CIRCLE-VALUED MORSE THEORY AND TUNNEL NUMBER OF KNOTS

A.V.PAJITNOV

ABSTRACT. Let L be a link in S^3; denote by $\mathcal{MN}(L)$ the Morse-Novikov number of L and by $t(L)$ the tunnel number of L. We prove that $\mathcal{MN}(L) \leqslant 2t(L)$ and deduce several corollaries.

1. INTRODUCTION

A) Background

Let L be a link in S^3. An arc γ in S^3 is called *a tunnel* for L if $\gamma \cap L$ consists of the two endpoints of γ. The *tunnel number* $t(L)$ is the minimal number m of disjoint tunnels $\gamma_1, \dots, \gamma_m$ such that the closure of

$$S^3 \setminus N(L \cup \gamma_1 \cup \ldots \cup \gamma_m)$$

is a handlebody. The tunnel number was introduced by B. Clark [1] in 1980; this invariant was studied in the works of T. Kobayashi, T. Kohno, H. Goda, K. Morimoto, Y. Rieck, M. Sakuma, M. Scharlemann, J. Schultens, M. Teragaito, Y. Yokota, and others. M. Scharlemann and J. Schultens [9] proved that $t(nK) \geqslant n$ for any n (here nK stands for the connected sum of n copies of the knot K). Also $t(nK) \geqslant \frac{2}{5}nt(K)$ if K is not a 2-bridge knot [10].

B. Morse-Novikov numbers

Pick an orientation preserving trivialisation of the normal bundle of L. The corresponding diffeomorphism of disc bundles $\phi : L \times D^2 \to N(L)$ will be called *framing* of L. Let C_L denote the closure of $S^3 \setminus N(L)$. A Morse function $f : C_L \to S^1$ is called *regular* if its restriction to the boundary $\partial N(L)$ is the canonical fibration over the circle: $(f \circ \phi)(l, z) = \frac{z}{|z|}$. The number of the critical points of index i of a regular Morse function f will be denoted by $m_i(f)$; the total number of critical points of f will be denoted by $m(f)$. The minimal value of $m(f)$ over all possible framings ϕ and regular Morse maps $f : C_L \to S^1$ is called *the Morse-Novikov number of the link L* and denoted by $\mathcal{MN}(L)$ [8]. The Morse-Novikov theory of circle-valued maps allows to obtain homological lower bounds for $\mathcal{MN}(L)$ as follows. Let $\bar{C}_L \to C_L = \overline{S^3 \setminus N(L)}$ be the infinite cyclic covering induced by f from the covering $\mathbf{R} \to S^1$. Denote the ring $\mathbf{Z}[t, t^{-1}]$ by Λ, and the ring $\mathbf{Z}((t))$ by $\widehat{\Lambda}$. The $\widehat{\Lambda}$-module

$$\mathcal{N}_*(L) = H_*(\bar{C}_L) \underset{\Lambda}{\otimes} \widehat{\Lambda}$$

is called *the Novikov homology* of L. The rank and torsion numbers of the $\widehat{\Lambda}$-module $\mathcal{N}_1(L)$ are denoted respectively by $b_1(L)$ and $q_1(L)$. Then

$$\mathcal{MN}(L) \geqslant 2(b_1(L) + q_1(L)).$$

(See the book [7] for a detailed exposition of the cicle-valued Morse theory.) As for the upper bounds for $\mathcal{MN}(L)$ H. Goda proved that $\mathcal{MN}(L) \leqslant 2$ for every prime link L with $\leqslant 10$ crossings [2]. M. Hirasawa proved that for every 2-bridge knot K we have $\mathcal{MN}(K) \leqslant 2$ (unpublished). Lee Rudolph and M. Hirasawa [4] proved that

$$\mathcal{MN}(K) \leqslant 4g_f(K)$$

where $g_f(K)$ is the *free genus* of K, that is, the minimal possible genus of a Seifert surface Σ bounding K such that $S^3 \setminus \Sigma$ is an open handlebody.

2. RESULTS

Main Theorem. *For every link L in S^3 we have*

$$\mathcal{MN}(L) \leqslant 2t(L).$$

The following corollaries are easily deduced.

Corollary 2.1. *For every g we have*

$$\mathcal{MN}(L) \leqslant 2(g + b_g(L) - 1),$$

(where $b_g(L)$ is the g-bridge number of L).

Corollary 2.2. *For every (1,1)-knot K we have* $\mathcal{MN}(K) \leqslant 2$.

Corollary 2.3. *For every link L we have*

$$q_1(L) + b_1(L) \leqslant t(L).$$

Proof of the main theorem: Let $m = t(L)$. Pick a framing $\phi : L \times D^2 \to N(L)$. Then the manifold $C_L = \overline{S^3 \setminus N(L)}$ is obtained from ∂C_L by attaching m one-handles and then attaching a handlebody of genus $(m+1)$ to the resulting cobordism. Thus we obtain a Morse function $g : C_L \to \mathrm{R}$ which is constant on ∂C_L and has the following Morse numbers:

$$m_0(g) = 0, \ m_1(g) = m, \ m_2(g) = m+1, \ m_3(g) = 1.$$

Pick any Morse map $h : C_L \to S^1$ such that $h|\partial C_L$ is the canonical fibration: $(h \circ \phi)(l, z) = \frac{z}{|z|}$. The 1-form induced by h from the canonical volume form on S^1 will be denoted by dh. Consider a closed 1-form $\omega_\epsilon = dg + \epsilon dh$. For $\epsilon > 0$ sufficiently small ω_ϵ is a Morse 1-form with the same Morse numbers as dg:

$$m_0(\omega_\epsilon) = 0, \ m_1(\omega_\epsilon) = m, \ m_2(\omega_\epsilon) = m+1, \ m_3(\omega_\epsilon) = 1.$$

The De Rham cohomology class of the 1-form

$$\frac{1}{\epsilon}\omega_\epsilon = \frac{1}{\epsilon}dg + dh$$

is the same as that of dh; therefore this form is the differential of a Morse map $g_1 : C_L \to S^1$ homotopic to h. The map g_1 is a regular Morse map; it has one local maximum, and the standard procedure of elimination of critical points gives us a regular Morse function $f : C_L \to S^1$ with

$$m_0(f) = 0, \ m_1(f) = m, \ m_2(f) = m, \ m_3(f) = 0.$$

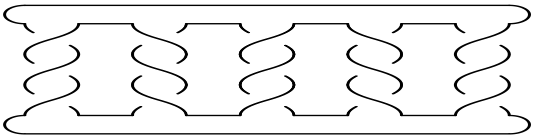
Thus $\mathcal{MN}(L) \leqslant 2m$.

3. EXAMPLES

A) Pretzel knots. Let q, r be positive integers; let

$$\mathcal{P} = P(2q+1, -2q-1, 2q+1, \ldots, 2q+1)$$

be the $(2r+1)$-stranded pretzel knot. (The knot $P(3, -3, 3, -3, 3)$ is depicted below.)



It is clear that $t(\mathcal{P}) \leqslant 2r$. An easy computation of the Alexander module via the Seifert matrix gives

$$\mathcal{N}_1(\mathcal{P}) \approx \left(\widehat{\Lambda}/XY\widehat{\Lambda}\right)^r$$

where $X = qt - (q+1)$, $Y = (q+1)t - q$. Thus $q_1(\mathcal{P}) = r$. Since $q_1(mK) = mq_1(K)$ for any knot K, we deduce that

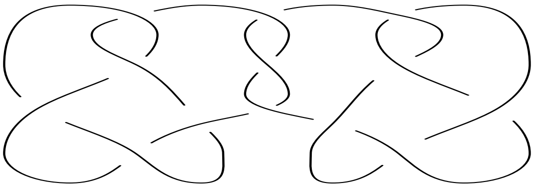
$$\frac{1}{2}nt(\mathcal{P}) \leqslant nq_1(\mathcal{P}) \leqslant t(n\mathcal{P}).$$

(The general theorem of M. Scharlemann and J. Schultens gives the estimate $\frac{2}{5}nt(\mathcal{P}) \leqslant t(n\mathcal{P})$.) In particular the growth rate of the knot satisfies

$$gr_t(\mathcal{P}) \geqslant -1/2.$$

B) A twisted $2 \cdot 5_2$.

Let K be the knot obtained from the connected sum $5_2 \# 5_2$ by twisting:



By a computation similar to the above:

$$\mathcal{N}_1(K) \approx \left(\widehat{\Lambda}/T\widehat{\Lambda}\right)^2$$

where $T = 2t^2 - 3t + 2$ is the Alexander polynomial of the knot 5_2. Thus $q_1(K) = 2$. Since $t(K) \leqslant 3$ we obtain:

$$\frac{2}{3}nt(K) \leqslant nq_1(K) \leqslant t(nK).$$

(M. Scharlemann and J. Schultens: $\frac{2}{5}nt(K) \leqslant t(nK)$.) We have also $gr_t(K) \geqslant -1/3$.

4. RELATIONS WITH PREVIOUSLY KNOWN RESULTS

A theorem of M. Hirasawa says that $\mathcal{MN}(K) \leqslant 2$ if K is a two-bridge knot. Since $t(K) \leqslant b(K) - 1$ our theorem implies this result. We deduce also the upper bound

$$\mathcal{MN}(K) \leqslant 4g_f(K)$$

obtained by Lee Rudolph and M. Hirasawa. Indeed Jung Hoon Lee [3] has shown that $t(K) \leqslant 2g_f(K)$.

Example: Consider the pretzel knot $K = P(-2l, q, r)$ where $l \geqslant 2$ and $q, r \geqslant 3$ are odd numbers. Then $t(K) \leqslant 2$, therefore $\mathcal{MN}(K) \leqslant 4$. The Alexander polynomial of the knot was computed by Dongseok Kim and Jaeun Lee [5], it equals

$$A(t) = lt^{q+r} - (2l-1)t^{q+r-1} + \ldots - (2l-1)t + l.$$

Further, $g(K) \geqslant \deg A(t)/2 = (q+r)/2$, therefore the free genus of K is not less than $(q+r)/2$.

5. CONJECTURES AND OPEN QUESTIONS

1. One of the main conjectures in the Morse-Novikov theory of knots and links is the following (M. Boileau, C. Weber):

$$\mathcal{MN}(K_1 \# K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2).$$

As for the tunnel number of knots we have

$$t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) + 1.$$

T. Kobayashi [6] proved that for every N there are knots K_1 and K_2 such that $t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) - N$. His work provides therefore a number of potential counter-examples to the conjecture above.

2. Pretzel knots. *Problem:* Compute the tunnel numbers and Morse-Novikov numbers of the pretzel knots and their multiples.

Example: For the knot $\mathcal{P} = P(3, -3, 3, -3, 3)$ we know that

$$4n = 2nq_1(\mathcal{P}) \leqslant \mathcal{MN}(n\mathcal{P}) \leqslant 2t(n\mathcal{P}).$$

Is it true that $2nq_1(\mathcal{P}) = \mathcal{MN}(n\mathcal{P})$? Is it true that $\mathcal{MN}(n\mathcal{P}) = 2t(n\mathcal{P})$?

REFERENCES

[1] B. Clark, *The Heegaard genus of manifolds obtained by surgery on links and knots*, Internat. J. Math. and Math. Sci. Vol.3 No.3 (1980), 583-589.

[2] H.Goda, *On handle number of Seifert surfaces in* S^3 , Osaka J.Math. **30**, (1993) 63 – 80.

[3] Jung Hoon Lee, *An upper bound for tunnel number of a knot using free genus*, Talk at 4th East Asian School of knots, http://faculty.ms.u-tokyo.ac.jp/~topology/EAS4slides/JungHoonLee.pdf

[4] M. Hirasawa, Lee Rudolph, *Constructions of Morse maps for knots and links, and upper bounds on the Morse-Novikov number*, math.GT/0311134, to appear in J. Knot Theory Ramifications.

[5] Dongseok Kim, Jaeun Lee, *Some invariants of pretzel links*, Bull. Austral. Math. Soc. 75 (2007), no. 2, 253–271.

[6] T. Kobayashi, *A construction of arbitrarily high degeneration of tunnel number of knots under connected sum,* J. Knot Theory Ramifications **3**, (1994) p. 179 – 186.

[7] A.V.Pajitnov, *Circle-valued Morse theory*, de Gruyter studies in Mathematics, vol. 32, (2006), 454 pages

[8] A.V.Pajitnov, C.Weber, L.Rudolph, *Morse-Novikov number for knots and links*, Algebra i Analiz, **13**, no.3 (2001), (in Russian), English translation: Sankt-Petersbourg Mathematical Journal. **13**, no.3 (2002), p. 417 – 426.

[9] M. Scharlemann, J. Schultens, *The tunnel number of the connect sum of n knots is at least n*, Topology **38** (1999) 265-270.

[10] M. Scharlemann, J. Schultens, *Annuli in generalized Heegaard splitting and degeneration of tunnel number*, Math. Ann **317** (2000) No. 4, 783–820.

LABORATOIRE MATHÉMATIQUES JEAN LERAY UMR 6629, UNIVERSITÉ DE NANTES, FACULTÉ DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX
E-mail address: `pajitnov@math.univ-nantes.fr`