of homogeneous filters (this is a particular case of Theorem 5 below when $d = 2$). Non-isotropic strong dependence also occurs when the filter is of the form $a(x,y) = |x+\theta y|^\alpha$, where $-1/2 < \alpha < 0$ and $\theta \in \mathbb{R}$, $\theta \neq 0$; indeed, the spectral density of the induced random field does not verify the assumptions of Definition 1 since it is proportional to $|x+\theta y|^{2\alpha}$. Furthermore Lemma 1 below shows that the covariance function of this random field has not the standard form (1.1).

Lemma 1. *Let $(X_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be a random field whose spectral density, defined on $[-\pi,\pi]^2$, is $f(x,y) = |x+\theta y|^{2\alpha}$, where $-1/2 < \alpha < 0$ and $\theta \in \mathbb{R}$. Then its covariance function does not verify (1.1).*

Proof of Lemma 1. One has

$$r(h,\ l) = \int_{[-\pi,\pi]^2} |x+\theta y|^{2\alpha} e^{i(hx+ly)}dxdy.$$

We restrict ourselves to the calculus of $r(h, \theta h)$, for h such that $\theta h \in \mathbb{Z}$, which suffices to conclude.

$$r(h,\ \theta h) = \int_{[-\pi,\pi]^2} |x+\theta y|^{2\alpha} e^{ih(x+\theta y)} dxdy.$$

We make the change of variable $u = x + \theta y$ and $v = \theta y - x$. Assume, without loss of generality, that $\theta \geq 1$; then one obtains the following new integration domain $I_1 \cup I_2 \cup I_3$ where :

$$I_1 = \begin{cases} -(\theta-1)\pi < u < (\theta-1)\pi \\ -2\pi + u < v < 2\pi + u, \end{cases}$$

$$I_2 = \begin{cases} -(\theta+1)\pi < u < -(\theta-1)\pi \\ -2\theta\pi - u < v < 2\pi + u, \end{cases}$$

$$I_3 = \begin{cases} (\theta-1)\pi < u < (\theta+1)\pi \\ -2\pi + u < v < 2\theta\pi - u, \end{cases}$$

and where $I_1 = \varnothing$ if $\theta = 1$. Hence one has

$$\begin{aligned} & r(h,\ \theta h) \\ &= \int_{-(\theta-1)\pi}^{(\theta-1)\pi} |u|^{2\alpha} e^{ihu} du + \int_{-(\theta+1)\pi}^{-(\theta-1)\pi} (2u + 2(\theta+1)\pi)|u|^{2\alpha} e^{ihu} du + \int_{-(\theta-1)\pi}^{(\theta+1)\pi} (2(\theta+1)\pi - 2u)|u|^{2\alpha} e^{ihu} du \\ &= 2\int_0^{(\theta-1)\pi} u^{2\alpha}\cos(hu)du + 4(\theta+1)\pi \int_{(\theta-1)\pi}^{(\theta+1)\pi} |u|^{2\alpha}\cos(hu)du - 4\int_{(\theta-1)\pi}^{(\theta+1)\pi} u|u|^{2\alpha}\cos(hu)du \\ &= \frac{1}{h^{2\alpha+1}}\left(2\int_0^{(\theta-1)\pi h} u^{2\alpha}\cos(u)du + 4(\theta+1)\pi\int_{(\theta-1)\pi h}^{(\theta+1)\pi h} |u|^{2\alpha}\cos(u)du - \frac{4}{h}\int_{(\theta-1)\pi h}^{(\theta+1)\pi h} u|u|^{2\alpha}\cos(u)du\right). \end{aligned}$$

The first integral converges to a non-zero limit and the two latest terms converge to 0. Hence

$$r(h, \theta h) \underset{h\to\infty}{\sim} ch^{-2\alpha-1}, \tag{3.3}$$

where c is a non-zero constant.

If the field X has a covariance function of the form (1.1), then, from (3.3), it would verify :

$$r(h,\ l) \underset{(h,\ l)\to\infty}{\sim} |(h,\ l)|^{-2\alpha-1} L(|(h,\ l)|)\, b\left(\frac{(h,\ l)}{|(h,\ l)|}\right).$$

One could then apply the partial sums convergence Theorem of Dobrushin and Major (1979) which requires the normalisation $n^{\alpha-1/2}L(n)^{-1/2}$. But Theorem 2 claims that the partial sums of X converge in law with normalisation n^{α}. Therefore, the covariance function of X is not of the form (1.1) and X is a non isotropic long memory random field. □

Proof of Theorem 2. We first prove (i) and we restrict ourselves to the proof of the convergence in law of $S_n(t)$, t being fixed. The convergence of the finite dimensional distributions follows easily. In order to apply Theorem 1 and the scheme explained in Section 2, we have to prove :

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left| a\left(\frac{x}{n}\right) \prod_{j=1}^d D_n(x_j, t_j) - a(0) \prod_{j=1}^d D(x_j, t_j) \right|^2 dx = 0. \tag{3.4}$$

Let us split the integral (3.4) :

$$\begin{aligned} & \int_{\mathbb{R}^d} \left| a\left(\frac{x}{n}\right) \prod_{j=1}^d D_n(x_j, t_j) - a(0) \prod_{j=1}^d D(x_j, t_j) \right|^2 dx \\ &= \int_{[-n\pi,n\pi]^d} \left| a\left(\frac{x}{n}\right) \prod_{j=1}^d D_n(x_j, t_j) - a(0) \prod_{j=1}^d D(x_j, t_j) \right|^2 dx + \int_{\cup_{j=1}^d \{|x_j|>n\pi\}} \left| a(0) \prod_{j=1}^d D(x_j, t_j) \right|^2 dx. \end{aligned}$$

The latest integral converges to 0, t being fixed, since the function $x_j \mapsto D(x_j, t_j)$ is continuous on $\mathbb{R}$ and verifies $|D(x_j,t_j)|^2 < 2x_j^{-2}$, so is integrable on $\mathbb{R}$. For the first integral :

$$\int_{[-n\pi,n\pi]^d} \left| a\left(\frac{x}{n}\right) \prod_{j=1}^{d} D_n(x_j,t_j) - a(0) \prod_{j=1}^{d} D(x_j,t_j) \right|^2 dx \leq$$

$$2\int_{[-n\pi,n\pi]^d} \left| a\left(\frac{x}{n}\right) - a(0)\right|^2 \prod_{j=1}^{d} |D_n(x_j,t_j)|^2 \, dx + 2\int_{[-n\pi,n\pi]^d} a^2(0) \left| \prod_{j=1}^{d} D_n(x_j,t_j) - \prod_{j=1}^{d} D(x_j,t_j) \right|^2 dx.$$

From Lemma 5 of Section 5, the latest integral vanishes when n tends to infinity. Finally, for the first integral, the change of variable $x/n \to x$ yields :

$$\int_{[-n\pi,n\pi]^d} \left| a\left(\frac{x}{n}\right) - a(0)\right|^2 \prod_{j=1}^{d} |D_n(x_j,t_j)|^2 \, dx = \int_{[-\pi,\pi]^d} |a(x)-a(0)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx, \quad (3.5)$$

with

$$\tilde{F}_{[nt_j]}(x_j) = 2\pi \frac{[nt_j]}{n} F_{[nt_j]}(x_j), \quad (3.6)$$

where F_n is the Fejer kernel :

$$F_n(x) = \begin{cases} \frac{1}{2\pi n} \frac{\sin^2(nx/2)}{\sin^2(x/2))} & \text{if } x \neq 0 \\ \frac{n}{2\pi} & \text{if } x = 0. \end{cases} \quad (3.7)$$

If $d = 1$, Theorem 7 of the appendix can be applied since $\tilde{F}_{[nt_1]}(x_1)$ is, up to a constant, a strong approximation of unity and because the function $|a(x)-a(0)|^2$ is continuous at $x = 0$. Therefore, according to (6.4), one obtains convergence (3.4). The same argument cannot be applied when $d \geq 2$ because the tensorial product $\prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j)$ is only a weak approximation of unity (cf Proposition 1 of the appendix). However, when $d = 2$, the result is still available as proved below.

We split the term (3.5) into :

$$\int_{[-\pi,\pi]^2} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx$$
$$= \int_{||x|| \leq \delta_n} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx + \int_{||x||>\delta_n} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx, \quad (3.8)$$

where the sequence $(\delta_n)_{n>0}$ shall be chosen below and where $||(x_1,x_2)|| = \max(|x_1|,|x_2|)$. From the continuity of $a(x)$ at $x = 0$, the first term tends to 0 as soon as $\delta_n \to 0$. Now,

$$\int_{||x||>\delta_n} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx \quad (3.9)$$
$$\leq \int_{|x_1|>\delta_n} \int_{-\pi}^{\pi} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx + \int_{|x_2|>\delta_n} \int_{-\pi}^{\pi} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx.$$

Both terms are treated in the same way, for example for the first one :

$$\int_{|x_1|>\delta_n} \int_{-\pi}^{\pi} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j) dx$$
$$= \int_{-\pi}^{\pi} \tilde{F}_{[nt_2]}(x_2) \left(\int_{|x_1|>\delta_n} |a(x)-a(0)|^2 \tilde{F}_{[nt_1]}(x_1) dx_1 \right) dx_2.$$

Proposition 2 of the appendix and Definition (3.6) of $\tilde{F}$ imply :

$$\sup_{|x_1|>\delta_n} \tilde{F}_{[nt_1]}(x_1) \leq \frac{\pi^2}{n\delta_n^2}.$$

Hence,

$$\int_{|x_1|>\delta_n} \int_{-\pi}^{\pi} |a(x)-a(0)|^2 \prod_{j=1}^{2} \tilde{F}_{[nt_j]}(x_j)dx \leq \frac{\pi^2}{\delta_n^2}\frac{1}{n}\int_{-\pi}^{\pi} \tilde{F}_{[nt_2]}(x_2)b(x_2)dx_2,$$

where $b(x_2) = \int_{[-\pi,\pi]} |a(x)-a(0)|^2 dx_1$ is integrable on $[-\pi,\pi]$.

One knows, from expression (3.6) of $\tilde{F}$ and Proposition 2 that

$$v_{2,n} = \frac{1}{n}\int_{-\pi}^{\pi} \tilde{F}_{[nt_2]}(x_2)b(x_2)dx_2 \underset{n\to\infty}{\longrightarrow} 0.$$

Then it suffices to choose $\delta_n^2 = (v_{1,n} \vee v_{2,n})^{1/2}$ where $v_{1,n}$ is defined like $v_{2,n}$ but with respect to t_1. One actually has $\lim_{n\to\infty} \delta_n = 0$ and, t_1 and t_2 being fixed, the convergence of each term of (3.9) to 0 is achieved.

We now investigate the proof of (ii), restricted to the convergence in law, t being fixed. We use Theorem 1 and prove the following convergence :

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left| n^{\alpha} a\left(\frac{x}{n}\right) \prod_{i=1}^{d} D_n(x_i,t_i) - \tilde{a}(x) \prod_{i=1}^{d} D(x_i,t_i) \right|^2 dx = 0. \tag{3.10}$$

We first give some properties of $\tilde{a}$

Lemma 2.

$$(i) \int_{[-\pi,\pi]^d} \tilde{a}^2(x)dx < \infty, \qquad (ii) \int_{\mathbb{R}^d} \tilde{a}^2(x) \prod_{i=1}^{d} (x_i^{-2} \wedge 1)dx < \infty.$$

Proof of Lemma 2. In the case $d=1$, the proof can be easily deduced from the case $d=2$. Hence we suppose $d=2$. Since $a(x)$ is equivalent to $\tilde{a}(x)$ when x goes to 0, it exists $0<\eta<\pi$ such that $||x|| < \eta$ yields $\tilde{a}^2(x)/a^2(x) < 2$. Thus,

$$\int_{||x||<\eta} \tilde{a}^2(x)dx = \int_{||x||<\eta} a^2(x)\frac{\tilde{a}^2(x)}{a^2(x)}dx \leq 2\int_{[-\pi,\pi]^d} a^2(x)dx < \infty.$$

As $\tilde{a}$ is homogeneous, using polar coordinates leads to

$$\int_{||x||<\eta} \tilde{a}^2(x)dx = \int_0^{\eta} r^{2\alpha+1}dr \int_0^{2\pi} \tilde{a}^2(\cos\theta,\sin\theta)d\theta = \eta^{2\alpha+2}\int_0^{2\pi} \tilde{a}^2(\cos\theta,\sin\theta)d\theta.$$

Therefore, the latest integral is finite and

$$\int_{[-\pi,\pi]^d} \tilde{a}^2(x)dx = \frac{\pi^{2\alpha+2}}{2\alpha+2}\int_0^{2\pi} \tilde{a}^2(\cos\theta,\sin\theta)d\theta < \infty,$$

that is (i) of Lemma 2.

For (ii),

$$\begin{aligned}
&\int_{\mathbb{R}^d} \tilde{a}^2(x) \prod_{i=1}^{d} (x_i^{-2} \wedge 1) dx \\
&\quad = \int_0^\infty \int_0^{2\pi} r^{2\alpha+1} \tilde{a}^2(\cos\theta, \sin\theta) \left((r^{-2}\cos^{-2}\theta) \wedge 1 \right) \left((r^{-2}\sin^{-2}\theta) \wedge 1 \right) d\theta dr \\
&\quad \leq \int_0^1 r^{2\alpha+1} dr \int_0^{2\pi} \tilde{a}^2(\cos\theta, \sin\theta) d\theta \\
&\qquad + \int_1^\infty \int_0^{2\pi} r^{2\alpha+1} \tilde{a}^2(\cos\theta, \sin\theta) \left((r^{-2}\cos^{-2}\theta) \wedge 1 \right) \left((r^{-2}\sin^{-2}\theta) \wedge 1 \right) d\theta dr
\end{aligned}$$

The first integral on $\{r \leq 1\}$ is finite. For the latest one, on $\{r > 1\}$, consider, without loss of generality, the integral with respect to θ on $[0, \pi]$. We split $[0, \pi]$ in two parts : on $\{|\theta - \pi/2| < \pi/4\}$, $\sin^{-2}\theta < 2$ and on $\{|\theta - \pi/2| > \pi/4\}$, $\cos^{-2}\theta < 2$, therefore

$$\begin{aligned}
&\int_1^\infty \int_0^{2\pi} r^{2\alpha+1} |\tilde{a}|^2(\cos\theta, \sin\theta) \left((r^{-2}\cos^{-2}\theta) \wedge 1 \right) \left((r^{-2}\sin^{-2}\theta) \wedge 1 \right) d\theta \\
&\qquad\qquad \leq c \int_1^\infty r^{2\alpha-1} dr \int_0^{2\pi} |\tilde{a}|^2(\cos\theta, \sin\theta) d\theta < \infty.
\end{aligned}$$

Consequently, *(ii)* of Lemma 2 is proved.

□

We return to the proof of (3.10).

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| n^\alpha a\left(\frac{x}{n}\right) \prod_{i=1}^{d} D_n(x_i, t_i) - \tilde{a}(x) \prod_{i=1}^{d} D(x_i, t_i) \right|^2 dx \leq \\
&2 \int_{[-n\pi, n\pi]^d} \left| n^\alpha a\left(\frac{x}{n}\right) - \tilde{a}(x) \right|^2 \prod_{i=1}^{d} |D_n(x_i, t_i)|^2 dx + 2 \int_{\mathbb{R}^d} \tilde{a}^2(x) \left| \prod_{i=1}^{d} D_n(x_i, t_i) - \prod_{i=1}^{d} D(x_i, t_i) \right|^2 dx.
\end{aligned} \tag{3.11}$$

The following lemma treats the convergence of the first integral in (3.11).

Lemma 3.

$$\lim_{n \to \infty} \int_{[-n\pi, n\pi]^d} \left| n^\alpha a\left(\frac{x}{n}\right) - \tilde{a}(x) \right|^2 \prod_{i=1}^{d} |D_n(x_i, t_i)|^2 dx = 0$$

Proof of Lemma 3. After a change of variables and thanks to the homogeneity of $\tilde{a}$, we have :

$$\int_{[-n\pi, n\pi]^d} \left| n^\alpha a\left(\frac{x}{n}\right) - \tilde{a}(x) \right|^2 \prod_{i=1}^{d} |D_n(x_i, t_i)|^2 dx = n^{2\alpha} \int_{[-\pi, \pi]^d} |a(x) - \tilde{a}(x)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx,$$

where $\tilde{F}_{[nt_j]}$ is defined in (3.6). Let $\alpha < \beta < 0$. We split the latest integral in the following way :

$$n^{2\alpha} \int_{[-\pi, \pi]^d} |a(x) - \tilde{a}(x)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx \tag{3.12}$$

$$= n^{2\alpha} \int_{||x|| \leq n^\beta} |a(x) - \tilde{a}(x)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx + n^{2\alpha} \int_{||x|| > n^\beta} |a(x) - \tilde{a}(x)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx.$$

Since $\beta > \alpha$ and from Lemma 2, the convergence to 0 of the second integral is proved exactly as the one of integral (3.9). For the first one, we use the fact that $a(x)$ is equivalent to $\tilde{a}(x)$ at $x = 0$. Let us fix $\varepsilon > 0$, then there exists n_0 such that for all $n > n_0$, $||x|| \leq n^\beta$ yields

$$|a(x) - \tilde{a}(x)| \leq \varepsilon |\tilde{a}(x)|.$$

Thus, for all $n > n_0$,

$$\begin{aligned} n^{2\alpha} \int_{||x|| \leq n^\beta} |a(x) - \tilde{a}(x)|^2 \prod_{j=1}^d \tilde{F}_{[nt_j]}(x_j) dx &\leq \varepsilon n^{2\alpha} \int_{||x|| \leq n^\beta} \tilde{a}^2(x) \prod_{j=1}^d \tilde{F}_{[nt_j]}(x_j) dx \\ &\leq c\varepsilon \int_{||x|| \leq n^{\beta+1}} \tilde{a}^2(x) \prod_{j=1}^d \frac{\sin^2\left(\frac{[nt_j]}{n}\frac{x_j}{2}\right)}{x_j^2} \frac{(\frac{x_j}{2n})^2}{\sin^2(\frac{x_j}{2n})} dx \\ &\leq c\varepsilon \int_{\mathbb{R}^d} \tilde{a}^2(x) \prod_{j=1}^d \frac{(1 \wedge t_j^2 x_j^2)}{x_j^2} dx, \end{aligned}$$

where c is a non-zero constant changing from line to line. The last integral is finite from (ii) of Lemma 2 and the convergence to O of (3.12) is proved. □

Let us return to the proof of (ii) of Theorem 2. From (3.11) and Lemma 3, it remains to prove :

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(x_i, t_i) - \prod_{i=1}^d D(x_i, t_i) \right|^2 dx = 0.$$

Let us split this integral as follows :

$$\begin{aligned} &\int_{\mathbb{R}^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(x_i, t_i) - \prod_{i=1}^d D(x_i, t_i) \right|^2 dx \\ &= \int_{[-n\pi, n\pi]^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(x_i, t_i) - \prod_{i=1}^d D(x_i, t_i) \right|^2 dx + \int_{\cup_i \{|x_i| > n\pi\}} \tilde{a}^2(x) \prod_{i=1}^d |D(x_j, t_j)|^2 dx. \end{aligned}$$

The second integral converges to 0. For the first one, the change of variable $x/n \to x$ yields :

$$\begin{aligned} &\int_{[-n\pi, n\pi]^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(x_i, t_i) - \prod_{i=1}^d D(x_i, t_i) \right|^2 dx \\ &\qquad = n^{2\alpha+d} \int_{[-\pi, \pi]^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(nx_i, t_i) - \prod_{i=1}^d D(nx_i, t_i) \right|^2 dx. \end{aligned}$$

From Lemma 5 of Section 5, one has

$$\sup_{x \in [-\pi, \pi]^d} \left| \prod_{i=1}^d D_n(nx_i, t_i) - \prod_{i=1}^d D(nx_i, t_i) \right|^2 = O(n^{-2}). \tag{3.13}$$

Since $\int_{[-\pi, \pi]^d} \tilde{a}^2(x) dx < \infty$, we obtain

$$\int_{[-n\pi, n\pi]^d} \tilde{a}^2(x) \left| \prod_{i=1}^d D_n(x_i, t_i) - \prod_{i=1}^d D(x_i, t_i) \right|^2 dx = O(n^{2\alpha+d-2}).$$

Since $d \leq 2$ and $\alpha < 0$, this term converges to 0.

□

4 Partial sums of random fields in dimension $d \geq 3$

As explained in Section 2, as far as our basic tool, for proving Theorem 2, is the spectral convergence Theorem 1, we need to establish the L^2-convergence of $a(x/n)\prod_{j=1}^{d} D_n(x_j, t_j)$. The following example proves that this method does not allow us to extend result (3.1), without further hypothesis, to dimensions $d \geq 3$. The following filter actually satisfies the hypothesis of (i) in Theorem 2 but the L^2-convergence of $a(x/n)\prod_{j=1}^{d} D_n(x_j, t_j)$ does not occur. Let filter a be defined on $[-\pi, \pi]^3$ by :

$$\begin{aligned} a(x, y, z) &= 1 && \text{if } |x| \leq c \text{ or if } yz = 0 \\ &= |y|^{-\alpha}|z|^{-\alpha} + 1 && \text{elsewhere,} \end{aligned}$$

where $1/4 < \alpha < 1/2$ and $0 < c < \pi$. One can prove that this filter is continuous at 0 with $a(0) = 1$ and that $a(x/n)\prod_{j=1}^{3} D_n(x_j, t_j)$ has no finite limit in $L^2([-\pi, \pi]^3)$.

Hereafter, we present two kind of results in dimension $d \geq 3$. The first ones relate to random fields constructed with a filter supposed to be continuous and non-zero at $x = 0$. Since this condition is not sufficient to obtain convergence results (as explained above), we suppose that this filter either is bounded or belongs to a particular class of functions allowing unboundedness. The second ones relate to random fields which lead to a non central limit theorem. They are constructed with a filter not necessary continuous at $x = 0$. It is actually supposed either to be a tensorial product or to be homogeneous. In this latter case, the spectral density of the random field can be unbounded all over a linear subspace of $\mathbb{R}^d$ yielding non-isotropic long memory.

4.1 Random fields constructed by a filter continuous at frequency 0

We first suppose that the filter leading to the field X is continuous and non-zero at $x = 0$ and that it is bounded. Then the partial sums S_n of X converge to the limit of the partial sums of ξ, in particular to the Brownian sheet if ξ is a strong white noise.

Theorem 3. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* **H 1**.

Let $a \in L^2[-\pi, \pi]^d$ be bounded on $[-\pi, \pi]^d$ and such that a is continuous at 0 and $a(0) \neq 0$.

Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (2.4), constructed by filtering ξ through a and define $S_n(t)$ by (2.3) for $t \in]0, \infty[^d$, thus

$$S_n(t) \xrightarrow[n\to\infty]{fidi} a(0)B(t),$$

where B is the limit of the partial sums of ξ introduced in hypothesis **H 1**.

Proof. We prove the convergence in law, t being fixed, the convergence of the finite dimensional distributions follows then easily. We only have to prove that :

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left| a\left(\frac{x}{n}\right) \prod_{j=1}^{d} D_n(x_j, t_j) - a(0) \prod_{j=1}^{d} D(x_j, t_j) \right|^2 dx = 0. \tag{4.1}$$

We split the above integral as in the proof of Theorem 2 and use the same argument, leading to the study of the integral

$$\int_{[-\pi,\pi]^d} |a(x) - a(0)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx.$$

For any fixed $t_j > 0$, $\tilde{F}_{[nt_j]}(x_j)$ is, up to a constant, a strong approximation of unity. From Proposition 1 of the appendix, $\prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j)$ is a weak approximation of unity. Since the function $|a(x) - a(0)|^2$ is continuous at $x = 0$ and is bounded on $[-\pi, \pi]^d$, Theorem 7 and (6.3) lead to

$$\lim_{n\to 0} \int_{[-\pi,\pi]^d} |a(x) - a(0)|^2 \prod_{j=1}^{d} \tilde{F}_{[nt_j]}(x_j) dx = 0.$$

□

Now, we restrict ourselves to a particular class of random fields, constructed with a filter continuous at $x = 0$ but non necessarily bounded.

Theorem 4. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* **H1**. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (2.4).*

Suppose that the filter $a \in L^2([-\pi,\pi]^d)$ is of the following form :

$$a(x_1,\ldots,x_d) = g\left(\sum_{i=1}^d \lambda_i x_i\right),$$

where the λ_i's are real constants and g is a function defined on a compact set of $\mathbb{R}$, square integrable and continuous at $x = 0$ with $g(0) \neq 0$.

Define the partial sums $S_n(t)$ by (2.3) for all $t \in]0,\infty[^d$. Then

$$S_n(t) \xrightarrow[n\to\infty]{fidi} g(0)B(t),$$

where B is the limit of the partial sums of ξ introduced in hypothesis **H 1**.

Proof. We have to prove convergence (3.4). We follow the same argument as in the proof of Theorem 2, leading to the study of :

$$\int_{[-\pi,\pi]^d} |a(x) - a(0)|^2 \prod_{j=1}^d \tilde{F}_{[nt_j]}(x_j)dx = \int_{[-\pi,\pi]^d} \left| g\left(\sum_{i=1}^d \lambda_i x_i\right) - g(0)\right|^2 \prod_{j=1}^d \tilde{F}_{[nt_j]}(x_j)dx.$$

We suppose, without loss of generality, that $\lambda_1 \neq 0$ and we make the change of variable $u = x_1 + \sum_{i=2}^d \frac{\lambda_i}{\lambda_1} x_i$, while the others variables remain unchanged. Hence,

$$\int_{[-\pi,\pi]^d} |a(x) - a(0)|^2 \prod_{j=1}^d \tilde{F}_{[nt_j]}(x_j)dx$$
$$\leq \int_{[-\tau,\tau]} |g(\lambda_1 u) - g(0)|^2 \left(\int_{[-\pi,\pi]^{d-1}} \tilde{F}_{[nt_1]}\left(u - \sum_{i=2}^d \frac{\lambda_i}{\lambda_1} x_i\right) \prod_{j=2}^d \tilde{F}_{[nt_j]}(x_j)dx_2\ldots dx_d\right) du,$$

where $[-\tau,\tau]$ is a compact set which contains the integration domain of u. Define

$$K_n(u) = \int_{[-\pi,\pi]^{d-1}} \tilde{F}_{[nt_1]}\left(u - \sum_{i=2}^d \frac{\lambda_i}{\lambda_1} x_i\right) \prod_{j=2}^d \tilde{F}_{[nt_j]}(x_j)dx_2\ldots dx_d,$$

one has then to prove that as $n \to \infty$,

$$\int_{[-\tau,\tau]} |g(\lambda_1 u) - g(0)|^2 K_n(u)du \longrightarrow 0. \tag{4.2}$$

In the particular case when for every i, $\lambda_i = t_i = 1$, K_n is, up to a multiplicative constant, the $(d-1)^{\text{th}}$ convolution product of the Fejer kernel F_n with itself. According to Proposition 1 of the appendix, K_n is therefore, up to a constant, a strong approximation of unity on $[-\tau,\tau]$.

Then Theorem 7 concludes the proof, since $|g(\lambda_1 u) - g(0)|^2$ is summable and continuous at $u = 0$.

In the general case when some λ_i's (or some t_i's) are not equal to 1, it is easy to prove that the above result still holds. □

4.2 Random fields with a spectral density unbounded at 0

We first suppose that the filter leading to the field X is a tensorial product. This situation is in fact very similar to the dimension 1. In the following theorem, the tensorial product is built from 1-dimensional filters corresponding either to (i) or to (ii) of Theorem 2.

Theorem 5. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* **H1**. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (2.4), where the filter a has the form :*

$$a(x_1,\ldots,x_d)=\prod_{j=1}^{d}a_j(x_j), \tag{4.3}$$

expression in which :

- *either $a_j\in L^2([-\pi,\pi])$ is continuous at 0 with $a_j(0)\neq 0$*
- *or $a_j\in L^2([-\pi,\pi])$ is equivalent at 0 to $\tilde{a}_j$ where $\tilde{a}_j$ is homogeneous of degree $\alpha_j\in\,]-1,0[$.*

Define the partial sums $S_n(t)$ by (2.3) for all $t\in\,]0,\infty[^d$. Denoting $\mathcal{J}$ the set of indexes j such that a_j is homogeneous of degree α_j and denoting $\mathcal{I}$ the others indexes, one has the following convergence in law :

$$n^{(\sum_{j\in\mathcal{J}}\alpha_j)}S_n(t)\xrightarrow[n\to\infty]{fidi}\left(\prod_{j\in\mathcal{I}}a_j(0)\right)\;I\left(\prod_{j\in\mathcal{J}}\tilde{a}_j(x_j)\prod_{j=1}^{d}D(x_j,t_j)\right),$$

where I is the linear application defined in Theorem 1.

Remark 6. If ξ is a strong white noise, one can write I as a stochastic integral (cf Remark 1 above). In this case, and when for all j, $\tilde{a}_j(x)=|x|^{-\alpha_j}$ with $0<\alpha_j<1/2$, the limit is the fractional Brownian sheet :

$$n^{(\sum_{j=1}^{d}\alpha_j)}S_n(t)\xrightarrow[n\to\infty]{fidi}\int_{\mathbb{R}^d}\prod_{j=1}^{d}\frac{e^{it_jx_j}-1}{ix_j|x_j|^{\alpha_j}}dW_0(x),$$

where W_0 is defined in Remark 1.

Proof of Theorem 5. The convergence of the finite dimensional distributions follows from the convergence in law, t being fixed. In order to apply Theorem 1, we have to prove :

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\left|n^{(\sum_{j\in\mathcal{J}}\alpha_j)}\prod_{j=1}^{d}a_j\left(\frac{x_j}{n}\right)D_n(x_j,t_j)-\prod_{j\in\mathcal{I}}a_j(0)\prod_{j\in\mathcal{J}}\tilde{a}_j(x_j)\prod_{j=1}^{d}D(x_j,t_j)\right|^2dx=0.$$

This result is not difficult to obtain by inference, using the decomposition :

$$AB-CD=(A-C)(B-D)+(A-C)D+(B-D)C.$$

□

Theorem 6 below relates to non-isotropic long memory fields whose spectral density is singular all over a linear subspace of $\mathbb{R}^d$.

Theorem 6. *Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be a stationary random field satisfying* **H1**. *Let $(X_k)_{k\in\mathbb{Z}^d}$ be the random field defined by (2.4).*

Define the partial sums $S_n(t)$ by (2.3) for all $t\in\,]0,\infty[^d$.

Suppose that a has the following form :

$$a(x)=\left|\sum_{i=1}^{d}\lambda_ix_i\right|^{\alpha},$$

where $-1/2 < \alpha < 0$ and the λ_i's are real constants. Then,

- If $d \leq 3$,

$$n^{\alpha} S_n(t) \xrightarrow[n\to\infty]{fidi} I\left(a(x)\prod_{i=1}^{d} D(x_i,t_i)\right), \tag{4.4}$$

where I is the linear application defined in Theorem 1.

- If $d \geq 4$ and $-\frac{1}{d-2} < 2\alpha < 0$ then convergence (4.4) holds.

Proof. We have to prove :

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(x_i,t_i)-\prod_{i=1}^{d} D(x_i,t_i)\right|^2 dx = 0.$$

We split this integral in the following way :

$$\int_{\mathbb{R}^d}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(x_i,t_i)-\prod_{i=1}^{d} D(x_i,t_i)\right|^2 dx$$
$$= \int_{[-n\pi,n\pi]^d}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(x_i,t_i)-\prod_{i=1}^{d} D(x_i,t_i)\right|^2 dx + \int_{\cup_i\{|x_i|>n\pi\}}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\prod_{i=1}^{d}|D(x_j,t_j)|^2 dx.$$

The second integral tends to 0 ; for the first one, the change of variable $x/n \to x$ yields :

$$\int_{[-n\pi,n\pi]^d}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(x_i,t_i)-\prod_{i=1}^{d} D(x_i,t_i)\right|^2 dx$$
$$= n^{2\alpha+d}\int_{[-\pi,\pi]^d}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(nx_i,t_i)-\prod_{i=1}^{d} D(nx_i,t_i)\right|^2 dx. \tag{4.5}$$

We now define the following set :

$$\mathcal{A}_n = \left\{x\in[-\pi,\pi]^d \ ; \ \left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha} \geq n^{\gamma} \text{ where } \gamma < 1-2\alpha\right\}.$$

Denoting $\overline{\mathcal{A}_n}$ the complementary set of $\mathcal{A}_n$ in $[-\pi,\pi]^d$, we split (4.5) according to $\mathcal{A}_n$ and $\overline{\mathcal{A}_n}$. Lemma 5 of Section 5 claims that

$$\int_{[-n\pi,n\pi]^d}\left|\prod_{j} D_n(x_j,t_j)-\prod_{j} D(x_j,t_j)\right|^2 dx = O(n^{-1}),$$

hence

$$\int_{[-\pi,\pi]^d}\left|\prod_{j} D_n(nx_j,t_j)-\prod_{j} D(nx_j,t_j)\right|^2 dx = O(n^{-d-1}). \tag{4.6}$$

Thus :

$$n^{2\alpha+d}\int_{\overline{\mathcal{A}_n}}\left|\sum_{i=1}^{d}\lambda_i x_i\right|^{2\alpha}\left|\prod_{i=1}^{d} D_n(nx_i,t_i)-\prod_{i=1}^{d} D(nx_i,t_i)\right|^2 dx$$
$$\leq n^{2\alpha+d+\gamma}\int_{[-\pi,\pi]^d}\left|\prod_{i=1}^{d} D_n(nx_i,t_i)-\prod_{i=1}^{d} D(nx_i,t_i)\right|^2 dx = O(n^{2\alpha-1+\gamma}).$$

Since $\gamma < 1 - 2\alpha$, this term tends to 0 when $n \to \infty$.

It remains to study integral (4.5) on $\mathcal{A}_n$. From (3.13)

$$n^{2\alpha+d} \int_{\mathcal{A}_n} \left|\sum_{i=1}^{d} \lambda_i x_i\right|^{2\alpha} \left|\prod_{i=1}^{d} D_n(nx_i, t_i) - \prod_{i=1}^{d} D(nx_i, t_i)\right|^2 dx \le O(n^{2\alpha+d-2}) \int_{\mathcal{A}_n} \left|\sum_{i=1}^{d} \lambda_i x_i\right|^{2\alpha} dx. \tag{4.7}$$

After a change of variables,

$$\int_{\mathcal{A}_n} \left|\sum_{i=1}^{d} \lambda_i x_i\right|^{2\alpha} dx = \int_{-n^{\frac{\gamma}{2\alpha}}}^{n^{\frac{\gamma}{2\alpha}}} |u|^{2\alpha} du \int_{[-\pi,\pi]^{d-1}} dx_2 \dots dx_d = c\, n^{\frac{\gamma}{2\alpha}(2\alpha+1)},$$

where c is a positive constant. In order to obtain the convergence to zero of the right hand side of (4.7), we need

$$\int_{\mathcal{A}_n} \left|\sum_{i=1}^{d} \lambda_i x_i\right|^{2\alpha} dx = o(n^{2-d-2\alpha}),$$

which is satisfied if

$$\frac{\gamma}{2\alpha}(2\alpha+1) - 2 + d + 2\alpha < 0. \tag{4.8}$$

If $d = 3$,

$$\frac{\gamma}{2\alpha}(2\alpha+1) + 2\alpha + 1 < 0 \Leftrightarrow \gamma > -2\alpha$$

and (4.4) is proved by choosing γ in $]-2\alpha;\ 1-2\alpha[$.

When $d \ge 4$, condition (4.8) is fulfilled if

$$\gamma > -2\alpha \frac{d + 2\alpha - 2}{2\alpha + 1},$$

which is possible as soon as $-\frac{1}{d-2} < 2\alpha < 0$, according to the initial condition on γ.

□

5 Proof of Theorem 1

Let's start with a first lemma stating the measurability of the field B in **H 1**.

Lemma 4. *Under* **H 1***, the field B is measurable and separable.*

The proof of Lemma 4 is relegated to the section 5.1.

We follow the scheme of Lang and Soulier (2000) whose proof deals with dimension 1. Let us define B_n on $\mathbb{R}^d$ by :

$$B_n(t_1, \dots, t_d) = \int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^{d} \frac{e^{it_j x_j} - 1}{ix_j} dW_n(x_1, \dots, x_d). \tag{5.1}$$

The integral is well defined since the integrand, being the Fourier transform of $\mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]}$ in $L^2(\mathbb{R}^d)$, is square integrable.

The spirit of the proof consists in defining the stochastic integral with respect to B_n in $L^2(\mathbb{R}^d)$. Then we write it in a form easily adaptable, via convergence arguments, to the random field B. This finally permits to define I.

The density of W_n is $f_n(x) = f(n^{-1}x)$ $(x \in [-n\pi, n\pi]^d)$ where f is the spectral measure of ξ. As f is uniformly bounded by M, so is f_n. Given $t_j \in \mathbb{R}$ and $x_j \in \mathbb{R}$, one has, with the help of Definition (2.7) of D

$$\begin{aligned} E(B_n(t_1,\ldots,t_d)^2) &= \int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^d |D(x_j,t_j)|^2 f_n(x)dx \\ &\le M \prod_{j=1}^d \int_{-\infty}^{\infty} |D(x_j,t_j)|^2 dx_j \\ &\le M \prod_{j=1}^d 4|t_j| \int_0^{\infty} \frac{1-\cos u}{u^2} du \le c \prod_{j=1}^d |t_j|, \end{aligned} \tag{5.2}$$

where c is a positive constant which may change in the sequel.

We first prove the convergence of the finite dimensional distributions of B_n to those of B. One can rewrite S_n^ξ of (2.1) in the following form :

$$\begin{aligned} S_n^\xi(t) &= n^{-d/2} \sum_{k=0}^{[nt]-1} \int_{[-\pi,\ \pi]^d} e^{i<k;\ x>} dW(x) \\ &= n^{-d} \sum_{k=0}^{[nt]-1} \int_{[-n\pi,\ n\pi]^d} e^{i<k;\ x/n>}\ dW_n(x) \\ &= \int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^d n^{-1} \left(\sum_{k_j=0}^{[nt_j]-1} e^{ik_j x_j/n} \right) dW_n(x) \\ &= \int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^d D_n(x_j,\ t_j)\ dW_n(x), \end{aligned}$$

with D_n defined by (2.6).

Hence,

$$\begin{aligned} &E\left(B_n(t_1,\ldots,t_d) - S_n^\xi(t_1,\ldots,t_d)\right)^2 \\ &\qquad = \int_{[-n\pi,\ n\pi]^d} \left| \prod_{j=1}^d D(x_j,t_j) - \prod_{j=1}^d D_n(x_j,t_j) \right|^2 f_n(x)dx. \end{aligned} \tag{5.3}$$

The sequence of functions f_n is uniformly bounded by M, thus, by Lemma 5 below, we obtain the convergence of (5.3) to zero. This convergence, joined with hypothesis **H 1**, yields the finite dimensional distributions convergence of B_n to B.

Lemma 5. *We have the following rates of convergence :*

$$\sup_{x\in[-n\pi,n\pi]^d} \left| \prod_{j=1}^d D_n(x_j,t_j) - \prod_{j=1}^d D_(x_j,t_j) \right|^2 \le O(n^{-2}) \tag{5.4}$$

$$\int_{[-n\pi,\ n\pi]^d} \left| \prod_{j=1}^d D_n(x_j,t_j) - \prod_{j=1}^d D(x_j,t_j) \right|^2 dx \le O(n^{-1}) \tag{5.5}$$

Proof of Lemma 5. For any fixed t and for $|x| < n\pi$:

$$\begin{aligned}|D_n(x,t)-D(x,t)| &= \left|\frac{e^{ix[tn]/n}-1}{n(e^{ix/n}-1)}-\frac{e^{itx}-1}{ix}\right| \\ &\leq \left|(e^{ix[tn]/n}-1)\left(\frac{1}{n(e^{ix/n}-1)}-\frac{1}{ix}\right)\right|+\left|\frac{e^{ix[tn]/n}-e^{itx}}{ix}\right| \\ &\leq 2\left|\frac{1}{n(e^{ix/n}-1)}-\frac{1}{ix}\right|+\left|\frac{e^{ix[tn]/n}-e^{itx}}{ix}\right|.\end{aligned}$$

On one hand we have :

$$\begin{aligned}\left|\frac{1}{n(e^{i\frac{x}{n}}-1)}-\frac{1}{ix}\right|^2 &= \frac{|ix-n(e^{i\frac{x}{n}}-1)|^2}{4n^2x^2\sin^2(\frac{x}{2n})} \\ &= \frac{(x-n\sin(\frac{x}{n}))^2+n^2\sin^4(\frac{x}{2n})}{4n^2x^2\sin^2(\frac{x}{2n})} \\ &= \frac{\sin^2(\frac{x}{2n})}{x^2}+\frac{(\frac{x}{n}-\sin(\frac{x}{n}))^2}{4x^2\sin^2(\frac{x}{2n})}.\end{aligned}$$

The first summand is not greater than $1/(4n^2)$ and the second one is an even function of $u = x/n$ which belongs to $[-\pi, \pi]$; furthermore,

$$\frac{(u-\sin(u))^2}{4n^2u^2\sin^2(\frac{u}{2})} \leq \frac{1}{n^2}\,\frac{1\wedge u^4}{4\sin^2(\frac{u}{2})} \leq c\frac{1}{n^2} \qquad \forall u \in [0,\pi],$$

where $1 \wedge u^4 = min(1, u^4)$. Hence,

$$\left|\frac{1}{n(e^{i\frac{x}{n}}-1)}-\frac{1}{ix}\right|^2 \leq c\frac{1}{n^2}. \tag{5.6}$$

On the other hand,

$$\begin{aligned}\left|\frac{e^{ix\frac{[tn]}{n}}-e^{itx}}{ix}\right|^2 &= \frac{\left|e^{ix(\frac{[tn]}{n}-t)}-1\right|^2}{x^2} \\ &= \frac{4}{x^2}\sin^2\left(\frac{x}{2}\left(\frac{[tn]}{n}-t\right)\right) \\ &\leq \left|\frac{[tn]}{n}-t\right|^2 = O\left(\frac{1}{n^2}\right).\end{aligned} \tag{5.7}$$

Hence,

$$\sup_{x\in[-n\pi,n\pi]}|D_n(x,t)-D(x,t)|^2 \leq O\left(\frac{1}{n^2}\right). \tag{5.8}$$

Moreover, (f_n) is uniformly bounded, so

$$\int_{-n\pi}^{n\pi}|D_n(t,x)-D(t,x)|^2 f_n(x)dx \leq O\left(\frac{1}{n^2}\right)\int_{-n\pi}^{n\pi}f_n(x)dx \leq O\left(\frac{1}{n}\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{5.9}$$

Inequalities (5.8) and (5.9) prove Lemma 5 for $d = 1$. Now when $d = 2$

$$\begin{aligned}&\sup_{x\in[-n\pi,n\pi]^2}|D_n(x_1,t_1)D_n(x_2,t_2)-D(x_1,t_1)D(x_2,t_2)|^2 \\ &\leq 3\sup_{x\in[-n\pi,n\pi]^2}|D_n(x_1,t_1)-D(x_1,t_1)|^2\sup_{x\in[-n\pi,n\pi]^2}|D_n(x_2,t_2)-D(x_2,t_2)|^2 \\ &\quad+3\sup_{x\in[-n\pi,n\pi]^2}|D_n(x_1,t_1)-D(x_1,t_1)|^2\sup_{x\in[-n\pi,n\pi]^2}|D(x_2,t_2)|^2 \\ &\quad+3\sup_{x\in[-n\pi,n\pi]^2}|D_n(x_2,t_2)-D(x_2,t_2)|^2\sup_{x\in[-n\pi,n\pi]^2}|D(x_1,t_1)|^2.\end{aligned}$$

For fixed t, $D(.,t)$ is bounded, from (5.8) we obtain the first inequality of Lemma 5. It is easy to extend this proof to $d > 2$.

The proof of (5.5) is similar, changing the sup norm for the L^1-norm. □

Let us return to the proof of Theorem 1. For every Φ in $L^2(\mathbb{R}^d)$, we denote by $\hat{\Phi}$ its Fourier transform defined in such a way that the transformation $\Phi \mapsto \hat{\Phi}$ is an isometry. Let us consider the linear mapping from $L^2(\mathbb{R}^d)$ into $L^2(\Omega)$ which transforms Φ into $(2\pi)^{d/2} \int \hat{\Phi} dW_n$.

Applying this mapping to the function $\mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]}$, we obtain (5.1). It remains to interpret $B_n(t_1, \ldots, t_d)$ as $\int \mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]} dB_n(t)$ and to extend this stochastic integral to $\Phi \in L^2(\mathbb{R}^d)$. This leads to

$$\int_{\mathbb{R}^d} \Phi(t) dB_n(t) = (2\pi)^{d/2} \int_{[-n\pi,\ n\pi]^d} \hat{\Phi}(x) dW_n(x). \tag{5.10}$$

We now investigate the convergence in law of this integral.

Suppose firstly that Φ is compactly supported and differentiable. We can then rewrite $\hat{\Phi}$: we transform

$$\hat{\Phi}(x_1, \ldots, x_d) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(t_1, \ldots, t_d) e^{it_1x_1} \ldots e^{it_dx_d} dt_1 \ldots dt_d$$

by d successive integrations by parts. The first step is the following :

$$\begin{aligned}\hat{\Phi}(x) =&(2\pi)^{-d} \int_{\mathbb{R}^{d-1}} \left[\Phi(t) \frac{e^{i(t_1x_1+\cdots+t_dx_d)}}{ix_1}\right]_{t_1\in\mathbb{R}} dt_2 \ldots dt_d \\ &- (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1} \left(\frac{e^{i(t_1x_1+\cdots+t_dx_d)}}{ix_1} + (-1)^{d-(d-1)} \frac{e^{i(t_2x_2+\cdots+t_dx_d)}}{ix_1} \right) dt,\end{aligned}$$

the first term vanishes since Φ vanishes outside a compact, yielding

$$\begin{aligned}\hat{\Phi}(x) &= -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1} \left(\frac{e^{i(t_1x_1+\cdots+t_dx_d)}}{ix_1} - \frac{e^{i(t_2x_2+\cdots+t_dx_d)}}{ix_1} \right) dt \\ &= -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1} \frac{e^{it_1x_1} - 1}{ix_1} e^{i(t_2x_2+\cdots+t_dx_d)} dt.\end{aligned}$$

The next integration by parts gives :

$$\begin{aligned}\hat{\Phi}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1 \partial t_2} \Bigg(&\frac{e^{i(t_1x_1+\cdots+t_dx_d)}}{ix_1ix_2} - \frac{e^{i(t_2x_2+\cdots+t_dx_d)}}{ix_1ix_2} \\ &+ \frac{e^{i(t_3x_3+\cdots+t_dx_d)}}{ix_1ix_2} - \frac{e^{it_1x_1} e^{i(t_3x_3+\cdots+t_dx_d)}}{ix_1ix_2} \Bigg) dt \\ = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial\Phi(t)}{\partial t_1 \partial t_2} &\frac{e^{it_1x_1} - 1}{ix_1} \frac{e^{it_2x_2} - 1}{ix_2} e^{i(t_3x_3+\cdots+t_dx_d)} dt.\end{aligned}$$

After d similar steps, we obtain

$$\hat{\Phi}(x_1, \ldots, x_d) = (-1)^d (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\partial\Phi(t_1, \ldots, t_d)}{\partial t_1 \ldots \partial t_d} \prod_{j=1}^{d} \frac{e^{it_jx_j} - 1}{ix_j} dt_1 \ldots dt_d.$$

According to the stochastic Fubini Theorem, which can be proved in our multidimensional setting similarly as in Lemma 3 of Lang and Soulier (2000), one can rewrite integral (5.10) :

$$\int_{\mathbb{R}^d} \Phi(t)dB_n(t) = (2\pi)^{d/2} \int_{[-n\pi,\ n\pi]^d} \hat{\Phi}(x)dW_n(x)$$

$$= (2\pi)^{d/2} \int_{[-n\pi,\ n\pi]^d} \left((-1)^d (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\partial \Phi(t_1,\dots,t_d)}{\partial t_1 \dots \partial t_d} \prod_{j=1}^d \frac{e^{it_j x_j} - 1}{i x_j} dt \right) dW_n(x)$$

$$= (-1)^d \int_{\mathbb{R}^d} \frac{\partial \Phi(t_1,\dots,t_d)}{\partial t_1 \dots \partial t_d} \left(\int_{[-n\pi,\ n\pi]^d} \prod_{j=1}^d \frac{e^{it_j x_j} - 1}{i x_j} dW_n(x) \right) dt$$

$$= (-1)^d \int_{\mathbb{R}^d} \frac{\partial \Phi(t_1,\dots,t_d)}{\partial t_1 \dots \partial t_d} B_n(t_1,\dots,t_d) dt_1 \dots dt_d. \tag{5.11}$$

Finally, we need a lemma which generalises a theorem of Grinblatt (1976).

Lemma 6. *Let $(Y_n(t))_{n\in\mathbb{N}}$ and $Y(t)$ be measurable processes defined for t in a compact set K of $\mathbb{R}^d$. Assume that sequence $(Y_n(t))_{n\in\mathbb{N}}$ converges to $Y(t)$ in the finite dimensional distributions sense. If $E|Y_n(t)|$ is uniformly bounded with respect to $n \in \mathbb{N}$ and to $t \in K$, and if as $n \to \infty$, $E|Y_n(t)| \to E|Y(t)|$ for all $t \in K$, then, for all continuous map H on $L^1(K)$, $H(Y_n)$ converges in law to $H(Y)$.*

The proof of this lemma follows the same lines as in Grinblatt (1976) and we omit it. More details can be found in Lavancier (2003a).

From (5.2), $E(B_n^2(t))$ is bounded with respect to n and t, hence so is $E|B_n(t)|$ and moreover the sequence B_n is uniformly integrable. This, and the fact that $B_n(t)$ converges in law to $B(t)$ imply that $E|B_n(t)|$ converges to $E|B(t)|$. By Lemma 4, B is measurable hence we can apply Lemma 6, with $Y_n = B_n$ and K a compact set of $\mathbb{R}^d$, to the map H defined by:

$$H(g) = \int_{\mathbb{R}^d} \frac{\partial \Phi(t_1,\dots,t_d)}{\partial t_1 \dots \partial t_d} g(t_1,\dots,t_d) dt_1 \dots dt_d,$$

with Φ differentiable and defined on a compact set. This map is actually continuous on $L^1(\mathbb{R}^d)$.

So, if Φ is differentiable and compactly supported, $H(B_n)$ converges in law to $H(B)$. Hence, by (5.11), $\int \Phi dB_n$ converges in law to $(-1)^d \int_{\mathbb{R}^d} \frac{\partial \Phi(t)}{\partial t_1 \dots \partial t_d} B(t)dt$. Let's denote I_B the linear application :

$$I_B(\Phi) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial \Phi(t)}{\partial t_1 \dots \partial t_d} B(t)dt.$$

Now, the set of all differentiable applications defined on a compact set is dense in $L^2(\mathbb{R}^d)$ and the linear application $I_B(\Phi)$ is bounded since

$$E\left(I_B(\Phi)\right)^2 = E\left(\int_{\mathbb{R}^d} (-1)^d \frac{\partial \Phi(t)}{\partial t_1 \dots \partial t_d} B(t)dt \right)^2$$

$$\leq \underline{\lim}\ E\left(\int_{\mathbb{R}^d} \frac{\partial \Phi(t)}{\partial t_1 \dots \partial t_d} B_n(t)dt \right)^2$$

$$= \underline{\lim}\ E\left((2\pi)^{d/2} \int_{[-n\pi,\ n\pi]^d} \hat{\Phi} dW_n \right)^2 \leq (2\pi)^d M ||\hat{\Phi}||_2^2 = (2\pi)^d M ||\Phi||_2^2. \tag{5.12}$$

Therefore, by Hahn Banach's Theorem, I_B can be extended to $L^2(\mathbb{R}^d)$ and (5.12) is still valid for all Φ in $L^2(\mathbb{R}^d)$:

$$E\left(I_B(\Phi)\right)^2 \leq (2\pi)^d M ||\Phi||_2^2. \tag{5.13}$$

We now define the application I of Theorem 1 by :

$$I(\Psi) = I_B(\check{\Psi}), \quad \forall \Psi \in L^2(\mathbb{R}^d)$$

where $\check{\Psi}$ is the inverse Fourier transform of Ψ.

We have

$$E\left(I(\Psi)\right)^2 = E\left(I_B(\check{\Phi})\right)^2 \leq (2\pi)^d M||\check{\Psi}||_2^2 = (2\pi)^d M||\Psi||_2^2,$$

which is (i) of Theorem 1.

Let us prove (iii). Let Ψ be a function of $L^2(\mathbb{R}^d)$, and a sequence Ψ_k converging to Φ in $L^2(\mathbb{R}^d)$. Suppose that for every k, $\hat{\Psi}_k$ is compactly supported and differentiable. Then,

(i) $\int \Psi_k dW_n$ converges in law to $I(\Psi_k)$, from Lemma 6

(ii) $E\left(I(\Psi_k) - I(\Psi)\right)^2 \leq M||\Psi_k - \Psi||_2^2 \longrightarrow 0$ when $k \to \infty$

(iii) $E(\int \Psi_k dW_n - \int \Psi dW_n)^2 \leq M||\Psi_k - \Psi||_2^2$, yielding

$$\lim_{k\to\infty} \overline{\lim}_{n\to\infty} E(\int \Psi_k dW_n - \int \Psi dW_n)^2 = 0.$$

Hence, the hypotheses of Theorem 4.2 of Billingsley (1968) are satisfied taking $X_{k,n} = \int \Psi_k dW_n$, $X_k = I(\Psi_k)$, $X = I(\Psi)$ and $Y_n = \int \Psi dW_n$. Consequently $\int \Psi dW_n$ converges in law to $I(\Psi)$.

If we finally consider a sequence of functions Ψ_n which converges to Ψ in $L^2(\mathbb{R}^d)$, we directly obtain that $\int \Psi_n dW_n$ converges in law to $I(\Psi)$.

In particular, taking $\check{\Psi} = \mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]}$:

$$I\left(\prod_{j=1}^{d} \frac{e^{it_j x_j} - 1}{ix_j}\right) = \lim_{n\to\infty} \int \prod_{j=1}^{d} \frac{e^{it_j x_j} - 1}{ix_j} dW_n(x) = \lim_{n\to\infty} B_n(t) = B(t),$$

which is (ii) of Theorem 1.

Let us conclude with the proof of (iv) where ξ is supposed to be a strong white noise. In this case, $B(t)$ is the Brownian sheet with covariance $\sigma(s,t) = \prod_{j=1}^{d} t_j \wedge s_j$. Then the norm of the application I is not greater than 1 from (i) of the theorem; moreover, the value 1 is achieved by the particular function considered in (ii). Hence I is an isometry. Let the measure W_0 be defined for all set A by $W_0(A) = I(\mathbb{1}_A)$. It is obviously an orthogonal measure since I preserves the scalar product. Furthermore

$$0 \leq E\left(W_0(A)\right) \leq \liminf_{n\to\infty} E\left(W_n(A)\right) = 0.$$

Therefore we can write I as a stochastic integral with respect to W_0 :

$$\forall \Phi \in L^2(\mathbb{R}^d)\ I(\Phi) = \int \Phi dW_0.$$

Notice that in this simple situation, (ii) of the theorem gives the harmonisable representation of the Brownian sheet and W_0 is the Gaussian white noise measure.

5.1 Proof of Lemma 4

It is known (see e.g. Gikhman and Skorokhod (1965)) that if the field $(B(t))_{t\in\mathbb{R}^d}$ is stochastically continuous almost everywhere, i.e. for almost every $t \in \mathbb{R}^d$, $\varepsilon > 0$,

$$\lim_{s\to t} P\left(|B(s) - B(t)| > \varepsilon\right) = 0, \tag{5.14}$$

then there exists a measurable and separable version of $B(t)$.

From hypothesis **H 1**, the joint distribution of $(S_n^\xi(s), S_n^\xi(t))$ converges to the distribution of $(B(s), B(t))$. Then, as $A = \left\{(x,y) \in \mathbb{R}^2 : |y - x| > \varepsilon\right\}$ is an open set, we have

$$P\left(|B(s) - B(t)| > \varepsilon\right) \leq \liminf_{n\to\infty} P\left(\left|S_n^\xi(s) - S_n^\xi(t)\right| > \varepsilon\right). \tag{5.15}$$

Let us assess $E(S_n^\xi(s)-S_n^\xi(t))^2$.

$$E(S_n^\xi(s)-S_n^\xi(t))^2=E\left(n^{-d/2}\sum_{k_1=0}^{[ns_1]}\cdots\sum_{k_d=0}^{[ns_d]}\xi_k-n^{-d/2}\sum_{k_1=0}^{[nt_1]}\cdots\sum_{k_d=0}^{[nt_d]}\xi_k\right)^2.$$

We split the domains of summation in the following way

$$\prod_{j=1}^d\Big\{0,\ldots,[ns_j]\Big\}=\prod_{j=1}^d\Big\{0,\ldots,[nt_j]\wedge[ns_j]\Big\}\cup\Big\{[nt_j]\wedge[ns_j]+1,\ldots,[ns_j]\Big\},$$

where $\Big\{[ns_j]+1,[ns_j]\Big\}=\emptyset$. Developping this latest expression yields

$$\prod_{j=1}^d\Big\{0,\ldots,[ns_j]\Big\}=\bigcup_{l=1}^{d-1}\bigcup_{C_d^l}\prod_{j\in C_d^l}\Big\{0,\ldots,[nt_j]\wedge[ns_j]\Big\}\prod_{j\in\overline{C_d^l}}\Big\{[nt_j]\wedge[ns_j]+1,\ldots,[ns_j]\Big\}$$
$$\bigcup\prod_{j=1}^d\Big\{[nt_j]\wedge[ns_j]+1,\ldots,[ns_j]\Big\}\bigcup\prod_{j=1}^d\Big\{0,\ldots,[nt_j]\wedge[ns_j]\Big\}$$

where C_d^l covers all the l-uple in $\{1,\ldots,d\}$ and where $\overline{C_d^l}$ is the complementary set of C_d^l in $\{1,\ldots,d\}$.

Now, we are able to write $S_n^\xi(s)-S_n^\xi(t)$ thanks to this decomposition : the terms associated to the latest union above will vanish. In the following sums, we agree that, if $l=0$, then the sum takes place only on $j'\in\overline{C}_d^0=\{1,\ldots,d\}$.

$$S_n^\xi(s)-S_n^\xi(t)=\frac{1}{n^{d/2}}\sum_{l=0}^{d-1}\sum_{C_d^l}\sum_{\substack{j\in C_d^l\\ j'\in\overline{C}_d^l}}\left(\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=[nt_{j'}]\wedge[ns_{j'}]+1}^{[ns_{j'}]}\xi_k-\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=[nt_{j'}]\wedge[ns_{j'}]+1}^{[nt_{j'}]}\xi_k\right).$$

From the convexity of $x\mapsto x^2$,

$$E\left(S_n^\xi(s)-S_n^\xi(t)\right)^2\le 2(2^d-1)\sum_{l=0}^{d-1}\sum_{C_d^l}E\left(n^{-d/2}\sum_{\substack{j\in C_d^l\\ j'\in\overline{C}_d^l}}\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=[nt_{j'}]\wedge[ns_{j'}]+1}^{[ns_{j'}]}\xi_k\right)^2$$
$$+2(2^d-1)\sum_{l=0}^{d-1}\sum_{C_d^l}E\left(n^{-d/2}\sum_{\substack{j\in C_d^l\\ j'\in\overline{C}_d^l}}\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=[nt_{j'}]\wedge[ns_{j'}]+1}^{[nt_{j'}]}\xi_k\right)^2.$$

The stationarity of ξ yields

$$E\left(S_n^\xi(s)-S_n^\xi(t)\right)^2\le 2(2^d-1)\sum_{l=0}^{d-1}\sum_{C_d^l}E\left(n^{-d/2}\sum_{\substack{j\in C_d^l\\ j'\in\overline{C}_d^l}}\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=0}^{[ns_{j'}]-[nt_{j'}]\wedge[ns_{j'}]-1}\xi_k\right)^2$$
$$+2(2^d-1)\sum_{l=0}^{d-1}\sum_{C_d^l}E\left(n^{-d/2}\sum_{\substack{j\in C_d^l\\ j'\in\overline{C}_d^l}}\sum_{k_j=0}^{[nt_j]\wedge[ns_j]}\sum_{k_{j'}=0}^{[nt_{j'}]-[nt_{j'}]\wedge[ns_{j'}]-1}\xi_k\right)^2.$$

From this point, let us note that for all $p_1,\dots,p_d$ belonging to $\{0,\dots,n\}$

$$\begin{aligned} E\left(n^{-d/2}\sum_{k_1=0}^{p_1}\dots\sum_{k_d=0}^{p_d}\xi_k\right)^2 &= n^{-d}\sum_{k_1,k_1'=0}^{p_1}\dots\sum_{k_d,k_d'=0}^{p_d}\int_{[-\pi,\pi]^d} f_\xi(\lambda)e^{i<k'-k,\lambda>}d\lambda \\ &\le Mn^{-d}\sum_{k_1,k_1'=0}^{p_1}\dots\sum_{k_d,k_d'=0}^{p_d}\int_{[-\pi,\pi]^d} e^{i<k'-k,\lambda>}d\lambda \\ &\le M\prod_{j=1}^{d}\frac{p_j+1}{n}. \end{aligned}$$

Therefore,

$$\begin{aligned} E\left(S_n^\xi(s)-S_n^\xi(t)\right)^2 \le & c\sum_{l=0}^{d-1}\sum_{C_d^l}\left(\prod_{j\in C_d^l}\frac{[nt_j]\wedge[ns_j]+1}{n}\right)\left(\prod_{j\in\overline{C_d^l}}\frac{[ns_j]-[nt_j]\wedge[ns_j]}{n}\right) \\ & + c\sum_{l=1}^{d-1}\sum_{C_d^l}\left(\prod_{j\in C_d^l}\frac{[nt_j]\wedge[ns_j]+1}{n}\right)\left(\prod_{j\in\overline{C_d^l}}\frac{[nt_j]-[nt_j]\wedge[ns_j]}{n}\right), \end{aligned}$$

where c is a positive constant. Finally

$$\begin{aligned} & E\left(S_n^\xi(s)-S_n^\xi(t)\right)^2 \le \\ & \quad c\sum_{l=0}^{d-1}\sum_{C_d^l}\prod_{j\in C_d^l}\left(t_j\wedge s_j+n^{-1}\right)\left(\prod_{j\in\overline{C_d^l}}\left(s_j-t_j\wedge s_j+n^{-1}\right)+\prod_{j\in\overline{C_d^l}}\left(t_j-t_j\wedge s_j+n^{-1}\right)\right). \end{aligned}$$

The last inequality proves that

$$\lim_{s\to t}\liminf_{n\to\infty} E\left(S_n^\xi(s)-S_n^\xi(t)\right)^2 = 0 \tag{5.16}$$

because $\overline{C_d^l}$ is never empty when $l\le d-1$.

Thanks to (5.16), applying the Tchebychev inequality in (5.15) leads to (5.14).

6 Appendix : properties of approximations of unity

In this Section, we summarise some properties of the approximations of unity needed in the proofs of the above Sections. Some of them are known or obvious, others are particular to the use we make of them and have not been found in the literature. This is in particular the case with properties of the tensorial product and the convolution product of approximations of unity. These ones depend on the nature of the approximations of unity that we consider. Indeed, we will distinguish two classes : the first one is the approximations of unity in a weak sense (that is the common sense of an approximation of unity), the other one is the approximations of unity in a strong sense. We apply these results to the Fejer kernel which is a strong approximation of unity ; we finally resume some specific properties of this kernel, which are useful in some of our proofs. According to the final point of view, we focus our study on approximations of unity defined on $[-\pi,\pi]^d$.

Definition 2. We say that a function $K_n : [-\pi,\pi]^d \to \mathbb{R}$ is a weak approximation of unity if $\forall n,\ K_n\ge 0,\ \int_{[-\pi,\pi]^d} K_n(x)dx = 1$ and if

$$\forall\delta>0,\ \lim_{n\to\infty}\int_{||x||>\delta} K_n(x)dx = 0. \tag{6.1}$$

We say that a function $K_n : [-\pi,\pi]^d \to \mathbb{R}$ is a strong approximation of unity if $\forall n,\ K_n \geq 0$, $\int_{[-\pi,\pi]^d} K_n(x)dx = 1$ and if

$$\forall \delta > 0,\ \lim_{n\to\infty} \sup_{||x||>\delta} K_n(x)dx = 0. \tag{6.2}$$

A strong approximation of unity is obviously a weak one. These functions are mainly used for the well known following property :

Theorem 7. *Let K_n be a weak approximation of unity on $[-\pi,\pi]^d$, then for any bounded function $g \in L^1([-\pi,\pi]^d)$, continuous at 0,*

$$\lim_{n\to\infty} \int_{[-\pi,\pi]^d} g(x)K_n(x)dx = g(0). \tag{6.3}$$

Let K_n be a strong approximation of unity on $[-\pi,\pi]^d$, then for any function $g \in L^1([-\pi,\pi]^d)$, continuous at 0,

$$\lim_{n\to\infty} \int_{[-\pi,\pi]^d} g(x)K_n(x)dx = g(0). \tag{6.4}$$

After a tensorial product or a convolution product, the quality of approximation of unity is preserved in the following way :

Proposition 1. *Let $K_n^{(1)},\dots,K_n^{(d)}$ be approximations of unity on $[-\pi,\pi]$.*
If the $K_n^{(i)}$'s are weak approximations of unity, then

1. *$K_n^{(1)} * \cdots * K_n^{(d)}(x)$ is still a weak approximation of unity on $[-\pi,\pi]$.*
2. *$P_n(x_1,\dots,x_d) = \prod_{i=1}^d K_n^{(i)}(x_i)$ is a weak approximation of unity on $[-\pi,\pi]^d$.*

If the $K_n^{(i)}$'s are strong approximations of unity, then

1. *$K_n^{(1)} * \cdots * K_n^{(d)}(x)$ is still a strong approximation of unity on $[-\pi,\pi]$.*
2. *$P_n(x_1,\dots,x_d) = \prod_{i=1}^d K_n^{(i)}(x_i)$ is no more a strong approximation of unity on $[-\pi,\pi]^d$, but only a weak one.*

Proof. The proof is straightforward, see Lavancier (2003b) for more details. Just notice that the tensorial product of two strong approximations of unity is not necessarily a strong approximation of unity. Indeed, consider for instance the Fejer kernel F_n defined in (6.5) below. It is a strong approximation of unity, but one has $F_n(0)F_n(\pi) = 0$ if n is even and $F_n(0)F_n(\pi) = \frac{1}{4\pi^2}$ if n is odd. Property (6.2) is therefore not verified by the tensorial product $(x,y) \mapsto F_n(x)F_n(y)$. □

We now focus more particularly on the Fejer kernel, defined on $[-\pi,\pi]$ by

$$F_n(x) = \begin{cases} \frac{1}{2\pi n}\frac{\sin^2(nx/2)}{\sin^2(x/2))} & \text{if } x \neq 0 \\ \frac{n}{2\pi} & \text{if } x = 0. \end{cases} \tag{6.5}$$

We summarise some of its properties, useful in the preceding Sections.

Proposition 2. *Let F_n be the Fejer kernel defined on $[-\pi,\pi]$ by (6.5), then*

1. *F_n is a strong approximation of unity from $[-\pi,\pi]$ into $\mathbb{R}$*
2. *$\forall \delta > 0\ \sup_{|x|>\delta} F_n(x) \leq \frac{\pi}{2n\delta^2}$*
3. *$\forall \delta > 0\ \int_{|x|>\delta} F_n(x)dx \geq \frac{1}{2\pi n}\left(\pi - \delta + \frac{\sin(n\delta)}{n}\right)$*
4. *$\forall g \in L^1([-\pi,\pi]),\ \int_{-\pi}^{\pi} g(x)F_n(x)dx = o(n)$ when $n \to \infty$*
5. *Let $\alpha > -1$, then $\int_{-\pi}^{\pi} |x|^\alpha F_n(x)dx \sim cn^{-\alpha}$ when $n \to \infty$, where c is a positive constant.*

Proof. The four first points can be easily checked. 5. is proved in Lemma 9 of Viano et al. (1995). □

References

Billingsley, P. (1968). *Convergence of probability measures.* John Wiley and Sons.

Breuer, P. and Major, P. (1983). Central limit theorems for non-linear functionals of gaussian fields. *Journal of Mulivariate Analysis*, 13:425–441.

Dobrushin, R. L. and Major, P. (1979). Non central limit theorems for non-linear functionals of gaussian fields. *Z. Warsch. verw. Geb.*, 50:27–52.

Doukhan, P., Lang, G., and Surgailis, D. (2002). Asymptotics of weighted empirical processes of linear fields with long-rang dependence. *Ann. Inst. Henri Poincaré*, 6:879–896.

Gikhman, I. I. and Skorokhod, A. V. (1965). *Introduction to the theory of random processes.* W. B. Saunders.

Grinblatt, L. (1976). A limit theorem for measurable random processes and its applications. *Proc. of the American Math. Society*, 61(2):371–376.

Lang, G. and Soulier, P. (2000). Convergence de mesures spectrales aléatoires et applications à des principes d'invariance. *Stat. Inf. for Stoch. Proc.*, 3:41–51.

Lavancier, F. (2003a). Principes d'invariance pour les champs aléatoires fortement dépendants, partie 1 : théorème de convergence de champs spectraux. Technical Report 60, XII, IRMA, Lille.

Lavancier, F. (2003b). Principes d'invariance pour les champs aléatoires fortement dépendants, partie 2 : application aux sommes partielles de champs aléatoires à longe mémoire. Technical Report 61, V, IRMA, Lille.

Viano, M.-C., Deniau, C., and Oppenheim, G. (1995). Long-range dependence and mixing for discrete time fractional processes. *J. Time Ser. Anal.*, 16(3):323–338.

Wainger, S. (1965). *Special trigonometric series in k-dimensions.* Number 59. AMS.