Inverse scattering for a Schrödinger operator with a repulsive potential.

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1 Introduction.

In this talk, we study an inverse scattering problem for a Schrödinger operator with a repulsive potential.

We consider a pair of Hamiltonians (H, H_0) where :

- H_0 is the free Hamiltonian.
- H is a perturbation of H_0.

Goals :

(1) We define the scattering operator $S = S(H, H_0)$, (S has a physical interpretation).

(2) Given this operator S, we answer the natural question :

can one determine and reconstruct the perturbation from the scattering operator S ?

The free Hamiltonian defined on $L^2(\mathbb{R}^n)$, $n \geq 2$, is given by :

$$H_0 \;=\; p^2 - x^2 \;, \tag{1.1}$$

where $p = -i\nabla$. H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

- **The classical flow :**

$\forall x, \xi \in \mathbb{R}^n,\ \forall t \in \mathbb{R}$,

$$\Phi_0^t(x,\xi) = (\cosh(2t)x + \sinh(2t)\xi,\ \sinh(2t)x + \cosh(2t)\xi) \;, \tag{1.2}$$

$\Longrightarrow \Phi_0^t(x,\xi)$ grows exponentially fast.

- **Some spectral results :**

If we denote by $A = x \cdot p + p \cdot x$ the generator of dilations, we have :

$$H_0\, U = U\, A \;, \tag{1.3}$$

where U is the unitary operator on $L^2(I\!R^n)$ defined by :

$$(1.4) \qquad U\Phi(x) = (\sqrt{2}\pi)^{-\frac{n}{2}} \ e^{-i\frac{x^2}{2}} \int_{I\!R^n} e^{i\sqrt{2}x\cdot y} \ e^{-i\frac{y^2}{2}} \Phi(y) \ dy.$$

$\Longrightarrow \sigma(H_0) = \sigma_{ac}(H_0) = I\!R.$

- **Mehler's formula :**

The free time evolution is given by :

$$(1.5) \qquad \forall t \neq 0 \ , \ e^{-itH_0} = M_t \ D_t \ \mathcal{F} \ M_t \ ,$$

where M_t is the multiplication operator :

$$(1.6) \qquad M_t\Phi(x) = e^{\frac{i}{2}\coth(2t) \ x^2} \ \Phi(x) \ ,$$

D_t is the dilation operator :

$$(1.7) \qquad D_t\Phi(x) = (i \sinh(2t))^{-\frac{n}{2}} \ \Phi\left(\frac{x}{\sinh(2t)}\right) \ ,$$

and $\mathcal{F}$ is the usual Fourier transform on $L^2(I\!R^n)$.

- **The full Hamiltonian** $H = H_0 + V(x)$ **:**

Intuitively, looking at the classical flow, a real-valued measurable electric potential V satisfying :

$$(H_1) \qquad | V(x) | \leq C \ (\ln < x >)^{-1-\epsilon}, \ \epsilon > 0 \ ,$$

where $< x >= (1+x^2)^{\frac{1}{2}}$, is of short-range for H_0.

With this hypothesis, H is essentially self-adjoint with domain $D(H) = D(H_0)$ and $\sigma_{ess}(H) = I\!R$. Moreover, H has no eigenvalues and $\sigma_{sc}(H) = \emptyset$, [Bony-Carles-Häfner-Michel], (2004).

- **The wave operators $W^{\pm}$:**

Under the asumption (H_1), Bony-Carles-Häfner-Michel showed that the wave operators :

$$W^{\pm} = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{1.8}$$

exist and are complete, (i.e $Ran\ W^{\pm} = \mathcal{H}^{(ac)}(H)$, the later being the subspace of absolute continuity of H).

We denote $S = S(V) = W^{+*}W^{-}$ the *scattering operator.*

In this talk, in order to solve the inverse problem, we need stronger hypotheses on the electric potential V.

We assume that $V \in C^{\infty}(\mathbb{R}^n)$ and it satisfies $\forall \alpha \in \mathbb{N}^n$ (with finite order) :

$$(H_2) \qquad | \partial_x^\alpha V(x) | \leq \; C_\alpha < x >^{-\rho-|\alpha|} \; , \; \rho \in]\frac{1}{2}, 1[\; .$$

The main result of this talk is :

Theorem 1.
Let $V_1, \; V_2$ be potentials satisfying (H_2). Then :

$$S(V_1) = S(V_2) \iff V_1 = V_2 \; .$$

We prove Theorem 1 by studying *the high energy limit* of S with *the Enss-Weder's time-dependent method.*

• High energy asymptotics of the scattering operator :

$\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ the set of the Schwartz functions, $\omega \in S^{n-1}$ is fixed, and $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x\cdot\omega}\Phi, \ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x\cdot\omega}\Psi$.

We have the following high energy asymptotics when $\lambda \to +\infty$, where $< \, , \, >$ is the usual scalar product in $L^2(\mathbb{R}^n)$:

Proposition 2.

$$< [S,p]\, \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > = \frac{1}{2\sqrt{\lambda}} < \left(\int_{-\infty}^{+\infty} \nabla V(x+t\omega)\ dt \right) \Phi \ , \ \Psi > + o\left(\lambda^{-\frac{1}{2}}\right) .$$

Using the inversion of the X-ray transform, Theorem 1 is proved.

• **Sketch of proof of Proposition 2 :**

Step 1 :

$$S - 1 = (W^+ - W^-)^* \; W^- = -i \int_{\mathbb{R}} e^{itH_0} \; V \; W^- \; e^{-itH_0} \; dt, \tag{1.9}$$

so we have :

$$[S, p] = [T, p] + [U, p], \tag{1.10}$$

where

$$T \;=\; -i \int_{\mathbb{R}} e^{itH_0} \; V \;\; e^{-itH_0} \; dt, \tag{1.11}$$

and

$$U \;=\; -i \int_{\mathbb{R}} e^{itH_0} \; V \; (W^- - 1) \; e^{-itH_0} \; dt. \tag{1.12}$$

Step 2 : the free evolution at high energies.

We denote by τ_a the translation operator of vector a on $L^2(\mathbb{R}^n)$: $\tau_a f(x) = f(x-a)$.

Lemma 3.
For $t \in \mathbb{R}$, we have :

$$e^{-itH_0}\Phi_{\lambda,\omega} = \tau_{\sinh(2t)\sqrt{\lambda}\omega}\; e^{\frac{i\lambda}{2}\cosh(2t)\;\sinh(2t)}\; e^{i\sqrt{\lambda}\cosh(2t)\;x\cdot\omega}\; e^{-itH_0}\Phi.$$

Using Lemma 3 and

$$(1.13) \qquad e^{itH_0}\; p\; e^{-itH_0} = \sinh(2t)\; x + \cosh(2t)\; p,$$

we show now that $[T,p]$ gives the leading term of Proposition 2.

One has :

$$< [T,p]\Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} > = \int_{\mathbb{R}} < \nabla V(x+\sqrt{\lambda}\sinh(2t)\omega)e^{-itH_0}\ \Phi,$$

$$e^{-itH_0}\ \Psi > \ \cosh(2t)\ dt$$

$$\sim \frac{1}{2\sqrt{\lambda}} < \left(\int_{-\infty}^{+\infty} \nabla V(x+t\omega)\ dt\right)\ \Phi\ ,\ \Psi >$$

by a simple change of variables.

Step 3 : technical lemmas.

Finally, we prove that $[U,p]$ gives a negligible contribution at high energies.

One can show :

$$(1.14) \qquad \int_{I\!R} \| \, V e^{-itH_0} \; \Psi_{\lambda,\omega} \, \| \; dt \; = O \; \left(\lambda^{-\frac{\rho}{2}} \right),$$

and

$$(1.15) \quad \| \; (W^{\pm} - 1) e^{-itH_0} \Phi_{\lambda,\omega} \; \| = \; O \; (\lambda^{-\frac{\rho}{2}}) \; , \text{ uniformly for } t \in I\!R \; .$$

So,

$$< [U, p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} > = \; O(\lambda^{-\rho}) = o(\lambda^{-\frac{1}{2}}).$$

since $\rho > \frac{1}{2}$.

2 Generalizations.

We can generalize Theorem 1 in the case where the free Hamiltonian H_0 is given by :

$$H_0 = p^2 - \sum_{k=1}^{n_-} a_k^2 \, x_k^2 + \sum_{k=n_-+1}^{n} a_k^2 \, x_k^2 \ , \tag{2.1}$$

with $a_k > 0$, $n_- \geq 1$ and with the convention $\sum_{j=a}^{b} = 0$ if $b < a$.

The main idea is to remark that the repulsive effects due to $-x_1^2$ overwhelm the confinement due to $+x_{n_-+1}^2$ and moreover, at high energies, the classical trajectories become straight lines.

For example, if $H_0 = p^2 - x_1^2 + x_2^2$ on $L^2(\mathbb{R}^n)$, one has for $\omega = (\omega_1, \omega_2)$ with $\omega_1 \neq 0$ and $p = (p_1, p_2)$,

$$< [S, p_2]\Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} > \sim < \int_{-\infty}^{+\infty} \partial_2 V(x_1 + \sqrt{\lambda}\sinh(2t)\omega_1, x_2 + \sqrt{\lambda}\sin(2t)\omega_2)\Phi,\ \Psi > \ \cos(2t)\ dt,$$

$$\sim \frac{1}{2\sqrt{\lambda}} < \int_{-\infty}^{+\infty} \partial_2 V(x + s\omega)\ ds\Phi, \Psi > \ ds,$$

when $\lambda \to +\infty$.