An inverse scattering problem for short-range systems in a time periodic electric field.

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1 Introduction.

In this talk, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t, x)$ (with the same period T).

For the sake of simplicity, we assume that the period $T = 1$.

The Hamiltonian is given on $L^2(\mathbb{R}^n)$ by :

$$H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x),$$

where $p = -i\partial_x$.

Physical interests.

- $E(t) = 0$: $H(t)$ describes an hydrogen atom placed in a linearly polarized monochromatic electric field.

- $E(t) = E_0$ with $E_0 \in \mathbb{R}^n$: Stark effect.

- $E(t) = \cos(2\pi t)\ E$ with $E \in \mathbb{R}^n$: AC-Stark effect in the E-direction.

Hypotheses

(A_1) $\quad t \to E(t) \in L^1_{loc}(I\!R; I\!R^n)$, $E(t+1) = E(t)$ $a.e$.

$V \in C^\infty(I\!R \times I\!R^n)$ is time-periodic with period 1, and satisfies the following estimations with $\delta > 0$:

(A_2) $\quad \forall \alpha,\ \forall k,\ |\,\partial_t^k \partial_x^\alpha V(t,x)\,| \leq\ C_{k,\alpha}\ <x>^{-\delta-|\alpha|}$,

where $<x>=\ (1+x^2)^{\frac{1}{2}}$.

Actually, we need (A_2) for only $k,\ \alpha$ with finite order .

2 The time evolution.

2.1 Definition.

We call *propagator* associated to H a family of unitary operators on $L^2(I\!R^n)$, $U(t,s),\ t,s \in I\!R$ such that :

1 - $U(t,s)$ is a strongly continuous w. r. to t, s.

2 - $U(t,s)\ U(s,r) = \ U(t,r)$ for all $t,s,r \in I\!R$.

3 - $U(t,s)\,(\mathcal{S}(I\!R^n)) \subset \mathcal{S}(I\!R^n)$ for all $t,s \in I\!R$.

4 - If $\Phi \in \mathcal{S}(I\!R^n)$, $U(t,s)\Phi$ is C^1 w. r. to t, s and :

$$i\ \frac{\partial}{\partial t} U(t,s)\ \Phi = \ H(t)\ U(t,s)\ \Phi\ ,$$

$$i\ \frac{\partial}{\partial s} U(t,s)\ \Phi = \ -U(t,s)\ H(s)\ \Phi\ .$$

Remark.

If $H(t) = H$ is time-independent, $U(t,s) = e^{-i(t-s)H}$.

2.2 Existence of the propagators.

The free Hamiltonian $H_0(t)$ is :

$$H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x \ .$$

Solving the Hamilton equations, we obtain that the classical trajectories associated to $H_0(t)$ are given by :

$$\phi_0^t(x,\xi) = (x + t\xi + \int_0^t \int_0^s E(u)\, du\, ds \ , \ \xi + \int_0^t E(u)\, du).$$

A particular case : the Stark Hamiltonian.

If we consider the Hamiltonian B_0 on $L^2(\mathbb{R}^n)$,

$$B_0 = \frac{1}{2}p^2 - E_0 \cdot x \ ,$$

we have the Avron-Herbst formula :

$$e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3} \ e^{itE_0\cdot x} \ e^{-i\frac{t^2}{2}E_0\cdot p} \ e^{-it\frac{p^2}{2}} \ .$$

Thus,

$$e^{-i(t-s)B_0} \ = \ T_0(t) \ e^{-i(t-s)\frac{p^2}{2}} \ T_0(s)^*$$

where

$$T_0(t) = e^{-i\frac{E_0^2}{6}t^3} \ e^{itE_0\cdot x} \ e^{-i\frac{t^2}{2}E_0\cdot p}.$$

The general case (I).

We make the following ansatz :

$$U_0(t,s) = \; T(t)\; e^{-i(t-s)\frac{p^2}{2}}\; T^*(s)\ .$$

with

$$T(t) = \; e^{-ia(t)}\; e^{-ib(t)\cdot x}\; e^{-ic(t)\cdot p}\ .$$

Since on the Schwartz space, $U_0(t,0)$ must satisfy :

$$i\ \frac{\partial}{\partial t}\ U_0(t,0) = H_0(t)\ U_0(t,0)$$

the functions $a(t),\ b(t), c(t)$ solve :

$$\dot{b}(t) = -E(t),\ \dot{c}(t) = -b(t),\ \dot{a}(t) = \frac{1}{2}\ b^2(t).$$

Then, we obtain :

$$b(t) = -\int_0^t E(s)\, ds$$

$$c(t) = -\int_0^t b(s)\, ds$$

$$a(t) = \frac{1}{2} \int_0^t b^2(s)\, ds \ .$$

In other words, conjugating with the unitary operator $T(t)$:

$$U_0(t,s) \quad \leftrightarrow \quad e^{-i(t-s)\frac{p^2}{2}},$$

and in the same way,

$$U(t,s) \quad \leftrightarrow \quad R(t,s),$$

where $R(t,s)$ is the propagator associated with $B(t) = \frac{1}{2}p^2 + V(t, x + c(t))$.

Note that the propagator $R(t,s)$ is well-defined since $V_1(t,x) = V(t, x + c(t))$ satisfies Yajima's hypotheses :

$$(i) \qquad \forall t \in I\!R,\ V_1(t,x) \text{ is } p^2 - \text{Kato small.}$$

$$(ii) \qquad t \longrightarrow V_1(t,x)(p^2 + i)^{-1} \text{ is differentiable.}$$

Nevertheless, this approach is not convenient from a scattering point of view. It is not clear that the new potential $V_1(t,x)$ is a short-range perturbation of $\frac{1}{2}\ p^2$ since $c(t) = O(t^2)$

The general case (II).

The basic idea is to rely $U_0(t,s)$ with $e^{-i(t-s)B_0}$

$$B_0 = \frac{1}{2}p^2 - E_0 \cdot x \ ,$$

for a suitable $E_0 \in I\!R^n$.

As in the previous approach, we make the following ansatz :

$$U_0(t,s) = T(t)\ e^{-i(t-s)B_0}\ T^*(s)\ .$$

with

$$T(t) = e^{-ia(t)}\ e^{-ib(t)\cdot x}\ e^{-ic(t)\cdot p}\ .$$

Now, the functions $a(t),\ b(t), c(t)$ solve :

$$\dot{b}(t) = -E(t) + E_0,\ \dot{c}(t) = -b(t),\ \dot{a}(t) = \frac{1}{2}\, b^2(t) - E_0 \cdot c(t).$$

We choose $E_0 = \int_0^1 \; E(t)\; dt$ the mean value of $E(t)$ and we define :

$$b(t) = \; - \int_0^t \; (E(s) - E_0)\; ds - \int_0^1 \int_0^t \; (E(s) - E_0)\; ds\; dt\; .$$

$$c(t) = \; - \int_0^t \; b(s)\; ds\; .$$

$$a(t) = \; \int_0^t \; \left(\frac{1}{2} \; b^2(s) - E_0 \cdot c(s) \right) \; ds\; .$$

We emphasize that $c(t)$ is periodic, so $c(t) = O(1)$.

In the same way, $U(t,s)$ (which exists, see the previous section) is given by :

$$(1) \qquad U(t,s) \ = \ T(t)\ R(t,s)\ T^*(s),$$

where $R(t,s)$ is the propagator associated with $B(t) = B_0 + V_1(t,x)$ with $V_1(t,x) = V(t, x + c(t))$.

Remarks :

To be precise, note that $R(t,s)$ is defined by (1). Moreover, we emphasize that the potential $V_1(t,x)$ has the following decay :

$$|\ V_1(t,x)\ | \leq \ C \ < x >^{-\delta} .$$

3 Quantum scattering.

For short-range potentials, the wave operators are defined for $s \in I\!R$ and $\Phi \in L^2(I\!R^n)$ by :

$$W^{\pm}(s)\ \Phi = \lim_{t \to \pm\infty} U(s,t)\ U_0(t,s)\ \Phi.$$

As we can guess, the short-range condition depends on the value of the mean of the external electric field :

$$E_0 \ = \ \int_0^1 \ E(t)\ dt\ .$$

3.1 The case $E_0 = 0$.

This case falls under the category of two-body systems with time-periodic potentials and it was studied by Kitada and Yajima (1982).

Theorem 1.

Assume that hypotheses $(A_1),\ (A_2)$ *are satisfied with* $\delta > 1$ *and with* $E_0 = 0$.

Then : (i) the wave operators $W^{\pm}(s)$ *exist.*

(ii) $W^{\pm}(s+1) = W^{\pm}(s)$

(iii) $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.

(iv) Ran $(W^{\pm}(s)) = \mathcal{H}_{ac}\,(U(s+1,s))$.

(v) $\mathcal{H}_{sc}\,(U(s+1,s)) = \emptyset$.

(vi) $\sigma_p(U(s+1,s))$ *is discrete outside* $\{1\}$.

Remarks.

1 - The unitary operators $U(s+1,s)$ are called the Floquet operators and they are mutually equivalent.

2 - If $H(t) = \frac{1}{2}p^2 + V(x)$, $U(s+1,s) = e^{-iH}$, so the assertions (v) and (vi) are well-known in this case.

3 - For $n =$ 3 and $\delta > 2$, $\mathcal{H}_p\left(U(s+1,s)\right)$ is finite dimensional : Galtbayar, Jensen and Yajima, (2004).

4 - For general $\delta > 0$, $W^{\pm}(s)$ do not exist and we have to define modified wave operators by solving an Hamilton-Jacobi equation.

3.2 The case $E_0 \neq 0$.

Using the unitary operators $T(t)$, we have to examine Hamiltonians with a constant external electric field E_0, (Stark Hamiltonians), but with a periodic potential $V_1(t, x)$.

The spectral and the scattering theory for Stark Hamiltonians with a time-independent potentiel V are well-known . In particular, if V satisfying (A_2), H has no eigenvalues.

The following theorem, due to Moller (2000), is a time-periodic version of these results.

Theorem 2.
Assume that hypotheses (A_1), (A_2) *are satisfied with* $\delta > \frac{1}{2}$ *and with* $E_0 \neq 0$.

Then : *(i)* $\mathcal{H}_{ac}(U(s+1,s)) = L^2(I\!R^n)$.

(ii) $W^{\pm}(s)$ *exist and are unitary.*

(iii) $W^{\pm}(s+1) = W^{\pm}(s)$.

(iv) $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.

3.3 The scattering operators.

For $s \in I\!R$, let $S(s) = W^{+*}(s)\ W^{-}(s)$ be the scattering operators. By Theorems 1 and 2, $S(s+1) = S(s)$.

4 Inverse scattering.

The inverse scattering problem consists to reconstruct $V(s,x)$ from $S(s)$, $s \in [0,1]$. We prove :

Theorem 3.
Assume that $E(t)$ satisfies (A_1) and let $V_j,\ j = 1,2$ be potentials satisfying (A_2). We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{1}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$).

Let $S_j(s)$ be the corresponding scattering operators.

Then :

$$\forall s \in [0,1],\ S_1(s) = S_2(s) \implies V_1 = V_2\ .$$

Remark.

If we know the free propagator $U_0(t,s)$, $s,t \in [0,1]$, by virtue of the following relation :

$$S(t) = U_0(t,s)\ S(s)\ U_0(s,t)\ ,$$

the potential $V(t,x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

Eh, what's up, Doc ?

1 - Weder (1997) : long-range time-dependent (not necessarily periodic) potentials but with $E(t) = 0$, (Enss-Weder's approach).

2 - Yajima (2004) : time-periodic potential with $\delta > \frac{n}{2} + 1$ and $E(t) = 0$ by studying the scattering matrices in a high energy regime.

3 - Weder (2004) : time-periodic potential that decays exponentially at infinity, with $E(t) = 0$, at a fixed quasi-energy.

Strategy of the proof.

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder's approach), but with time-dependent potentials.

Notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where Π_{E_0} is the orthogonal hyperplane to E_0.
- $\Phi_{\lambda,\omega} = \ e^{i\sqrt{\lambda} x.\omega}\Phi, \ \Psi_{\lambda,\omega} = \ e^{i\sqrt{\lambda} x.\omega}\Psi.$

Proposition 4.

$$\lim_{\lambda \to +\infty} \sqrt{\lambda} \ < [S(s), p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > =$$

$$< \left(\int_{-\infty}^{+\infty} \ \partial_x V(s, x + t\omega) \ dt \right) \ \Phi \ , \ \Psi > .$$

Comments.

We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform on the orthogonal hyperplane to E_0.

Actually, for the case $n = 2$, $E_0 \neq 0$ and $\delta > \frac{3}{4}$, Proposition 4 is also valid for $\omega \in S^{n-1}$ satisfying $|\, \omega \cdot E_0 \,| < |\, E_0 \,|$.

Then, Theorem 3 follows from Proposition 4 and the inversion of the Radon transform.

Sketch of proof of Proposition 4.

Let us prove Proposition 4 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar.

We consider the Hamiltonians :

$$B_0 = \frac{1}{2}\, p^2 - E_0 \cdot x\ ,$$

$$B(t) = B_0 + V_1(t,x)\ ,\ V_1(t,x) = V(t, x + c(t)).$$

$V_1(t,x)$ is a short-range perturbation of B_0 since $c(t) = O(1)$, ($c(t)$ is a periodic function).

Then, we can define the usual wave operators for the pair $(B(t), B_0)$:

$$\Omega^{\pm}(s) = \ \mathrm{s} - \lim_{t \to \pm\infty}\ R(s,t)\ e^{-i(t-s)B_0}\ .$$

where $R(t,s)$ is the propagator associated with $B(t)$.

Let $S_1(s) = \Omega^{+*}(s)\ \Omega^{-}(s)$ be the scattering operators.

By construction,

$$S(s) = T(s)\ S_1(s)\ T^*(s)\ .$$

Using the fact that $e^{-ib(s)\cdot x}\ p\ e^{ib(s)\cdot x} = p + b(s)$, we have :

$$[S(s), p] = [S(s), p + b(s)] = T(s)\ [S_1(s), p]\ T^*(s).$$

On the other hand,

$$T^*(s)\ \Phi_{\lambda,\omega} = \ e^{i\sqrt{\lambda}x.\omega}\ e^{ic(s)\cdot(p+\sqrt{\lambda}\omega)}\ e^{ib(s)\cdot x}\ e^{ia(s)}\ \Phi.$$

So, we obtain :

$$< [S(s), p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = < [S_1(s), p]\ f_{\lambda,\omega},\ g_{\lambda,\omega} >,$$

where :

$$f = \ e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Phi \text{ and } g = \ e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Psi\ .$$

Then, we finish the proof by using similar arguments as in the Stark effect case, (Weder (1996), N. (2003)), but *with a time-dependent potential.*