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PROOF OF NOVIKOV'S CONJECTURE ON HOMOLOGY WITH LOCAL COEFFICIENTS OVER A FIELD OF FINITE CHARACTERISTIC

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1. Let M be a smooth manifold, and let ξ be a closed 1-form on M with nondegenerate zeros. For each $t \in \mathbf{C}$ the formula $\rho_t(\gamma) = \exp(t\int_\gamma \xi)$ defines a representation ρ_t of the group $\pi_1(M)$ in $\mathbf{C}$ and correspondingly a local system of groups of $\mathbf{C}$ on M. In [1] and [2] it was shown that for almost all $t \in \mathbf{C}$ the number $c_m(\xi)$ of zeros of index m of the form ξ is bounded below by the number $\dim H^m(M, \rho_t)$. In [1] $\dim H^m(M, \rho_t)$ was computed in terms of the action on $H^*(M, \mathbf{C})$ of the Massey operations corresponding to the form ξ.

S. P. Novikov conjectured that for an algebraically closed field k of any characteristic and any m the homology of M with coefficients in a local system close to the identity representation $\rho: \pi_1(M) \to \mathrm{GL}(m, k)$ of general position can be explicitly computed on the basis of the usual homology. It was also suggested that for the case $m = 1$, $\pi_1(M) = (\mathbf{Z})^n$ the homology of general position can be obtained by the usual sweep-out procedure by means of differentials expressed in terms of the Massey brackets in analogy to [1].

The present note contains a precise formulation and proof of the corresponding theorem for the case $m = 1$, $\pi_1(M) = (\mathbf{Z})^n$. Homology classes with coefficients in more special local systems are also computed in terms of the Massey operations.

Namely, let X be a cell complex, let $\pi_1(X) = (\mathbf{Z})^n$, and let k be an infinite field of any characteristic; then the representation space of the group $\pi_1(X)$ in k is $(k^*)^n \subset k^n$. For any algebraic curve $\gamma(t)$ in k^n whose coordinates are polynomials in t which passes through the point $\mathbf{1} = (1, \dots, 1)$ (the trivial representation) there exists a spectral sequence $\{\mathscr{E}_r(\gamma)\}$ starting from $H_*(X, k)$ and converging to homology with coefficients in the local system determined by a general point of the curve γ (Proposition 2). Let $\xi \in H^1(X, k)$ be a cohomology class corresponding to the tangent vector to the curve γ at the point $\mathbf{1}$; let $x \in H_*(X, k)$. It is then possible to reduce the indeterminacy of the Massey brackets $\langle x, \xi, \xi, \dots, \xi\rangle$ (see part 2 for precise formulations) so that the differentials d_r in the spectral sequence $\{\mathscr{E}_r(\gamma)\}$ are given precisely by the formula $d_r(x) = \langle x, \xi, \dots, \xi\rangle$ (Theorem 1). These spectral sequences depend on the choice of the curve (in this regard see Remark 5 in §5).

We proceed to precise formulations.

2. Massey brackets. We recall the definition of the higher Massey operations (see [3]). We denote by $|x|$ the grading of an element x of a graded group, and by $\bar{x}$ we denote the element $(-1)^{|x|}x$. Throughout this note, R is a commutative ring with identity. Let X be a space, and let $x_i \in H^{n_j}(X, R)$, $1 \le i \le r$. We say that a cochain $y \in C^*(X, R)$ belongs to the Massey product $\langle x_1, \dots, x_r\rangle$ if $|y| = n_1 + \dots + n_r + 2 - r$ and there exist cochains c_{ij} such that $|c_{ij}| = n_j + \dots + n_{i+j-1} + 1 - i$, $i = 1, \dots, r-1$, $j = 1, \dots, r-(i-1)$ and 1) $c_{1j} \in x_j$, 2) $\delta c_{ij} = \bar{c}_{1j}c_{i-1,j+1} + \bar{c}_{2j}c_{i-2,j+2} + \dots + \bar{c}_{i-1,j}c_{1,i+j-1}$, 3) $y = \bar{c}_{11}c_{r-1,2} + \bar{c}_{21}c_{r-2,3} + \dots + \bar{c}_{r-1,1}c_{1r} + \delta u$ (here and below we consider singular cochains; the multiplication sign $\cup$ is omitted). It is convenient to consider the cochains

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c_{ij} in the form of a triangle with no lower vertex (an incomplete Massey triangle of dimension r) :

$$\begin{array}{c} c_{11} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot c_{1r} \\ c_{21} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot c_{2,r-1} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ c_{r-1,1} c_{r-1,2} \end{array}$$

If $y = \delta c_{r,1}$, then we place $c_{r,1}$ at the bottom (a complete Massey triangle with vertex $c_{r,1}$). By definition, the incomplete Massey triangle possesses the property that all its subtriangles with vertices c_{ij}, $i \leq r-1$, are complete Massey triangles.

If in each row of a Massey triangle all cochains are the same ($c_{ij} = \xi_i$) we call it a *symmetric Massey triangle* (an s-*triangle* for short) and write $(\xi_1, \xi_2, \ldots)$. The corresponding product is called a *symmetric Massey product* and is denoted by $\langle \xi_1 \rangle_r$. These products were introduced and studied by Kraines [3].

LEMMA 1. *Let X be a space. Suppose R has a monomorphism $\varphi: R \to K$, where K is a field of characteristic 0, and $\varphi^*: H^*(X, R) \to H^*(X, K)$ is monomorphic. Then any incomplete s-triangle consisting of odd-dimensional normalized cochains over R can be completed over R.*

Thus, for spaces without torsion and odd-dimensional cocycles x the brackets $\langle x \rangle^n$ are equal to 0.

PROOF. We denote by $\bar{C}^*(X)$ the complex of normalized cochains over K, by $A^*(X)$ the algebra of polynomial differential forms over K, and by ρ_1 the integration mapping (see [4]). The mapping ρ_1 is not multiplicative, but there exist "higher homotopies" $\rho_i: [A^*(X)]^{\otimes i} \to \bar{C}^*(X)$, $\deg \rho_i = 1 - i$, which satisfy the corresponding differential identity ([4], Proposition 3.3). Suppose the odd-dimensional normalized cochains $(\xi_1, \ldots, \xi_k)$ define an s-triangle over R. It suffices for us to prove that it can be filled out over K. By induction on m we can show that there exist cocycles $a_m \in A^*(X)$ such that

$$\xi_1 = \rho_1(a_1), \ldots, \xi_k = \sum_I (-1)^{\varepsilon(m)} \rho_m(a_{i_1} \otimes \cdots \otimes a_{i_m})$$

(summation over all multi-indices $I = (i_1, \ldots, i_m)$, $m \geq 1$, $i_j > 0$, $|I| = i_1 + \cdots + i_m = k$; $\varepsilon(m) = (m-1)(m-2)/2$). Then the cochain

$$\xi_{k+1} = \sum_J (-1)^{\varepsilon(m)} \rho_m(a_{j_1} \otimes \cdots \otimes a_{j_m})$$

(summation over $J = (j_1, \ldots, j_m)$, $m \geq 2$, $|J| = k+1$) fills out our triangle.

COROLLARY 1. *Suppose $X = \mathbf{T}^n$, R is arbitrary, and $\xi \in H^1(\mathbf{T}^n, R)$. Then there exists an infinite s-triangle $(\xi = \xi_1, \xi_2, \ldots)$, and the values of the cochains ξ_i on the generating circles of the torus can be chosen arbitrarily.*

Suppose now that $X = \mathbf{R}^n$, $\pi: \mathbf{R}^n \to \mathbf{T}^n$ is a projection, $\mathbf{1}$ is the unit 0-cochain on $\mathbf{R}^n$, $S: C^1(\mathbf{R}^n) \to C^0(\mathbf{R}^n)$ is the canonical chain homotopy, e_i are the coordinate unit vectors in $\mathbf{R}^n$, and t_i is the operator of translation by e_i. Suppose $(\xi_1, \xi_2, \ldots)$ is any infinite s-triangle, $\xi_i \in C^1(\mathbf{T}^n, R)$. We define the infinite Massey triangle [1] on $\mathbf{R}^n$

$$\begin{array}{c} h_1 = \mathbf{1}, \xi_1, \xi_1, \cdots \\ h_2, \xi_2, \cdots \\ \cdot \cdot \cdot \cdot \cdot \end{array} \tag{1}$$

by induction, setting $h_1 = \mathbf{1}$, $\mu_1 = \xi_1$, and $h_2 = S\xi_1, \ldots, h_m = S\mu_{m-1}$, where μ_{m-1} is the corresponding Massey cocycle (in the notation we do not distinguish the cochains ξ_i and $\pi^*(\xi_i)$). We set $A_{ij} = \langle \mu_i, e_j \rangle$.

LEMMA 2. 1) $t_j\mu_m = \mu_m + A_{1j}\mu_{m-1} + \cdots + A_{m-1,j}\mu_1$.

2) $t_j h_m = h_m + A_{1j}h_{m-1} + \cdots + A_{m-1,j}h_1$.

The proof is by induction on m with use of the commutation formula: for 1-cocycles λ we have $[t_i, S](\lambda) = \langle \lambda, e_i \rangle$.

REMARK. It follows from Corollary 1 that the elements $A_{ij} \in R$ can be chosen arbitrarily.

We introduce Massey brackets of the form $\langle x_1, \dots, x_r \rangle$, where $x_1 \in H_*(X, R)$ and $x_2, \dots, x_r \in H^*(X, R)$. The definition is the same as above, only the $c_{k,1}$ are chains while $\bar{c}_{k,1}c_{r-k,k+1}$ is understood as $\bar{c}_{k,1} \cap c_{r-k,k+1}$.

We fix an infinite s-triangle $\Delta = (\xi_1, \xi_2, \dots)$, $\xi_i \in H^1(X, R)$. We say that $y \in D_r x$ if x and y are cycles, and the Massey triangle

$$\begin{array}{llll} x_1 = x, \xi_1, \xi_1, \dots & & & \\ & x_2, \xi_2, \cdots & & \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \\ & & & x_r, \xi_r \end{array}$$

exists, and y is cohomologous to $\sum_{i=1}^{r-1} x_i \xi_{r-i+1}$.

PROPOSITION 1. *There exists a spectral sequence* $\{E^r_*, d_r\}$ *such that* 1) $E^1_* = H_*(X, R)$ *and* $d_1(x) = x \cap \xi_1 = \langle x, \xi_1 \rangle$, *and* 2) $|d_r| = -1$; *if* $\bar{x}$ *and* $\bar{y}$ *are cochains representing elements* $x, y \in E^r_*$, *then* $y = d_r(x)$ *is equivalent to* $\bar{y} \in D_r \bar{x}$.

3. Complexes over polynomial rings. We fix n polynomials $P_i \in R[t]$ and denote by Λ the ring $S^{-1}R[t]$, where $S = \{P_1, \dots, P_i\}$. There is the exact sequence

$$0 \to \Lambda \xrightarrow{t} \Lambda \xrightarrow{\varepsilon} R \to 0, \tag{2}$$

where ε is the argumentation (the value at the point $t = 0$). We denote by Q the ring of Laurent polynomials over R in the variables t_i and t_i^{-1}, $1 \leq i \leq n$. We define a ring morphism $\varphi\colon Q \to \Lambda$ by the formula $\varphi(t_i) = P_i$. It makes Λ a Q-module.

Suppose we have a complex D_* of free, finitely generated Q-modules. Multiplying it (over Q) by the exact sequence (2) and passing to homology, we obtain the exact pair

$$\begin{tikzcd} H_*(D_* \otimes \Lambda) \arrow[rr, "t"] & & H_*(D_* \otimes \Lambda) \arrow[dl, "\varepsilon"] \\ & H_*(D_* \otimes R) \arrow[ul, "\delta"] & \end{tikzcd}$$

Let $R = k$ be an infinite field. Any point α with nonzero coordinates in the space k^n determines a homomorphism $\alpha\colon Q \to k$ and a complex $D_* \otimes_\alpha k$. Its homology is called homology of D_* with coefficients at the point α (notation: $H^\alpha_*(D, k)$). The polynomials P_i define a curve in k^n. We have the simple

PROPOSITION 2. *The spectral sequence* $\{\mathscr{E}^r_*, \partial_r\}$ *obtained from* (3) *begins from the homology* $H^1_*(D, k)$ *and converges to the homology* $H^\alpha_*(D, k)$, *where* α *is a general point of the curve* γ.

4. The main theorem. Let X be a complex, and suppose that $\pi_1(X) = (\mathbf{Z})^n$. Let R be an integral principal ideal ring, and suppose there are given polynomials $P_i(t) = 1 + a_{1i}t + \cdots + a_{Ni}t^N$, $1 \leq i \leq n$, $a_{ik} \in R$, where the elements a_{1i} are relatively prime. In the notation of §3 we set $D_* = C_*(\tilde{X}, R)$. According to §3, we have the spectral sequence $\{\mathscr{E}^r_*, \partial_r\}$, where $\mathscr{E}^r_* = H_*(X, R)$.

There is the mapping $f\colon X \to \mathbf{T}^n$ which induces an isomorphism in π_1. On $\mathbf{T}^n$ we construct the infinite s-triangle $\Delta = (\xi_1, \xi_2, \dots)$ so that in the Massey triangle of the form (1) on $\mathbf{R}^n$ corresponding to it the relation $A_{ij} = a_{ij}$ is satisfied. Inducing the triangle Δ on X, we obtain (see part §2) the spectral sequence $\{E^r_*, d_r\}$, where $E^1_* = H_*(X, R)$.

THEOREM 1. *The spectral sequences* $\{\mathscr{E}_*^r, \partial_r\}$ *and* $\{E_*^r, -d_r\}$ *coincide.*

PROOF. Let $p\colon \tilde{X} \to X$ be a universal covering. Let $x \in \mathscr{E}_*^r$. We take any cycle $x' \in C_*(X, R)$ such that $x' \in x$. Using the fact that the a_{1j} are relatively prime, we can find a chain $\tilde{X} \in C_*(\tilde{X}, R)$ such that $p_*(\tilde{x}) = x$ and in the module $C_*(\tilde{X}, R) \otimes \Lambda$ we have $\partial\tilde{x} \otimes 1 = t^r y$ and $\varepsilon(y) = \partial_r x$. We show that x' lives through to E^r and $-y \in D_r x'$. By means of the mapping $\tilde{f}\colon \tilde{X} \to \mathbf{R}^n$ we induce the cochains h_i, ξ_i. We claim that

$$\begin{array}{llll} x = p_*(\tilde{x}), & \xi_1, \quad \xi_1 & & \\ & \rho_*(\tilde{x} \cap h_2), & \xi_2 & \\ & \cdots\cdots\cdots\cdots & & \\ & & p_*(\tilde{x} \cap h_r), & \xi_r \end{array}$$

is a Massey triangle (we denote the corresponding Massey cycles by M_k, $k \le r - 1$) and that M_{r-1} is homologous to $-y$. Indeed, we consider the free k-dimensional R-module $L_k = \{h_1, \dots, h_k\}$ in $C^*(\tilde{X}, R)$. From Lemma 2 and the choice $A_{ij} = a_{ij}$ it follows that L_k is a Q-module in $C^*(\tilde{X}, R)$ which is isomorphic as a Q-module to $\Lambda/t^k\Lambda$ (under the isomorphism $h_m \to t^{k-m}$). From the properties of the operation $\cap$ it follows that the mapping $p_*(\cdot \cap \cdot)\colon C_*(\tilde{X}, R) \otimes L_k \to C_*(X, R)$ factors through $C_*(\tilde{X}, R) \otimes_Q L_k \approx C_*(\tilde{X}, R) \otimes_Q (\Lambda/t^k\Lambda)$. It is now evident that the element $\partial p_*(\tilde{x} \cap h_i) = p_*(\partial\tilde{x} \cap h_k) + M_{k-1}$ for $k \le r$ is equal to M_{k-1}. For $k = r + 1$ we find that $-M_r$ and $\varepsilon(y) = \partial_r(x)$ are cohomologous. The theorem is proved.

5. Further remarks. 1) The results of §§3 and 4 generalize immediately to the case of curves passing through an arbitrary point (not the identity) of the representation space.

2) Let M be a manifold, let $\pi_1(M) = (\mathbf{Z})^n$, and let ξ be a closed Morse 1-form. We call ξ *rational* if its periods are rational. Approximating the form ξ by rational forms (and using [5] and [6]), we easily find that $c_i(\xi) \ge H_i^{9.p.}(M, k)$. We denote the vector of periods for ξ by $\hat{\xi}$.

PROPOSITION 3. *Suppose* γ *is a curve in* $\mathbf{C}^n$, *the rational points are dense in* γ, *and* $\hat{\xi} \in \gamma$. *Then* $c_i(\xi) \ge H_i^\alpha(M, k)$, *where* α *is a general point of the cone* $C(\gamma, 0)$.

3) Suppose $\pi_1(X) = \mathbf{Z}$ and the polynomial $P_1(t)$ is $1 + t$. In this case $\{\mathscr{E}_r, \partial_r\}$ is the Milnor spectral sequence [7] of the cyclic covering $\tilde{X} \to X$. Here $\{E_r, d_r\}$ does not depend on the choice of the s-triangle. For Proposition 2 in this case, see [6].

4) There is a cohomological version of our constructions (in place of chains in $\tilde{X}$ it is necessary to take cochains in $\tilde{X}$ with compact support), and there is also a version with de Rham cohomology. Novikov [1] considered the de Rham case. In our case his construction corresponds to an analytic curve $\gamma(t) = \exp(t \int \xi)$, and in the corresponding s-triangle all $\xi_i = 0$ for $i > 1$.

5) Of course, the sequences $\{\mathscr{E}_r, \partial_r\}$ depend on the choice of the curve γ. We shall consider the case of lines γ which pass through $\mathbf{1}$.

PROPOSITION 4. *For all* $\gamma \in \mathbf{P}^n(k)$ *not belonging to some proper projective submanifold* $S \subset \mathbf{P}^n(k)$, *the spectral sequences* $\{\mathscr{E}_r(\gamma), \partial_r\}$ *are isomorphic.*

6. We here announce the proof of Novikov's conjecture for $k = \mathbf{C}$ and any m. Let M be a smooth compact manifold. Suppose in the representation space R of the group $\pi_1(M)$ in $\mathrm{GL}(m, \mathbf{C})$ the point $\mathbf{1}$ is not isolated. We consider any analytic curve $\gamma(t)$ in R such that $\gamma(0) = \mathbf{1}$.

THEOREM 2. *There exists an infinite s-triangle* $(\theta_1, \theta_2, \dots)$ *consisting of* $m \times m$ *matrix* 1*-forms such that the spectral sequence* (E_r, d_r) *generated by this triangle*

($E_1 = H^*(M, \mathbf{C})$, $d_1(x) = x \wedge \theta_1$) converges to the homology $H^*(M, \rho)$, where ρ is a local system corresponding to a general point of the curve γ.

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