# ERRATUM to "Recovering the mass and the charge of a Reissner-Nordström black hole by an inverse scattering experiment" (Inverse Problems 24 no. 2, (2008))

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This note contains an erratum to the paper cited above. In what follows, we shall use freely the same notations and the objects as introduced in [1]. The erratum is that the reconstrution formula (3.4) in Theorem 3.2 in [1] is false. A miscalculation at the very end of our paper (precisely in the line 18, p 17) leads to an incorrect formula. Using exactly the same notations as in [1], the correct reconstruction formula in Thm 3.2 is:

$$F_{\text{out}}(\lambda) =< \psi_1, \psi_2 > + \frac{1}{\lambda} < \psi_1, L_1(x, D_x)\psi_2 > + \frac{1}{\lambda^2} < \psi_1, L_2(x, D_x)\psi_2 > + O(\lambda^{-3}), \tag{1}$$

where $L_j(x, D_x)$ are differential operators given by

$$L_1(x, D_x) = \frac{i(l+\frac{1}{2})^2}{2r_+} \text{ (unchanged)}, \qquad L_2(x, D_x) = -\frac{(l+\frac{1}{2})^4}{8r_+^2} - \frac{i(l+\frac{1}{2})^2 D_x}{2r_+} \quad \text{(corrected)}$$

and

$$r_+ = M + \sqrt{M^2 - Q^2}. \tag{2}$$

Regrettably an essential term used in the proof of our main Theorem (see Theorem 3.1) has disappeared in the corrected formula (1) and thus unvalidates Theorem 3.1. Indeed the term of order $O(\lambda^{-1})$ and the term of order $O(\lambda^{-2})$ in (1) only permit us to recover the physical quantity $r_+$, the radius of the event horizon of the RN black hole. Moreover it is clear from (2) that this information is not enough to recover uniquely the mass $M$ and the square of the charge $Q^2$ as stated in Theorem 3.1. We now correct this error using the same global strategy as in [1]. We first give a refined construction formula and then state the corresponding (and equivalent to Theorem 3.1) main result.

We first compute the asymptotic of $F_{\text{out}}(\lambda)$ (see (3.7) of [1] for the definition) up to the order $O(\lambda^{-4})$. For this we need to slightly improve the order of accuracy of the modifiers $J^{\pm}(\lambda)$ introduced in section 3.2 of [1]. We thus define $J^{\pm}(\lambda)$ as Fourier Integral Operators (FIO) with scalar phases

$$\varphi^{\pm}(x, \xi, \lambda) = x\xi + \frac{1}{2(\xi+\lambda)} \int_0^{\pm\infty} a^2(x+s)ds, \tag{3}$$

and the following matrix-valued amplitudes

$$\begin{aligned} P^{\pm}(x, \xi, \lambda) \quad = \quad & \Big(p(x, \xi+\lambda) + \frac{1}{\lambda^2}\big(P_{\text{in}}k_1(x, \xi) + p(x, \xi+\lambda)l_1(x, \xi)\big) \\ & + \frac{1}{\lambda^3}\big(P_{\text{in}}k_2(x, \xi) + p(x, \xi+\lambda)l_2^{\pm}(x, \xi)\big) + \frac{1}{\lambda^4}P_{\text{in}}k_3(x, \xi)\Big)\; g(\xi), \end{aligned} \tag{4}$$

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where the additional terms relative to (3.50) in [1] are

$$l_2^{\pm}(x,\xi) = -\frac{\xi a^2}{4} + \frac{i}{8}\int_x^{\pm\infty} \big(a^4(s) - a''(s)a(s)\big)ds,$$

$$k_3(x,\xi) = \frac{i11a'a^2}{32}\Gamma^2 + \frac{i3\xi^2 a'}{4}\Gamma^2 + \frac{3\xi a''}{8}\Gamma^2 - \frac{ia'''}{16}\Gamma^2.$$

These new modifiers are constructed exactly as in section 3.2 of [1] and in such a way that the operators $C^{\pm}(\lambda) = H(\lambda)J^{\pm}(\lambda) - J^{\pm}(\lambda)(D_x+\lambda)$ (which appears in (3.10) in [1]) be FIOs with phases $\varphi^{\pm}(x,\xi,\lambda)$ and amplitudes $c^{\pm}(x,\xi,\lambda)$ satisfying the estimates:

$$\forall R\in\mathbb{R},\ \exists C\in\mathbb{R},\quad |c^{\pm}(x,\xi,\lambda)| \le C\langle x\rangle^{-2}\lambda^{-4},\quad \forall \pm x \ge \pm R,\ \forall \xi\in K \text{ compact},$$

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta} c^{\pm}(x,\xi,\lambda)| \le C\langle x\rangle^{-2-\alpha}\lambda^{-4},\quad \forall \alpha,\beta\in\mathbb{N}, \alpha\ge 1,\ \forall x\in\mathbb{R},\ \forall\xi\in K \text{ compact}.$$

Note that these estimates improved and replaced those in (3.49) of [1]. Finally, using the modifiers defined by (3), (4) as well as the previous estimates we can prove the next two properties exactly as in [1] (see Lemma 3.2 and Lemma 3.3). For any $\psi\in\mathcal{H}^0_{\text{out}}$ such that $\hat{\psi}\in C_0^{\infty}(\mathbb{R};\mathbb{C}^2)$ and for $\lambda$ large, we have

$$W^{\pm}(\lambda)\psi = \lim_{t\to\pm\infty} e^{itH(\lambda)}J^{\pm}(\lambda)e^{-it(D_x+\lambda)}\psi.$$

$$\|(W^{\pm}(\lambda) - J^{\pm}(\lambda))\psi\| = O(\lambda^{-4}). \tag{5}$$

Now we compute the asymptotic of $F_{\text{out}}(\lambda)$ up to the order $O(\lambda^{-4})$ exactly as in section 3.3 of [1] with the help of (3), (4) and (5). After a long but easy calculation we obtain

**Theorem 0.1 (Reconstruction formula)** *Let $\psi_1,\psi_2\in\mathcal{H}^0_{out}$ such that $\hat{\psi}_1,\hat{\psi}_2\in C_0^{\infty}(\mathbb{R};\mathbb{C}^2)$. Then for $\lambda$ large, we obtain*

$$\begin{aligned} F_{out}(\lambda) \;=\;& <\psi_1,\psi_2> + \frac{1}{\lambda} <\psi_1, L_1(x,D_x)\psi_2> + \frac{1}{\lambda^2} <\psi_1, L_2(x,D_x)\psi_2> \\ & + \frac{1}{\lambda^3} <\psi_1, L_3(x,D_x)\psi_2> + O(\lambda^{-4}), \end{aligned} \tag{6}$$

*where $L_j(x,D_x)$ are differential operators given by*

$$L_1(x,D_x) = \frac{iI_l}{2},\quad L_2(x,D_x) = -\frac{I_l^2}{8} - \frac{iI_lD_x}{2},$$

$$L_3(x,D_x) = \frac{i(J_l+K_l)}{8} - \frac{iI_l^3}{48} + \frac{I_l^2}{4}D_x + \frac{iI_l}{2}D_x^2,$$

*and*

$$I_l = \int_{\mathbb{R}} a_l^2(x)dx,\quad J_l = \int_{\mathbb{R}} a_l^4(x)dx,\quad K_l = \int_{\mathbb{R}} (a_l')^2(x)dx.$$

The quantities $I_l, J_l, K_l$ in Theorem 0.1 can be explicitly computed in terms of the parameters $M$ and $Q$. We obtain

$$I_l = \frac{(l+\frac{1}{2})^2}{r_+},\quad J_l = (l+\frac{1}{2})^4\Big(\frac{1}{3r_+^3} - \frac{M}{2r_+^4} + \frac{Q^2}{5r_+^5}\Big), \tag{7}$$

$$K_l = (l+\frac{1}{2})^2\Big(\frac{1}{3r_+^3} - \frac{3M}{2r_+^4} + \frac{9M^2+4Q^2}{5r_+^5} - \frac{2MQ^2}{r_+^6} + \frac{4Q^4}{7r_+^7}\Big). \tag{8}$$

Le us also recall here that the reconstruction formula (6) is valid on each spin-weighted spherical harmonics indexed here by $l\in\mathbb{N}$. Using Theorem 0.1, we prove the uniqueness result

**Theorem 0.2** *The parameters $M$ and $Q^2$ are uniquely determined by the knowledge of the scattering operator $S_{out}$ or $S_{in}$.*

*Proof*: We only prove the result for $S_{\text{out}}$ since the proof for $S_{\text{in}}$ is the same (see [1]). Consider two RN black holes with mass $M_1$ (resp. $M_2$) and charge $Q_1$ (resp. $Q_2$). We shall distinguish the objects associated to these two black holes by an upper index 1 or 2 in the usual notations. Assume now that the scattering operators $S^1_{\text{out}}$ and $S^2_{\text{out}}$ are equal. Then $F^1_{\text{out}} = F^2_{\text{out}}$ on each spin-weighted spherical harmonic by definition of $F_{\text{out}}$. Equating the terms of same orders in the high-energy expansion (6), we get inductively

$$r^1_+ = r^2_+ = r_+, \tag{9}$$

$$J^1_l + K^1_l = J^2_l + K^2_l. \tag{10}$$

By (9) the radii $r^1_+$ and $r^2_+$ coincide and the common value $r_+$ can be recovered from $S_{\text{out}}$. Moreover, since $r_+$ is a zero of the function $F(r)$ (see (2.2) in [1]), we can express $Q^2$ in term of $M$ by the formula

$$Q^2 = -r_+^2 + 2r_+M. \tag{11}$$

Plugging (11) into (7) and (8), the equality in (10) becomes

$$\Big( - \frac{(l+\frac{1}{2})^2}{10} - \frac{13}{70} + \frac{3(M_1+M_2)}{35r_+} \Big)(M_1 - M_2) = 0. \tag{12}$$

But using (2) and (9), we see immediately that $M_1, M_2 < r_+$. Hence the equation $-\frac{13}{70} + \frac{3(M_1+M_2)}{35r_+} = \frac{(l+\frac{1}{2})^2}{10} > 0$ has no solution since $-\frac{13}{70} + \frac{3(M_1+M_2)}{35r_+} < -\frac{1}{70} < 0$. We conclude from (12) that $M_1 = M_2$ and then from (11) that $Q_1^2 = Q_2^2$ which proves the Theorem.
$\diamondsuit$

**Remark 0.1** *Explicit formulae for $M$ and $Q^2$ can be obtained in term of the high-energy scattering data $I_l, J_l$. Assume indeed that $S_{out}$ is known on each spin-weighted spherical harmonic. Then so is the function $F_{out}(\lambda)$ of Theorem 0.1. The term of order $O(\lambda^{-1})$ in (6) permits to recover directly $I_l$ and thus $r_+$ by (7). The term of order $O(\lambda^{-3})$ in turn permits to recover $J_l + K_l$ for all $l$ in $\mathbb{N}$. By homogeneity (see (7) and (8)), we can recover $J = \frac{1}{3r_+^3} - \frac{M}{2r_+^4} + \frac{Q^2}{5r_+^5}$ and $K = \frac{1}{3r_+^3} - \frac{3M}{2r_+^4} + \frac{9M^2+4Q^2}{5r_+^5} - \frac{2MQ^2}{r_+^6} + \frac{4Q^4}{7r_+^7}$ separately. Consider for instance that $r_+$ and $J$ are known. Using (7) and the preceding expression for $J$, we have*

$$-\frac{2}{r_+}M + \frac{1}{r_+^2}Q^2 = -1, \quad -\frac{1}{2r_+^4}M + \frac{1}{5r_+^5}Q^2 = J - \frac{1}{3r_+^3}. \tag{13}$$

*Hence the parameters $(M, Q^2)$ are solution of the system of linear equations (13) whose determinant $\frac{1}{10r_+^6}$ is nonzero. Thus $M$ and $Q^2$ can be determined from the scattering data $r_+$ and $J$ and we obtain the formulae*

$$M = \frac{4r_+}{3} - 10Jr_+^4, \quad Q^2 = \frac{5r_+^2}{3} - 20Jr_+^5.$$

Eventually, we would like to correct another and minor error at the beginning of our paper. The error is the statement that the scattering operator $S$ leaves invariant the incoming and outgoing subspaces $\mathcal{H}^0_{\text{in}}$ and $\mathcal{H}^0_{\text{out}}$ (see the paragraph after (2.17) in [1] for the definition of these subspaces). More precisely, denoting the projections onto $\mathcal{H}^0_{\rm in}$ and $\mathcal{H}^0_{\rm out}$ by $P_{\rm in}$ and $P_{\rm out}$, it is written erroneously in (2.19) that $S = S_{\rm in} \oplus S_{\rm out}$ where

$$S_{\rm in} = P_{\rm in} S P_{\rm in}, \quad S_{\rm out} = P_{\rm out} S P_{\rm out}. \tag{14}$$

This result was based on Lemma 2.1 whose proof turns out to be false. In fact, $S$ can be decomposed into the sum

$$S = S_{\rm in} + S_{\rm out} + P_{\rm out} S P_{\rm in} + P_{\rm in} S P_{\rm out},$$

where the last two terms are nonzero. We emphasize however that the strategy and techniques used in our paper [1] as well as the new main results stated in the first part of this note are not affected by this error and can be summarized as follows: RN black holes are uniquely determined by the knowledge of $S_{\rm in}$ or $S_{\rm out}$ (defined in (14)) at high-energies.

**Remark 0.2** *It would be maybe more relevant to assume the operator $P_{in} S P_{out}$ (or $P_{out} S P_{in}$) to be known and try to recover the parameters of the black hole from it. It would correspond indeed to an inverse scattering experiment of radar type, a signal being emitted from the earth in the remote past and the reflected part of this signal being capted on the earth in the future. (Notice that in our result, the signal is emitted from the earth in the past and capted at the event horizon of the black hole in the future). The high-energy approach used in [1] however is not well adapted to this situation. At high-energy indeed, it can be shown by our methods that the reflected part of the signal is of order $O(\lambda^{-\infty})$ when the energy $\lambda$ is large, (see also [2] where a stationary representation of the scattering operator in terms of transmission and reflexion coefficients and for short-range potentials is used). In consequence, no information can be recovered from the high-energies of $P_{in} S P_{out}$.*

# References


[1] Daudé T., Nicoleau F., *Recovering the mass and the charge of a Reissner-Nordström black hole by an inverse scattering experiment*, Inverse Problems 24 no. 2, (2008).

[2] Grébert B., *Inverse scattering for the Dirac operator on the real line*, Inverse Problems 8 no. 5, (1992).