

A constructive procedure to recover asymptotics of short-range or long-range potentials.

François Nicoleau

Département de Mathématiques
Laboratoire Jean Leray
U.M.R 6629 - Université de Nantes
2, rue de la Houssinière BP 92208
F-44322 Nantes cedex 03
e-mail : nicoleau@math.univ-nantes.fr

Abstract

We study an inverse scattering problem for a pair of Hamiltonians (H, H_0) on $L^2(\mathbb{R}^n)$, where $H_0 = -\Delta$ and $H = H_0 + V$, V being a short-range or long-range potential. By an elementary constructive method, we show that the scattering operator S , which is localized near a fixed energy $\lambda > 0$, determines the asymptotics of the potential V at infinity, in dimension $n \geq 3$. This is done by studying the action of the scattering operator on suitable wave packets.

1 Introduction.

In this short note, we study an inverse scattering problem for the pair (H, H_0) on $L^2(\mathbb{R}^n)$, $n \geq 2$, where the free operator is $H_0 = -\Delta$ with domain $D(H_0) = H^2(\mathbb{R}^n)$, and

$$(1.1) \quad H = H_0 + V,$$

with a potential $V \in C^\infty(\mathbb{R}^n)$ satisfying for each $\alpha \in \mathbb{N}^n$,

$$(H_1) \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \quad \rho > 1,$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Under the hypothesis (H_1) , the wave operators :

$$(1.2) \quad W^\pm(H, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0},$$

exist and they are complete, i.e $Ran W^\pm(H, H_0) = \mathcal{H}_{ac}(H) =$ subspace of absolute continuity of H , (see [13] for example).

Let S be the scattering operator defined by :

$$(1.3) \quad S = W^{+*}(H, H_0) W^-(H, H_0).$$

In order to localize the scattering operator near a fixed energy $\lambda > 0$, we introduce a cut-off function $\chi \in C_0^\infty(]0, +\infty[)$, $\chi = 1$ in a neighborhood of $\lambda > 0$.

The goal of this note is to obtain information about the potential from $S\chi(H_0)$. We show that for $n \geq 3$, the operator $S\chi(H_0)$ determines the asymptotics of the potential at infinity, by studying :

$$(1.4) \quad F(h) = \langle S\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} \rangle ,$$

where \langle, \rangle is the usual scalar product in $L^2(\mathbb{R}^n)$, $\Phi_{h,\omega}$, $\Psi_{h,\omega}$ are suitable test functions, (see section 2), and $h \rightarrow 0$.

Our method is close to [12] where a scattering process with an obstacle and electromagnetic potentials at high energy is studied. As in [11]-[12], we use a stationary approach and we obtain the asymptotics of $F(h)$. We emphasize that the complete asymptotic expansion of $F(h)$ is necessary to recover the complete asymptotics of the potential V . Note that in [1], the authors study a scattering problem with potentials at high energies with a nice time-dependent method, but they only obtain the leading term of the high energies asymptotics.

In [4], it is shown using Fadeev's approach, under the hypothesis (H_1) with $\rho > n$, that the scattering amplitude $T(\mu, \theta, \omega)$, $\mu \in [\lambda, \lambda + \delta]$, $\delta > 0$, $\theta, \omega \in S^{n-1}$, determines the Fourier transform of V on $\{ \xi \in \mathbb{R}^n, | \xi | \leq 2\sqrt{\lambda} \}$. In [5], this result is extended to the case $\rho > \frac{3}{2}$.

In ([8], Corollary 1.1), M. S. Joshi and A. Sa Barreto show that the scattering amplitude at a fixed energy $\lambda > 0$ determines the asymptotics of the potential. More precisely, they show that if $V_1, V_2 \in S_{cl}^{-2}(\mathbb{R}^n)$ for $n \geq 3$, and if the associated matrices at some non-zero fixed energy are equal up to smooth terms, then $V_1 - V_2 \in S^{-\infty}(\mathbb{R}^n)$. Their proof uses Hörmander's calculus of Fourier operators and the calculus of Legendrian distributions developed by Melrose and Zworski [10]. In [9], this approach is used to recover asymptotics of metrics from fixed energy scattering data. In [7], a more explicit approach for short-range potentials on \mathbb{R}^n is proposed, based on the resolution of an eikonal equation.

We emphasize that the approach used in [7]-[8] for classical symbols is more general than our method : the authors only assume that the scattering matrices associated to the potentials $V_j \in S_{cl}^{-2}(\mathbb{R}^n)$ are equal at some fixed energy $\lambda > 0$ up to smooth terms.

Nevertheless it seems that our method is very elementary and constructive : the complete asymptotics of the potential appears simply and entirely in Theorem 2. Moreover, it allows us to study generic short-range potentials, (see Theorem 1 and Corollary 6).

Finally, working with modified wave operators, it is not difficult to extend our results to the case of long-range potentials ($\rho > 0$), (see section 4 for details).

2 Asymptotics for the localized scattering operator.

2.1 Definition of the test functions.

First, let us define $U(h)$, $h > 0$, as an unitary operator on $L^2(\mathbb{R}^n)$ by :

$$(2.1) \quad U(h)\Phi(x) = h^{\frac{n}{2}} \Phi(hx).$$

We also need a technical energy cut-off function $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_0(\xi) = 1$ if $|\xi| \leq 1$, $\chi_0(\xi) = 0$ if $|\xi| \geq 2$. As in [12], χ_0 is only introduced to control the spreading of the wave packets $\Phi_{h,\omega}$.

For $\omega \in S^{n-1}$, we write $x \in \mathbb{R}^n$ as $x = y + t\omega$, $y \in \Pi_\omega =$ orthogonal hyperplane to ω and we consider :

$$(2.2) \quad X_\omega = \{x = y + t\omega \in \mathbb{R}^n : |y| \geq 1\}.$$

Definition

For $\Phi \in C_0^\infty(X_\omega)$ and suitable $\epsilon > 0$,

$$(2.3) \quad \Phi_{h,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} U(h) \chi_0(h^\epsilon D) \Phi,$$

where $D = -i\nabla$, ($\Psi_{h,\omega}$ is defined in the same way with $\Psi \in C_0^\infty(X_\omega)$).

2.2 Asymptotic expansion of the scattering operator.

In this section, we prove the following theorems :

Theorem 1

Assume that $\epsilon < 1$. We have, when $h \rightarrow 0$,

$$(2.4) \quad \langle (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} \rangle = \frac{1}{2i\sqrt{\lambda}} \langle \int_{-\infty}^{+\infty} V(h^{-1}x + t\omega) dt \Phi, \Psi \rangle + O(h^\mu),$$

where $\mu = \min(2(\rho-1), \rho)$.

Remarks

a - Using (H_1) , it is easy to see that the first term of the (R.H.S) of (2.4) is equal to $O(h^{\rho-1})$ since Φ, Ψ have their support in X_ω .

b - In section 3, using (2.4), we shall recover the leading term of the asymptotics of V at infinity.

If $V \in S_{cl}^{-2}(\mathbb{R}^n)$, a similar approach allows us to obtain a complete asymptotic expansion of $\langle (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} \rangle$.

Recall that the class $S_{cl}^{-2}(\mathbb{R}^n)$ of classical symbols of order -2 is the set of the smooth functions $V(x)$ such that there exist a sequence of smooth homogeneous functions f_j of order $-j$ such that

$$(2.5) \quad \left| \partial_x^\alpha \left(V(x) - \sum_{j=2}^{N-1} f_j(x) \right) \right| \leq C_{\alpha,N} \langle x \rangle^{-N-|\alpha|}$$

Now, we can formulate :

Theorem 2

Assume that $V \in S_{cl}^{-2}(\mathbb{R}^n)$. We have the following asymptotic expansion when $h \rightarrow 0$:

$$(2.6) \quad \langle (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} \rangle \sim \sum_{j \geq 1} h^j \langle \Phi, A_j(x, \omega, D) \Psi \rangle,$$

where $A_j(x, \omega, D)$ is a differential operator given by :

$$(2.7) \quad A_j(x, \omega, D) = \frac{i}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} f_{j+1}(x+t\omega) dt + B_j(x, \omega, D),$$

with $B_1 = 0$ and for $j \geq 2$, $B_j(x, \omega, D)$ is a differential operator only depending on the functions f_k , $2 \leq k \leq j$.

With this asymptotic expansion, we shall recover V modulo $S^{-\infty}$, (see section 3.2).

2.3 Proof of Theorem 1.

• **Step 1 :**

Let us begin by an elementary lemma.

Lemma 3

For $\epsilon < 1$ and $h \ll 1$, we have :

$$(2.8) \quad \chi(H_0)\Phi_{h,\omega} = \Phi_{h,\omega}.$$

Proof

We easily obtain :

$$(2.9) \quad \mathcal{F} [\chi(H_0)\Phi_{h,\omega}](\xi) = h^{-\frac{n}{2}} \chi(\xi^2) \chi_0(h^{\epsilon-1}(\xi - \sqrt{\lambda}\omega)) \mathcal{F} \Phi(h^{-1}(\xi - \sqrt{\lambda}\omega)),$$

where \mathcal{F} is the usual Fourier transform. Then, on $Supp \chi_0$, we have $|\xi - \sqrt{\lambda}\omega| \leq 2h^{1-\epsilon}$. So, for $\epsilon < 1$ and h small enough, we have $\chi(\xi^2) = 1$. \square

Then, by Lemma 3, we obtain,

$$(2.10) \quad F(h) = \langle W^-(H, H_0)\Phi_{h,\omega}, W^+(H, H_0)\Psi_{h,\omega} \rangle.$$

An easy calculation shows that :

$$(2.11) \quad F(h) = \langle \Omega^-(h, \omega) \chi_0(h^\epsilon D) \Phi, \Omega^+(h, \omega) \chi_0(h^\epsilon D) \Psi \rangle,$$

where

$$(2.12) \quad \Omega^\pm(h, \omega) = s - \lim_{t \rightarrow \pm\infty} e^{itH(h, \omega)} e^{-itH_0(h, \omega)},$$

with

$$(2.13) \quad H_0(h, \omega) = (D + \sqrt{\lambda} h^{-1} \omega)^2,$$

and

$$(2.14) \quad H(h, \omega) = H_0(h, \omega) + h^{-2} V(h^{-1} x).$$

So, by (2.11), we have to find the asymptotics of $\Omega^\pm(h, \omega) \chi_0(h^\epsilon D) \Phi$. We follow the same strategy as in ([11], [12]), and we only treat the case (+).

• **Step 2 :**

As in ([11], [12]), we construct a modifier $J^+(h, \omega)$ in the form :

$$(2.15) \quad J^+(h, \omega) = 1 + d^+(h^{-1} x, \omega),$$

We denote :

$$(2.16) \quad T^+(h, \omega) = H(h, \omega) J^+(h, \omega) - J^+(h, \omega) H_0(h, \omega).$$

A direct calculation shows that :

$$(2.17) \quad \begin{aligned} T^+(h, \omega) &= h^{-2} V(h^{-1} x) - 2i\sqrt{\lambda} h^{-2} \omega \cdot \nabla d^+(h^{-1} x, \omega) \\ &\quad + h^{-2} V(h^{-1} x) d^+(h^{-1} x, \omega) - h^{-2} \Delta d^+(h^{-1} x, \omega) \\ &\quad - 2h^{-1} \nabla d^+(h^{-1} x, \omega) \cdot \nabla. \end{aligned}$$

We solve the transport equation :

$$(2.18) \quad \omega \cdot \nabla d^+(x, \omega) = \frac{1}{2i\sqrt{\lambda}} V(x).$$

The solution of (2.18) is given by :

$$(2.19) \quad d^+(x, \omega) = \frac{i}{2\sqrt{\lambda}} \int_0^{+\infty} V(x + t\omega) dt.$$

We easily obtain :

$$(2.20) \quad |\partial_x^\alpha d^+(x, \omega)| \leq C_\alpha \langle x \rangle^{1-\rho-|\alpha|}, \quad \forall x \in \Gamma^+(\omega),$$

where

$$\Gamma^+(\omega) = \{ x \in \mathbb{R}^n : |x| \geq R, x \cdot \omega \geq -\sigma |x| \}, \quad \sigma \in]-1, 1[.$$

From this, we deduce the following lemma, see ([11], Proposition 9) for details.

Lemma 4

For $\epsilon < 1$ and $h \ll 1$,

$$(2.21) \quad \Omega^+(h, \omega) \chi_0(h^\epsilon D) \Phi = \lim_{t \rightarrow +\infty} e^{itH(h, \omega)} J^+(h, \omega) e^{-itH_0(h, \omega)} \chi_0(h^\epsilon D) \Phi.$$

• Step 3 :

Recall that $\mu = \min(2(\rho - 1), \rho)$. Now, we can formulate :

Lemma 5

Assume $\epsilon < 1$. We have :

$$(2.22) \quad \| (\Omega^+(h, \omega) - J^+(h, \omega)) \chi_0(h^\epsilon D) \Phi \| = O(h^\mu).$$

Proof

First, we write :

$$(2.23) \quad (\Omega^+(h, \omega) - J^+(h, \omega)) \chi_0(h^\epsilon D) \Phi = i \int_0^{+\infty} e^{itH(h, \omega)} T^+(h, \omega) e^{-itH_0(h, \omega)} \chi_0(h^\epsilon D) \Phi dt.$$

We want to construct a cut-off χ^+ in space variables which localizes far from the origin. Let V_ω be a neighborhood of ω in S^{n-1} . We define :

$$(2.24) \quad O^+ = \{ x + t\omega'; x \in \text{Supp } \Phi, t \geq 0, \omega' \in V_\omega \}.$$

If V_ω is rather small, it is clear that $O^+ \subset \mathbb{R}^n \setminus B$ where B is the unit ball.

Let $\chi^+ \in C^\infty(\mathbb{R}^n \setminus B)$ be a cut-off function such that $\chi^+ = 1$ in a conical neighborhood of O^+ .

Now, we have the following estimation which is obtained by using a standard non stationary phase argument :

$$(2.25) \quad \forall \epsilon < 1, \forall N \geq 1, (\chi^+ - 1) e^{-itH_0(h, \omega)} \chi_0(h^\epsilon D) \Phi = O(\langle t \rangle^{-N} h^N),$$

in the sense of the L^2 -norm. For more details, see ([12], lemma 10).

So, we deduce using (2.17) and (2.25) :

$$(2.26) \quad \| (\Omega^+(h, \omega) - J^+(h, \omega)) \chi_0(h^\epsilon D) \Phi \| \leq \int_0^{+\infty} \| \chi^+ T^+(h, \omega) e^{-itH_0(h, \omega)} \chi_0(h^\epsilon D) \Phi \| dt,$$

modulo $O(h^\infty)$. Since $\text{Supp } \chi^+ \subset \mathbb{R}^n \setminus B$, using (2.17), (2.20) and (2.26), as in ([12], Lemma 11, step 4), we obtain :

$$(2.27) \quad \| (\Omega^+(h, \omega) - J^+(h, \omega)) \chi_0(h^\epsilon D) \Phi \| = O(h^{2(\rho-1)}) + O(h^\rho) = O(h^\mu).$$

• **Step 4 :**

Using the following estimation on $L^2(\mathbb{R}^n) : \forall N \geq 1$,

$$(2.28) \quad \| (\chi_0(h^\epsilon D) - 1)\Phi \| = O(h^N),$$

and Lemma 5, we obtain :

$$(2.29) \quad \Omega^+(h, \omega)\chi_0(h^\epsilon D)\Phi = \Phi + \frac{i}{2\sqrt{\lambda}} d^+(h^{-1}x, \omega) \Phi + O(h^\mu).$$

This concludes the proof of Theorem 1. \square

2.4 Proof of Theorem 2.

We only sketch the proof. In this case, since V is a classical symbol of order -2 , we construct the modifier $J^+(h, \omega)$, (see section 2.3) as a pseudodifferential operator, (actually a differential operator), having the asymptotic expansion :

$$(2.30) \quad J^+(h, \omega) = op \left(\sum_{k \geq 0} h^k d_k^+(x, \xi, \omega) \right) , \quad d_0^+ = 1.$$

Roughly speaking, we construct d_k^+ in order to obtain $T^+(h, \omega) = O(h^\infty)$. An easy calculation shows us we need to solve the transport equations :

$$(2.31) \quad -2i\sqrt{\lambda}\omega \cdot \nabla d_1^+ + f_2 = 0,$$

and for $m \geq 1$,

$$(2.32) \quad -2i\sqrt{\lambda}\omega \cdot \nabla d_{m+1}^+ - 2i\xi \cdot \nabla d_m^+ - \Delta d_m^+ + \sum_{k=0}^m f_{m-k+2} d_k^+ = 0.$$

Then Theorem 2 follows as in section 2.2, (see also [11], [12] for details). \square

3 Recovering asymptotics of the potential.

3.1 A first application.

In this section, we use (2.4) in the particular case where V is an asymptotic homogeneous function.

Let V_j , $j = 1, 2$ be two potentials satisfying (H_1) and when $|x| \rightarrow +\infty$,

$$(3.1) \quad V_j(x) = f_j(x) + o(|x|^{-\rho}), \quad \rho > 1,$$

where f_j is a homogeneous function of order $-\rho$.

We denote S_j the scattering operator associated with the pair $(H_0 + V_j, H_0)$. We have the following result :

Corollary 6

Assume that the dimension $n \geq 3$ and $S_1\chi(H_0) = S_2\chi(H_0)$.

Then : $f_1 = f_2$.

Proof

By Theorem 1, we have for $\epsilon < 1$ and $h \rightarrow 0$,

$$(3.2) \quad \langle (S_j - 1) \chi(H_0) \Phi_{h,\omega}, \Psi_{h,\omega} \rangle = \frac{h^{\rho-1}}{2i\sqrt{\lambda}} \langle \int_{-\infty}^{+\infty} f_j(x + t\omega) dt \Phi, \Psi \rangle + o(h^{\rho-1}).$$

So, if $S_1\chi(H_0) = S_2\chi(H_0)$, we deduce :

$$(3.3) \quad \forall \omega \in S^{n-1}, \forall x \in X_\omega, \int_{-\infty}^{+\infty} V_0(x + t\omega) dt = 0,$$

where $V_0 = f_1 - f_2$. So, using the uniqueness theorem for the Radon transform [3] in dimension $n \geq 3$, we obtain $V_0(x) = 0, |x| \geq 1$. \square .

Comments

For $n = 2$, since f_j is a homogeneous function, it is not difficult to show using [5] that Corollary 6 is still true if $\rho > \frac{3}{2}$.

In [14], P. Sabatier constructed examples of smooth potentials decreasing at infinity as $|x|^{-\frac{3}{2}}$ with zero amplitude at a fixed energy. These examples do not contradict our results for two reasons. First, in [14], the potentials do not satisfy (3.1), (the leading term of the potential is oscillating). Secondly, in Corollary 6, the energy is not really fixed but belongs to a small neighborhood of $\lambda > 0$.

3.2 A constructive procedure to recover complete asymptotics.

As in section 3.1, if V is a classical symbol of order -2 and if $n \geq 3$, using the uniqueness theorem for the Radon transform [3] and Theorem 2, we reconstruct all the f_j 's for $j \geq 2$.

Thus, if we denote $S^{-\infty}(\mathbb{R}^n)$ the set of functions rapidly decreasing at infinity, we recover a result close to ([9], Corollary 1.1 - Corollary 1.2), (actually, our result is weaker).

Corollary 7

Assume that $V_1, V_2 \in S_{cl}^{-2}(\mathbb{R}^n)$, $n \geq 3$ and $S_1\chi(H_0) = S_2\chi(H_0)$.

Then : $V_1 - V_2 \in S^{-\infty}(\mathbb{R}^n)$.

In particular, if V_1, V_2 are rational potentials of order less than or equal to -2 , then $V_1 = V_2$.

Comments

Let us conclude this note by some remarks in the two dimensional case. As for Corollary 6, using the results of [5], Corollary 7 is true for $n = 2$.

Nevertheless, as it was pointed by P. Grinevich [2], for $n = 2$, we can construct examples of smooth rational potentials decaying at infinity as $|x|^{-2}$, with zero amplitude at a fixed energy.

4 The long-range case.

In this section, we generalize to the long-range case the results obtained in the preceding section for short-range interactions.

We consider generic long-range potentials V , (i.e V satisfies (H_1) with $\rho > 0$). Obviously, under such hypotheses, in general the usual wave operators do not exist. So, we have to define modified wave operators. For $\rho > \frac{1}{2}$, we can use the well-known Dollard wave operators. Since we consider general long-range interactions, we have to define other modified wave operators.

These modified wave operators, close to Isozaki-Kitada's wave operators [6], have been constructed in [10]. Roughly speaking, we define :

$$(4.1) \quad W^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} J^\pm e^{-itH_0} ,$$

where J^\pm is a Fourier integral operator with phase $\varphi^\pm(x, \xi)$ satisfying an approximate eikonal equation :

$$(4.2) \quad |\partial_x \varphi^\pm(x, \xi)|^2 + V(x) \approx \xi^2 ,$$

and with amplitude 1, (see [11] for details). When $\rho > \frac{1}{2}$, our modified wave operators are equal to the Dollard's one up to an energy phase ([11], Corollary 13), and in suitable incoming or outgoing zones,

$$(4.3) \quad \varphi^\pm(x, \xi) = x \cdot \xi + \int_0^{\pm\infty} (V(x + t\xi) - V(t\xi)) dt .$$

The modified scattering operator is defined as in (1.3) by : $S = W^{+*}W^-$.

Using the same approach as in Theorem 1, replacing $J^+(h, \omega)$ in (2.30) with a Fourier integral operator as in [11]-[12], we easily obtain :

Theorem 8

We have, when $h \rightarrow 0$,

$$(4.4) \quad \langle [S, D]\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} \rangle = \frac{1}{2\sqrt{\lambda}} \langle \int_{-\infty}^{+\infty} \nabla V(h^{-1}x + t\omega) dt \Phi, \Psi \rangle + o(h^\rho),$$

As in Corollary 6, in the particular case where V is an asymptotic homogeneous function, we can recover the leading term of the potential at infinity :

Let V_j , $j = 1, 2$ be two potentials satisfying (H_1) and (3.1) with $\rho > 0$. We denote S_j the modified scattering operator associated with the pair $(H_0 + V_j, H_0)$. We have the following result :

Corollary 9

Assume that the dimension $n \geq 3$ and $S_1\chi(H_0) = S_2\chi(H_0)$.

Then : $f_1 = f_2$.

Moreover, using the same approach as in Theorem 2 - Corollary 7, we have :

Corollary 10

Assume that $V_1, V_2 \in S_{cl}^{-1}(\mathbb{R}^n)$, $n \geq 3$ and $S_1\chi(H_0) = S_2\chi(H_0)$.

Then : $V_1 - V_2 \in S^{-\infty}(\mathbb{R}^n)$.

In particular, if V_1, V_2 are rational potentials of order less than or equal to -1 , then $V_1 = V_2$.

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