STABLE FACTORIZATION OF THE CALDERÓN PROBLEM VIA THE BORN APPROXIMATION

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Abstract. In this article we prove the existence of the Born approximation in the context of the radial Calderón problem for Schrödinger operators. This is the inverse problem of recovering a radial potential on the unit ball from the knowledge of the Dirichlet-to-Neumann map (DtN map from now on) of the corresponding Schrödinger operator. The Born approximation naturally appears as the linear component of a factorization of the Calderón problem; we show that the non-linear part, obtaining the potential from the Born approximation, enjoys several interesting properties. First, this map is local, in the sense that knowledge of the Born approximation in a neighborhood of the boundary is equivalent to knowledge of the potential in the same neighborhood, and, second, it is Hölder stable. This shows in particular that the ill-posedness of the Calderón problem arises solely from the linear step, which consists in computing the Born approximation from the DtN map by solving a Hausdorff moment problem. Moreover, we present an effective algorithm to compute the potential from the Born approximation and show a result on reconstruction of singularities. Finally, we use the Born approximation to obtain a partial characterization of the set of DtN maps for radial potentials. The proofs of these results do not make use of Complex Geometric Optics solutions or its analogues; they are based on results on inverse spectral theory for Schrödinger operators on the half-line, in particular on the concept of A-amplitude introduced by Barry Simon.

1. Introduction

1.1. The problem and the setting. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a smooth bounded domain, denote by ∂_ν the outward normal unit vector field on $\partial\Omega$. The Calderón problem in Ω is the inverse problem of reconstructing a positive conductivity function γ in the equation

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = & 0 \quad \text{in } \Omega \subset \mathbb{R}^d, \\ u = & f, \quad \text{on } \partial\Omega, \end{cases}$$

from the knowledge of the Dirichlet to Neumann map (in what follows, the DtN map)

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \ni f \longmapsto \gamma \partial_\nu u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

This problem goes back to Calderón who considered it since the fifties and published his results in 1980 in [Cal80].[1] It is well known that this inverse problem is closely related to the problem of reconstructing a real-valued potential V in the Schrödinger

[1]This reference has been reprinted in [Cal06].

equation

$$
\begin{cases} -\Delta v + Vv = 0 & \text{in } \Omega, \\ v = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

from the corresponding DtN map (provided it is well-defined)

$$\Lambda_V : H^{1/2}(\partial\Omega) \ni g \longmapsto \partial_\nu v|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

In fact, in the case of regular conductivities, the conductivity problem can be reduced to the Schrödinger problem by the Liouville transform $v = \sqrt{\gamma}u$, $g = \sqrt{\gamma}f$ and with the potential $V = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$, (see [Uhl92] for instance).

One can interpret the Calderón problem for the Schrödinger equation (1.1) as the problem of inverting of the non-linear map

$$
\begin{aligned}
\Phi : \mathcal{V} &\longrightarrow \mathcal{L}(L^2(\partial\Omega)), \\
V &\longmapsto \Lambda_V - \Lambda_0,
\end{aligned}
$$

where $\mathcal{V}$ is some class of real-valued potentials defined on Ω, Λ_0 is the DtN map associated to the free Laplacian ($V = 0$), and $\mathcal{L}(L^2(\partial\Omega))$ is the space of linear bounded operators on $L^2(\partial\Omega)$.

The uniqueness question for the Calderón problem amounts to showing the injectivity of the map Φ. This has been established, both for conductivities and potentials, by many authors at different levels of regularity, starting with Kohn and Vogelius [KV84] who showed that the DtN map determines uniquely smooth conductivities/potentials and all their derivatives on $\partial\Omega$. This was one of the ingredients used by Sylvester and Uhlmann in [SU87] to solve the Calderón problem for dimension $d \geq 3$ in the smooth setting. The other major ingredient was the use of complex geometrical optics (CGO) solutions which, since then, have played an important role to prove uniqueness and stability results for less regular potentials and conductivities in dimensions $d \geq 3$, see for instance [Nac88, Cha90, Nac92, Bro96, HT13, Hab15, CR16]. The two-dimensional case is quite different mathematically, and complete results were obtained later. The planar Calderón problem for the conductivity equation was solved by Nachman [Nac96] and by Astala and Päivärinta [AP06] for $\mathcal{C}^2$ and L^∞ conductivities respectively. The Calderón problem for the Schrödinger equation was solved in [Buk08] (for $\mathcal{C}^1$ potentials) and [BIY15] (L^p with $p > 2$). These results also rely on exponentially growing solutions of the equations, as in the case $d \geq 3$.

The inverse map Φ^{-1} is in general never globally continuous. Alessandrini [Ale88] showed the existence of sequences of potentials at distance one in L^∞ such that their DtN maps are arbitrarily close in operator norm (for L^p results see [AC08, FKR14]). In spite of the fact that the Calderón problem is an ill-posed inverse problem, it has been shown that it is conditionally stable. This means that Φ is an homeomorphism when restricted to compact subsets of potentials K and in particular, that Φ^{-1} has a modulus of continuity on $\Phi(K)$. The stability of the reconstruction process can be stated as the question of estimating this modulus of continuity. For example, Alessandrini proved in [Ale88] for $d \geq 3$ that Φ^{-1} has a logarithmic modulus of continuity when one assumes certain *a priori* regularity and boundedness assumptions on the conductivities or the potentials to provide the required compactness (this was extended to $d = 2$ by [BFR07]). This result is sharp, as showed by Mandache[2] in the case of the unit ball [Man01, Theorem 1 and Corollary 2] or [KRS21] in a more general setting. In conclusion, even assuming *a priori* regularity on potentials, the problem of the determination of the potential V from the DtN map Λ_V is still highly unstable.

In this work we show that when $\Omega = \mathbb{B}^d := \{x \in \mathbb{R}^d : |x| < 1\}$ is the unit ball in Euclidean space, with $\partial\Omega = \mathbb{S}^{d-1}$, $d \geq 2$, and $\mathcal{V}$ is a suitable, yet very general, class of real-valued radial potentials, one can factor the inverse map Φ^{-1} as:

$$
\begin{tikzcd}
\mathcal{M} := \Phi(\mathcal{V}) \arrow[rr, "\Phi^{-1}"] \arrow[dr, "d\Phi_0^{-1}"'] & & \mathcal{V} \\
& \mathcal{B} \arrow[ur, "\Phi_{\mathrm{B}}^{-1}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Lambda_V - \Lambda_0 \arrow[rr, mapsto, "\Phi^{-1}"] \arrow[dr, mapsto, "d\Phi_0^{-1}"'] & & V \\
& V^{\mathrm{B}} \arrow[ur, mapsto, "\Phi_{\mathrm{B}}^{-1}"'] &
\end{tikzcd}
\tag{1.2}
$$

The map $d\Phi_0$ is the Fréchet differential of Φ at $V = 0$, which is known to be injective, as it was proved originally by [Cal80] in the conductivity problem. We show that $d\Phi_0^{-1}$ can be extended to $\mathcal{M} := \Phi(\mathcal{V})$, the set of all operators $\Lambda_V - \Lambda_0$ arising from potentials in $\mathcal{V}$, and that it maps each $\Lambda_V - \Lambda_0$ to a function V^{B} that is supported on $\mathbb{B}^d$. It turns out that this function is the solution of a certain moment problem and its moments, to be defined later on, are the eigenvalues of $\Lambda_V - \Lambda_0$ (see Theorem 1). In short, $d\Phi_0^{-1}$ is the solution operator for a moment problem, and is unbounded as a map from $\mathcal{L}(L^2(\mathbb{S}^{d-1}))$ to any Lebesgue space, although it enjoys conditional logarithmic-type stability.

The map Φ_{B} is a non-linear bijection from $\mathcal{V}$ onto $\mathcal{B}$, the image $d\Phi_0^{-1}(\mathcal{M})$. We will show that inverting Φ_{B} one can reconstruct V directly from V^{B} with Hölder stability in certain L^1 weighted spaces, and locally from the boundary—see Theorem 2 and Theorem 3. More graphically:

$$
\Lambda_V - \Lambda_0 \xmapsto[\text{Linear bijection}]{\text{Conditional log stability}} V^{\mathrm{B}} \xmapsto[\text{Non-linear}]{\text{(local) Hölder stability}} V. \tag{1.3}
$$

Therefore, in a certain sense the linear part of the factorization absorbs completely the instability and the ill-posedness of the inverse problem. This suggests that V^{B} should play a important role in the numerical reconstruction of V from the DtN map. Another remarkable consequence of the previous factorization is that it implies a partial characterization result for the DtN maps of radial potentials, see Remark 1.1 and Section 7 for more details.

In the previous diagrams we have used DtN maps for conceptual simplicity but, as we will see later on, the map $d\Phi_0^{-1}$ is well defined even in cases in which the boundary value problem has no uniqueness, and $\Lambda_V - \Lambda_0$ is replaced by the Cauchy data of (1.1) (defined in Section 2). Also, even if the previous results are relative to the Schrödinger case (1.1), they can be carried to the conductivity case, at least in the case of regular conductivities. This will be the subject of a forthcoming paper.

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[2] It is relevant to note, in connection with our results, that the potentials given in [Man01] are not necessarily radial but Mandache claims that even radial potentials give counterexamples to stability (see the remark before [Man01, Lemma 4]).

The function V^{B} plays an analogous role to the Born approximation in scattering problems. Therefore, from now on we will refer to V^{B} as the Born approximation of the potential V. It also shares some common traits with its counterparts in scattering theory. For example, in spite of the fact that it naturally appears when performing a linearization of the problem, it is globally well-defined (not just in a neighborhood of the potential where the linearization is made). It also contains qualitative information on the potential since in general $V - V^{\mathrm{B}}$ can be expected to be 2 derivatives more regular than V (see Theorem 5). This property of the Born approximation is known as recovery of singularities, and it is well known in scattering problems, see for example [PS91, Rui01, Mer18, Mer19]. For results of recovery of singularities in the context of the Calderón problem see [GLS+18].

The proofs of the uniqueness and stability results in this paper do not involve the construction of exponentially growing solutions like the CGOs commonly used in the Calderón problem. Instead they arise from the approach to 1-d inverse spectral theory introduced by Simon in [Sim99] and the follow-up papers [GS00, RS00] together with Gesztesy and Ramm respectively, and later improved by Avdonin Mikhaylov and Rybkin [AMR07, AM10] using the boundary control approach of the 1-d wave equation. This approach has been applied in the context of the Steklov problem for warped product manifolds in [DHN21, DKN21, DKN23, Gen20, Gen22]. In particular, the results in [DKN21] imply stability and uniqueness results for the radial Calderón problem both for the conductivity and Schrödinger cases. We also mention that spectral theory methods had already been used in the context of the radial Calderón problem in the work [KV85, Section 6], and that they have been used to produce convergent reconstruction algorithms in [Syl92] in the 2-d conductivity case.

1.2. **Existence of the Born approximation.** In the previous discussion and in diagrams (1.2) and (1.3) we have introduced the Born approximation as the object satisfying the identity

$$V^{\mathrm{B}} := d\Phi_0^{-1}(\Lambda_V - \Lambda_0). \tag{1.4}$$

The definition (1.4) is formal, since it is not clear that the map $d\Phi_0^{-1}$ can be extended from its natural domain —the "tangent space" to $\mathcal{M}$, following the analogy from differential geometry— to the whole set $\mathcal{M}$ of operators $\Lambda_V - \Lambda_0$. This approach is widely used when building practical reconstruction algorithms in numerical methods for EIT; in that context it is known under different names such as one step linearization method (see [HS10] for references) or Calderón method (see [BM08, MLSM20, SM20]). The Born approximation or similar objects also appear in [BKM11, KM11, DHK11, DK14, HIK+21].

As far as we know there are no rigorous results that justify identity (1.4), except from partial results in [HS10] for the conductivity case, and in general this problem remains open. Our first result shows that, under the radial assumption on the potentials, (1.4) admits a simple reformulation in terms of a Hausdorff moment problem and, in particular, that V^{B} is a well-defined object.

Let us first give some heuristics to motivate why such a reformulation can be achieved and shed some light to the inherent difficulty of rigorously defining the Born approximation. Suppose for the moment that $V \in L^\infty(\mathbb{B}^d)$ is radial and that Λ_V is well-defined. In that case, the spectral theory of Λ_V is easy to describe. Denote the subspace of spherical harmonics of degree k by $\mathfrak{H}_k$ (these are the restrictions to the sphere $\mathbb{S}^{d-1}$ of the complex homogeneous polynomials of degree k on $\mathbb{R}^d$ that are harmonic). The spaces $\mathfrak{H}_k$ are mutually orthogonal in $L^2(\mathbb{S}^{d-1})$, and consist of eigenfunctions of the DtN map: by separation of variables one can show that

$$\Lambda_V|_{\mathfrak{H}_k} = \lambda_k[V]\,\mathrm{Id}_{\mathfrak{H}_k}, \tag{1.5}$$

for every $k \in \mathbb{N}_0$, where $\mathbb{N}_0$ stands for the set of non-negative integers. For example, when $V = 0$ a direct computation gives that

$$\Lambda_0(Y_k) = kY_k \quad \text{for all } Y_k \in \mathfrak{H}_k,$$

and hence, $\lambda_k[0] = k$ for every $k \in \mathbb{N}_0$.

For a radial function $F = F_0(|\cdot|)$ and any $k \in \mathbb{N}_0$, we define the moments:

$$\sigma_k[F] := \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{B}^d} F(x)|x|^{2k}\,dx = \int_0^1 F_0(r)r^{2k+d-1}\,dr. \tag{1.6}$$

This definition makes sense as soon as[3] $F \in L^1(\mathbb{B}^d, |x|^{2\kappa}dx)$ and $k \geq \kappa \geq 0$, and extends by duality to the subspace of $\mathcal{E}'(\mathbb{R}^d)$ formed by the compactly supported radial distributions supported in $\overline{\mathbb{B}^d}$ (this is recalled in Section 6).

In [BCMM22, Theorem 2] it is proved that there exits $C_d > 0$ such that

$$|\lambda_k[V] - k - \sigma_k[V]| \leq C_d k^{-3}\|V\|^2_{L^\infty(\mathbb{B}^d)}, \qquad \forall k > \|V\|^{1/2}_{L^\infty(\mathbb{B}^d)} - \tfrac{d-2}{2}.$$

This identity implies that the Fréchet derivative of Φ at the zero potential verifies that $d\Phi_0(V)$ is a bounded operator in $L^2(\mathbb{S}^{d-1})$ satisfying:

$$d\Phi_0(V)|_{\mathfrak{H}_k} = \sigma_k[V]\,\mathrm{Id}_{\mathfrak{H}_k}, \qquad \forall k \in \mathbb{N}_0.$$

Since $d\Phi_0(V^{\mathrm{B}}) = \Lambda_V - \Lambda_0$, identity (1.4) implies that formally

$$\sigma_k[V^{\mathrm{B}}] = \lambda_k[V] - k, \qquad \forall k \in \mathbb{N}_0. \tag{1.7}$$

Therefore, V^{B} should be a function/distribution in $\mathbb{B}^d$ whose moments are the eigenvalues of $\Lambda_V - \Lambda_0$, *i.e.* V^{B} is the solution to a Hausdorff moment problem. This formal statement has been obtained through different means in [BCMM22, BCMM24], and implicitly in [DKN21]. While uniqueness is in general guaranteed (see Section 6), the existence of solutions to (1.7) is a subtle issue. In fact, most sequences of complex numbers are not sequences of moments of any function (see Section 7).

In order to state precisely the main result in this work, it is convenient to introduce the following norm on the class of measurable functions $F : \mathbb{B}^d \longrightarrow \mathbb{C}$:

$$\|F\|_{\mathcal{V}_d} := \sup_{j\in\mathbb{N}_0}\int_{2^{-(j+1)}<|x|<2^{-j}} |F(x)||x|^{2-d}\,dx. \tag{1.8}$$

[3] In what follows, $L^p(\mathbb{B}^d, |x|^\kappa dx)$ will denote the spaces of measurable functions F such that $\int_{\mathbb{B}^d}|F(x)|^p|x|^\kappa dx < \infty$.

We define the associated Banach space of radial and real-valued functions:

$$\mathcal{V}_d := \{V \in L^1_{\mathrm{loc}}(\mathbb{B}^d \setminus \{0\}; \mathbb{R}) : V = q(|\cdot|),\ \|V\|_{\mathcal{V}_d} < \infty\}.$$

This space contains and is strictly larger than the set of radial potentials in $L^{d/2}(\mathbb{B}^d)$, since it includes the critical potential $V(x) = c|x|^{-2}$ with $c \in \mathbb{R}$ (in fact, it contains the radial functions in the Lorentz space $L^{d/2,\infty}(\mathbb{B}^d)$, with $d > 2$, see Appendix B).

Note that the DtN map is not always well-defined for every potential $V \in \mathcal{V}_d$, for it could happen, for instance, that 0 is in the Dirichlet spectrum of $-\Delta + V$ or that $-\Delta + V$ is not essentially self-adjoint. However, working in the radial setting allows to give a meaningful definition of $\lambda_k[V]$ in terms of separation of variables (see Definition 2.4) that coincides with the spectral definition when the DtN map exists. When the standard weak formulation of (1.1) with $\Omega = \mathbb{B}^d$ is well-defined, then the values $\lambda_k[V]$ for $k > k_V$, where

$$k_V := \beta_V - (d-2)/2, \qquad \beta_V := \frac{2}{|\mathbb{S}^{d-1}|} \max\left(\sqrt{6|\mathbb{S}^{d-1}|\|V\|_{\mathcal{V}_d}}, 3e\ \|V\|_{\mathcal{V}_d}\right), \tag{1.9}$$

can be determined from a section of the Cauchy data of $-\Delta + V$ (see Remark 2.3). The eventual ambiguity in the definition of $\lambda_k[V]$ for some indices $k \le k_V$ will have no effect on the definition of the Born approximation outside of the origin.

Theorem 1 (Existence)**.** *Let $d \ge 2$ and $V \in \mathcal{V}_d$; then the following hold.*

i) There exists a unique radial function $V^{\mathrm{B}} \in L^1(\mathbb{B}^d, |x|^{2k_0}dx)$ for some $k_0 \in \mathbb{N}_0$ such that $k_0 \le k_V + 1$ and

$$\sigma_k[V^{\mathrm{B}}] = \lambda_k[V] - k, \qquad \text{for all } k \ge k_0. \tag{1.10}$$

ii) There exits a unique distribution $V_r^{\mathrm{B}} \in \mathcal{E}'(\mathbb{R}^d)$, radially symmetric and supported in $\overline{\mathbb{B}^d}$, such that

$$\sigma_k[V_r^{\mathrm{B}}] = \lambda_k[V] - k, \qquad \text{for all } k \in \mathbb{N}_0, \tag{1.11}$$

In addition, $V_r^{\mathrm{B}} = V^{\mathrm{B}}$ in $\mathbb{B}^d \setminus \{0\}$ in the sense of distributions.

This theorem shows that the Born approximation V^{B} is a well defined function that exists for all the potentials $V \in \mathcal{V}_d$. However, V^{B} can be in some cases a strongly singular function in $x = 0$, which explains why (1.10) only holds in general for $k > k_V$, (see Section 3.4 for explicit examples that present such behavior). This motivates the introduction of the distribution V_r^{B}, which is a *regularization* of the Born approximation in the sense of [GS64, p. 11], since it is a distribution that coincides identically with V^{B} when $x \ne 0$, and vanishes outside the ball. Either way, (1.10) and (1.11) provide two rigorous interpretations of the formal identity (1.7).

One advantage of introducing the regularized Born approximation V_r^{B} is that in Theorem 6.1 we show that there is an explicit expression to compute V_r^{B} from the spectrum of the DtN map:

$$\widehat{V_r^{\mathrm{B}}}(\xi) = 2\pi^{d/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+d/2)} \left(\frac{|\xi|}{2}\right)^{2k} (\lambda_k[V] - k), \tag{1.12}$$

where the following convention for the Fourier transform of integrable functions is used

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\, dx,$$

with its natural extension to $\mathcal{E}'(\mathbb{R}^d)$. Identity (1.12) originates from a solution formula for the moment problem for compactly supported distributions (see Lemma 6.2 and [BCMM22, Section 3]) and it serves to reconstruct V_r^{B} explicitly, and hence V^{B} by restriction to $\mathbb{B}^d \setminus \{0\}$, from the eigenvalues of Λ_V. This formula appeared originally in [BCMM22, Theorem 1] as a formal expression obtained by linearizing a well known formula to reconstruct V from the DtN map using CGOs, and has been used to numerically reconstruct V^{B} in [BCMM24].

Remark 1.1. As we have already mentioned, the existence of solutions of the Hausdorff moment problem is a subtle issue, since a sequence of moments must satisfy certain non-trivial necessary conditions (see, for example, [Hau23] and [Bor78]). The existence of the Born approximation obtained by Theorem 1 can be considered then as a kind of partial characterization of DtN operators. This is discussed in more detail in Section 7.

1.3. **Stable reconstruction of a potential from its Born approximation.** We start by establishing that the correspondence $\Phi_{\mathrm{B}}(V) = V^{\mathrm{B}}$ depicted in (1.2) is injective. In fact, the Born approximation contains all the necessary information to reconstruct V *locally* from the boundary.

Theorem 2 (Uniqueness)**.** *Let $d \geq 2$ and $0 < b < 1$. Assume that $V \in \mathcal{V}_d$ is a radial potential. Then*

$$V_1^{\mathrm{B}}(x) = V_2^{\mathrm{B}}(x) \quad \text{a.e. for } b < |x| < 1 \iff V_1(x) = V_2(x) \quad \text{a.e. for } b < |x| < 1.$$

A simple consequence of Theorem 2 is that $V(x) = 0$ for $b < |x| < 1$ iff $V^{\mathrm{B}}(x) = 0$ for $b < |x| < 1$. The ability to recover the exterior support of V from V^{B} is connected to the results obtained in [HS10] for the non-radial conductivity problem, and can be clearly observed in numerical reconstructions of V^{B} obtained in [BCMM24] using (1.12).

A remarkable feature of this uniqueness result, apart from the interesting local behaviour which was already exhibited in [DKN21, Theorem 4], is that the proof does not involve the use of exponentially growing solutions or CGOs. Instead, it is based on the approach to 1-dimensional inverse spectral theory originally introduced by Simon in [Sim99]: the Born approximation is the exact counterpart in the radial Calderon problem of the A-amplitude introduced in that paper. Previous uniqueness results for the radial Calderón problem using techniques from inverse spectral theory can be found in [KV85, Syl92].

We now turn to the question of stability. The following theorem shows that the non-linear map Φ_{B} in the factorization (1.3) is Hölder-stable.

Theorem 3 (Stability)**.** *Let $d \geq 2$. Consider two radial potentials $V_1, V_2 \in \mathcal{V}_d$ with $V_i = q_i(|\cdot|)$, and their respective Born approximations V_1^{B} and V_2^{B}. Fix some $0 < b < 1$,* *and let* $\varepsilon_0 < \min(1, -\log b)$. *Let* $1 < p \leq \infty$ *and assume that there is a constant* $K_p(b) < \infty$ *such that*

$$\begin{aligned} \max_{i=1,2} \left(\int_b^1 r^{2p-1} |q_i(r)|^p \, dr \right)^{1/p} &< K_p(b) \quad \text{if } 1 < p < \infty, \\ \max_{i=1,2} \left(\sup_{b<r<1} |q_i(r)| \right) &< K_\infty(b) \quad \text{if } p = \infty. \end{aligned} \tag{1.13}$$

Then, if p' *is the Hölder conjugate exponent of* p, *and*

$$\int_{b<|x|<1} \left| V_1^{\mathrm{B}}(x) - V_2^{\mathrm{B}}(x) \right| |x|^{2-d} \, dx < \varepsilon_0^{(1+p')/p'}, \tag{1.14}$$

it holds that

$$\begin{aligned} &\int_{b<|x|<1} |V_1(x) - V_2(x)| \, |x|^{2-d} \, dx \\ &\qquad < C(b, K_p(b)) \left(\int_{b<|x|<1} \left| V_1^{\mathrm{B}}(x) - V_2^{\mathrm{B}}(x) \right| |x|^{2-d} \, dx \right)^{1/(p'+1)}, \end{aligned} \tag{1.15}$$

where the constant $C(b, K_p(b)) > 0$ *can be computed explicitly in terms of* b *and* $K_p(b)$.

An important feature of this result is that the stability estimate of the map $\Phi_{\mathrm{B}}^{-1} : V^{\mathrm{B}} \longmapsto V$ is not a conditional stability estimate: simple integrability conditions (1.13) are required on the potentials (which do not imply they lie in a compact set). Theorem 3 implies that Φ_{B}^{-1} is Hölder continuous, at least in a local sense in the annuli $b < |x| < 1$. We conclude from this that the linear operator $d\Phi_0^{-1} : \Lambda_V - \Lambda_0 \longmapsto V^{\mathrm{B}}$ is decompressing the information on V contained in the DtN map, and transforming the ill-posed inverse problem of inverting $\Phi : V \longmapsto \Lambda_V$ in a well posed inverse problem of inverting the map $\Phi_{\mathrm{B}} : V \longmapsto V^{\mathrm{B}}$. Unfortunately, the fact that V^{B} may be singular at $x = 0$ means that the stability estimate (1.15) cannot hold in the whole ball $\mathbb{B}^d$ (*i. e.* with $b = 0$), at least not without imposing extra assumptions on the potentials, (see the examples in Section 3.4). The instability and ill-posedness of the Calderón problem must be then caused solely by the map $d\Phi_0^{-1}$, the solution operator to the Hausdorff moment problem, which is a notoriously ill-posed problem.

Theorem 3 follows from Theorem 5.2, a stability result for a problem in inverse spectral theory from Schrödinger operators in the half-line. The proof of this result is based on the work [Sim99], where uniqueness for a problem in inverse spectral theory is proved using a method that is close to constitute an explicit reconstruction algorithm. We analyze this in Section 4, where we elaborate these results in a proper reconstruction method for the radial Calderón problem. We present next a simplified version of this method for radial $\mathcal{C}^1(\mathbb{B}^d)$ potentials (see Algorithm 4.9 for the general version).

Algorithm 4 (Reconstruction)**.** *Given* $(\lambda_k[V])_{k\in\mathbb{N}_0}$ *one reconstructs* V *as follows.*

1) *Using* (1.12) *and that* $V^{\mathrm{B}} = V_r^{\mathrm{B}}$ *in* $\mathbb{B}^d \setminus \{0\}$, *one can reconstruct* V^{B} *from the eigenvalues of the DtN map* $(\lambda_k[V])_{k\in\mathbb{N}_0}$ *(or the Cauchy data, when the DtN map is not well-defined).*

2) Find the unique $\mathcal{C}^1$ solution of

$$r\frac{\partial W}{\partial r}(r,s) - s\frac{\partial W}{\partial s}(r,s) = s^2\int_r^1 W\left(\frac{r}{\nu}, s\right) W(\nu, s)\frac{d\nu}{\nu}, \qquad r,s \in (0,1),$$

such that $W(|x|,1) = V^{\mathrm{B}}(x)$. As it turns out, $W(|x|,s) = s^{-2}[V_s]^{\mathrm{B}}(x)$ where $V_s(x) = s^2V(sx)$. In other words, $W(\cdot,s)$ is the radial profile of the Born approximation of the dilated potential V_s.

3) Once $W(r,s)$ is known, use that $W(1,|x|) = V(x)$. This holds since, by Theorem 5 below, the Born approximation always coincides with the potential at the boundary of $\mathbb{B}^d$.

In this algorithm one reconstructs the potential layer by layer: the information on V that is already known is taken outside the ball by dilating the potential. This is reminiscent of the so-called layer stripping methods used in EIT, see [CNS20] for references.

Nonetheless, the previous reconstruction method has the strong disadvantage of being based on a non-linear integro-differential equation. Using the boundary control approach for the wave equation, a much simpler reconstruction method for the same inverse spectral problem studied by Simon has been developed in [AM10]. This provides provides an alternative reconstruction method for the radial Calderón problem which is based on solving a much simpler linear integral equation. Details will be given in a forthcoming article.

1.4. **Structure and approximation properties of the Born approximation.** In the previous discussion we have analyzed the uniqueness, stability and reconstruction properties of V^{B} and the maps introduced in (1.2). We now turn to investigate the qualitative behavior of V^{B} and its connections to the potential V.

Theorem 5 (Approximation properties)**.** *Let $V \in \mathcal{V}_d$, $d \geq 2$, such that $V = q(|\cdot|)$, and let*

$$\alpha(r) := \min\left(\beta_V, \int_r^1 s|q(s)|\,ds\right).$$

Then $V^{\mathrm{B}} = V + F(|\cdot|)$, where F is a continuous function in $(0,1]$ that satisfies:

$$|F(r)| \leq \frac{1}{r^{\alpha(r)+2}}\left(\int_r^1 s|q(s)|\,ds\right)^2, \qquad F(1) = 0, \quad F'(1^-) = 0. \tag{1.16}$$

In addition, if q is $\mathcal{C}^m$ in $(b,1]$ with $m \in \mathbb{N}_0$, then F is in $\mathcal{C}^{m+2}$ in $(b,1]$.

This theorem shows that V^{B} approximates V when $b < |x| < 1$ with an error that only depends on b and the size of the potential in that region. It also implies that V^{B} contains all the singularities and discontinuities of the potential. The recovery of singularities from the Born approximation is a well known phenomenon in certain scattering problems (see the references given previously), and can be observed in the numerical reconstructions of V^{B} in [BCMM24]. The Born approximation also enjoys a simple monotonicity property.

Theorem 6 (Monotonicity)**.** *Let $d \geq 2$ and $V \in \mathcal{V}_d$. Then,*

$$V_1(x) \leq -V_2(x) \quad on \ \{b < |x| < 1\} \Longrightarrow V_1^{\mathrm{B}}(x) \leq -V_2^{\mathrm{B}}(x) \quad on \ \{b < |x| < 1\},$$

for any $0 < b < 1$.

1.5. **Structure of the paper.** In Section 2 we introduce necessary background on the radial Calderón Problem. Theorem 1(i), Theorem 2 and Theorem 5 are based in the works [Sim99, AMR07], and the monotonicity result Theorem 6 on [GS00]. All these results are proved in Section 3. In Section 4 we discuss a reconstruction method for the radial Calderón problem based on the Born approximation, in particular the validity of Algorithm 4 is proved there. We then use this to address in Section 5 the stability result for the Born approximation Theorem 3, which is based on a new stability result for the A-amplitude of inverse spectral theory given in Theorem 5.2. The regularization of the Born approximation is introduced in Section 6, together with the proof of Theorem 1(ii). Finally, Section 7 presents the consequences of our result towards giving a characterization of the set of radial DtN maps.

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2. The direct problem in the radial case

In this section present several useful facts on the Dirichlet problem for Schrödinger operators with potentials $V \in \mathcal{V}_d$, which may have low regularity, as well as the precise definition of the sequence $(\lambda_k[V])_{k\in\mathbb{N}_0}$ when the DtN map is not well-defined.

2.1. The DtN map and Cauchy data.

Consider the Dirichlet problem

$$\begin{cases} -\Delta u + Vu = & 0 \quad \text{on } \mathbb{B}^d, \\ \quad u|_{\mathbb{S}^{d-1}} = & f, \end{cases} \tag{2.1}$$

where $f \in H^{1/2}(\mathbb{S}^{d-1})$ and $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$.

The general version of the Calderón Problem for the Schrödinger equation consists in determining the potential V in (2.1) from the Cauchy data

$$\mathcal{C}(V) = \{(f, \partial_\nu u|_{\mathbb{S}^{d-1}}) \in H^{1/2}(\mathbb{S}^{d-1}) \times H^{-1/2}(\mathbb{S}^{d-1}) : \\ u \in H^1(\mathbb{B}^d) \text{ is a solution of (2.1)}\}. \tag{2.2}$$

Here $\partial_\nu u|_{\mathbb{S}^{d-1}}$ is defined in the weak sense using that for all $f, g \in H^{1/2}(\mathbb{S}^{d-1})$

$$\langle g, \partial_\nu u|_{\mathbb{S}^{d-1}}\rangle_{H^{1/2}(\mathbb{S}^{d-1})\times H^{-1/2}(\mathbb{S}^{d-1})} \\ = \int_{\mathbb{B}^d} \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\mathbb{B}^d} V(x)u(x)v(x)\, dx, \tag{2.3}$$

where u solves (2.1) and $v \in H^1(\mathbb{B}^d)$ is any function such that $v|_{\mathbb{S}^{d-1}} = g$.

The weak formulation of (2.1) and the expression (2.3) are well defined if one assumes

$$V \in L^p(\mathbb{B}^d) \text{ with } p > 1 \text{ and } p \geq d/2. \tag{2.4}$$

In addition, if one also requires that

$$0 \notin \operatorname{Spec}_{H_0^1(\mathbb{B}^d)}(-\Delta + V), \tag{2.5}$$

then there is a unique solution $u \in H^1(\mathbb{B}^d)$ of (2.1) for each $f \in H^{1/2}(\mathbb{S}^{d-1})$. When (2.5) holds, one can define the Dirichlet to Neumann (or DtN map), a bounded operator

$$\Lambda_V : H^{1/2}(\mathbb{S}^{d-1}) \to H^{-1/2}(\mathbb{S}^{d-1}),$$

by $\Lambda_V(f) := \partial_\nu u|_{\mathbb{S}^{d-1}}$. In this case, the Cauchy data $\mathcal{C}(V)$ coincides with the graph of the DtN map.

Let $u_j \in H^1(\mathbb{B}^d)$, $j = 1, 2$, be the solutions of (2.1) with $V = V_j$ and $f = f_j$, and assume (2.5) holds for both potentials. Then from (2.3) it follows that

$$\langle f_1, (\Lambda_{V_1} - \Lambda_{V_2}) f_2 \rangle_{H^{1/2}(\mathbb{S}^{d-1}) \times H^{-1/2}(\mathbb{S}^{d-1})} = \int_{\mathbb{B}^d} (V_1(x) - V_2(x)) u_1(x) u_2(x)\, dx. \tag{2.6}$$

For smooth potentials V, the DtN map Λ_V is a classical pseudo-differential operator (modulo a smoothing operator) of order one and its symbol can be calculated explicitly. For such potentials, the difference $\Lambda_{V_1} - \Lambda_{V_2}$ always belongs to the space of compact operators $\mathcal{K}(L^2(\mathbb{S}^{d-1})) \subset \mathcal{L}(L^2(\mathbb{S}^{d-1}))$ since its principal symbol is of order -1, (see for instance [NSU95] where calculations are done in the case of Schrödinger operators with magnetic fields).

2.2. **Generalized DtN eigenvalues in the radial case.**

Note that $\mathcal{V}_d$ contains very singular potentials, for which that the weak formulation (2.3) does not make sense (and thus even the Cauchy data (2.2) are not well-defined); the operator $-\Delta + V$ could even fail to be essentially self-adjoint (for instance, when $V(x) = c|x|^{-2}$ with $0 \leq c + \frac{(d-1)(d-3)}{4} < \frac{3}{4}$, see ([RS75], Theorem X.11)). The rest of this section is devoted to show that even in these situations one has a well-defined sequence $(\lambda_k[V])_{k \in \mathbb{N}_0}$ from which the Born approximation can be defined.

Assume for the moment that $V(x) = q(|x|)$ for some measurable function $q : (0, 1) \to \mathbb{R}$ such that (2.4) holds. Recall that, as in the introduction, $\mathfrak{H}_k$ stands for the subspace of spherical harmonics of degree k in $\mathbb{S}^{d-1}$. Let $Y_k \in \mathfrak{H}_k$ then:

$$-\Delta_{\mathbb{S}^{d-1}} Y_k(\omega) = k(k + d - 2) Y_k(\omega), \qquad \omega \in \mathbb{S}^{d-1}.$$

Taking $f = Y_k$ in (2.1) and using a Fourier expansion in spherical harmonics of u, it yields that $u(x) = b_k(|x|) Y_k(x/|x|)$, where b_k is a solution of

$$-\frac{1}{r^{d-1}} \frac{d}{dr} \left(r^{d-1} \frac{d}{dr} b_k(r) \right) + \left(\frac{k(k + d - 2)}{r^2} + q(r) \right) b_k(r) = 0, \tag{2.7}$$

subject to the boundary condition $b_k(1) = 1$.

Note that if, in addition to (2.4), the potential satisfies (2.5) then there exists a unique solution b_k such that $b_k(|x|) Y_k(x/|x|)$ belongs to $H^1(\mathbb{B}^d)$. Since the normal derivative coincides with the radial derivative in spherical coordinates, it turns out that the DtN map is well-defined and satisfies that

$$\Lambda_V(Y_k) = \partial_r b_k(1) Y_k.$$

This shows that Y_k is an eigenfunction of Λ_V, and that the space $\mathfrak{H}_k$ is an invariant subspace of the DtN map operator with eigenvalue

$$\lambda_k[V] := \partial_r b_k(1). \tag{2.8}$$

We now want understand how this can be generalized when conditions (2.4) or (2.5) fail and the DtN map is not well-defined. The key technical point to achieve this is contained in the following result. We recall that k_V is the constant defined in (1.9).

Lemma 2.1. *Let $d \geq 2$, $V \in \mathcal{V}_d$. Then, for every $k > k_V$ there is a unique solution b_k of (2.7) with $b_k(1) = 1$ such that for every $Y_k \in \mathfrak{H}_k$, the function $u_k(x) = b_k(|x|) Y_k(x/|x|)$ satisfies that*

$$u_k \in L^2(\mathbb{B}^d, |x|^{-1} dx). \tag{2.9}$$

Remark 2.2. When the conclusion of Lemma 2.1 holds, then u_k is an H^1_{loc} solution to

$$\begin{cases} -\Delta u_k + V u_k = & 0 \qquad \text{on } \mathbb{B}^d \setminus \{0\}, \\ \quad u_k|_{\mathbb{S}^{d-1}} = & Y_k. \end{cases} \tag{2.10}$$

In fact $u_k \in \mathcal{C}^1(\mathbb{B}^d \setminus 0)$, since $b_k \in \mathcal{C}^1(0,1)$ and the solid harmonic is a polynomial.

Remark 2.3. In addition, if $V \in L^p(\mathbb{B}^d)$ with $p > 1$ and $p \geq d/2$, then the solutions u_k obtained in Lemma 2.1 are a proper weak $H^1(\mathbb{B}^d)$ solutions of (2.10); and given any other solution $u \in H^1(\mathbb{B}^d)$ of (2.10) with the same k it holds that

$$\partial_r b_k(1) = (Y_k, \partial_\nu u|_{\mathbb{S}^{d-1}})_{L^2(\mathbb{S}^{d-1})}, \tag{2.11}$$

for all $Y_k \in \mathfrak{H}_k$ with $\|Y_k\|_{L^2(\mathbb{S}^{d-1})} = 1$. This assertion is proved using standard arguments from the theory of linear elliptic equations, see Lemma A.1 in Appendix A.

The proof of Lemma 2.1 is given in Section 2.3. This result shows, in particular, that for a potential satisfying (1.8), problem (2.10) always possesses a unique solution of separation of variables, at least for k large enough (even when (2.10) is not well-posed). This motivates the following definition.

Definition 2.4. *Let $V \in \mathcal{V}_d$ and denote by $\mathcal{B}_V$ the set of indices $k \in \mathbb{N}_0$ such that the conclusion of Lemma 2.1 fails. We define:*

$$\lambda_k[V] := \begin{cases} \partial_r b_k(1), & k \in \mathbb{N}_0 \setminus \mathcal{B}_V \\ k, & k \in \mathcal{B}_V. \end{cases} \tag{2.12}$$

Note that Lemma 2.1 states that $\mathcal{B}_V$ is at most finite. In addition, if (2.4) holds, solutions u_k with $k > k_V$ constitute a well-defined section of the Cauchy data (2.2), and for each of those k, Remark 2.3 ensures that $\lambda_k[V]$ can be determined from *any* solution of (2.10) via (2.11). If in addition (2.5) holds, then $(\lambda_k[V])_{k \in \mathbb{N}_0}$ coincides exactly with the spectrum of the DtN map.

2.3. **Reduction to a Schrödinger operator on the half-line.**
In this section we prove Lemma 2.1. Let $r = e^{-t}$; then, writing

$$Q(t) := e^{-2t} q(e^{-t}), \tag{2.13}$$

so that $q(r) = r^{-2} Q(-\log r)$, we have the following. A function b_k is a solution to (2.7) if and only if

$$v_k(t) := e^{-\frac{d-2}{2}t} b_k(e^{-t}), \tag{2.14}$$

solves the following boundary value problem on the half-line:

$$\begin{cases} -v_k'' + Q v_k = & -\left(k + \frac{d-2}{2}\right)^2 v_k \quad \text{on } \mathbb{R}^+, \\ v_k(0) = & 1. \end{cases} \tag{2.15}$$

We will show that for potentials Q satisfying that

$$|||Q||| := \sup_{y>0} \int_y^{y+1} |Q(t)|\, dt < \infty, \tag{2.16}$$

problem (2.15) posseses a unique solution provided that k is big enough (see Lemma 2.6). Performing a change of variables will yield Lemma 2.1.

Assumption (2.16) comes from [AMR07, Theorem 2] and essentially corresponds to case 2 in [RS00]. In particular it implies that the operator

$$-\frac{d^2}{dt^2} + Q, \tag{2.17}$$

(with Dirichlet boundary condition at $t = 0$), is in the limit point case at infinity. This operator is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^+)$ and bounded from below, (see e.g [Eas72], [RS75], Theorem X.7). This condition motivates the introduction of the norm $\|\cdot\|_{\mathcal{V}_d}$ in (1.8).

Remark 2.5. Let $V(x) = q(|x|)$ on $\mathbb{B}^d$ and $Q(t) = e^{-2t} q(e^{-t})$. Then

$$\frac{1}{3} |\mathbb{S}^{d-1}| |||Q||| \le \|V\|_{\mathcal{V}_d} \le |\mathbb{S}^{d-1}| |||Q|||. \tag{2.18}$$

In fact it follows from (1.8) that

$$\|V\|_{\mathcal{V}_d} = |\mathbb{S}^{d-1}| \sup_{j\in\mathbb{N}_0} \int_{j\log 2}^{(j+1)\log 2} |Q(t)|\, dt, \qquad j \in \mathbb{N}_0.$$

The space $\mathcal{V}_d$ contains the radial functions in Lorentz space $L^{d/2,\infty}(\mathbb{B}^d)$ with $d > 2$, (the weak $L^{d/2}(\mathbb{B}^d)$ space), see Appendix B.

Define the constant

$$\beta_Q := 2 \max\left(\sqrt{2|||Q|||}, e|||Q|||\right). \tag{2.19}$$

Lemma 2.6. *Let $Q \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that $|||Q||| < \infty$, and consider the equation*

$$-u_z'' + Q u_z = z u_z.$$

Then, for all $z \in \mathbb{C} \setminus [-\beta_Q{}^2, \infty)$ there exists a unique solution u_z such that $u_z(0) = 1$ and $u_z \in L^2(\mathbb{R}^+)$.

Proof. Under condition (2.16), it is known that $-\frac{d^2}{dx^2}+Q$ is limit point at infinity, (see [Sim99]). Thus, for $\operatorname{Im} z \neq 0$, there exists a unique solution u_z with $u_z(0)=1$ which is L^2 at infinity. Moreover, for $\operatorname{Im} z \neq 0$ one has, (see e.g [Tes14], Lemma 9.14):

$$\operatorname{Im} M(z) = \operatorname{Im} z \int_0^{+\infty} |u_z(x)|^2 \, dx, \tag{2.20}$$

where $M(z)$ is the so-called Weyl-Titchmarsh function (see Section 3 for details). One has $\overline{M(z)} = M(\bar z)$ and under the assumption (2.16), the map $k \to M(-k^2)$ has an analytic continuation to $\operatorname{Re} k > \beta_Q$, (see [AMR07], Section 5, Algorithm 1).

For a fixed $k > \beta_Q$, (k real), and for $\epsilon > 0$ small enough, we set

$$f(\epsilon) := \operatorname{Im} M(-k^2 + i\epsilon). \tag{2.21}$$

Clearly, f is smooth, $f(0)=0$ and using (2.20), one gets

$$\frac{f(\epsilon)}{\epsilon} = \int_0^{+\infty} |u_{-k^2+i\epsilon}(x)|^2 \, dx.$$

Taking $\epsilon \to 0$ and using Fatou's lemma, we see that u_{-k^2} is L^2 at infinity. Uniqueness follows from the fact that $-\frac{d^2}{dx^2}+Q$ is limit point at infinity. □

We can now prove Lemma 2.1.

Proof of Lemma 2.1. As we have seen, using the change of variables (2.14) in (2.7), the function $v_k(t) = e^{-\frac{d-2}{2}t} b_k(e^{-t})$ satisfies (2.15) with $Q(t) = e^{-2t} q(e^{-t})$. Since $V = q(|\cdot|)$, by (2.18) we know that $\|V\|_{\mathcal{V}_d} < \infty$ implies $|||Q||| < \infty$. Therefore, by Lemma 2.6, for all $k+(d-2)/2 > \beta_Q$ there exists a unique solution v_k of (2.15) such that $v_k \in L^2(\mathbb{R}^+)$. Also, from (1.9), (2.19) and (2.18), it follows that $\beta_Q \le \beta_V$.

Now, using that $b_k(r) = r^{-\frac{d-2}{2}} v_k(-\log r)$ it follows that

$$\|v_k\|^2_{L^2(\mathbb{R}^+)} = \int_0^1 |b_k(r)|^2 \frac{1}{r}\, dr = \int_{\mathbb{B}^d} |u_k(x)|^2 \frac{1}{|x|}\, dx,$$

where $u_k(x) = b_k(x) Y_k(x/|x|)$. Thus, b_k is the unique solution of (2.7) such that (2.9) holds. □

3. Connection with inverse spectral theory and Simon's A-amplitude

This section is devoted to the proofs of Theorems 1(i), 2, 5 and 6. This will be done by establishing a link between Simon's approach to inverse spectral theory for Schrödinger operators on the half-line and the radial Calderón problem.

3.1. The DtN map and Weyl-Titchmarsh function.

Recall that $\beta_Q = 2\max\left(\sqrt{2|||Q|||}, e|||Q|||\right)$. By Lemma 2.6 if Q satisfies (2.16), the Schrödinger equation

$$-u_z'' + Q u_z = z u_z, \qquad \text{on } \mathbb{R}^+,$$

has a unique solution $u_z \in L^2(\mathbb{R}^+)$ up to a multiplicative constant whenever $\operatorname{Im}(z) > 0$.

The Weyl-Titchmarsh function $M(z)$ associated to the half-line Schrödinger operator is defined as

$$M(z) := \frac{u_z'(0)}{u_z(0)}, \qquad z \in \mathbb{C}_+ := \{\mathrm{Im}(z) > 0\}.$$

$M(z)$ is analytic in $\mathbb{C}_+$. and, under the assumption $|||Q||| < \infty$, $M(z)$ has an analytic continuation to $\mathbb{C} \setminus [-\beta_Q^2, \infty)$, (see Lemma 2.6). Therefore, by (2.15) we have that

$$v_k'(0) = M\left(-\kappa_k^2\right),$$

where, for simplicity, we introduce the notation

$$\kappa_k = k + \frac{d-2}{2}.$$

On the other hand, using (2.8) and inverting the change of variables (2.14) one obtains

$$\lambda_k[V] = \partial_r b_k(1) = \partial_r \left[r^{-\frac{d-2}{2}} v_k(-\log r) \right] \Big|_{r=1} = -\frac{d-2}{2} - v_k'(0).$$

From this, it follows that

$$\lambda_k[V] = -\frac{d-2}{2} - M(-\kappa_k^2). \tag{3.1}$$

This shows that when (2.13) holds, the eigenvalues of the DtN map of V coincide with the values of the M-function of Q on a certain discrete set.

3.2. **Simon's A-amplitude.**

Simon proved in [Sim99], and was later refined in [AMR07] assuming just that Q satisfies (2.16), that there exists a function $A \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(t) e^{-2\kappa t}\, dt \quad \text{ for } \mathrm{Re}(\kappa) > \beta_Q, \tag{3.2}$$

where the integral is absolutely convergent. The function A is called the A-amplitude of Q. This function enjoys a series of interesting properties.

Theorem 3.1 (Theorem 1.5 [Sim99] and [AMR07])**.** *Under the assumption* (2.16)*, Q on $[0,a]$ is only a function of A on $[0,a]$. More precisely*

$$Q_1(t) = Q_2(t) \;\; \text{a.e. on } [0,a] \iff A_1(t) = A_2(t) \;\; \text{a.e. on } [0,a].$$

Theorem 3.2 (Simon [Sim99] and [AMR07])**.** *Assume Q satisfies* (2.16)*. Then $A \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, and*

$$A(t) = Q(t) + E(t),$$

where $E \in \mathcal{C}(\mathbb{R}^+)$ satisfies, for every $t > 0$,

$$|E(t)| \le \frac{1}{2} \left(\int_0^t |Q(s)|\, ds \right)^2 \left\{ e^{2\sqrt{2}|||Q|||t} + \frac{1}{\sqrt{2\pi}} e^{2e|||Q|||t} \right\}, \tag{3.3}$$

and

$$|E(t)| \le \left(\int_0^t |Q(s)|\, ds \right)^2 \exp\left(t \int_0^t |Q(s)|\, ds \right). \tag{3.4}$$

In addition, if Q is of class $\mathcal{C}^m$, $m \ge 1$ in $(0,a)$, then E is of class $\mathcal{C}^{m+2}$ in $(0,a)$.

The previous theorem implies that the difference between $E = A - Q$ is small close to the origin and that eventually $E(0) = 0$ and $E'(0^+) = 0$. It also provides a recovery of singularities result. Estimate (3.4) shows that the error only depends locally on Q, while (3.3) provides a global control of the growth of the exponential factor when $|||Q||| < \infty$.

3.3. **From the A-amplitude to the Born approximation.**

Proof of Theorem 1(i). We can combine the relation between the eigenvalues of the DtN map and the M function given by (3.1) with the representation of the M as a Laplace transform given in (3.2) to obtain that

$$\lambda_k[V] = k + \int_0^\infty A(t)e^{-2\left(k+\frac{d-2}{2}\right)t}\,dt, \qquad \text{for all } \ k > k_Q, \tag{3.5}$$

where $k_Q := \beta_Q - \frac{d-2}{2}$. Using the change of variables $r = e^{-t}$, we have

$$\lambda_k[V] - k = \int_0^1 A(-\log r) r^{2k+d-3}\,dr, \qquad \text{for all } \ k > k_Q,$$

which can also be written as

$$\lambda_k[V] - k = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{B}^d} |x|^{2k}\frac{A(-\log|x|)}{|x|^2}\,dx \qquad \text{for all } \ k > k_Q. \tag{3.6}$$

Recall that, *formally*, V^{B} should be a solution of the moment problem (1.7). The previous expression implies that there exists such a solution for all $k > k_Q$, since we can take

$$V^{\mathrm{B}}(x) := \frac{A(-\log|x|)}{|x|^2}. \tag{3.7}$$

Notice that this is a actual solution of the problem, since the fact that (3.5) converges absolutely implies that also (3.6) is absolutely convergent. Thus, we finally have that

$$\lambda_k[V] - k = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{B}^d} |x|^{2k} V^{\mathrm{B}}(x)\,dx \qquad \text{for all } \ k > k_Q. \tag{3.8}$$

Uniqueness is proved in Section 7 (see in particular identity (7.1)), and the identity (1.10) follows from (3.8), since by Remark 2.5 one always has $\beta_Q < \beta_V$ and $k_Q < k_V$ when (2.13) holds. □

Proof of Theorem 2. Is a direct consequence of Theorem 3.1 using that

$$V(x) = |x|^{-2}Q(-\log|x|), \quad \text{and} \quad V^{\mathrm{B}}(x) = |x|^{-2}A(-\log|x|). \tag{3.9}$$ □

Proof of Theorem 5. Using the change of variables $q(r) = r^{-2}Q(-\log r)$, it follows that

$$\int_0^t |Q(u)|\,du = \int_r^1 s|q(s)|\,ds.$$

Therefore, since $F(r) = r^{-2}E(-\log r)$, and $|||Q||| = \|V\|_{\mathcal{V}_d}$, estimate (3.3) becomes

$$|F(r)| \le \frac{1}{2}\left(\int_r^1 s|q(s)|\,ds\right)^2 \left\{r^{-2\sqrt{2}\|V\|_{\mathcal{V}_d}} + r^{-2e\|V\|_*}\right\},$$

and (3.4) becomes

$$|F(r)| \leq \left(\int_r^1 s|q(s)|\, ds \right)^2 r^{-\int_r^1 s|q(s)|\, ds},$$

which together yield (1.16). The recovery of singularities statement follows directly from (2.16). □

Proof of Theorem 6. It is a direct consequence of [GS00, Theorem 10.2] together with (3.9). □

3.4. **Some explicit examples.**

We present two examples for which the Born approximation V^{B} can be computed explicitly. They show in particular that the Born approximation can effectively be more singular at the origin than the potential V.

First, let us consider the so-called Bargmann potentials in $\mathbb{R}^+$:

$$Q(t) = -8\mu^2 \frac{\mu-\nu}{\mu+\nu} \frac{e^{-2\mu t}}{\left(1 + \frac{\mu-\nu}{\mu+\nu} e^{-2\mu t}\right)^2},$$

where $\mu > 0$, and $\nu \geq 0$. Then, in [GS00, Section 11] it is shown that for $s \geq 0$,

$$A(s) = 2(\nu^2 - \mu^2) e^{-2\nu s}.$$

Therefore, using (3.9) one gets:

$$V(x) = -8\mu^2 \frac{\mu-\nu}{\mu+\nu} \frac{|x|^{2(\mu-1)}}{\left(1 + \frac{\mu-\nu}{\mu+\nu} |x|^{2\mu}\right)^2},$$

and

$$V^{\mathrm{B}}(x) = 2(\nu^2 - \mu^2)|x|^{2(\nu-1)}.$$

When $\mu \geq 1$, V is a continuous function on $\mathbb{B}^d$ whereas the Born approximation V^{B} has a singularity at the origin if $\nu < 1$.

Secondly, let us consider the potential defined in the unit ball $\mathbb{B}^d$ by

$$V(x) = \frac{q_0}{|x|^2}, \quad q_0 \in \mathbb{R}.$$

As discussed previously (see also Appendix B), $V \in L^{\frac{d}{2},\infty}(\mathbb{B}^d)$, (and if q_0 is small enough, the DtN map is well defined, see Remark B.2). This potential corresponds, by the change of variables (2.13), to the case $Q(t) = q_0$, $t \in \mathbb{R}^+$, which was studied in [GS00, Theorem 10.1] to conclude the following.

If $q_0 > 0$, the Born approximation is given by

$$V^{\mathrm{B}}(x) = -\frac{\sqrt{q_0}}{|x|^2 \log|x|} J_1(-2\sqrt{q_0} \log|x|). \tag{3.10}$$

Using the well-known asymptotics for the Bessel functions at infinity, (see [Leb65, Eq. (5.11.6)] we see that:

$$V^{\mathrm{B}}(x) = O\left(\frac{1}{|x|^2 |\log|x||^{\frac{3}{2}}}\right), \quad |x| \to 0. \tag{3.11}$$

In particular, we see that the singularity at the origin for the potential V^{B} is more or less the same as the one for the initial potential V.

In the same way, if $q_0 < 0$, we obtain:

$$V^{\mathrm{B}}(x) = -\frac{\sqrt{-q_0}}{|x|^2 \log|x|} I_1(-2\sqrt{-q_0}\log|x|), \tag{3.12}$$

where I_1 is the corresponding modified Bessel function of order one, and we have the following asymptotics (see [Leb65, Eq. (5.11.10)]):

$$V^{\mathrm{B}}(x) = O\left(\frac{1}{|x|^{2+2\sqrt{-q_0}} |\log|x||^{\frac{3}{2}}}\right), \quad |x| \to 0. \tag{3.13}$$

So, in this case the singularity at $x = 0$ for V^{B} is stronger than the one for V.

4. Effective reconstruction algorithms

The proof of Theorem 3.1 in [Sim99] is close to constitute an explicit reconstruction algorithm for the potential Q in terms of the A-amplitude. In this section we show that this approach can be adapted to the Calderón problem, and together with formula (6.4), yields Algorithm 4.9, a method to reconstruct a radial potential V from its Cauchy Data or DtN map.

4.1. **Simon's approach to reconstruction.**

The idea introduced by Simon in [Sim99] is to study the A-amplitudes of the translated potentials $Q_s(t) = Q(t+s)$. Notice that this removes a part from Q from the domain $\mathbb{R}^+$, since we are translating Q to the left. The key is that one can read the value of $Q_s(0) = Q(s)$ from the corresponding A-amplitudes of Q_s as the potential is translated out of the domain. It also will be essential to use that the A-amplitudes of Q_s are related by a certain equation.

Let $Q : \mathbb{R}^+ \to \mathbb{R}$ be a potential satisfying assumption (2.16). Let A be the A-amplitude associated to Q. For every fixed $s \geq 0$, consider the potential $Q_s(t) = Q(t+s)$, where $t \in \mathbb{R}^+$, and denote by $A(t,s)$ the corresponding A-amplitude of Q_s. Since Q has just local L^1 regularity, and so does A, it is not clear if $A(t,s)$ is well defined. The simplest way overcome this difficulty, given a specific realization of Q, is to pick the realization of $A(t,s)$ such that $A(t,s) - Q_s(t)$ is a continuous function for every fixed s (this is possible by Theorem 3.2). This is enough to properly define $A(t,s)$, as the following lemma shows.

Lemma 4.1. *Assume that Q satisfies (2.16). Then, for all $t, s \in \mathbb{R}^+$ it holds that*

$$|A(t,s) - Q(t+s)| \leq \alpha(t,s)^2 e^{t\,\alpha(t,s)},$$

where

$$\alpha(t,s) = \int_0^t |Q(y+s)|\,dy = \int_s^{t+s} |Q(y)|\,dy.$$

Moreover $A(t,s) - Q(t+s)$ *is a jointly continuous function on* $[0,\infty)\times[0,\infty)$.

The first statement follows applying Theorem 3.2 to the potential $Q_s(t)$. We postpone momentarily the proof of the continuity statement.

Let $Q \in \mathcal{C}^1(\mathbb{R}^+)$ satisfying (2.16). In [Sim99] it is shown that $A(t,s)$ satisfies the initial value problem

$$\begin{aligned}
&\frac{\partial A}{\partial s}(t,s) = \frac{\partial A}{\partial t}(t,s) + \int_0^t A(w,s)A(t-w,s)\,dw, \qquad (t,s)\in\mathbb{R}^+\times\mathbb{R}^+,\\
&A(t,0) = A(t), \qquad t\in\mathbb{R}^+,
\end{aligned} \tag{4.1}$$

where $A(t)$ denotes the A-amplitude of Q. If $Q \in \mathcal{C}^1(\mathbb{R}^+)$ this equation holds in the strong sense, and also in the general case under a suitable weak formulation (see Theorem 4.2). Then, it follows from Theorem 3.2 that

$$\lim_{t\to 0^+} A(t,s) = Q(s), \tag{4.2}$$

where the convergence holds in $L^1(0,T)$ for all $T>0$. If Q is continuous, then the convergence holds also point-wise, and in general will hold at any point of right Lebesgue continuity of Q (see [Sim99]). Therefore (4.2) together with (4.1) give a procedure to reconstruct the potential Q from its A-amplitude, provided that (4.1) can be uniquely solved under certain assumptions. This will be proved in Lemma 4.4 below, and in Section 5 we will analyze the stability of this reconstruction procedure. We start by stating a weak version of Equation (4.1).

Theorem 4.2 (Theorem 6.3 of [Sim99])**.** *Let* Q *such that* (2.16) *holds.*
If $K(t,s) = A(t-s,s)$ *then*

$$K(t,s_2) = K(t,s_1) + \int_{s_1}^{s_2}\int_{y_2}^{t} K(y_1,y_2)K(t-y_1+y_2,y_2)\,dy_1dy_2, \tag{4.3}$$

with $0<s_1<s_2<t<\infty$.

Remark 4.3. The previous theorem implies that K satisfies an initial value problem with $K(t,0) = A(t,0) = A(t)$, where $A(t)$ is the A-amplitude of Q. Moreover, by Lemma 4.1 we know that $K(t,s)-A(t) = K(t,s)-Q(t)+(A(t)-Q(t))$ is continuous for $0\le s\le t<\infty$.

The previous conditions are enough to obtain a uniqueness result for the initial value problem for (4.3), as the following lemma shows.

Lemma 4.4. *Let* $a\in\mathbb{R}^+$ *and* $f\in L^1(0,a)$. *There is at most one solution of* (4.3) *in* $\{0<s_1<s_2<t<a\}$ *such that*

$$K(t,s) = f(t) + K_0(t,s),$$

where $K_0(t,s)$ *is a continuous function on* $0<s\le t\le a$ *and* $K_0(t,0)=0$.

This lemma is a consequence of the estimates proved by Simon in [Sim99, Section 7]. Since the result is not explicitly stated in [Sim99], we give a proof here for completeness.

Proof. Assume that $K(t,s) = f(t) + K_0(t,s)$ and $\tilde{K}(t,s) = f(t) + \tilde{K}_0(t,s)$ are two solutions satisfying the conditions of the statement of the lemma. Let

$$g(s) := \int_s^a \left| K(t,s) - \tilde{K}(t,s) \right| dt = \int_s^a \left| K_0(t,s) - \tilde{K}_0(t,s) \right| dt.$$

By the previous assumptions, g is a continuous function in $[0,a]$. Moreover

$$\begin{aligned} D := \sup_{0\le s<a} \int_s^a \left(|K(t,s)| + \left|\tilde{K}(t,s)\right| \right) dt \\ < \int_s^a \left(|K_0(t,s)| + \left|\tilde{K_0}(t,s)\right| + 2\,|f(t)| \right) dt < \infty \end{aligned} \tag{4.4}$$

since the function of s obtained from last integral is continuous on $[0,a]$. Using this in (4.3) it follows that

$$g(s_2) \le g(s_1) + D \int_{s_1}^{s_2} g(y)\,dy.$$

We now define $h_z(s) = \sup_{z\le y\le s} g(y)$. The previous estimate implies that

$$h_{s_1}(s_2) \le h_{s_1}(s_1) + D h_{s_1}(s_2) \int_{s_1}^{s_2} dy.$$

Therefore, if $h_{s_1}(s_1) = g(s_1) = 0$ and $(s_2 - s_1)D < 1$, then it follows that $h_{s_1}(s_2) = 0$. This shows that if $g(s_1)$ vanishes, then $g(s)$ vanishes in $(s_1, s_1 + 1/D)$.

Since $g(0) = 0$ one can apply the previous argument a finite number of times to deduce that $g(s) = 0$ in $[0,a]$, and therefore that $K = \tilde{K}$ on $\{0 < s_1 < s_2 < t < a\}$. □

We can now prove that $A(t,s) - Q(t+s)$ is a jointly continuous function, as stated previously.

Proof of Lemma 4.1. The first estimate is a direct application of (3.4).

To prove that $A(t,s) - Q(t+s)$ is continuous in t and s, we start from the estimate

$$|A_1(t) - Q_1(t) - (A_2(t) - Q_2(t))| \le e^{t(\alpha_{Q_1}(t)^2 + \alpha_{Q_2}(t)^2)} \int_0^t |Q_1(s) - Q_2(s)|\,ds, \tag{4.5}$$

where

$$\alpha_Q(t) = \int_0^t |Q(s)|\,ds, \quad j = 1,2.$$

This shows that $A - Q$ is a continuous function of Q in $L^1(0,T)$ for all $T > 0$. This is proved in [Sim99, Theorem 2.1] for $L^1(\mathbb{R}^+)$ potentials[4], but the extension for general potentials is immediate due to the local dependence of the A-amplitude from Q (see Theorem 3.1).

[4]Note that [Sim99, equation (2.4)] contains a typograhpical error, the correct left hand side is the one in (4.5) instead of just $|A_1(t) - A_2(t)|$.

Let $T > 0$ and $s_1, s_2 \in [0, T)$ with $s_1 < s_2$. We now apply (4.5) with $Q_1(t) = Q(t+s_1)$ and $Q_2(t) = Q(t+s_2)$. With this choice we have $A_1(t) = A(t, s_1)$, $A_2(t) = A(t, s_2)$ and

$$\sup_{t\in[0,T]} |A(t, s_1) - Q(t + s_1) - (A(t, s_2) - Q(t + s_2))| \leq e^{2T\alpha_Q(2T)^2} \int_0^{2T} |Q(s) - Q(s + s_2 - s_1)|\, ds, \tag{4.6}$$

where we have used a change of variable $s = s' + s_1$ in the integral term. Since translations are continuous in the L^1 norm, we have that $\lim_{\varepsilon\to 0^+} \omega(\varepsilon) = 0$ where

$$\omega(\varepsilon) = \int_0^{2T} |Q_1(s) - Q_2(s + \varepsilon)|\, ds.$$

From Theorem 3.2 it follows that for any fixed $s \in [0, T)$ the function $A(t, s) - Q(t+s)$ is continuous in t for $t \in [0, T]$. Combining this with the estimate

$$\sup_{t\in[0,T]} |A(t, s_1) - Q(t + s_1) - (A(t, s_2) - Q(t + s_2))| \leq C_T\omega(s_2 - s_1),$$

that follows from (4.6), we obtain that $A(t, s) - Q(t+s)$ is a jointly continuous function in $[0, T)^2$. Since T is arbitrary, this finishes the proof of the lemma. □

4.2. **Reconstruction for the radial Calderón problem.**

It is not difficult to adapt the prevous reconstruction method to the radial Calderón problem using the transformation $V(x) = q(|x|) = |x|^{-2}Q(-\log |x|)$ for the potentials, as we now show.

Let $V(x) = q(|x|)$ and define

$$V_s(x) := s^2V(sx), \quad q_s(r) := s^2q(sr) \quad s \in [0, 1].$$

If $V_s^{\mathrm{B}} := [V_s]^{\mathrm{B}}$ we introduce the W function given by

$$W(|x|, s) := \frac{1}{s^2}V_s^{\mathrm{B}}(x), \quad s \in [0, 1]. \tag{4.7}$$

It will be convenient to use the notation

$$V^{\mathrm{B}}(x) = q^{\mathrm{B}}(|x|), \quad \text{and} \quad V_s^{\mathrm{B}}(x) = q_s^{\mathrm{B}}(|x|),$$

for the radial profiles of the Born approximations. Thus $W(r, s) = s^{-2}q_s^{\mathrm{B}}(r)$.

In terms of the A-amplitude it holds that

$$A(t, s) = e^{-2(t+s)}W(e^{-t}, e^{-s}), \qquad W(r, s) = \frac{1}{r^2s^2}A(-\log r, -\log s). \tag{4.8}$$

We restate Lemma 4.1 in this context as follows.

Lemma 4.5. *Assume that $V = q(|\cdot|)$ with $V \in \mathcal{V}_d$. Then, for all $r, s \in (0, 1]$ it holds that*

$$|F_s(r)| \leq \frac{s^{-2}}{r^{2+g(r,s)}}g(r, s)^2,$$

where

$$F_s(r) := W(r,s) - q(rs), \quad \textit{and} \quad g(r,s) := \int_r^1 ws^2|q(sw)|\,dw = \int_{rs}^s t|q(t)|\,dw.$$

Moreover $F_s(r)$ *is a jointly continuous function on* $(0,1] \times (0,1]$.

The proof is straightforward using (4.8) and Lemma 4.1. We can now prove the analogue of (4.2).

Proposition 4.6. *Assume that* $V \in \mathcal{V}_d$ *with* $V = q(|\cdot|)$ *and fix* b *such that* $0 < b < 1$. *Then, if* W *is given by* (4.7) *it holds that*

$$\lim_{r\to 1^-} W(r,\cdot) = q(\cdot), \quad \textit{in } L^1(b,1).$$

Proof. We use that

$$\begin{aligned} W(r,s) &= q(s) + (W(r,s) - q(sr)) + (q(sr) - q(s)) \\ &= q(s) + F_s(r) + (q(sr) - q(s)), \end{aligned}$$

where $F_s(r) = W(r,s) - q(rs)$. From Lemma 4.5 it follows that

$$|F_s(r)| \le \frac{s^{-2}}{r^{g(r,s)+2}} g(r,s)^2, \quad \text{with} \quad g(r,s) \le \int_{br}^1 t|q(t)|\,dt.$$

Hence $\lim_{r\to 1^-} \sup_{s\in(b,1)} F_s(r) = 0$, which finishes the proof of the proposition. □

Using the transformation (4.8) in (4.1), one can show that W satisfies also a first order PDE with a non-linear integral term:

$$r\frac{\partial W}{\partial r}(r,s) - s\frac{\partial W}{\partial s}(r,s) = s^2 \int_r^1 W\left(\frac{r}{\nu}, s\right) W(\nu,s)\frac{d\nu}{\nu}, \tag{4.9}$$

for all $(r,s) \in (0,1)\times(0,1)$. This holds in the classical sense for $\mathcal{C}^1(\mathbb{B}^d)$ potentials, since in this case $A(t,s)$—and hence $W(r,s)$—is a jointly $\mathcal{C}^1$ function, as shown in [Sim99, Section 2].

With the change variables $U(r,s) = W\left(\frac{r}{s}, s\right)$ the equation (4.9) becomes

$$\frac{\partial U}{\partial s}(r,s) = -s\int_r^s U(\nu,s)U\left(r\frac{s}{\nu}, s\right)\frac{d\nu}{\nu}, \qquad 0 < r < s < 1.$$

In the general case $V \notin \mathcal{C}^1(\mathbb{B}^d)$, $W(r,s)$ can be shown to satisfy the integral version of the previous equation.

Proposition 4.7. *Let* $V \in \mathcal{V}_d$ *and let* W *be given by* (4.7). *Define*

$$U(r,s) := W\left(\frac{r}{s}, s\right), \qquad 0 < r < s < 1.$$

Then, we have that

$$U(r,s_2) = U(r,s_1) + \int_{s_2}^{s_1} y_2 \int_r^{y_2} U(y_1,y_2)U\left(r\frac{y_2}{y_1}, y_2\right)\frac{dy_1}{y_1}\,dy_2, \tag{4.10}$$

for all $0 < r < s_2 < s_1 < 1$.

In addition, it holds that $U(r,s)$ *is the unique solution of* (4.10) *in* $0<r<s<1$ *such that:*

i) $U_0(r,s) := U(r,s) - U(r,1)$ *is a continuous function for* $0 \le r \le s < 1$ *and* $U_0(r,1) = 0$.
ii) $U(r,1) = q^{\mathrm{B}}(r)$ *for* $0<r<1$.

The equation (4.10) has a strong local behaviour even if it contains a non-local term: the value $U(s_0, r_0)$ of a solution only depends on the values of U in the triangular region

$$D_{(r_0,s_0)} = \{(r,s) \in (0,1)^2 : r \le s,\ r \ge r_0,\ s \ge s_0\}.$$

To see this notice that taking $s_2 = s_0$ and $r = r_0$ in the integral term in (4.8) we have $y_2 \ge s_0$, $y_1 \ge r_0$ and $r_0 \frac{y_2}{y_1} \ge r_0$. This gives the equation a local behaviour that is in turn reflected in Theorem 2 and other results.

The proof of Proposition 4.7 is based on the fact that the initial value problem for (4.10) has at most one solution that is a continuous perturbation of a free solution, as the next lemma states.

Lemma 4.8. *Let* $b \in (0,1)$ *and* $f \in L^1(b,1)$. *There is a unique solution of*

$$U(r,s_2) = U(r,s_1) + \int_{s_2}^{s_1} y_2 \int_r^{y_2} U(y_1,y_2) U\left(r\frac{y_2}{y_1}, y_2\right) \frac{dy_1}{y_1}\, dy_2,$$

with $b<r<s_1<s_2<1$ *such that*

$$U(r,s) = f(r) + U_0(r,s),$$

where $U_0(r,s)$ *is a continuous function for* $b<r\le s\le 1$ *and* $U_0(r,1)=0$.

Proof. Is an immediate consequence of Lemma 4.4 and (4.8). which implies that

$$U(r,s) = W\left(\frac{r}{s}, s\right) = \frac{1}{r^2 s^2} A(-\log(r) + \log s, -\log s) = \frac{1}{r^2 s^2} K(-\log r, -\log s).$$

□

Proof of Proposition 4.7. That (4.10) holds for all $V \in \mathcal{V}_d$ follows directly from (4.8) and Theorem 4.2.

The second statement follows from Lemma 4.8, provided that we show that $U_0(r,s) = W\left(\frac{r}{s}, s\right) - W(r,1)$ is a continuous function on $\{0<r\le s\le 1\}$. We have that

$$\begin{aligned} U_0(sr,s) &= W(r,s) - W(rs,1) \\ &= F_s(r) - \left(q^{\mathrm{B}}(rs) - q(rs)\right). \end{aligned} \tag{4.11}$$

By Lemma 4.5 we know that that $F_s(r)$ is continuous on $(0,1]^2$. On the other hand, $q^{\mathrm{B}}(r') - q(r') = F_1(r')$ is continuous on $(0,1]$, so, taking $r' = rs$, the second term in (4.11) is also continuous on $(0,1]^2$. Replacing r by r/s in (4.11) we conclude that $U_0(r,s)$ is a continuous function on $\{0<r\le s\le 1\}$. □

We can finally state the algorithm to reconstruct $V \in \mathcal{V}_d$ form V^{B}.

Algorithm 4.9. *Given V^{B} and $0 < b < 1$, it is possible to reconstruct $V \in \mathcal{V}_d$ in the region $b < |x| < 1$ with the following three steps:*

1) Using (1.12) and that $V^{\mathrm{B}} = V_r^{\mathrm{B}}$ on $\mathbb{B}^d \setminus \{0\}$, reconstruct V^{B} from $(\lambda_k[V])_{k\in\mathbb{N}_0}$.

2) Find the unique solution $U(r,s)$ of (4.10) such that $U_0(r,s) = U(r,s) - q^{\mathrm{B}}(r)$ is a continuous function with $U_0(r,1) = 0$ and $q^{\mathrm{B}}(|x|) = V^{\mathrm{B}}(x)$.

3) Use that $\lim_{r\to 1^-} U(r|x|,|x|) = V(x)$ in the sense of Proposition 4.6.

5. A local Hölder stability estimate

The goal of this section is to present a Hölder stability estimate for the map that associates the A-amplitude to the potential Q; this is presented in Theorem 5.2. We then show how Theorem 3 follows from this result.

5.1. A local stability estimate for the A-amplitude.

Let $Q_s(t)$ and $A(t,s)$ as in Section 4. We now consider two potentials, Q_1 and Q_2. Denote by $A_1(t,s)$ and $A_2(t,s)$ the corresponding A-amplitudes of the translated potentials. Also, let $a > 0$ be fixed, and define

$$g_a(s) := \int_0^{a-s} |A_2(t,s) - A_1(t,s)|\, dt, \quad s \in [0,a], \tag{5.1}$$

and the constant

$$D(a) := \sup_{0\le s<a} \int_0^{a-s} [|A_1(t,s)| + |A_2(t,s)|]\, dt. \tag{5.2}$$

Lemma 5.1. *Let Q_1, Q_2 satisfying (2.16), and fix $a > 0$. Then $g_a(s)$ is a continuous function on $[0,a]$ and $D(a) < \infty$. In addition,*

$$g_a(s) \le g_a(0) e^{sD(a)}. \tag{5.3}$$

Proof of Lemma 5.1. The proof follows from arguments in [Sim99, Section 7]. For convenience of the reader we give a proof here. Recall the definitions of $g_a(s)$ and $D(a)$ given, respectively, in (5.2) and (5.1), and define $K_j(t,s) = A_j(t-s,s)$ for $j = 1,2$. The proof is similar to that of Lemma 4.4.

Since $K_j(t,s) - A_j(t)$ is a continuous function (see Remark 4.3), it follows that $g_a(s)$ is continuous on $[0,a]$ (notice that g_a is the analogue of g in the proof of Lemma 4.4).

On the other hand, $K_j(t,s)$ satisfies (4.3) for $j = 1,2$, so taking the difference of the equation for $j = 1$ and $j = 2$, and making a simple estimate, one can show that

$$g_a(s_2) \le g_a(s_1) + D(a) \int_{s_1}^{s_2} g_a(y)\, dy, \qquad s_1 < s_2 < a.$$

Therefore for $s_1 = 0$ and $s_2 = s$ it reduces to

$$g_a(s) \le g_a(0) + D(a) \int_0^s g_a(y)\, dy,$$

so, since g_a is continuous, a direct application of Grönwall's inequality yields (5.3).

It remains to justify that $D(a) < \infty$, but this can be done in the same way as in (4.4). $\square$

Using the previous lemma we can now prove a local stability estimate for the map $A \longmapsto Q$.

Theorem 5.2. *Consider two potentials Q_1, Q_2 satisfying assumption (2.16) and their respective A-amplitudes A_1 and A_2. Fix any $a > 0$ and take any $\varepsilon_0 < \min(1, a)$. Let $1 < p \le \infty$, denote by p' its Hölder conjugate exponent, and assume that there is a constant $M_p(a) < \infty$ such that*

$$\begin{aligned} \max_{i=1,2} \left(\int_0^a |Q_i(t)|^p \, dt \right)^{1/p} &< M_p(a) \quad \text{if } 1 < p < \infty, \\ \max_{i=1,2} \left(\sup_{t \in [0,a]} |Q_i(t)| \right) &< M_\infty(a) \quad \text{if } p = \infty. \end{aligned} \tag{5.4}$$

Then, if

$$\int_0^a |A_1(t) - A_2(t)| \, dt < \varepsilon_0^{(1+p')/p'}, \tag{5.5}$$

then

$$\int_0^a |Q_1(t) - Q_2(t)| \, dt < C_1(a, M_p(a), D(a)) \left(\int_0^a |A_1(t) - A_2(t)| \, dt \right)^{1/(p'+1)},$$

where $D(a)$ is the constant defined in (5.2), and $C_1(a, M_p(a), D(a))$ is a constant that can be computed explicitly in terms of a, $D(a)$, and $M_p(a)$.

Remark 5.3. The constant $D(a)$ may seem impractical since it depends on the A-amplitudes of the translated potentials. This can be overcome using (5.4) in (5.2) together with the rough bound provided by Lemma 4.1. This implies that

$$D(a) < 2a^3 M_p(a)^2 e^{a^2 M_p(a)} + 2a M_p(a),$$

and gives

$$C_1(a, M_p(a), D(a)) < C_2(a, M_p(a)),$$

for an appropriate constant $C_2(a, M_p(a))$ that depends only on a and $M_p(a)$.

Proof of Theorem 5.2. First, assume that $s, t \ge 0$. Then we have that

$$\begin{aligned} &Q_1(s+t) - Q_2(s+t) = \\ &\qquad Q_1(s+t) - A_1(t,s) - (Q_2(s) - A_2(t,s)) + (A_1(t,s) - A_2(t,s)) \,. \end{aligned}$$

In particular, for any $0 < \varepsilon < \varepsilon_0$ and $s \ge 0$

$$\begin{aligned} \int_0^\varepsilon |Q_1(s+t) - Q_2(s+t)| \, dt \le &\int_0^{s+\varepsilon-s} |A_1(t,s) - A_2(t,s)| \, dt \\ &+ \int_0^\varepsilon |A_1(t,s) - Q_1(s+t)| \, dt + \int_0^\varepsilon |A_2(t,s) - Q_2(s+t)| \, dt. \end{aligned} \tag{5.6}$$

We now assume that $0 \le s < a - \varepsilon_0$. By (5.1), the first term on the right satisfies

$$\int_0^{s+\varepsilon-s} |A_1(t,s) - A_2(t,s)| \, dt = g_{s+\varepsilon}(s) < g_a(s), \tag{5.7}$$

since $s+\varepsilon < s+\varepsilon_0 < a$. If $i = 1,2$, applying Lemma 4.1 with $Q = Q_i$, the remaining terms satisfy

$$\begin{aligned}(5.8)\qquad \int_0^\varepsilon |A_i(t,s) - Q_i(s+t)|\, dt &\le \int_0^\varepsilon \alpha(t,s)^2 e^{t\alpha(t,s)}\, dt \\ &\le \varepsilon\alpha(\varepsilon,s)^2 e^{\varepsilon\alpha(\varepsilon,s)} = \varepsilon\left(\int_s^{s+\varepsilon} |Q_i(y)|\, dy\right)^2 e^{\varepsilon\int_s^{s+\varepsilon}|Q_i(y)|\,dy},\end{aligned}$$

since $0 \le t \le \varepsilon$ implies $\alpha(t,s)^2 e^{t\alpha(t,s)} \le \alpha(\varepsilon,s)^2 e^{\varepsilon\alpha(\varepsilon,s)}$. By (5.4) and using Hölder inequality together with the fact that $s+\varepsilon < a$, we have

$$\alpha(\varepsilon,s) \le \varepsilon^{1/p'} M_p(a),$$

where $1 \le p' < \infty$ is the conjugate exponent of p. Then (5.8) becomes

$$(5.9)\qquad \int_0^\varepsilon |A_i(t,s) - Q_i(s+t)|\, dt < \varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}}.$$

Inserting (5.7) and (5.9) in (5.6) gives

$$\int_0^\varepsilon |Q_1(s+t) - Q_2(s+t)|\, dt < g_a(s) + 2\varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}},$$

and hence, Lemma 5.1 yields

$$\int_0^\varepsilon |Q_1(s+t) - Q_2(s+t)|\, dt < g_a(0)e^{sD(a)} + 2\varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}}.$$

We now integrate both sides in the s variable:

$$\begin{aligned}\int_0^\varepsilon \int_0^{a-\varepsilon} &|Q_1(s+t) - Q_2(s+t)|\, ds\, dt \\ &< \frac{g_a(0)}{D(a)}\left(e^{(a-\varepsilon)D(a)} - 1\right) + 2(a-\varepsilon)\varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}} \\ &< \frac{g_a(0)}{D(a)} e^{aD(a)} + 2a\varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}},\end{aligned}$$

which, changing variables in the integrals, becomes

$$(5.10)\qquad \int_0^\varepsilon \int_t^{a+t-\varepsilon} |Q_1(s) - Q_2(s)|\, ds\, dt < \frac{g_a(0)}{D(a)} e^{aD(a)} + 2a\varepsilon^{1+2/p'} M_p(a)^2 e^{M_p(a)\varepsilon^{1+1/p'}}.$$

We now want to get rid of the dependence in t of the limits of the second integral. Now, by Hölder inequality

$$\int_0^\varepsilon \int_0^t |Q_1(s) - Q_2(s)|\, ds\, dt \le 2\int_0^\varepsilon t^{1/p'} M_p(a)\, dt = \frac{2}{1+1/p'}\varepsilon^{1+1/p'} M_p(a),$$

and, analogously,

$$\int_0^\varepsilon \int_{a+t-\varepsilon}^a |Q_1(s) - Q_2(s)|\, ds\, dt \le 2\int_0^\varepsilon (\varepsilon - t)^{1/p'} M_p(a)\, dt = \frac{2}{1+1/p'}\varepsilon^{1+1/p'} M_p(a).$$

Combining these two observations we obtain that

$$\begin{aligned}\int_0^{\varepsilon}\int_t^{a+t-\varepsilon} &|Q_1(s)-Q_2(s)|\,ds\,dt\\ &> \int_0^{\varepsilon}\int_0^{a} |Q_1(s)-Q_2(s)|\;ds\,dt - \frac{4}{1+1/p'}\varepsilon^{1+1/p'}M_p(a)\\ &= \varepsilon\int_0^{a} |Q_1(s)-Q_2(s)|\;ds - \frac{4}{1+1/p'}\varepsilon^{1+1/p'}M_p(a).\end{aligned}$$

Inserting this estimate in (5.10) gives

$$\begin{aligned}\varepsilon\int_0^{a} |Q_1(s)-Q_2(s)|\;ds < g_a(0)\frac{e^{aD(a)}}{D(a)} &+ \varepsilon^{1+2/p'}2aM_p(a)^2e^{M_p(a)\varepsilon^{1+1/p'}}\\ &+ \varepsilon^{1+1/p'}\frac{4}{1+1/p'}M_p(a).\end{aligned}$$

Choosing $\varepsilon = g_a(0)^{\frac{p'}{p'+1}}$ to optimize the inequality, and using that $\varepsilon < \varepsilon_0 < 1$, gives

$$\int_0^{a} |Q_1(s)-Q_2(s)|\;ds < g_a(0)^{\frac{1}{p'+1}}\left(\frac{e^{aD(a)}}{D(a)} + 2aM_p(a)^2e^{M_p(a)} + 2M_p(a)\right).$$

This concludes the proof of the theorem, since, by (5.1)

$$g_a(0) = \int_0^{a} |A_2(t)-A_1(t)|\;dt. \qquad \square$$

5.2. **Proof of Theorem 3.** By Theorem 1 for each potential V_j, $i=1,2$ the Born approximation is well defined. In addition, by (3.7) we have that

$$V_j^{\mathrm{B}}(x) = |x|^{-2}A_j(-\log|x|), \quad i=1,2, \tag{5.11}$$

where A_j is the A-function associated to the potential Q_j, defined by

$$Q_j(t) = e^{-2t}q_j(e^{-t}), \quad i=1,2.$$

Here Q_j satisfies (2.16) by (2.18) for $j=1,2$.

Fix $a=-\log b$. A direct change of variables $r=e^{-t}$ in (1.13) implies that (5.4) holds with $M_p(a)=K_p(b)$. Also, using the same change of variables and (5.11) in (1.14) means that (5.5) holds for $\varepsilon_0 < \min(1,-\log b) = \min(1,a)$. Thus we can apply Theorem 5.2 which, taking into account Remark 5.3, gives the estimate

$$\int_0^{-\log b} |Q_1(t)-Q_2(t)|\;dt < C_2(-\log b, K_p(b))\left(\int_0^{-\log b} |A_1(t)-A_2(t)|\;dt\right)^{1/(p'+1)}.$$

Using the change of variables $t=-\log r$ this becomes (1.15) with $C(b,K_p(b)) := C_2(-\log b, K_p(b))$. $\square$

6. Canonical regularization of the Born approximation

Here we denote by $\mathcal{E}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, respectively, the spaces of compactly supported and tempered distributions, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. We also denote by $\mathcal{D}'(U)$ the space of distributions on an open set $U \subset \mathbb{R}^d$. The Fourier transform is an isomorphism of $\mathcal{S}'(\mathbb{R}^d)$; and the Paley-Wiener theorem ensures that given any $f \in \mathcal{E}'(\mathbb{R}^d)$, its Fourier transform:

$$\widehat{f}(\xi) := \langle f, e_{-i\xi}\rangle_{\mathcal{E}'\times\mathcal{C}^\infty}, \qquad e_\xi(x) := e^{\xi\cdot x}, \qquad x,\xi \in \mathbb{R}^d, \tag{6.1}$$

extends to an entire function in $\mathbb{C}^d$. The moments $\sigma_k[f]$ of a distribution $f \in \mathcal{E}'(\mathbb{R}^d)$ are defined by

$$\sigma_k[f] := |\mathbb{S}^{d-1}|^{-1}\, \langle f, m_k\rangle_{\mathcal{E}'\times\mathcal{C}^\infty}, \quad m_k(x) := |x|^{2k}, \qquad \forall k \in \mathbb{N}_0. \tag{6.2}$$

A distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ is is *radially symmetric* (or just radial) if and only if

$$\langle f, \varphi\circ\rho\rangle_{\mathcal{D}'\times\mathcal{C}_c^\infty} = \langle f, \varphi\rangle_{\mathcal{D}'\times\mathcal{C}_c^\infty}, \qquad \forall\rho \in \mathrm{SO}(d), \quad \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

In what follows, given $V \in \mathcal{V}_d$, we will denote by V_e^{B} the extension by zero of V^{B} to $\mathbb{R}^d$. Since Theorem 1(i) ensures that $V^{\mathrm{B}} \in L^1_{\mathrm{loc}}(\mathbb{B}^d \setminus \{0\})$, one automatically has $V_e^{\mathrm{B}} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \subset \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$.

Theorem 1(ii) follows directly from the following result.

Theorem 6.1. *Let $d \geq 2$ and $V \in \mathcal{V}_d$. There exists a unique compactly supported radial distribution $V_r^{\mathrm{B}} \in \mathcal{E}'(\mathbb{R}^d)$ such that*

$$\sigma_k[V_r^{\mathrm{B}}] = \lambda_k[V] - k, \qquad \forall k \in \mathbb{N}_0. \tag{6.3}$$

In addition, V_r^{B} is a regularization of V_e^{B}, namely,

$$\left\langle V_r^{\mathrm{B}}, \varphi\right\rangle_{\mathcal{E}'\times\mathcal{C}^\infty} = \left\langle V_e^{\mathrm{B}}, \varphi\right\rangle_{\mathcal{E}'\times\mathcal{C}^\infty}, \qquad \forall\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\}),$$

(and in particular $\operatorname{supp} V_r^{\mathrm{B}} \subseteq \overline{\mathbb{B}^d}$*) and the Fourier transform of V_r^{B} satisfies the following identities:*

i) For all $\xi \in \mathbb{R}^d$

$$\widehat{V_r^{\mathrm{B}}}(\xi) = 2\pi^{d/2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+d/2)}\left(\frac{|\xi|}{2}\right)^{2k}(\lambda_k[V]-k). \tag{6.4}$$

ii) If Λ_V is well defined then, for every $\xi \in \mathbb{R}^d\setminus\{0\}$, and for all $\zeta_1,\zeta_2 \in \mathbb{C}^d$ such that $\zeta_1\cdot\zeta_1 = \zeta_2\cdot\zeta_2 = 0$ and $\zeta_1+\zeta_2 = -i\xi$ the following holds

$$\widehat{V_r^{\mathrm{B}}}(\xi) = (\overline{e_{\zeta_1}}, (\Lambda_V - \Lambda_0)e_{\zeta_2})_{L^2(\mathbb{S}^{d-1})}, \tag{6.5}$$

where, for $\zeta \in \mathbb{C}^d$, we have written $e_\zeta(x) := e^{\zeta\cdot x}$.

Note that the fact that $(\lambda_k[V]-k)_{k\in\mathbb{N}_0}$ is the sequence of moments of a unique radial distribution in $\mathcal{E}'(\mathbb{R}^d)$ is a non-trivial information on the structure of DtN maps (Section 7 delves on this topic). Since V_r^{B} coincides exactly with V^{B} outside the origin, formula (6.4) offers an explicit method to reconstruct V_r^{B} and, therefore V^{B}, from $(\lambda_k[V]-k)_{k\in\mathbb{N}_0}$. Identity (6.5) connects the concept of the Born approximation with the method of Complex Geometrical Optics solutions of the Schrödinger equation of [SU87], was used in [BCMM22] to introduce the Born approximation in the context of the Calderón problem. In fact, assertions (i) and (ii) are proven in [BCMM22, Theorem 1] (that result shows the equality between the right-hand sides of (6.4) and (6.5)).

To prove Theorem 6.1 we need to show that the Fourier transform of a radial compactly supported distribution can always be reconstructed from the moments (6.2) by an explicit formula. The following extends [BCMM22, Identity (1.20)] to distributions.

Lemma 6.2. *Let $f \in \mathcal{E}'(\mathbb{R}^d)$ be radially symmetric. Then*

$$\widehat{f}(\xi) = 2\pi^{d/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+d/2)} \left(\frac{|\xi|}{2}\right)^{2k} \sigma_k[f]. \tag{6.6}$$

Proof. If $f \in \mathcal{E}'(\mathbb{R}^d)$ then the Paley-Wiener theorem ensures that $\widehat{f}$ is an entire function on $\mathbb{R}^d$ (see, for example, [Ḧ90, Theorem 7.3.1]). Moreover f is radial if and only if $\widehat{f}$ is radial. Therefore it must hold that

$$\widehat{f}(\xi) = \sum_{k=0}^{\infty} a_k |\xi|^{2k},$$

for some appropriate coefficients $a_k \in \mathbb{C}$.

On the other hand, $a_k = b_k(-\Delta)^k \widehat{f}(0)$ for all $k \in \mathbb{N}_0$, where $(b_k)_{k\in\mathbb{N}_0}$ are some coefficients independent of f—notice that b_k is essentially a coefficient of the Taylor expansion of the radial profile function of $\widehat{f}$. Using (6.1) one can show that

$$(-\Delta)^k \widehat{f}(0) = \langle f, m_k \rangle_{\mathcal{E}' \times \mathcal{C}^\infty}.$$

Hence we conclude that

$$\widehat{f}(\xi) = \sum_{k=0}^{\infty} b_k |\xi|^{2k} \sigma_k[f], \tag{6.7}$$

where the $(b_k)_{k\in\mathbb{N}_0}$ coefficients are independent of $f \in \mathcal{E}'(\mathbb{R}^d)$. Formula (6.6) is proved in [BCMM22, p. 19] for compactly supported $f \in L^1(\mathbb{R}^d)$. Using this, and the fact that $(b_k)_{k\in\mathbb{N}_0}$ are universal, concludes the proof the lemma. ☐

Proof of Theorem 6.1. By Theorem 5 we know that $V_e^{\mathrm{B}} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ and that at $x = 0$ it has a singularity of order $|x|^{-\alpha}$ for some $\alpha > 0$ that depends on V. Then, by [GS64, Proposition 1 p. 11] there is always an extension $F \in \mathcal{D}'(\mathbb{R}^d)$ of V_e^{B} such that

$$\langle F, \varphi \rangle_{\mathcal{D}' \times \mathcal{C}_c^\infty} = \left\langle V_e^{\mathrm{B}}, \varphi \right\rangle_{\mathcal{D}' \times \mathcal{C}_c^\infty} \qquad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\}).$$

Such an extension is called a regularization of the singular function V_e^{B}. In particular, we have that $F \in \mathcal{E}'(\mathbb{R}^d)$ since it coincides with V_e^{B} outside the origin, and hence vanishes outside $\mathbb{B}$. Notice that two different regularizations of V_e^{B} differ in a distribution supported at $x = 0$, or in other words, in a finite linear combination of derivatives of the Dirac delta distribution δ_0 which is supported at $x = 0$.

We now claim that, since there exists an $N \in \mathbb{N}_0$ large enough such that $m_N V_e^{\mathrm{B}} \in L^1(\mathbb{R}^d)$, for every regularization F of V_e^{B} one can always find an $N' \in \mathbb{N}_0$ such that

$$\langle m_k F, \varphi\rangle_{\mathcal{E}'\times\mathcal{C}^\infty} = \left\langle m_k V_e^{\mathrm{B}}, \varphi\right\rangle_{\mathcal{E}'\times\mathcal{C}^\infty} \qquad \text{for all } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d), \text{ and for all } k \geq N'.$$

In other words, $m_k F = m_k V_e^{\mathrm{B}}$ as distributions in $\mathcal{E}'(\mathbb{R}^d)$, for all $k \geq N'$.

As an immediate consequence of this, we obtain that

$$\sigma_k[F] = \sigma_k[V^{\mathrm{B}}] = \lambda_k[V] - k \quad \text{for all } k \geq N'. \tag{6.8}$$

To prove the claim, start by observing that $m_N F$ and $m_N V^{\mathrm{B}}$ are both compactly supported distributions that are identical outside $x = 0$, so they differ only in a finite linear combination of derivatives of δ_0. Let M be the maximum order of the derivatives of δ_0. Therefore, for any $N' \geq N$ large enough

$$\begin{aligned} m_{N'}F - m_{N'}V^{\mathrm{B}} &= m_{N'-N}(m_N F - m_N V^{\mathrm{B}}) \\ &= m_{N'-N} \sum_{|\alpha|\leq M} c_\alpha \partial_x^\alpha \delta_0 = 0, \end{aligned}$$

where the right hand side will vanish provided $N' - N > M$.

Now, define formally V_r^{B} by

$$\widehat{V_r^{\mathrm{B}}}(\xi) := 2\pi^{d/2} \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+d/2)} \left(\frac{|\xi|}{2}\right)^{2k} (\lambda_k[V] - k).$$

An immediate consequence of Lemma 6.2 and (6.8) is that

$$\widehat{F}(\xi) = \widehat{V_r^{\mathrm{B}}}(\xi) + P(|\xi|^2),$$

where P is a polynomial of order N' at most. This proves that $\widehat{V_r^{\mathrm{B}}}$ is a tempered distribution and that $V_r^{\mathrm{B}} \in \mathcal{S}'(\mathbb{R}^d)$ is well defined. Moreover, since the inverse Fourier transform of $P(|\xi|^2)$ is a linear combination of derivatives of δ_0, we actually have that $V_r^{\mathrm{B}} \in \mathcal{E}'(\mathbb{R}^d)$ is supported in the ball and that $V_r^{\mathrm{B}} = V^{\mathrm{B}}$ outside $x = 0$. Therefore, V_r^{B} is a regularization of V^{B}. As an extra property, we also get from the formula defining $\widehat{V_r^{\mathrm{B}}}(\xi)$ that V_r^{B} is a radial distribution.

To summarize, we have proved that there exits a radial distribution $V_r^{\mathrm{B}} \in \mathcal{E}'(\mathbb{R}^d)$ supported in the closed unit ball, such that (6.3) holds. This distribution is uniquely determined by (6.4) and it is a regularization of V^{B}. The fact that V_r^{B} satisfies the identity (ii) of the statement is a consequence of [BCMM22, Theorem 1]. □

7. A partial characterization of DtN operators

We recall (see (3.7)) that the Born approximation $V^{\mathrm{B}}(x) \in L^1(\mathbb{B}^d, |x|^{2k_0}dx)$ with $k_0 := \lfloor k_V \rfloor + 1 > \lfloor k_Q \rfloor + 1$ is given by

$$V^{\mathrm{B}}(x) = \frac{A(-\log|x|)}{|x|^2}.$$

Let $q^{\mathrm{B}}(r) := r^{-2}A(-\log r)$. It follows that for $k \geq k_0$,

$$\sigma_k[V^{\mathrm{B}}] = \int_0^1 q^{\mathrm{B}}(r) r^{2k+d-1}\, dr = \int_0^1 \left(\frac{1}{2} q^{\mathrm{B}}(\sqrt{t})\; t^{\frac{d}{2}-1}\right)\; t^k\; dt.$$

Clearly,

$$Q^{\mathrm{B}}(t) := \frac{1}{2} q^{\mathrm{B}}(\sqrt{t})\; t^{\frac{d}{2}-1+k_0},$$

verifies $Q^{\mathrm{B}} \in L^1((0,1))$. For $f \in L^1((0,1), dt)$, we define the Hausdorff moments by

$$\mu_k[f] = \int_0^1 t^k f(t)\; dt, \qquad \forall k \in \mathbb{N}_0.$$

so that one has

$$\sigma_{k+k_0}[V^{\mathrm{B}}] = \mu_k[Q^B], \qquad \forall k \in \mathbb{N}_0. \tag{7.1}$$

Since the classical Hausdorff moment problem possesses a unique solution in $L^1((0,1))$ (see [Wid41, Chapter III]), it follows that V^{B} is the *unique* function in $L^1(\mathbb{B}^d, |x|^{2k_0}dx)$ such that $\sigma_k[V^{\mathrm{B}}] = \lambda_k[V] - k$ for all $k \geq k_0$.

Moreover, we can give a partial characterization of DtN operators Λ_V (through their eigenvalues $\lambda_k[V]$). To this end, we set $\mu_n := \lambda_{n+k_0}[V] - (n + k_0)$ with $n \geq 0$, and following [Wid41, p. 101], we introduce several definitions:

$$\begin{aligned} \Delta^k \mu_n &:= \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}, \quad k \geq 0, \\ \lambda_{k,m} &:= \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m, \quad k \geq m \geq 0. \end{aligned}$$

Finally, for $k \geq 1$, we define

$$L_k(t) := (k+1)\lambda_{k,\lfloor kt \rfloor}, \quad t \in [0,1].$$

It is showed in [Wid41, p. 112] that $(\lambda_{n+k_0}[V]-(n+k_0))_{n\in\mathbb{N}_0}$ are the Hausdorff moments of $Q^{\mathrm{B}} \in L^1((0,1))$ if and only if the sequence $(L_n(t))_{n\in\mathbb{N}}$ converges in $L^1((0,1))$. This is a partial characterization of the DtN operators since the aforementioned condition only characterizes the Born approximation V^{B}. Moreover, this result is not easily interpreted, independently of the functions $L_n(t)$, in terms of the sequence $(\mu_n)_{n\in\mathbb{N}_0}$.

In contrast, if only potentials $V \in L^p(\mathbb{B}^d) \subset \mathcal{V}_d$ with $p > d/2$ are considered, one can obtain a clearer partial characterization involving directly the eigenvalues $\lambda_n[V]$. Indeed, using Hölder's inequality, we easily see that the function $F(r)$ introduced in (1.16) satisfies the following estimate:

$$|F(r)| \leq \frac{C}{r^{\alpha+2}},$$

where C is a suitable constant and

$$\alpha := \min\left(\beta_V, \left(\frac{1}{1+p'(1+\frac{1-d}{p})}\right)^{\frac{1}{p'}} \frac{||V||_{L^p(\mathbb{B}^d)}}{|\mathbb{S}^{d-1}]|^{\frac{1}{p}}}\right),$$

p' being the conjugate exponent of p. With the help of Theorem 5, one gets $q^{\mathrm{B}}(r) = q(r) + F(r)$, thus clearly $q^{\mathrm{B}}(r)$ satisfies

$$\int_0^1 |q^{\mathrm{B}}(r)|^p \, r^{d-1+2k_1} \, dr < \infty,$$

where

$$k_1 := \lfloor \frac{p(\alpha+2)-d}{2} \rfloor + 1.$$

It follows that

$$\tilde{Q}^{\mathrm{B}}(t) := \frac{1}{2} q^{\mathrm{B}}(\sqrt{t}) \, t^{\frac{d}{2}-1+k_1}$$

is a function in $L^p((0,1))$ and we have $\sigma_{k+k_1}[V^{\mathrm{B}}] = \mu_k[\tilde{Q}^{\mathrm{B}}]$, $k \geq 0$. As previously, we define the sequence $(\tilde{\mu}_n)_{n\geq 0}$ by $\tilde{\mu}_n := \lambda_{n+k_1}[V] - (n+k_1)$, and we set

$$\tilde{\lambda}_{k,m}[V] := \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \tilde{\mu}_m \quad , \quad k \geq m \geq 0.$$

Now, using [Wid41, Theorem 5, p. 110], we see that $(\lambda_{n+k_1}[V] - (n+k_1))_{n\in\mathbb{N}_0}$ are the Hausdorff moments of $\tilde{Q}^{\mathrm{B}} \in L^p((0,1))$ if and only if

$$\sup_{k\in\mathbb{N}_0} (k+1)^{p-1} \sum_{m=0}^{k} |\tilde{\lambda}_{k,m}[V]|^p < \infty, \qquad \forall k \in \mathbb{N}_0. \tag{7.2}$$

Summarizing, we have shown that the eigenvalues of any DtN map issued from a radial potential in $L^p(\mathbb{B}^d)$, $p > d/2$, must satisfy condition (7.2).

Remark 7.1. Condition (7.2) can be viewed as a partial characterization of DtN operators for radial potentials $V \in L^p(\mathbb{B}^d)$, $p > d/2$. Nonetheless, a total characterization should involve additional conditions. This is due to the fact that not every locally integrable function is the A-amplitude of a Schrödinger operator on the half-line, as has been shown by Remling [Rem03].The characterization problem for radial DtN maps will be addressed in a forthcoming work.

Appendix A. Solutions by separation of variables

Lemma A.1. *Let $d \geq 2$ and $V \in L^p(\mathbb{B}^d)$ with $p > 1$ and $p \geq d/2$ be a radial function. Then, for every $k > k_V$ there is a unique solution b_k of (2.7) with $b_k(1) = 1$ such that the function $u_k(x) = b_k(|x|)Y_k(x/|x|)$ is a proper weak $H^1(\mathbb{B}^d)$ solution of (2.10). Moreover, for any other solution $u \in H^1(\mathbb{B}^d)$ of (2.10) it holds that*

$$\partial_r b_k(1) = (Y_k, \partial_\nu u|_{\mathbb{S}^{d-1}})_{L^2(\mathbb{S}^{d-1})},$$

for all $Y_k \in \mathfrak{H}_k$ with $\|Y_k\|_{L^2(\mathbb{S}^{d-1})} = 1$.

Proof. Let u_0 be an $H_0^1(\mathbb{B}^d)$ solution of

$$\begin{cases} -\Delta u_0 + V u_0 = & 0 \quad \text{on } \mathbb{B}^d, \\ \quad u_0|_{\mathbb{S}^{d-1}} = & 0, \end{cases} \tag{A.1}$$

Since V is radial, using a Fourier expansion in spherical harmonics of u_0, one can show that $u_0(x) = \sum_{k=0}^{\infty} a_k(|x|)Y_k(x/|x|)$, for some $Y_k \in \mathfrak{H}_k$, where $a_k(r)$ is a solution of

(2.7) with $a_k(1) = 0$. Defining $v_k(t) := e^{-\frac{d-2}{2}t} a_k(e^{-t})$ and Q from the radial profile of V by (2.13), we obtain that v_k solves

$$\begin{cases} -v_k'' + Qv_k = & -\left(k + \frac{d-2}{2}\right)^2 v_k \quad \text{on } \mathbb{R}^+, \\ v_k(0) = & 0. \end{cases}$$

This implies that v_k is a Dirichlet eigenfunction of the 1-d operator (2.17) with eigenvalue $-\left(k + \frac{d-2}{2}\right)^2$. By Lemma 2.6, we know that $k + \frac{d-2}{2} \le \beta_Q$, and therefore

$$u_0(x) = \sum_{0 \le k \le k_V} a_k(|x|) Y_k(x/|x|), \tag{A.2}$$

where we have used that $k_V = \beta_V + \frac{d-2}{2}$ and $\beta_Q \le \beta_V$, and imposed $a_k = 0$ whenever $k + \frac{d-2}{2} > \beta_Q$. Notice that any choice of $Y_k \in \mathfrak{H}_k$ in (A.2) gives a solution (not necessarily distinct) of the homogeneous problem (A.1).

Problem (2.1) can be reduced to

$$\begin{cases} -\Delta w + Vw = & -Vw_0 \quad \text{on } \mathbb{B}^d, \\ w|_{\mathbb{S}^{d-1}} = & 0, \end{cases} \tag{A.3}$$

using the change of variables $u = w + w_0$, where w_0 is the unique harmonic function in $\mathbb{B}^d$ satisfying $w_0|_{\mathbb{S}^{d-1}} = f$. Notice that

$$w_0(x) = \sum_{k=0}^{\infty} |x|^k Y_k(x/|x|), \qquad Y_k = \Pi_{\mathfrak{H}_k} f, \tag{A.4}$$

where $\Pi_{\mathfrak{H}_k}$ stands for the $L^2(\mathbb{S}^{d-1})$ projector to the subspace $\mathfrak{H}_k$ of spherical harmonics.

If $V \in L^p(\mathbb{B}^d)$ with $p > 1$ and $p \ge d/2$, the standard theory of elliptic equations implies that (A.3) has a solution w if and only if

$$(Vw_0, u_0)_{L^2(\mathbb{B}^d)} = 0,$$

for all u_0 that are solutions of the homogeneous problem (A.1), see for example[5] [Eva98, Section 6.2.3]. By the previous discussion we know that u_0 must satisfy (A.2). Therefore, using (A.4) and that V is radial, one can verify that $(Vw_0, u_0)_{L^2(\mathbb{B}^d)} = 0$ holds if we require $\Pi_{\mathfrak{H}_k} f = 0$ for all $0 \le k \le k_V$.

Therefore, given f such that

$$f = \sum_{k > k_V} Y_k, \qquad Y_k \in \mathfrak{H}_k,$$

there always exits a solution $w \in H^1(\mathbb{B}^d)$ of (A.3) and, as a consequence, a solution $u = w + w_0$ in $H^1(\mathbb{B}^d)$ of (2.1), even if (2.5) does not necessarily hold. Using a Fourier expansion in spherical harmonics of u in (2.1), one can show that u must satisfy

$$u(x) = \sum_{k > k_V}^{\infty} b_k(|x|) Y_k(x/|x|) + u_0,$$

[5]This is proved for bounded potentials, but the case $L^{d/2}(\mathbb{B}^d)$ can be proved using the same arguments.

where b_k solves (2.7) with boundary conditions $b_k(1) = 1$ if $k > k_V$, and u_0 is any homogeneous solution. In particular it clearly holds by (A.2) that

$$\partial_r b_k(1) = \frac{1}{\|Y_k\|^2_{L^2(\mathbb{S}^{d-1})}} (Y_k, \partial_\nu u|_{\mathbb{S}^{d-1}})_{L^2(\mathbb{S}^{d-1})}.$$

Also, since we can choose $u_0 = 0$ there is a unique solution $u_f \in H^1(\mathbb{B}^d)$ of (2.1) such that

$$u_f(x) = \sum_{k > k_V} b_k(|x|) Y_k(x/|x|).$$

In the particular case of $f = Y_k$ one obtains that $u_k(x) := u_f(x) = b_k(|x|)Y_k(x/|x|)$. This is the only solution of separation of variables since any other solution differs in a homogeneous solution satisfying (A.2). This finishes the proof of the lemma. □

Appendix B. The space $\mathcal{V}_d$

It is simple to show that the space $\mathcal{V}_d$ defined in (1.2) contains the radial $L^{d/2}(\mathbb{B}^d)$. Note first that

$$\|V\|_{L^{d/2}(\mathbb{B}^d)} = |\mathbb{S}^{d-1}|^{2/d} \|Q\|_{L^{d/2}(\mathbb{R}^+)}.$$

Since

$$\int_y^{y+1} |Q(t)|\, dt \le \left(\int_y^{y+1} |Q(t)|^{d/2}\, dt \right)^{2/d} \le \|Q\|_{L^{d/2}(\mathbb{R}^+)},$$

it follows that $|||Q||| \le \|Q\|_{L^{d/2}(\mathbb{R}^+)}$, and as a consequence we obtain that

$$\|V\|_{\mathcal{V}_d} \le |\mathbb{S}^{d-1}|^{(d-2)/d} \|V\|_{L^{d/2}(\mathbb{B}^d)}. \tag{B.1}$$

In fact, a stronger estimate holds for $d > 2$.

Lemma B.1. *Let $d > 2$ and let $V \in L^{d/2,\infty}(\mathbb{B}^d)$ be a (not necessarily radial[6]) potential. Then*

$$\|V\|_{\mathcal{V}_d} \le C_d \|V\|_{L^{d/2,\infty}(\mathbb{B}^d)},$$

where $C_d > 0$ only depends of d.

Proof. Let $A_j = \{x \in \mathbb{R}^d : 2^{-j-1} < |x| < 2^{-j}\}$ and denote by χ_{A_j} the characteristic function of the set A_j. From (1.8), since $\frac{d}{d-2}$ is the Hölder conjugate exponent of $\frac{d}{2}$, it follows using Hölder inequality for Lorentz spaces [Hun66] that

$$\begin{aligned} \|V\|_{\mathcal{V}_d} &\le C_d \sup_{j \in \mathbb{N}_0} \|\chi_{A_j} |\cdot|^{2-d}\|_{L^{\frac{d}{d-2},1}(\mathbb{B}^d)} \|\chi_{A_j} V\|_{L^{\frac{d}{2},\infty}(\mathbb{B}^d)} \\ &\le C_d \left(\sup_{j \in \mathbb{N}_0} \|\chi_{A_j} |\cdot|^{2-d}\|_{L^{\frac{d}{d-2},1}(\mathbb{B}^d)} \right) \|V\|_{L^{\frac{d}{2},\infty}(\mathbb{B}^d)}. \end{aligned}$$

[6]Notice that $\|V\|_{\mathcal{V}_d}$ is well defined for non-radial potentials in (1.8) even if, for convenience, we have included the radial assumption in the definition of $\mathcal{V}_d$.

To finish, we need to show that the factor with the $\sup_{j\in\mathbb{N}_0}$ is finite. The norm of the Lorentz space $L^{\frac{d}{d-2},1}(\mathbb{B}^d)$ is given by

$$\|\chi_{A_j}|\cdot|^{2-d}\|_{L^{\frac{d}{d-2},1}(\mathbb{B}^d)} = \frac{d}{d-2}\int_0^\infty |g_j(t)|^{\frac{d-2}{d}}\,dt,$$

where $g_j(t)$ is the distribution function of $\chi_{A_j}|x|^{2-d}$, *i. e.*

$$g_j(t) = |\{x\in\mathbb{B}^d : 2^{-j-1} < |x| < 2^{-j},\ |x|^{2-d} > t\}|.$$

From an explicit computation of $g_j(t)$ it follows that

$$g_j(t) \le \frac{1}{d}\left(1-\frac{1}{2^d}\right)|\mathbb{S}^{d-1}| \begin{cases} 2^{-dj} & 0 < t \le 2^{(j+1)(d-2)}, \\ 0 & 2^{(j+1)(d-2)} < t < \infty. \end{cases}$$

Hence

$$\|\chi_{A_j}|\cdot|^{2-d}\|_{L^{\frac{d-2}{d},1}(\mathbb{B}^d)} \le C_d\left(2^{-dj}\right)^{\frac{d-2}{d}} 2^{(j+1)(d-2)} = 2^{d-2}C_d,$$

which proves that

$$\sup_{j\in\mathbb{N}_0}\|\chi_{A_j}|\cdot|^{2-d}\|_{L^{\frac{d}{d-2},1}(\mathbb{B}^d)} = C_d < \infty,$$

and finishes the proof of the lemma. □

The previous lemma shows that the set of radial potentials in the Lorenz space $L^{d/2,\infty}(\mathbb{B}^d)$ are contained in $\mathcal{V}_d$ with $d>2$. Among other things, in the radial case this implies that all potentials $V(x)=|x|^{-2}f(|x|)$ with f bounded, belong to $\mathcal{V}_d$. The inclusion $L^{d/2,\infty}(\mathbb{B}^d)\subset\mathcal{V}_d$ is strict, since $\mathcal{V}_d$ also contains any radial $L^1(\mathbb{B}^d)$ potential which vanishes in a neighborhood of the origin. It is not clear that this inclusion holds in dimension $d=2$. Indeed, we can not apply Hölder inequality for Lorentz spaces in this case, (since $L^{\infty,q}(\mathbb{B}^2)=\{0\}$ for $q\neq\infty$). Nevertheless, note that the critical potential $V(x)=c|x|^{-2}$ belongs to $\mathcal{V}_2$ (and also to $L^{1,\infty}(\mathbb{B}^2)$). In principle one can only guarantee in this case the trivial inclusion $L^1(\mathbb{B}^2)\subset\mathcal{V}_2$.

Remark B.2. In general, the Schrödinger equation (1.1) is not (uniquely) solvable for potentials in $L^{d/2,\infty}(\mathbb{B}^d)$. Nevertheless, for potentials V with a *small norm* living in the so-called Fefferman-Phong class $F_p\supset L^{\frac{d}{2},\infty}(\mathbb{R}^d)$ with $\frac{d-1}{2}<p<\frac{d}{2}$ and $d\ge 3$, it is shown in [Cha90, Proof of Lemma 2], that the DtN map Λ_V is always well defined : using a Poincaré-type inequality, we see that the bilinear form related to the operator $H=-\Delta+V$ is continuous and coercive in $H^1_0(\mathbb{B}^d)$. It follows that 0 is not a Dirichlet eigenvalue of H. Moreover, Chanillo shows that the map $V\mapsto\Lambda_V$ is injective. This last result is closely related to the unique continuation principle (UCP). Generically, (UCP) does not hold for potentials belonging to these Lorentz spaces, (see the nice counterexamples in [KT02]), except for potentials with a small norm ([JK85]).

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