

# Recovering the mass and the charge of a Reissner-Nordström black hole by an inverse scattering experiment

Thierry Daudé <sup>\*</sup> and François Nicoleau <sup>†</sup>

## Abstract

In this paper, we study inverse scattering of massless Dirac fields that propagate in the exterior region of a Reissner-Nordström black hole. Using a stationary approach we determine precisely the leading terms of the high-energy asymptotic expansion of the scattering matrix that, in turn, permit us to recover uniquely the mass of the black hole and its charge up to a sign.

## 1 Introduction

Black hole spacetimes are probably among the most fascinating objects whose existence is predicted by Einstein's general relativity theory. In the last twenty years, numerous mathematical studies have been achieved in order to better understand their properties. Propagation of fields (wave, Klein-Gordon, Dirac, Maxwell) in these peculiar geometries and direct scattering results have been extensively studied in [2, 3, 7, 9, 10, 17, 19, 22] and used for instance to give rigorous interpretation of the Hawking effect [4, 18, 23]. Other original phenomena related to black hole spacetimes are for instance superradiance (the possibility of extracting energy from a rotating black hole) [5, 12, 13, 14] or causality violation (the existence of time machine inside rotating black hole that allows to travel through time) [6]. Despite the richness of these original phenomena, a striking feature of black holes spacetimes is their simplicity. Here we refer to the fact that, whatever might be the initial configuration of a collapsing stellar body, the resulting black hole spacetime can be eventually described by “at most” three parameters - its mass, its electric charge and its angular momentum - an important uniqueness result summarized by the well-known formula: *Black holes have no hair*. A natural question arises thus: assume that we are observers living in the exterior region of a black hole, at rest with respect to it and located far from it (such observers are called static at infinity and can be typically thought as a telescope on earth aimed at the black hole), can we in these conditions measure the defining parameters of the black hole by an inverse scattering experiment?

The result contained in this paper is a first step in this direction. Here we focus our attention on Reissner-Nordström black holes, that is black holes that are spherically symmetric and electrically charged and thus described by only two parameters: their mass  $M$  and their charge  $Q$ . We consider massless Dirac fields which evolve in the exterior region of a Reissner-Nordström black hole and we use the direct scattering results, already obtained in [24, 22, 7], to define the wave operators  $W^\pm$  and the corresponding scattering operator  $S$ . The main result of this paper is then the following. Suppose that the scattering operator  $S$  is known to observers static at infinity (accessible in theory by physical experiments), then we show that such observers can recover uniquely the mass  $M$  of the black hole as well as its charge  $Q$ ,

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<sup>\*</sup>Department of Mathematics and Statistics, McGill University, 805 Sherbrooke South West, Montréal QC, H3A 2K6

<sup>†</sup>Département de Mathématiques, Université de Nantes, 2, rue de la Houssinière, BP 92208, 44322 Nantes Cedex 03

up to a sign. Note here that we cannot completely determine the charge  $Q$  of the black hole because of the simplicity of our model. The propagation of massless Dirac fields indeed is only influenced by the geometry (*i.e.* the metric) of the black hole which depends on  $Q^2$  (cf. (2.1)). The consideration of massive charged Dirac fields should solve this undetermination since an explicit term corresponding to the interaction between the charge of the black hole and that of the field appear then in the equation (see [7]). The choice of dealing with massless Dirac fields is nevertheless meaningful since it is the simplest model we could study (in particular no long-range terms appear in the equation) that already provides significant informations. Moreover, we expect to be able to extend this result to much more complicated models such as rotating black holes (or Kerr black holes) for which there already exists a direct scattering theory [19].

The strategy we adopt to prove our main result is based on a high-energy asymptotic expansion of the scattering operator  $S$ . Such a technique was introduced by Enss and Weder in [11] and used successfully to recover the potential of multidimensional Schrödinger operators (note that the case of multidimensional Dirac operators in flat spacetime was treated later by Jung in [21]). They showed that the first term of the high-energy asymptotics is exactly the Radon transform of the potential they are looking for. Since they work in dimension greater than two, this Radon transform can be inverted and the potential thus recovered. In our problem however, due to the spherical symmetry of the black hole, we are led to study a one dimensional Dirac equation and the Radon transform cannot be inverted in this case. Nevertheless the first term of the high-energy asymptotics leads to an integral that can fortunately be explicitly computed and gives exactly the radius of the event horizon, already a physically relevant information. In order to obtain the mass and the charge of the black hole, we must then calculate the second term of the asymptotics. We follow here the stationary technique introduced by one of us [25] which is close in spirit to the Isozaki-Kitada method used in long-range scattering theory [20]. The basic idea is to replace the wave operators (and thus the scattering operator) by explicit Fourier Integral Operators, called modifiers, from which we are able to compute the high-energy asymptotic expansion readily. The construction of these modifiers and the precise determination of their phase and amplitude is the crux of this paper. While this method was well-known for Schrödinger operators and applied successfully to various situations (see [1, 25, 26, 27]), it has required some substantial modifications when applied to our model due to the specific properties of Dirac operators, essentially caused by the matrix-valued nature of the equation. A good starting point to deal with this difficulty was the paper of Gâtel and Yafaev [15] where direct scattering theory of massive Dirac fields in flat spacetime was studied and modifiers (at fixed energy) were constructed.

## 2 Reissner-Nordström black hole and Dirac equation

In this section, we first briefly describe the geometry of Reissner-Nordström black holes. In particular we point out the main features and properties that are meaningful from scattering theory viewpoint. We then write down the partial differential equation that governs the evolution of massless Dirac fields in the exterior region of Reissner-Nordström black holes and recall some known results in direct scattering theory.

### 2.1 Reissner-Nordström black holes

In Schwarzschild coordinates a Reissner-Nordström black hole is described by a four dimensional smooth manifold

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r^+ \times S_\omega^2,$$

equipped with the lorentzian metric

$$g = F(r) dt^2 - (F(r))^{-1} dr^2 - r^2 d\omega^2, \quad (2.1)$$

where

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (2.2)$$

and  $d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$  is the euclidean metric on the sphere  $S^2$ . The quantities  $M > 0$  and  $Q \in \mathbb{R}$  appearing in (2.2) are interpreted as the mass and the electric charge of the black hole.

The metric (2.1) has two types of singularities. Firstly, the point  $\{r = 0\}$  for which the function  $F$  is singular. This is a true singularity or *curvature singularity*<sup>1</sup>. Secondly, the spheres whose radii are the roots of  $F$  (note that the coefficient of the metric  $g$  involving  $F^{-1}$  explodes in this case). The number of these roots depends on the respective values of the constants  $M$  and  $Q$ . In this paper we only consider the case  $M > |Q|$  for which the function  $F$  has two zeros at the values  $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ . The spheres  $\{r = r_+\}$  and  $\{r = r_-\}$  are called the *exterior* and *interior event horizons* of the Reissner-Nordström black hole. Both exterior and interior horizons are not true singularities in the sense given for  $\{r = 0\}$ , but *coordinate singularities*. It turns out that, using appropriate coordinate systems, these horizons can be understood as regular null hypersurfaces that can be crossed one way but would require speeds greater than that of light to be crossed the other way. Hence their name: event horizons (we refer to [30] for a general introduction to black hole spacetimes).

Throughout this paper we shall nevertheless use the Schwarzschild coordinates as the coordinates of our analysis. The reasons are twofold. First in this coordinate system the coefficients of the metric (2.1) don't depend on the variables  $t$  and  $\omega$ , reflecting in fact the various symmetries of the black hole. We shall see in the next subsection 2.2 that the Dirac equation will take a very convenient form, namely an evolution equation (with respect to  $t$ ) with time-independent Hamiltonian acting on the spacelike hypersurface  $\Sigma = \mathbb{R}_r^+ \times S_\omega^2$ , a nice formalism when dealing with scattering theory. Second, it corresponds implicitly to the natural notion of observers static at infinity mentioned in the introduction. Such observers are located far away from the exterior horizon of the black hole and live on the integral curves of their velocity 4-vector

$$U = \frac{1}{F(r)} \frac{\partial}{\partial t}.$$

In particular, the time coordinate  $t$  is the proper time of such observers and thus corresponds to their true experience of time. We refer to [28] for a more complete discussion of this notion.. This choice of coordinates appears then quite natural with the idea of scattering experiments we have in mind.

We point out now a remarkable property of the exterior horizon when described in Schwarzschild coordinates, property that has important consequences for scattering theory. From the point of view of observers static at infinity, the exterior horizon is perceived as an *asymptotic region* of spacetime. Precisely, it means that the exterior horizon is never reached in a finite time by incoming and outgoing null radial geodesics, *i.e* the trajectories followed by light-rays aimed radially at the black hole or at infinity. These geodesics are given by the integral curves of  $F(r)^{-1} \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ . Along the outgoing geodesics we can express the time  $t$  as a function of  $r$  by the formula

$$t(r) = \int_{r_0}^r F(\tau)^{-1} d\tau, \quad (2.3)$$

where  $r_0 > r_+$  is fixed. From (2.2) and (2.3), we see immediately that  $t(r) \rightarrow -\infty$  when  $r \rightarrow r_+$ . An analogue formula holds for incoming geodesics.

<sup>1</sup>It means that certain scalars obtained by contracting the Riemann tensor blow up when  $r \rightarrow 0$ .

In consequence we can restrict our attention to the exterior region  $\{r > r_+\}$  of a Reissner-Nordström black hole and study the inverse problem for massless Dirac fields there. We just have to keep in mind that we won't need any boundary conditions on the exterior event horizon  $\{r = r_+\}$  since this horizon is an asymptotic region spacetime. To make this point even clearer, let us introduce a new radial coordinate  $x$ , called the Regge-Wheeler coordinate, which has the property of straightening the null radial geodesics and will greatly simplify the later analysis. It is defined implicitly by the relation

$$\frac{dr}{dx} = F(r), \quad (2.4)$$

and explicitly by

$$x = r + \frac{r_+^2}{r_+ - r_-} \log(r - r_+) + \frac{r_-^2}{r_+ - r_-} \log(r - r_-). \quad (2.5)$$

In the coordinate system  $(t, x, \omega)$ , the horizon  $\{r = r_+\}$  is pushed away to  $\{x = -\infty\}$  and thanks to (2.4), the metric takes the form

$$g = F(r)(dt^2 - dx^2) - r^2 d\omega^2. \quad (2.6)$$

Observe that the incoming and outgoing null radial geodesics are now generated by the vector fields  $\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$  and take the simple form

$$\gamma^\pm(t) = (t, x_0 \pm t, \omega_0), \quad t \in \mathbb{R}, \quad (2.7)$$

where  $(x_0, \omega_0) \in \mathbb{R} \times S^2$  are fixed. These are simply straight lines with velocity  $\pm 1$  mimicking, at least in the  $t - x$  plane, the situation of a one-dimensional Minkowski spacetime.

From now on we shall work on the manifold  $\mathcal{B} = \mathbb{R}_t \times \mathbb{R}_x \times S_\omega^2$  equipped with the metric (2.6).

## 2.2 Dirac equation and direct scattering results

Scattering theory for Dirac equations on the spacetime  $\mathcal{B}$  has been the object of several papers [24, 22, 7] (chronological order) in the last fifteen years. We use the following convenient form for a massless Dirac equation obtained therein. Under Hamiltonian form the equation eventually reads

$$i\partial_t \psi = H\psi, \quad (2.8)$$

where  $\psi$  is a 2-components spinor belonging to the Hilbert space  $\mathcal{H} = L^2(\mathbb{R} \times S^2; \mathbb{C}^2)$  and the Hamiltonian  $H$  is given by

$$H = \Gamma^1 D_x + a(x) D_{S^2}, \quad (2.9)$$

where  $a(x) = \frac{\sqrt{F(r)}}{r}$ ,  $D_x = -i\partial_x$ ,  $D_{S^2}$  denotes the Dirac operator on  $S^2$ , *i.e.*

$$D_{S^2} = -i\Gamma^2 \left( \partial_\theta + \frac{\cot \theta}{2} \right) - \frac{i}{\sin \theta} \Gamma^3 \partial_\varphi, \quad (2.10)$$

and  $\Gamma^1, \Gamma^2, \Gamma^3$  appearing in (2.9) and (2.10) are usual  $2 \times 2$  Dirac matrices that satisfy the anticommutator relations  $\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\text{Id}$ .

We can simplify further the expression of the Hamiltonian by using the spherical symmetry of the equation. The operator  $D_{S^2}$  has compact resolvent and can be diagonalized into an infinite sum of matrix-valued multiplication operators. The eigenfunctions associated to  $D_{S^2}$  are a generalization of usual spherical harmonics called *spin-weighted harmonics*. We refer to I.M. Gel'Fand and Z.Y. Sapiro [16] for a detailed presentation of these generalized spherical harmonics.

There exists thus a family of function  $F_n^l$  with the indexes  $(l, n)$  running in the set  $\mathcal{I} = \{(l, n), l - |\frac{1}{2}| \in \mathbb{N}, l - |n| \in \mathbb{N}\}$  which is a Hilbert basis of  $L^2(S^2; \mathbb{C}^2)$  and such that  $\forall (l, n) \in \mathcal{I}$ ,

$$D_{S^2} F_n^l = -(l + \frac{1}{2}) \Gamma^2 F_n^l.$$

The Hilbert space  $\mathcal{H}$  can then be decomposed into the infinite direct sum

$$\mathcal{H} = \bigoplus_{(l,n) \in \mathcal{I}} \left[ L^2(\mathbb{R}_x; \mathbb{C}^2) \otimes F_n^l \right] := \bigoplus_{(l,n) \in \mathcal{I}} \mathcal{H}_{ln},$$

where  $\mathcal{H}_{ln} = L^2(\mathbb{R}_x; \mathbb{C}^2) \otimes F_n^l$  is identified with  $L^2(\mathbb{R}; \mathbb{C}^2)$ . We obtain the orthogonal decomposition for the Hamiltonian  $H$

$$H = \bigoplus_{(l,n) \in \mathcal{I}} H^{ln},$$

with

$$H^{ln} := H|_{\mathcal{H}_{ln}} = \Gamma^1 D_x + a_l(x) \Gamma^2, \quad (2.11)$$

and  $a_l(x) = -a(x)(l + \frac{1}{2})$ . The operator  $H^{ln}$  is a selfadjoint operator on  $\mathcal{H}_{ln}$  with domain  $D(H^{ln}) = H^1(\mathbb{R}; \mathbb{C}^2)$ . Note also that we use the following representation for the Dirac matrices  $\Gamma^1$  and  $\Gamma^2$ .

$$\Gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (2.12)$$

In this paper it will be enough to restrict our study to a fixed harmonics. To simplify notation we thus simply write  $\mathcal{H}$ ,  $H$  and  $a(x)$  instead of  $\mathcal{H}_{ln}$ ,  $H^{ln}$  and  $a_l(x)$  respectively.

We summarize now the direct scattering results obtained in [24, 22, 7]. We define  $H_0 = \Gamma^1 D_x$  as the free Hamiltonian. It is clearly a selfadjoint operator on  $\mathcal{H}$  with domain  $D(H_0) = H^1(\mathbb{R}; \mathbb{C}^2)$ . The main results concerning the spectral properties of the pair  $(H, H_0)$  are the following. The Hamiltonian  $H$  has no eigenvalue and no singular continuous spectrum, *i.e.*

$$\sigma_{pp}(H) = \emptyset, \quad \sigma_{sc}(H) = \emptyset, \quad (2.13)$$

and the standard wave operators

$$W^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}, \quad (2.14)$$

exist and are complete on  $\mathcal{H}$ . Let us make a few comments on these results. It follows directly from (2.13) and (2.14) that  $H$  has purely absolutely continuous spectrum and thus all states scatter in the asymptotic regions of spacetime where they obey a simpler equation given by the comparison dynamics  $e^{itH_0}$ . Notice that this comparison dynamics is particularly simple for massless Dirac fields. Due to the diagonal form of the matrix  $\Gamma^1$ , it becomes simply a system of transport equations with velocity  $+1$  for the second component of the spinor and velocity  $-1$  for the first. At last, observe that the potential  $a$  has the following asymptotics as  $x \rightarrow \pm\infty$ :

$$\begin{aligned} \exists \kappa > 0, \quad |a(x)| &= O(e^{\kappa x}), \quad x \rightarrow -\infty, \\ |a(x)| &= O(|x|^{-1}), \quad x \rightarrow +\infty. \end{aligned}$$

Eventhough  $a$  is apparently a long-range potential having Coulomb decay at  $+\infty$ , we don't need to modify the comparison dynamics in the definition of the wave operators. We refer to [7], Section 7 for a proof of this point.

The scattering operator  $S$  is defined by

$$S = (W^+)^*W^-. \quad (2.15)$$

We can simplify the operator  $S$  by considering incoming and outgoing initial data separately<sup>2</sup>. In order to define these subspaces of initial data precisely, let us introduce the asymptotic velocity operators

$$P_0^\pm = s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH_0} \frac{x}{t} e^{-itH_0},$$

and

$$P^\pm = s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \frac{x}{t} e^{-itH}.$$

We refer to [8] for a detailed presentation of these operators and their usefulness in scattering theory. By a direct calculation, we get easily  $P_0^\pm = \Gamma^1$  while it was shown in [7] that the operators  $P^\pm$  exist, have also spectra equal to  $\{-1, +1\}$  and satisfy the intertwining relations

$$P^\pm W^\pm = W^\pm \Gamma^1. \quad (2.16)$$

Let us denote  $P_{\text{in}} = \mathbf{1}_{\mathbb{R}^-}(\Gamma^1)$  and  $P_{\text{out}} = \mathbf{1}_{\mathbb{R}^+}(\Gamma^1)$  the projections onto the negative and positive spectral subspaces of the “free” asymptotic velocity  $\Gamma^1$ . Then we define  $\mathcal{H}_{\text{in}}^0 = P_{\text{in}}\mathcal{H}$  and  $\mathcal{H}_{\text{out}}^0 = P_{\text{out}}\mathcal{H}$  and refer to these spaces as incoming and outgoing initial data for the free dynamics<sup>3</sup>. In a similar way, we define  $\mathcal{H}_{\text{in}}^\pm = \mathbf{1}_{\mathbb{R}^-}(P^\pm)\mathcal{H}$  and  $\mathcal{H}_{\text{out}}^\pm = \mathbf{1}_{\mathbb{R}^+}(P^\pm)\mathcal{H}$  the spaces of incoming and outgoing initial data for the full dynamics when  $t \rightarrow \pm\infty$ . In our particular case it turns out that we have equalities between some of these spaces. Precisely we have

**Lemma 2.1** *With the same notations as above, we have*

$$\begin{aligned} \mathcal{H}_{\text{in}}^+ &= \mathcal{H}_{\text{in}}^- &=: \mathcal{H}_{\text{in}}, \\ \mathcal{H}_{\text{out}}^+ &= \mathcal{H}_{\text{out}}^- &=: \mathcal{H}_{\text{out}}. \end{aligned}$$

**Proof:** We only prove the case “out” since the proof for “in” is similar. We shall use the following characterization of the asymptotic velocity operators obtained in [7]

$$P^\pm = s - C^\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \Gamma^1 e^{-itH}. \quad (2.17)$$

We compute  $\mathbf{1}_{\mathbb{R}^-}(P^-)\mathbf{1}_{\mathbb{R}^+}(P^+)$ . By (2.17) we get

$$\mathbf{1}_{\mathbb{R}^-}(P^-)\mathbf{1}_{\mathbb{R}^+}(P^+) = s - \lim_{t \rightarrow \pm\infty} e^{-itH} \mathbf{1}_{\mathbb{R}^-}(\Gamma^1) e^{2itH} \mathbf{1}_{\mathbb{R}^+}(\Gamma^1) e^{-itH}.$$

Since  $\Gamma^1 H = \tilde{H} \Gamma^1$  with  $\tilde{H} = \Gamma^1 D_x - a(x) \Gamma^2$ , we have

$$\begin{aligned} \mathbf{1}_{\mathbb{R}^-}(P^-)\mathbf{1}_{\mathbb{R}^+}(P^+) &= s - \lim_{t \rightarrow \pm\infty} e^{-itH} e^{2it\tilde{H}} \mathbf{1}_{\mathbb{R}^-}(\Gamma^1) \mathbf{1}_{\mathbb{R}^+}(\Gamma^1) e^{-itH}, \\ &= 0. \end{aligned}$$

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<sup>2</sup>The terms incoming and outgoing data have to be understood here in reference to the black hole, *i.e.* as the data whose associated solutions propagate, when  $t$  increases, in direction of the black hole (incoming data) or escape from it (outgoing data)

<sup>3</sup>Observe that  $\mathcal{H}_{\text{in}}^0$  (resp.  $\mathcal{H}_{\text{out}}^0$ ) are simply the spinors with nonvanishing first component (resp. second component) only

Hence  $\mathbf{1}_{\mathbb{R}^+}(P^+) = (\mathbf{1}_{\mathbb{R}^-}(P^-) + \mathbf{1}_{\mathbb{R}^+}(P^-))\mathbf{1}_{\mathbb{R}^+}(P^+)\mathbf{1}_{\mathbb{R}^+}(P^-)\mathbf{1}_{\mathbb{R}^+}(P^+)$  from which we deduce immediately that  $\mathcal{H}_{\text{out}}^+ \subset \mathcal{H}_{\text{out}}^-$ . Analogously we can prove  $\mathcal{H}_{\text{out}}^- \subset \mathcal{H}_{\text{out}}^+$  and thus the equality  $\mathcal{H}_{\text{out}}^+ = \mathcal{H}_{\text{out}}^-$  holds.  $\diamond$

Thanks to the intertwining relations (2.16) and Lemma 2.1 we see that the wave operators  $W^\pm$  are partial isometries from  $\mathcal{H}_{\text{in/out}}^0$  into  $\mathcal{H}_{\text{in/out}}$ . We define

$$W_{\text{out}}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} P_{\text{out}},$$

$$W_{\text{in}}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} P_{\text{in}},$$

the corresponding outgoing and incoming wave operators. Remark here that by the relations  $H_0 \mathbf{1}_{\mathbb{R}^\pm}(\Gamma^1) = \pm D_x \mathbf{1}_{\mathbb{R}^\pm}(\Gamma^1)$  the Hamiltonian  $H_0$  becomes “scalar” when projected on  $\mathcal{H}_{\text{in/out}}^0$ . Hence we can simplify the expression of the outgoing and incoming wave operators by

$$W_{\text{out}}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itD_x} P_{\text{out}},$$

$$W_{\text{in}}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{itD_x} P_{\text{in}}.$$

This minor remark will turn out to be important in the next section. Now from the definition (2.15) of the scattering operator, we deduce that  $S$  leaves invariant the incoming and outgoing subspaces  $\mathcal{H}_{\text{in/out}}^0$  and thus can be decomposed into the direct sum

$$S = S_{\text{in}} \oplus S_{\text{out}}, \quad (2.18)$$

where

$$S_{\text{in}} = (W_{\text{in}}^+)^* W_{\text{in}}^-, \quad S_{\text{out}} = (W_{\text{out}}^+)^* W_{\text{out}}^-. \quad (2.19)$$

Clearly  $S_{\text{in}}$  and  $S_{\text{out}}$  are isometries on  $\mathcal{H}_{\text{in}}^0$  and  $\mathcal{H}_{\text{out}}^0$ .

### 3 The inverse problem at high energy

#### 3.1 The main result and the strategy of the proof

We are now ready to study the inverse problem for massless Dirac fields satisfying (2.8). The question we address is: can we determine the mass  $M$  and the charge  $Q$  of the black hole from the knowledge of the scattering operator  $S$ ? The main result of this paper is summarized in the following Theorem

**Theorem 3.1** *Assume that the scattering matrix  $S_{\text{out}}$  (or  $S_{\text{in}}$ ) is known. Then the corresponding potential  $a(x)$  can be entirely determined. Moreover the mass  $M$  and the square of the charge  $Q^2$  of the black hole can be recovered by the formulas*

$$M = \lim_{x \rightarrow +\infty} \frac{x - x^3 a^2(x)}{2}, \quad (3.1)$$

$$Q^2 = \lim_{x \rightarrow +\infty} (x^4 a^2(x) - x^2 + 2Mx). \quad (3.2)$$

The strategy we adopt to prove this Theorem is based on a high-energy asymptotic expansion of  $S_{\text{out}}$ , a well-known technique initially developed in the case of multidimensional Schrödinger operators by Enss, Weder [11]. This method can be used to study Hamiltonians with electric and magnetic potentials [1], the Dirac equation [21] and Stark Hamiltonians [31, 27].

Precisely, we consider  $\psi_1, \psi_2 \in \mathcal{H}_{\text{out}}^0$  such that  $\hat{\psi}_1, \hat{\psi}_2 \in C_0^\infty(\mathbb{R}; \mathbb{C}^4)$  and we define the map

$$F_{\text{out}}(\lambda) = \langle S_{\text{out}} e^{i\lambda x} \psi_1, e^{i\lambda x} \psi_2 \rangle, \quad (3.3)$$

for  $\lambda \in \mathbb{R}$ . Following the ideas of [25, 26], we shall obtain an asymptotic expansion of  $S(\lambda)$  when  $\lambda \rightarrow \infty$  with the exact expression of the first three leading terms. Exactly we get the following *reconstruction formula*:

**Theorem 3.2 (Reconstruction formula)** *Let  $\psi_1, \psi_2 \in \mathcal{H}_{\text{out}}^0$  such that  $\hat{\psi}_1, \hat{\psi}_2 \in C_0^\infty(\mathbb{R}; \mathbb{C}^2)$ . Then for  $\lambda$  large, we obtain*

$$F_{\text{out}}(\lambda) = \langle \psi_1, \psi_2 \rangle + \frac{1}{\lambda} \langle \psi_1, L_1(x, D_x) \psi_2 \rangle + \frac{1}{\lambda^2} \langle \psi_1, L_2(x, D_x) \psi_2 \rangle + O(\lambda^{-3}), \quad (3.4)$$

where  $L_j(x, D_x)$  are differential operators given by

$$L_1(x, D_x) = L_1 = \frac{i(l + \frac{1}{2})^2}{2r_+},$$

$$L_2(x, D_x) = \frac{a_l^2(x)}{2} + \frac{(l + \frac{1}{2})^4}{8r_+^2} - \frac{i(l + \frac{1}{2})^2 D_x}{2r_+},$$

and  $r_+ = M + \sqrt{M^2 - Q^2}$ .

Let us admit temporarily this result and use it to prove the main theorem. If we assume that the scattering matrix  $S_{\text{out}}$  is known, then we can use inductively the high-energy expansion (3.4) to recover first, the radius  $r_+$  of the exterior event horizon (term of order  $\lambda^{-1}$ ) and second, the potential  $a(x)$ . Indeed, the term of order  $\lambda^{-2}$  (since  $r_+$  is already determined) gives the quantity

$$\langle \frac{a_l^2(x)}{2} \psi_1, \psi_2 \rangle. \quad (3.5)$$

Since (3.5) can be determined for any  $\psi_1, \psi_2$  in a dense subset in  $\mathcal{H}$ , we recover completely the potential  $a_l(x)$  and thus  $a(x)$ . Now the formulae for the mass (3.1) and for the charge (3.2) follow directly from the definition of  $a(x)$

$$a^2(x) = \frac{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}{r^2},$$

and the fact that

$$r(x) \sim x, x \rightarrow +\infty$$

an immediate consequence of (2.5). This ends the proof of our main Theorem and proves that observers static at infinity who can measure the scattering matrix are able to recover the mass of the black hole and its charge, up to a sign. Note that the first term of the expansion (3.4) permits us to determine the radius  $r_+$  of the exterior event horizon, an information of physical relevance.

The main step in the proof of Theorem 3.1 is clearly the reconstruction formula 3.4 stated above. In order to prove it, we first rewrite  $F_{\text{out}}(\lambda)$  as follows

$$\begin{aligned} F_{\text{out}}(\lambda) &= \langle S_{\text{out}} e^{i\lambda x} \psi_1, e^{i\lambda x} \psi_2 \rangle, \\ &= \langle W_{\text{out}}^- e^{i\lambda x} \psi_1, W_{\text{out}}^+ e^{i\lambda x} \psi_2 \rangle, \\ &= \langle W_{\text{out}}^-(\lambda) \psi_1, W_{\text{out}}^+(\lambda) \psi_2 \rangle, \end{aligned} \quad (3.6)$$

where

$$W_{\text{out}}^\pm(\lambda) = s - \lim_{t \rightarrow \pm\infty} e^{itH(\lambda)} e^{-it(D_x + \lambda)} P_{\text{out}},$$

and

$$H(\lambda) = \Gamma^1(D_x + \lambda) + a(x)\Gamma^2.$$

In order to obtain an asymptotic expansion of the operators  $W_{\text{out}}^\pm(\lambda)$ , we follow the procedure exposed in [25, 26], procedure inspired by the well-known Isozaki-Kitada method [20] developed in the setting of long-range stationary scattering theory. It consists simply in replacing the wave operators  $W_{\text{out}}^\pm(\lambda)$  by “well-chosen” energy modifiers  $J_{\text{out}}^\pm(\lambda)$ , defined as Fourier Integral Operators (FIO) with explicit phase and amplitude. Well-chosen here means practically that we look for  $J_{\text{out}}^\pm(\lambda)$  satisfying for  $|\lambda|$  large

$$W_{\text{out}}^\pm(\lambda)\psi = \lim_{t \rightarrow \pm\infty} e^{itH(\lambda)} J_{\text{out}}^\pm(\lambda) e^{-it(D_x + \lambda)} \psi, \quad (3.7)$$

and

$$\|(W_{\text{out}}^\pm(\lambda) - J_{\text{out}}^\pm(\lambda))\psi\| = O(\lambda^{-3}), \quad (3.8)$$

for a fixed  $\psi \in \mathcal{H}_{\text{out}}^0$  such that  $\hat{\psi} \in C_0^\infty(\mathbb{R}; \mathbb{C}^2)$ . In particular if we manage to construct  $J_{\text{out}}^\pm(\lambda)$  satisfying (3.8) then we obtain by (3.6)

$$F_{\text{out}}(\lambda) = \langle J_{\text{out}}^-(\lambda) \psi_1, J_{\text{out}}^+(\lambda) \psi_2 \rangle + O(\lambda^{-3}),$$

from which we can calculate the first terms of the asymptotics easily.

The rest of this paper is devoted to the construction of the modifiers  $J_{\text{out}}^\pm(\lambda)$  and to the proof of Theorem 3.2. For simplicity of notations we omit in the next subsections the underscored “out” when naming the wave operators  $W_{\text{out}}^\pm$  and modifiers  $J_{\text{out}}^\pm$ .

### 3.2 Construction of the modifier

Let us give a hint on how to construct the modifiers  $J^\pm(\lambda)$  a priori defined as an FIO with “scalar” phase  $\varphi^\pm(x, \xi, \lambda)$  and “matrix-valued” amplitude  $p^\pm(x, \xi, \lambda)$ . If we assume that (3.7) is true then we can write

$$(W^\pm(\lambda) - J^\pm(\lambda))\psi = i \int_0^{\pm\infty} e^{itH(\lambda)} C^\pm(\lambda) e^{-it(D_x + \lambda)} \psi dt,$$

where

$$C^\pm(\lambda) := H(\lambda)J^\pm(\lambda) - J^\pm(\lambda)(D_x + \lambda), \quad (3.9)$$

is an FIO with phase  $\varphi^\pm(x, \xi, \lambda)$  and amplitude  $c^\pm(x, \xi, \lambda)$ . Hence we get the simple estimate

$$\|(W^\pm(\lambda) - J^\pm(\lambda))\psi\| \leq \int_0^{\pm\infty} \|C^\pm(\lambda) e^{-itD_x} \psi\| dt. \quad (3.10)$$

In order that (3.8) be true it is then clear that the FIO  $C(\lambda)$  has to be “small” in some sense. Precisely we shall need that the amplitude  $c^\pm(x, \xi, \lambda)$  be short-range in the variable  $x$  and of order  $\lambda^{-3}$  in the variable  $\lambda$ .

### 3.2.1 Construction of the modifiers at fixed energy

We first look at the problem at fixed energy (*i.e.* we take  $\lambda = 0$  in the previous formulae). Hence we aim to construct modifiers  $J^\pm$  with scalar phase  $\varphi^\pm(x, \xi)$  and matrix-valued amplitude  $p^\pm(x, \xi)$  such that the amplitude  $c^\pm(x, \xi)$  of the operator  $C^\pm = HJ^\pm - J^\pm D_x$  be short-range in  $x$ . We follow here the treatment given by Gâtel and Yafaev in [15] where a similar problem for massive Dirac Hamiltonians in Minkowski spacetime was considered.

The operator  $C^\pm$  is clearly an FIO with phase  $\varphi^\pm(x, \xi)$  and amplitude

$$c^\pm(x, \xi) = B^\pm(x, \xi)p^\pm(x, \xi) - i\Gamma^1 \partial_x p^\pm(x, \xi), \quad (3.11)$$

where

$$B^\pm(x, \xi) = \Gamma^1 \partial_x \varphi^\pm(x, \xi) + a(x)\Gamma^2 - \xi. \quad (3.12)$$

As usual we look for a phase  $\varphi^\pm$  close to  $x\xi$  and an amplitude  $p^\pm$  close to 1. So the term  $\partial_x p^\pm$  in (3.11) should be short-range et can be neglected. Moreover if  $p^\pm$  was exactly 1 we would have to determine  $\varphi^\pm$  so that

$$B^\pm = 0. \quad (3.13)$$

A direct calculation of  $B^\pm = 0$  leads to a matrix-valued phase  $\varphi^\pm$  whereas we look for a scalar one. But if  $\varphi^\pm$  satisfies (3.13) then it should also satisfy the equation  $B^2 = 0$ . Using the anticommutation properties of the Dirac matrices we have

$$(B^\pm)^2(x, \xi) = (\partial_x \varphi^\pm)^2 + a^2(x) - \xi^2 - 2\xi B^\pm(x, \xi) = 0. \quad (3.14)$$

Using (3.13) again, we eventually get the scalar equation

$$r^\pm(x, \xi) := (\partial_x \varphi^\pm)^2 + a^2(x) - \xi^2 = 0, \quad (3.15)$$

in fact an eikonal equation of Schrödinger type. Let us set  $\varphi^\pm(x, \xi) = x\xi + \phi^\pm(x, \xi)$  where  $\phi^\pm(x, \xi)$  should be a priori small in the variable  $x$ . Then we must solve  $2\xi \partial_x \phi^\pm + (\partial_x \phi^\pm)^2 + a^2(x) = 0$ . Neglecting  $(\partial_x \phi^\pm)^2$  in this last equation, we obtain

$$2\xi \partial_x \phi^\pm + a^2(x) = 0. \quad (3.16)$$

For  $\xi \neq 0$ , we get the following two solutions of (3.16) by the formulae

$$\phi^\pm(x, \xi) = \frac{1}{2\xi} \int_0^{\pm\infty} a^2(x+s) ds, \quad (3.17)$$

and with this choice, we obtain for  $\xi \neq 0$ ,

$$r^\pm(x, \xi) = \frac{1}{4\xi^2} a^4(x). \quad (3.18)$$

In our derivation of the phase, it is important to keep in mind that we didn't find an approximate solution of (3.13) but instead of (3.14). Therefore we cannot expect to take  $p^\pm = 1$  as a first approximation and we have to work a bit more. In order to construct the amplitude  $p^\pm$  we follow here particularly closely [15].

So we look for  $p^\pm$  such that  $B^\pm p^\pm$  is as small as possible. What we know is that  $(B^\pm)^2$  is already small thanks to our choice of  $\varphi^\pm$ . Hence let's try to find a relation between  $B^\pm$  and  $(B^\pm)^2$ . Observe first that  $B^\pm$  have the following expression on  $\Gamma^\pm$

$$B^\pm(x, \xi) = B(x, \xi) = B_0(x, \xi) + 2\xi K(x, \xi), \quad (3.19)$$

where

$$B_0(x, \xi) = -2\xi P_{\text{in}}, \quad (3.20)$$

$$K(x, \xi) = \frac{1}{2\xi} \left( -\frac{a^2(x)}{2\xi} \Gamma^1 + a(x) \Gamma^2 \right). \quad (3.21)$$

In particular,  $B^\pm$  don't depend on  $\pm$  in  $\Gamma^\pm$ . Taking the square of (3.19) we get

$$B^2 = B_0^2 + 2\xi B_0 K + 2\xi K B \quad (3.22)$$

But from (3.20) we see that  $B_0^2 = -2\xi B_0$ . Whence (3.22) becomes

$$B^2 = -2\xi B_0(1 - K) + 2\xi K B. \quad (3.23)$$

If we add  $2\xi B$  to both sides of the equality in (3.23) then we obtain by (3.14),

$$r(x, \xi) := B^2 + 2\xi B = -2\xi B_0(1 - K) + 2\xi(1 + K)B, \quad (3.24)$$

or equivalently, if we isolate  $B$  on the left hand side

$$2\xi(1 + K)B = r(x, \xi) - 4\xi^2 P_{\text{in}}(1 - K). \quad (3.25)$$

It follows immediately from the definition of  $K$  that  $(1 \pm K)$  are invertible for  $\xi \in X := \{ \xi \in \mathbb{R} : |\xi| \geq R \}$ ,  $R \gg 1$ . In consequence we can write for  $\xi \in X$

$$B(1 - K)^{-1} = \frac{1}{2\xi} (1 + K)^{-1} r(x, \xi) (1 - K)^{-1} - 2\xi P_{\text{in}}. \quad (3.26)$$

The first term in the right hand side of (3.26) is small thanks to our choice of phase but the second one is not. We choose  $p^\pm$  in such a way that it cancels this term. According to (3.26), a natural choice for  $p^\pm$  is thus

$$p^\pm(x, \xi) = p(x, \xi) = (1 - K)^{-1} P_{\text{out}}, \quad (3.27)$$

for which we have

$$q(x, \xi) := B(x, \xi) p(x, \xi) = \frac{1}{2\xi} (1 + K)^{-1} r(x, \xi) (1 - K)^{-1}. \quad (3.28)$$

Note that  $p$  defined by (3.27) is independent of  $\pm$ .

Let us summarize the situation at this stage. For  $\xi \neq 0$ , we have defined the phase  $\varphi^\pm(x, \xi) = x\xi + \phi^\pm(x, \xi)$  by (3.17) and for  $\xi \in X$ , the amplitude  $p$  is given by (3.27). Directly from the definitions, the following estimates hold.

**Lemma 3.1 (Estimates on the phase and the amplitude)**

$$\forall (x, \xi) \in \mathbb{R} \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta \varphi^\pm(x, \xi)| \leq C_{\alpha\beta} (1 + |x| + |\xi|)^{(2-\alpha-\beta)_+}. \quad (3.29)$$

$$\forall(x, \xi) \in \mathbb{R} \times X, \quad |\partial_{x, \xi}^2(\varphi^\pm(x, \xi) - x\xi)| \leq \frac{C}{R^2} \quad (3.30)$$

$$\forall(x, \xi) \in \mathbb{R}^\pm \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta \phi^\pm(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1-\alpha} \langle \xi \rangle^{-1-\beta}. \quad (3.31)$$

$$\forall(x, \xi) \in \mathbb{R} \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta K(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1-\alpha} \langle \xi \rangle^{-1-\beta}. \quad (3.32)$$

$$\forall(x, \xi) \in \mathbb{R} \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta (p(x, \xi) - P_{out})| \leq C_{\alpha\beta} \langle x \rangle^{-1-\alpha} \langle \xi \rangle^{-1-\beta}. \quad (3.33)$$

$$\forall(x, \xi) \in \mathbb{R} \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta r(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-4-\alpha} \langle \xi \rangle^{-2-\beta}. \quad (3.34)$$

$$\forall(x, \xi) \in \mathbb{R} \times X, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-4-\alpha} \langle \xi \rangle^{-3-\beta}. \quad (3.35)$$

Thanks to (3.29), (3.30) and (3.33), for  $R$  large enough, we can define precisely our modifier  $J^\pm$  as a bounded operator on  $\mathcal{H}$  ([29], Corollary IV.22): let  $\theta \in C^\infty(\mathbb{R})$  be a cutoff function such that  $\theta(\xi) = 0$  if  $|\xi| \leq \frac{1}{2}$  and  $\theta(\xi) = 1$  if  $|\xi| \geq 1$ . For  $R$  large enough,  $J^\pm$  is the Fourier Integral Operator with phase  $\varphi^\pm(x, \xi)$  and amplitude

$$P(x, \xi) = p(x, \xi) \theta\left(\frac{\xi}{R}\right) \quad (3.36)$$

Moreover, we have

**Proposition 3.1** *For any  $\psi \in \mathcal{H}_{out}^0$  such that  $\hat{\psi} \in C_0^\infty(X; \mathbb{C}^2)$ , we have*

$$W^\pm \psi = \lim_{t \rightarrow \pm\infty} e^{itH} J^\pm e^{-itD_x} \psi. \quad (3.37)$$

**Proof:** We only consider the case (+). It is clear that it suffices to show

$$\lim_{t \rightarrow +\infty} \left( J^+ - \theta\left(\frac{D_x}{R}\right) P_{out} \right) e^{-itD_x} \psi = 0. \quad (3.38)$$

Let  $\eta \in C^\infty(\mathbb{R})$  such that  $\eta(x) = 1$  if  $x \geq 1$  and  $\eta(x) = 0$  if  $x \leq 0$ . We write  $e^{-itD_x} \psi$  as :

$$e^{-itD_x} \psi = \eta(x) e^{-itD_x} \psi + (1 - \eta(x)) e^{-itD_x} \psi \quad (3.39)$$

Clearly,  $(1 - \eta(x)) e^{-itD_x} \psi = e^{-itD_x} (1 - \eta(x+t)) \psi \rightarrow 0$  on  $\mathcal{H}$ .

Now, using (3.29), (3.31), (3.33) and the standard pseudodifferential calculus, we see that

$T := \left( J^+ - \theta\left(\frac{D_x}{R}\right) P_{out} \right) \eta(x)$  is a  $\Psi$ do with symbol  $k^+(x, \xi)$  which satisfies

$$\forall(x, \xi) \in \mathbb{R} \times \mathbb{R}, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta k^+(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1-\alpha} \langle \xi \rangle^{-1-\beta}. \quad (3.40)$$

Then  $T$  is a compact operator on  $\mathcal{H}$  and since  $e^{-itD_x} \psi \rightarrow 0$  weakly, Proposition 3.1 is proved.

◇

### 3.2.2 Construction of the modifiers at high energy

Let's come back to the construction of  $J^\pm(\lambda)$  the modifiers at high energy. Recall that we look for modifiers that satisfy (3.7) and (3.8). Proposition 3.1 suggests to construct  $J^\pm(\lambda)$  close to  $e^{-i\lambda x} J^\pm e^{i\lambda x}$ . Clearly,  $e^{-i\lambda x} J^\pm e^{i\lambda x}$  are the FIOs with phases  $\varphi^\pm(x, \xi, \lambda) = x\xi + \left(\frac{1}{2(\xi+\lambda)} \int_0^{\pm\infty} a^2(x+s) ds\right)$  and amplitude  $P(x, \xi + \lambda)$ . Since the modifiers  $J^\pm(\lambda)$  will be applied on functions  $\psi$  having compact support in the Fourier variable, the cut-off in the  $\xi$  variable in the definition of  $P(x, \xi + \lambda)$  disappears for  $\lambda$  sufficiently large enough.

Thus, if we take exactly  $J^\pm(\lambda) = e^{-i\lambda x} J^\pm e^{i\lambda x}$  as the modifiers, according to (3.11), for  $\xi$  in a compact set and  $\lambda \gg 1$ , the amplitude of the operator  $C^\pm(\lambda)$  becomes

$$c^\pm(x, \xi, \lambda) = c(x, \xi, \lambda) = B(x, \xi + \lambda)p(x, \xi + \lambda) - i\Gamma^1 \partial_x p(x, \xi + \lambda).$$

We want the amplitude  $c(x, \xi, \lambda)$  to be of order  $O(\langle x \rangle^{-2} \lambda^{-3})$ . By (3.35) the first term  $B(x, \xi + \lambda)p(x, \xi + \lambda)$  has order  $O(\langle x \rangle^{-4} \lambda^{-3})$ . By (3.33) the second term  $\partial_x p(x, \xi + \lambda)$  has order  $O(\langle x \rangle^{-2} \lambda^{-1})$  which is not a sufficient decay in  $\lambda$  for our purpose.

So, we have to work a bit more. We look for the modifiers  $J^\pm(\lambda)$ , with  $\lambda$  large enough, as FIOs with phases  $\varphi^\pm(x, \xi, \lambda)$  and with a new amplitude  $P(x, \xi, \lambda)$  under the form

$$P(x, \xi, \lambda) = \left( p(x, \xi + \lambda) + \frac{1}{\lambda^2} \left( P_{\text{in}} k_1(x, \xi) + p(x, \xi + \lambda) l_1(x, \xi) \right) + \frac{1}{\lambda^3} P_{\text{in}} k_2(x, \xi) \right) g(\xi),$$

where  $g \in C_0^\infty(\mathbb{R})$  with  $g \equiv 1$  on  $\text{Supp } \hat{\psi}$ , the functions  $k_1, k_2, l_1$  (that can be matrix-valued) will be short-range of order  $O(\langle x \rangle^{-2})$ . Observe that there are two different types of correcting terms in this new amplitude: on one hand the terms which contain  $p(x, \xi + \lambda) \simeq P_{\text{out}}$  (when  $\lambda \rightarrow \infty$ ) are correcting terms that "live" in  $\mathcal{H}_{\text{out}}^0$ ; on the other hand the terms which contain  $P_{\text{in}}$  are clearly correcting terms that "live" in  $\mathcal{H}_{\text{in}}^0$ . Now, noting that

$$B(x, \xi + \lambda) = -2(\xi + \lambda)P_{\text{in}} + M(x, \xi + \lambda), \quad M(x, \xi + \lambda) = -\frac{a^2(x)}{2(\xi + \lambda)}\Gamma^1 + a(x)\Gamma^2,$$

the amplitude  $c(x, \xi, \lambda)$  can be written for  $\xi \in \text{Supp } \hat{\psi}$  as

$$\begin{aligned} c(x, \xi, \lambda) &= B(x, \xi + \lambda)P(x, \xi + \lambda) - i\Gamma^1 \partial_x P(x, \xi + \lambda), \\ &= B(x, \xi + \lambda)p(x, \xi + \lambda) - i\Gamma^1 \partial_x p(x, \xi + \lambda) \\ &\quad + \frac{1}{\lambda^2} \left( -2(\xi + \lambda)P_{\text{in}}k_1(x, \xi) + M(x, \xi + \lambda)P_{\text{in}}k_1(x, \xi) - i\Gamma^1 P_{\text{in}}\partial_x k_1(x, \xi) \right) \\ &\quad + \frac{1}{\lambda^2} \left( B(x, \xi + \lambda)p(x, \xi + \lambda)l_1(x, \xi) - i\Gamma^1 (\partial_x p(x, \xi + \lambda)l_1 + p(x, \xi + \lambda)\partial_x l_1(x, \xi)) \right) \\ &\quad + \frac{1}{\lambda^3} \left( -2(\xi + \lambda)P_{\text{in}}k_2(x, \xi) + M(x, \xi + \lambda)P_{\text{in}}k_2(x, \xi) - i\Gamma^1 P_{\text{in}}\partial_x k_2(x, \xi) \right). \end{aligned} \quad (3.41)$$

Directly from the definitions we get the following asymptotics when  $\lambda \rightarrow \infty$

$$\begin{aligned} \partial_x p(x, \xi + \lambda) &= +\frac{a'(x)}{2\lambda}\Gamma^2 P_{\text{out}} - \frac{\xi a'(x)}{2\lambda^2}\Gamma^2 P_{\text{out}} + O(\langle x \rangle^{-2} \lambda^{-3}), \\ p(x, \xi + \lambda) &= P_{\text{out}} + O(\langle x \rangle^{-1} \lambda^{-1}), \\ B(x, \xi + \lambda)p(x, \xi + \lambda) &= O(\langle x \rangle^{-4} \lambda^{-3}). \end{aligned}$$

This leads to

$$\begin{aligned}
c(x, \xi, \lambda) &= -i \frac{a'(x)}{2\lambda} \Gamma^1 \Gamma^2 P_{\text{out}} + i \frac{\xi a'(x)}{2\lambda^2} \Gamma^1 \Gamma^2 P_{\text{out}} - \frac{2}{\lambda} P_{\text{in}} k_1 - \frac{2\xi}{\lambda^2} P_{\text{in}} k_1 \\
&+ \frac{a(x)}{\lambda^2} \Gamma^2 P_{\text{in}} k_1 - i \frac{1}{\lambda^2} \Gamma^1 P_{\text{in}} \partial_x k_1 - i \frac{1}{\lambda^2} \Gamma^1 P_{\text{out}} \partial_x l_1 - \frac{2}{\lambda^2} P_{\text{in}} k_2 \\
&+ O(\langle x \rangle^{-2} \lambda^{-3}).
\end{aligned} \tag{3.42}$$

But the anticommutation properties of the Dirac matrices entail the relations  $\Gamma^1 \Gamma^2 P_{\text{out}} = -P_{\text{in}} \Gamma^2$  and  $\Gamma^2 P_{\text{in}} = P_{\text{out}} \Gamma^2$ . Hence the amplitude  $c(x, \xi, \lambda)$  takes the form

$$\begin{aligned}
c(x, \xi, \lambda) &= \frac{1}{\lambda} P_{\text{in}} \left( i \frac{a'(x)}{2} \Gamma^2 - 2k_1 \right) \\
&+ \frac{1}{\lambda^2} P_{\text{in}} \left( -i \frac{\xi a'(x)}{2} \Gamma^2 - 2\xi k_1 + i \partial_x k_1 - 2k_2 \right) + \frac{1}{\lambda^2} P_{\text{out}} \left( a(x) \Gamma^2 k_1 - i \partial_x l_1 \right) \\
&+ O(\langle x \rangle^{-2} \lambda^{-3}).
\end{aligned} \tag{3.43}$$

Using (3.43) we can cancel the terms of order less than  $O(\lambda^{-3})$  inductively. We choose

$$k_1(x, \xi) = i \frac{a'(x)}{4} \Gamma^2, \tag{3.44}$$

which is obviously short-range of order  $O(\langle x \rangle^{-2})$  and cancel the first term in (3.43). It remains then

$$\begin{aligned}
c(x, \xi, \lambda) &= \frac{1}{\lambda^2} P_{\text{in}} \left( -i \frac{\xi a'(x)}{2} \Gamma^2 - i \frac{2\xi a'(x)}{4} \Gamma^2 - \frac{a''(x)}{4} \Gamma^2 - 2k_2 \right) \\
&+ \frac{1}{\lambda^2} P_{\text{out}} \left( i \frac{a'(x) a(x)}{4} - i \partial_x l_1 \right) + O(\langle x \rangle^{-2} \lambda^{-3}),
\end{aligned} \tag{3.45}$$

for  $\xi \in \text{Supp } \hat{\psi}$ . We choose

$$l_1(x, \xi) = \frac{a^2(x)}{8}, \tag{3.46}$$

and

$$k_2(x, \xi) = -i \frac{\xi a'(x)}{2} \Gamma^2 - \frac{a''(x)}{8} \Gamma^2, \tag{3.47}$$

The three correcting terms  $k_1, l_1, k_2$  are short-range of order  $O(\langle x \rangle^{-2})$  for  $\xi$  in a compact set and cancel the two first terms in (3.45). This leads to

$$\forall(x, \xi) \in \mathbb{R}^2, \forall \alpha, \beta \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta c(x, \xi, \lambda)| \leq C_{\alpha\beta} \langle x \rangle^{-2-\alpha} \lambda^{-3}. \tag{3.48}$$

Eventually the new amplitude  $P(x, \xi, \lambda)$  of  $J^\pm(\lambda)$  is defined as

$$\begin{aligned}
P(x, \xi, \lambda) &= \left( p(x, \xi + \lambda) + \frac{1}{\lambda^2} \left( i \frac{a'(x)}{4} P_{\text{in}} \Gamma^2 + p(x, \xi + \lambda) \frac{a^2(x)}{8} \right) \right. \\
&\quad \left. + \frac{1}{\lambda^3} \left( -i \frac{\xi a'(x)}{2} P_{\text{in}} \Gamma^2 - \frac{a''(x)}{8} P_{\text{in}} \Gamma^2 \right) \right) g(\xi),
\end{aligned}$$

or equivalently

$$P(x, \xi, \lambda) = \left( p(x, \xi + \lambda) + \frac{1}{\lambda^2} \left( i \frac{a'(x)}{4} \Gamma^2 P_{\text{out}} + p(x, \xi + \lambda) \frac{a^2(x)}{8} \right) \right)$$

$$+ \frac{1}{\lambda^3} \left( -i \frac{\xi a'(x)}{2} \Gamma^2 P_{\text{out}} - \frac{a''(x)}{8} \Gamma^2 P_{\text{out}} \right) g(\xi). \quad (3.49)$$

Mimicking the proof of Proposition 3.1, we have

**Lemma 3.2** *For any  $\psi \in \mathcal{H}_{\text{out}}^0$  such that  $\hat{\psi} \in C_0^\infty(\mathbb{R})$  and for  $\lambda$  large, we have*

$$W^\pm(\lambda)\psi = \lim_{t \rightarrow \pm\infty} e^{itH(\lambda)} J^\pm(\lambda) e^{-it(D_x + \lambda)} \psi. \quad (3.50)$$

Now, we can state the main result of this section:

**Lemma 3.3** *When  $\lambda$  tends to infinity, the following estimate holds:*

$$\|(W^\pm(\lambda) - J^\pm(\lambda))\psi\| = O(\lambda^{-3}).$$

**Proof:** Let us consider the case (+). Thanks to (3.10), we have to estimate  $\|D^+(t, \lambda)\psi\|$  where  $D^+(t, \lambda)$  is the FIO with phase

$$\varphi^+(x, \xi, \lambda, t) = x\xi + \frac{1}{2(\xi + \lambda)} \int_{x+t}^{+\infty} a^2(s) ds$$

(which is uniformly bounded with respect to  $t$ ) and with amplitude  $c^+(x+t, \xi, \lambda)$ .

Let  $\chi \in C_0^\infty(\mathbb{R})$  be a cut-off function defined by  $\chi(x) = 1$  if  $|x| \leq \frac{1}{2}$ ,  $\chi(x) = 0$  if  $|x| \geq 1$  and let  $\zeta = 1 - \chi$ . We have

$$\|D^+(t, \lambda)\psi\| \leq \|\chi(\frac{2x}{t})D^+(t, \lambda)\psi\| + \|\zeta(\frac{2x}{t})D^+(t, \lambda)\psi\| := (1) + (2). \quad (3.51)$$

First, we estimate the contribution (1). On  $\text{Supp } \chi(\frac{2x}{t})$ ,  $|x+t| \geq \frac{t}{2}$ . Using (3.48) and the continuity of FIOs, we have

$$\|\chi(\frac{2x}{t})D^+(t, \lambda)\| = O(\langle t \rangle^{-2} \lambda^{-3}). \quad (3.52)$$

Now, let us estimate the contribution (2). It is clear that

$$(2) \leq \|\zeta(\frac{2x}{t})D^+(t, \lambda)\zeta(\frac{8x}{t})\psi\| + \|\zeta(\frac{2x}{t})D^+(t, \lambda)\chi(\frac{8x}{t})\psi\| := (a) + (b). \quad (3.53)$$

On  $\text{Supp } \zeta(\frac{8x}{t})$ ,  $|x| \geq \frac{t}{16}$ . Since  $\psi \in \mathcal{S}(\mathbb{R})$ ,  $\|\zeta(\frac{8x}{t})\psi\| = O(\langle t \rangle^{-N})$  for all  $N \geq 0$ . Thus, using the continuity of FIOs again, we obtain  $(a) = O(\langle t \rangle^{-N} \lambda^{-3})$ .

Now, we evaluate the contribution (b) by using a standard non-stationary phase argument. We have

$$\zeta(\frac{2x}{t})D^+(t, \lambda)\chi(\frac{8x}{t})\psi(x) = (2\pi)^{-2} \zeta(\frac{2x}{t}) \int_{\mathbb{R}^2} e^{i\Psi^+(t, x, y, \xi, \lambda)} c(x+t, \xi, \lambda) \chi(\frac{8y}{t}) \psi(y) d\xi dy, \quad (3.54)$$

where

$$\Psi^+(t, x, y, \xi, \lambda) = (x-y)\xi + \frac{1}{2(\xi + \lambda)} \int_{x+t}^{+\infty} a^2(s) ds. \quad (3.55)$$

In order to investigate possible critical point, we calculate

$$\partial_\xi \Psi^+(t, x, y, \xi, \lambda) = x-y - \frac{1}{2(\xi + \lambda)^2} \int_{x+t}^{+\infty} a^2(s) ds. \quad (3.56)$$

On  $\text{Supp } \zeta(\frac{2x}{t})$ ,  $|x| \geq \frac{t}{4}$  and on  $\text{Supp } \chi(\frac{8y}{t})$ ,  $|y| \leq \frac{t}{8}$ . Moreover the integral which appears on the (RHS) of (3.56) is uniformly bounded with respect to  $\lambda, x$  and  $t$ . Thus,  $|\partial_\xi \Psi^+(t, x, y, \xi, \lambda)| \geq C(1+t+|x|)$ . We conclude by a standard argument of non-stationary phase that for all  $N \geq 0$ ,  $(b) = O(\langle t \rangle^{-N} \lambda^{-3})$ . Integrating over  $(0, +\infty)$ , we obtain Lemma 3.3.

◇

### 3.3 High energy asymptotics of the scattering operator

In this section we use the previous construction of the modifiers  $J^\pm(\lambda)$  to prove the reconstruction formula (3.4). Recall that we want to find the asymptotics as  $\lambda \rightarrow \infty$  of the function (see (3.6))

$$F(\lambda) = \langle W^-(\lambda)\psi_1, W^+(\lambda)\psi_2 \rangle,$$

where  $\hat{\psi}_1, \hat{\psi}_2$  have compact support. Using Lemma 3.3, we get

$$F(\lambda) = \langle J^-(\lambda)\psi_1, J^+(\lambda)\psi_2 \rangle + O(\lambda^{-3}).$$

Let us first give the asymptotic expansion of  $J^\pm(\lambda)$  at high energy. By the constructions above, we have for  $\lambda$  large

$$\begin{aligned} J^\pm(\lambda)\psi &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\varphi^\pm(x,\xi,\lambda)} \left( p(x, \xi + \lambda) + \frac{1}{\lambda^2} \left( i \frac{a'(x)}{4} \Gamma^2 + p(x, \xi + \lambda) \frac{a^2(x)}{8} \right) \right. \\ &\quad \left. + \frac{1}{\lambda^3} \left( -i \frac{\xi a'(x)}{2} \Gamma^2 - \frac{a''(x)}{8} \Gamma^2 \right) \right) \hat{\psi}(\xi) d\xi. \end{aligned} \quad (3.57)$$

Thus  $J^\pm(\lambda)$  can be seen as pseudodifferential operators whose symbols  $j^\pm$  are given by

$$\begin{aligned} j^\pm(x, \xi, \lambda) &= e^{\frac{i}{2(\xi+\lambda)} \int_0^{\pm\infty} a^2(x+s) ds} \left( p(x, \xi + \lambda) + \frac{1}{\lambda^2} \left( i \frac{a'(x)}{4} \Gamma^2 + p(x, \xi + \lambda) \frac{a^2(x)}{8} \right) \right. \\ &\quad \left. + \frac{1}{\lambda^3} \left( -i \frac{\xi a'(x)}{2} \Gamma^2 - \frac{a''(x)}{8} \Gamma^2 \right) \right). \end{aligned}$$

Hence, using Taylor expansion of  $e^t$  at  $t = 0$  and using the expansion

$$p(x, \xi + \lambda) = \left( 1 + \frac{a(x)}{2\lambda} \Gamma^2 - \frac{\xi a(x)}{2\lambda^2} \Gamma^2 \right) P_{\text{out}} + O(\lambda^{-3}),$$

we get

$$\begin{aligned} j^\pm(x, \xi, \lambda) &= \left( 1 + \frac{i}{2\lambda} I^\pm - \frac{i\xi}{2\lambda^2} I^\pm - \frac{1}{8\lambda^2} (I^\pm)^2 + O(\lambda^{-3}) \right) \\ &\quad \left( 1 + \frac{a(x)}{2\lambda} \Gamma^2 - \frac{\xi a(x)}{2\lambda^2} \Gamma^2 + i \frac{a'(x)}{4\lambda^2} \Gamma^2 + \frac{a^2(x)}{8\lambda^2} + O(\lambda^{-3}) \right), \end{aligned}$$

where we denoted  $I^\pm(x) = \int_0^{\pm\infty} a^2(x+s) ds$ . If we put together the terms with same order, we obtain

$$\begin{aligned} j^\pm(x, \xi, \lambda) &= 1 + \left( \frac{iI^\pm}{2\lambda} + \frac{a(x)}{2\lambda} \Gamma^2 \right) + \left( -\frac{i\xi I^\pm}{2\lambda^2} - \frac{(I^\pm)^2}{8\lambda^2} + \frac{a^2(x)}{8\lambda^2} \right) \\ &\quad + \left( \frac{ia(x)I^\pm}{4\lambda^2} - \frac{\xi a(x)}{2\lambda^2} + i \frac{a'(x)}{4\lambda^2} \right) \Gamma^2 + O(\lambda^{-3}). \end{aligned}$$

We denote  $R^\pm$  the differential operators

$$R^\pm = \left( -\frac{iI^\pm D_x}{2} - \frac{(I^\pm)^2}{8} + \frac{a^2(x)}{2} \right) + \left( \frac{ia(x)I^\pm}{4} - \frac{a(x)D_x}{2} + i \frac{a'(x)}{4} \right) \Gamma^2, \quad (3.58)$$

as a shorthand. Then the high energy asymptotics of  $J^\pm(\lambda)$  are

$$J^\pm(\lambda)\psi = \psi + \frac{1}{\lambda} \left( \frac{iI^\pm}{2} + \frac{a(x)}{2} \Gamma^2 \right) \psi + \frac{1}{\lambda^2} R^\pm \psi + O(\lambda^{-3}). \quad (3.59)$$

We now can prove Theorem 3.2. We have

$$\begin{aligned}
F(\lambda) &= \langle J^-(\lambda)\psi_1, J^+(\lambda)\psi_2 \rangle + O(\lambda^{-3}) \\
&= \left\langle \left(1 + \frac{iI^-}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2 + \frac{1}{\lambda^2}R^-\right)\psi_1, \left(1 + \frac{iI^+}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2 + \frac{1}{\lambda^2}R^+\right)\psi_2 \right\rangle \\
&= \langle \psi_1, \psi_2 \rangle + \left\langle \left(\frac{iI^-}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2\right)\psi_1, \psi_2 \right\rangle + \left\langle \psi_1, \left(\frac{iI^+}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2\right)\psi_2 \right\rangle \\
&\quad + \frac{1}{\lambda^2} \left( \langle R^-\psi_1, \psi_2 \rangle + \langle \psi_1, R^+\psi_2 \rangle \right) + \left\langle \left(\frac{iI^-}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2\right)\psi_1, \left(\frac{iI^+}{2\lambda} + \frac{a(x)}{2\lambda}\Gamma^2\right)\psi_2 \right\rangle, \\
&= \langle \psi_1, \psi_2 \rangle + \frac{M}{\lambda} + \frac{N}{\lambda^2}.
\end{aligned}$$

Let's compute the first order term  $M$ . Since  $\Gamma^2\psi_1 \in \mathcal{H}_{\text{in}}^0$ , the following terms vanish

$$\langle \Gamma^2\psi_1, \psi_2 \rangle = \langle \psi_1, \Gamma^2\psi_2 \rangle = 0. \quad (3.60)$$

In consequence

$$M = \langle \psi_1, \frac{i}{2}(I^+ - I^-)\psi_2 \rangle = \frac{i}{2} \int_{\mathbb{R}} a^2(s) ds \langle \psi_1, \psi_2 \rangle.$$

But a direct calculation using (2.2) and (2.4) shows that  $\int_{\mathbb{R}} a^2(s) ds = \frac{(l + \frac{1}{2})^2}{r_+}$ . Whence

$$M = \frac{i(l + \frac{1}{2})^2}{2r_+} \langle \psi_1, \psi_2 \rangle. \quad (3.61)$$

We compute now the second order term  $N$ . Notice first that by (3.58) and (3.60), we get

$$\langle R^\pm\psi_1, \psi_2 \rangle = \left\langle \left( -\frac{iI^\pm D_x}{2} - \frac{(I^\pm)^2}{8} - \frac{a^2(x)}{8} \right) \psi_1, \psi_2 \right\rangle,$$

and

$$\left\langle \left( \frac{iI^-}{2} + \frac{a(x)}{2}\Gamma^2 \right) \psi_1, \left( \frac{iI^+}{2} + \frac{a(x)}{2}\Gamma^2 \right) \psi_2 \right\rangle = \langle \psi_1, \frac{1}{4}I^-I^+\psi_2 \rangle + \langle \psi_1, \frac{a^2(x)}{4}\psi_2 \rangle.$$

Hence we obtain

$$N = -\langle \psi_1, \frac{i}{2}(I^+ - I^-)D_x\psi_2 \rangle + \frac{1}{8} \langle \psi_1, (I^+ - I^-)^2\psi_2 \rangle + \langle \psi_1, \frac{a^2(x)}{2}\psi_2 \rangle.$$

Using  $\int_{\mathbb{R}} a^2(s) ds = \frac{(l + \frac{1}{2})^2}{r_+}$  again,

$$N = -\frac{i(l + \frac{1}{2})^2}{2r_+} \langle \psi_1, D_x\psi_2 \rangle + \langle \psi_1, \frac{a^2(x)}{2}\psi_2 \rangle + \frac{(l + \frac{1}{2})^4}{8r_+^2} \langle \psi_1, \psi_2 \rangle. \quad (3.62)$$

The two leading terms (3.61) and (3.62) give exactly the reconstruction formula (3.4). Hence our main result is proved.

◇

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