An inverse to the antisymmetrization map of Cartan & Eilenberg

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1 A contracting homotopy for the Chevalley-Eilenberg resolution

1.1 Convolution, Cofree coalgebras and coderivations

Definition 1.1.1. *Let (A, μ) be an algebra object and (C, Δ) be a coalgebra object in some monoidal category $(\mathcal{M}, \otimes)$. The set $\mathrm{Hom}_{\mathcal{M}}(C, A)$ can be endowed with an associative composition product $\star$, called* ***convolution product****, defined by*

$$f \star g := \mu \circ (f \otimes g) \circ \Delta$$

for all f and g in $\mathrm{Hom}_{\mathcal{M}}(C, A)$.

Let R be a commutative ring, (C, ϵ) be a cocommutative counital coalgebra in the category of graded R-modules, and V a graded R-submodule of C. Denote by $\Delta : C \to C \otimes_R C$ the coproduct of C, $\epsilon : C \to R$ its counit.

Definition 1.1.2. *C is said to be* ***connected*** *if there exists morphism of coalgebras $\eta : R \to C$ such that $\epsilon\eta = \mathrm{Id}_R$ and $\bar{C} := C/\mathrm{Im}\eta$ is a conilpotent coalgebra.*

Assume that C is connected. C is said to be ***cofreely generated*** *by V if there exists a morphism of graded R-modules $p : C \to V$ such that for every connected graded R-coalgebra D and every morphism of graded R-modules $\bar{f} : D \to V$, there exists a unique morphism of coalgebras $f : D \to C$ such that the following diagramm*

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "\bar{f}"'] & C \arrow[d, "p"] \\ & V \end{tikzcd}$$

commutes.

Denote by $|x|$ the degree of an homogenous element in V.

Proposition 1.1.3. *Let V be a graded R-module.*

- *Two cofree connected cocommutative coalgebras cogenerated by V are isomorphic.*
- *Moreover, one of them is given by the connected cocommutative coalgebra $\mathrm{proj} : S_*V \to V$, where S_*V is the quotient of the (graded) tensor algebra $T_*V := \oplus_{n \geq 0} V^{\otimes_R n}$ by the ideal generated by relations of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$. S_*V is a graded commutative algebra and can be equipped with a graded cocommutative coproduct $\Delta : S_*V \to S_*V \otimes_R S_*V$ turning it into a Hopf algebra such that every element in $V \subset S_*V$ is primitive. The projection morphism $\mathrm{proj} : S_*V \to V$ is induced by the canonical projection of TV on its length 1 term.*
- *In particular, the unique morphism of coalgebras $f : D \to C$ lifting a given linear map $\bar{f} : D \to V$, where D is any connected cocommutative coalgebra, can be defined thanks to convolution in $\mathrm{Hom}_R(D, S_*V)$ (see 1.1.1) via*

$$f := \exp_\star(\bar{f}) := \sum_{n \geq 0} \frac{1}{n!} \bar{f}^{\star n}$$

with $\bar{f}^{\star 0} := \eta\epsilon$.

Definition 1.1.4. *Let (C, Δ) be a coalgebra in some monoidal category, and $\phi : C \to C$ be an endomorphism of coalgebra. A **coderivation of** C **along** ϕ is a morphism $d : C \to C$ such that*

$$\Delta \circ d = (\phi \otimes d + d \otimes \phi) \circ \Delta$$

When $\phi = \mathrm{Id}_C$, we simply say that d is a coderivation.

Proposition 1.1.5. *Let $\bar{d} : S_*V \to V$ be a graded R-linear map. Then*

- *There exists a unique coderivation $d : S_*V \to S_*V$ along ϕ such that $\bar{d} = \mathrm{proj} \circ d$.*
- *d is given by $d := \bar{d} \star \phi$.*

Proposition 1.1.6. *Let $\phi : C \to C$ and $\psi : C \to C$ be two coalgebra endomorphisms of a given coalgebra C in the category of graded R-modules, and d (resp. D) be a coderivation of C along ϕ (resp. along ψ). Then*

- *$\psi \circ d$ is a coderivation of C along $\psi \circ \phi$.*
- *Suppose that $\phi \circ \psi = \psi \circ \phi$. Then the graded bracket*

  $$[d, D] := d \circ D - (-1)^{|d||D|} D \circ d$$

  is a coderivation of C along $\phi \circ \psi$.

1.2 The Chevalley-Eilenberg resolution

Let L be a Lie algebra over some commmutative ring R of characteristic 0 with Lie bracket $[-,-] : L \wedge_R L \to L$ (Here Λ_R stands for the exterior product of R-modules). Denote by UL its universal enveloping algebra, that is the algebra obtained by quotienting the tensor algebra $TL := \oplus_{n \geq 0} L^{\otimes n}$ by the ideal generated by relations of the form $g \otimes g' - g' \otimes g - [g, g']$ when g and g' run over L. The product of two elements x and y of UL will be written xy. Recall that UL can be endowed with

- a comultiplication $\Delta : UL \to UL \otimes_R UL$ determined by saying that every element of $L \subset UL$ is primitive,
- a counit $\epsilon : UL \to R$ and a unit $\eta : R \to UL$, both induced by the canonical ones of TL,
- an antipode $S : UL \to UL$ which is the only algebra antimorphism such that $S(g) = -g$ for all g in L,

turning it into Hopf algebra.

Following 1.1.1, this Hopf algebra structure gives rise to a **convolution product** $\star$ on $\mathrm{End}_R(UL)$, the R-module of linear endomorphism of UL, such that

$$f \star h := \mu(f \otimes h)\Delta$$

for all f and h in $\mathrm{End}_R(UL)$, where μ denotes the associative product of UL.

Definition 1.2.1. *The **first eulerian idempotent** of L is the R-linear endomorphism $\mathrm{pr} : UL \to UL$ defined by*

$$\mathrm{pr} := \sum_{i \geq 0} \frac{(-1)^i}{i+1} (\mathrm{Id} - \eta\epsilon)^{\star i+1}$$

Theorem 1.2.2. [PBW] *The first eulerian idempotent* pr *takes its values in L. Moreover,* $\mathrm{pr} : UL \to L$ *is a cofree connected cocommutative coalgebra cogenerated by the R-module L.*

Proposition 1.2.3. [Eulerian idempotents] *For all k and l in $\mathbb{N}$*

$$\frac{1}{k!\, l!} \mathrm{pr}^{\star k} \circ \mathrm{pr}^{\star l} = \begin{cases} \frac{1}{k!} \mathrm{pr}^{\star k} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Notice that L can be seen as a graded R-module concentrated in degree 0. When $V = \{V_i\}_{i \geq 0}$ is a graded module, denote by $V[1]$ the shifted module whose degree i component is $V[1]_i := V_{i-1}$.

Definition 1.2.4. *The* ***Chevalley-Eilenberg resolution*** *of L is the chain complex of R-modules $C_*(L) := UL \otimes_R SL[1]$ with differential $d : C_*(L) \to C_{*-1}(L)$ of degree -1 defined by*

$$\begin{aligned} d(x \otimes g_1 \wedge \cdots \wedge g_n) := & \sum_{i=1}^{n} (-1)^{i+1} x g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \\ & + \sum_{i<j} (-1)^{j+1} x \otimes g_1 \wedge \cdots \wedge [g_i, g_j] \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n \end{aligned}$$

for all x in UL and g_1,..., g_n in L, where $\hat{g}_i$ means that g_i has been omitted.

Remark 1.2.5. *When $V = \{V_n\}_{n\geq 0}$ is a graded module concentrated in degree 1, we will always identify $S_n V$ with the n-th exterior power $\Lambda^n V_1$.*

Proposition 1.2.6. *Define $\mathrm{PR} : C_*(L) \to L \otimes_R R \oplus R \otimes_R L[1] \cong L \oplus L[1]$ by*

$$\mathrm{PR} := \mathrm{pr} \otimes \epsilon + \epsilon \otimes \mathrm{proj}$$

Then $(C_*(L), \mathrm{PR})$ is a cofree cocommutative connected (graded) coalgebra generated by $L \oplus L[1]$. Moreover, the differential d is the unique coderivation generated by

$$\begin{array}{rcl} \bar{d} : C_*(L) & \to & L \oplus L[1] \\ x \otimes y & \mapsto & \mathrm{pr}(x \, \mathrm{proj} y) + \epsilon(x) B(y) \end{array}$$

for all x in UL and y in $S_* L[1]$, where $B : S_* L[1] \to L[1]$ coincides with the Lie bracket in degree 2 and is zero elsewhere.

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring $\mathbb{K}$ containing $\mathbb{Q}$, and denote by $\mathfrak{g}[t]$ the $\mathbb{K}[t]$-Lie algebra $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[t]$. An element of $\mathfrak{g}[t]$ is just a polynomial expression in t with coefficients in $\mathfrak{g}$. We have obvious isomorphisms

$$U(\mathfrak{g}[t]) \cong U\mathfrak{g}[t] := U\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[t]$$

and

$$C_*(\mathfrak{g}[t]) \cong C_*(\mathfrak{g})[t] := C_*(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[t]$$

Moreover, "formal integration on $[0,1]$" gives a $\mathbb{K}$-linear map $I_{[0,1]} : \mathbb{K}[t] \to \mathbb{K}$, sending each t^n to $\frac{1}{n+1}$, providing a morphism of chain complexes

$$I := \mathrm{Id} \otimes I_{[0,1]} : C_*(\mathfrak{g})[t] \to C_*(\mathfrak{g})$$

which behaves with respect to "formal derivation" $\frac{d}{dt} : t^n \mapsto n t^{n-1}$ as in the usual real case. The inclusion $\mathbb{K} \subset \mathbb{K}[t]$ induces an inclusion of chain complexes

$$C_*(\mathfrak{g}) \hookrightarrow C_*(\mathfrak{g})[t]$$

Given a $\mathbb{K}$-module V, $V[t]$ will always denote the $\mathbb{K}[t]$-module $V \otimes \mathbb{K}[t]$, and $\mathbb{K}[t]$-linear morphism from $V[t]$ to some other $\mathbb{K}[t]$-module will always be defined on V and extended to $V[t]$ by linearity. Note that all previous considerations can be easily generalized to the case when one replaces $\mathbb{K}[t]$ by $\mathbb{K}[t_1, t_2, \cdots, t_n]$, the algebra of polynomials in n indeterminates t_1, t_2, ..., t_n. From sequel, $\otimes$ will always mean $\otimes_{\mathbb{K}}$.

Notation 1.2.7. *We'll make an intensive use of Sweedler's notation to write iterated comultiplications in cocommutative coalgebras:*

$$\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(n)}$$

will stand for

$$(\Delta \otimes \mathrm{Id}^{\otimes(n-2)}) \circ (\Delta \otimes \mathrm{Id}^{\otimes(n-3)}) \circ \cdots \circ (\Delta \otimes \mathrm{Id}) \circ \Delta(x)$$

Definition 1.2.8. *Define two $\mathbb{K}[t]$-linear maps $\phi_t : U\mathfrak{g}[t] \to U\mathfrak{g}[t]$ and $A_t : U\mathfrak{g} \otimes \mathfrak{g}[t] \to U\mathfrak{g}[t]$ by*

$$\phi_t := \sum_{k \geq 0} \frac{t^k}{k!} \mathrm{pr}^{\star k}$$

and

$$A_t(x, g) := A_t(x \otimes g) \sum_{(x)} \phi_{-t}(x^{(1)}) \phi_t(x^{(2)} g)$$

for all x in $U\mathfrak{g}$ and g in $\mathfrak{g}$.

Proposition 1.2.9. • *As endomorphisms of* $U\mathfrak{g}[t_1,t_2] := U\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[t_1,t_2]$*:*

$$\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1 t_2}$$

and

$$\phi_{t_1} \star \phi_{t_2} = \phi_{t_1+t_2}$$

- A_t *takes its values in* $\mathfrak{g}[t]$ *i.e.*

$$A_t(x,g) \in \mathfrak{g}[t]$$

for all x *in* $U\mathfrak{g}$ *and* g *in* $\mathfrak{g}$*.*

- $\frac{d\phi_t}{dt} = \phi_t \star \mathrm{pr}$ *as a* $\mathbb{K}[t]$*-linear endomorphism of* $U\mathfrak{g}[t]$*.*

Definition 1.2.10. *Define* $\mathbb{K}[t]$*-linear morphisms of graded modules* $a_t : C_*(\mathfrak{g})[t] \to C_*(\mathfrak{g})[t]$ *and* $b_t : C_*(\mathfrak{g})[t] \to C_{*+1}(\mathfrak{g})[t]$ *by*

$$a_t(x \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} \phi_t(x^{(1)}) \otimes A_t(x^{(2)}, g_1) \wedge \cdots \wedge A_t(x^{(n+1)}, g_n)$$

and

$$b_t(x \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} \phi_t(x^{(1)}) \otimes \mathrm{pr}(x^{(2)}) \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n)$$

Proposition 1.2.11. a_t *is an endomorphism of coalgebra and* b_t *is a degree* $+1$ *coderivation of* $C_*(\mathfrak{g})[t]$ *along* a_t*.*

The following theorem implies that the Chevalley-Eilenberg resolution is indeed a resolution:

Theorem 1.2.12. *The degree* 1 $\mathbb{K}$*-linear map* $s : C_*(\mathfrak{g}) \to C_{*+1}(\mathfrak{g})$ *defined by*

$$s := I \circ b_t$$

is a contracting homotopy of the chain complex $(C_*(\mathfrak{g}), d)$*.*

Proof. The theorem is a direct consequence of the three following facts:

- $\frac{d}{dt}a_t$ is a coderivation along a_t: Proposition 1.2.11 asserts that a_t is a coalgebra endomorphism i.e.

  $$\Delta a_t = (a_t \otimes a_t)\Delta$$

  Thus

  $$\Delta \frac{d}{dt} a_t = \frac{d}{dt}\Delta a_t = \frac{d}{dt}(a_t \otimes a_t)\Delta = (\frac{d}{dt}a_t \otimes a_t + a_t \otimes \frac{d}{dt}a_t)\Delta$$

  which exactely means that $\frac{d}{dt}a_t$ is a coderivation along a_t.

- Proposition 1.2.11 (resp. 1.2.6) tells us that b_t (resp. d) is a coderivation along a_t (resp. the identity map of $C_*(\mathfrak{g})[t]$). By proposition 1.1.6, since the identity map obviously commutes with a_t, the graded bracket $[d, b_t] = db_t + b_t d$ is a coderivation along a_t.

- The two preceeding coderivations are equal:

  $$db_t + b_t d = \frac{d}{dt} a_t \tag{1}$$

  As both sides of this equation are coderivations along a_t, propositions 1.1.5 and 1.2.6 imply that all we need to check is wether their postcompositioms by PR are equal. Since PR vanishes on $U\mathfrak{g} \otimes S_{\geq 2}\mathfrak{g}[1]$, we can restrict to length lower than 2. Let x be an element of $U\mathfrak{g}$ and g be in $\mathfrak{g}$:

  $$(db_t + b_t d)(x) = \sum_{(x)} \phi_t(x^{(1)})\mathrm{pr}(x^{(2)}) = \phi_t \star \mathrm{pr}(x)$$

  But the last point of proposition 1.2.9 tells us that $\frac{d}{dt}\phi_t = \phi_t \star \mathrm{pr}$ so that

  $$\mathrm{PR}(db_t + b_t d)(x) = \mathrm{pr}(\frac{d}{dt}\phi_t(x)) = \mathrm{PR}\frac{d}{dt}a_t(x)$$

which proves that (1) holds in length 0. For length 1, we have, thanks to the cocommutativity of the coproduct and the properties of ϕ_t listed in proposition 1.2.9:

$$\begin{aligned}(db_t + b_t d)(x \otimes g) = &\sum_{(x)} \phi_t(x^{(1)})\mathrm{pr}(x^{(2)}) \otimes A_t(x^{(3)}, g) - \phi_t(x^{(1)})A_t(x^{(2)}, g) \otimes \mathrm{pr}(x^{(3)}) \\ &- \phi_t(x^{(1)}) \otimes [\mathrm{pr}(x^{(2)}), A_t(x^{(3)}, g)] + \sum_{(xg)} \phi_t((xg)^{(1)}) \otimes \mathrm{pr}((xg)^{(2)}) \\ = &\sum_{(x)} \frac{d}{dt}\phi_t(x^{(1)}) \otimes A_t(x^{(2)}, g) + \phi_t(x^{(1)}) \otimes \mathrm{pr}(x^{(2)}g) - \sum_{(x)} \phi_t(x^{(1)}) \otimes [\mathrm{pr}(x^{(2)}), A_t(x^{(3)}, g)]\end{aligned}$$

But for any y in $U\mathfrak{g}$

$$\begin{aligned}\frac{d}{dt}A_t(y, g) &= -\sum_{(y)} \mathrm{pr}(y^{(1)})A_t(y^{(2)}, g) + \sum_{(y)} \phi_{-t}(y^{(1)})\phi_t((yg)^{(2)})\mathrm{pr}((yg)^{(3)}) \\ &= -\sum_{(y)}[\mathrm{pr}(y^{(1)}), A_t(y^{(2)}, g)] + \mathrm{pr}(yg)\end{aligned}$$

Thus

$$\begin{aligned}(db_t + b_t d)(x \otimes g) = &\sum_{(x)} \frac{d}{dt}\phi_t(x^{(1)}) \otimes A_t(x^{(2)}, g) + \phi_t(x^{(1)}) \otimes \frac{d}{dt}A_t(x^{(2)}, g) \\ = &\frac{d}{dt}a_t(x \otimes g)\end{aligned}$$

which obviously implies the desired equality by applying PR.

Finally, we have

$$sd + ds = I(b_t d + db_t) = I\frac{d}{dt}a_t = a_1 - a_0 = \mathrm{Id}_{C_*(\mathfrak{g})}$$

on $C_*(\mathfrak{g}) \subset C_*(\mathfrak{g})[t]$. □

1.3 The Koszul resolution

The Chevalley-Eilenberg resolution of $U\mathfrak{g}$ enables one to build a new chain-complex, this time consisting of $U\mathfrak{g}$-bimodules:

Definition 1.3.1. *The* ***Koszul resolution*** *of $U\mathfrak{g}$ is the complex of $U\mathfrak{g}$-bimodules $CK_*(\mathfrak{g})$ defined by*

$$CK_*(\mathfrak{g}) := U\mathfrak{g} \otimes S_*\mathfrak{g}[1] \otimes U\mathfrak{g}$$

with differential $d^K : CK_*(\mathfrak{g}) \to CK_{*-1}(\mathfrak{g})$ defined by

$$\begin{aligned}d^K(1 \otimes g_1 \wedge \cdots \wedge g_n \otimes 1) := &\sum_{i=1}^{n}(-1)^{i+1}(g_i \otimes g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge g_n \otimes 1 - 1 \otimes g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge g_n \otimes g_i) \\ &+ \sum_{1 \le i < j \le n} (-1)^{j+1} 1 \otimes g_1 \wedge \cdots \wedge [g_i, g_j] \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge g_n \otimes 1\end{aligned}$$

for all g_1, g_1, ..., g_n in $\mathfrak{g}$.

Proposition 1.3.2. *The degree +1 map $h : CK_*(\mathfrak{g}) \to CK_{*+1}(\mathfrak{g})$ defined in degree n by*

$$h(x \otimes g_1 \wedge \cdots \wedge g_n \otimes y) := \sum_{(x)} \int_0^1 dt\, \phi_t(x^{(1)}) \otimes \mathrm{pr}(x^{(2)}) \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n) \otimes \phi_{1-t}(x^{(n+3)})y$$

for all x, y in $U\mathfrak{g}$ and g_1, g_1, ..., g_n in $\mathfrak{g}$, is a contracting homotopy.

As a corollary, we recover the well known following fact (at least when $\mathfrak{g}$ is free over $\mathbb{K}$):

Corollary 1.3.3. *If $\mathfrak{g}$ is projective over $\mathbb{K}$, the Koszul resolution of $U\mathfrak{g}$ is a projective resolution of the $U\mathfrak{g}$-bimodule $U\mathfrak{g}$ via the product map*

$$\begin{array}{rcl} CK_0(\mathfrak{g}) = U\mathfrak{g}^{\otimes 2} & \to & U\mathfrak{g} \\ x \otimes y & \mapsto & xy \end{array}$$

2 An inverse to the antisymmetrization map

In this section, we drop the symbol $\sum_{(x)}$ in Sweedler's notation of iterated coproducts so that

$$x^{(1)} \otimes \cdots \otimes x^{(n)}$$

will stand for

$$\sum_x x^{(1)} \otimes \cdots \otimes x^{(n)}$$

2.1 The antisymmetrization morphism F_*

Definition 2.1.1. *The* ***bar resolution*** *of $U\mathfrak{g}$ is the complex of $U\mathfrak{g}$-bimodules $B_*(U\mathfrak{g})$ defined in degree n by*

$$B_n(U\mathfrak{g}) := U\mathfrak{g} \otimes U\mathfrak{g}^{\otimes n} \otimes U\mathfrak{g}$$

with differential $d^B : B_*(U\mathfrak{g}) \to B_{*-1}(U\mathfrak{g})$ defined by

$$\begin{aligned} d^B(a < x_1|\cdots|x_n > b) :=& ax_1 < x_2|\cdots|x_n > b + \sum_{i=1}^{n-1}(-1)^i a < x_1|\cdots|x_i x_{i+1}|\cdots|x_n > b \\ &+ (-1)^n a < x_1|\cdots|x_{n-1} > x_n b \end{aligned}$$

for all a, b, x_1, ..., x_n in $U\mathfrak{g}$. The notation $a < x_1|\cdots|x_n > b$ stands for the element $a\otimes x_1\otimes\cdots\otimes x_n\otimes b$ in $B_n(U\mathfrak{g}) = U\mathfrak{g}^{\otimes(n+2)}$ and $1 < x_1|\cdots|x_n > 1$ will be abbreviated in $< x_1|\cdots|x_n >$ in the sequel.

Proposition 2.1.2. *If $\mathfrak{g}$ is projective over $\mathbb{K}$, the bar resolution defined above is a projective resolution of the $U\mathfrak{g}$-bimodule $U\mathfrak{g}$ via the same map as $CK_*(\mathfrak{g})$.*

Definition 2.1.3. *The* ***antisymmetrization map*** *$F_* : CK_*(\mathfrak{g}) \to B_*(U\mathfrak{g})$ is the morphism of graded $U\mathfrak{g}$-bimodules defined in degree n by*

$$F_n(1 \otimes g_1 \wedge \cdots \wedge g_n \otimes 1) := \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) < g_{\sigma(1)}|\cdots|g_{\sigma(n)} >$$

for all g_1, ..., g_n in $\mathfrak{g}$, where Σ_n denotes the n-th symmetric group and $\operatorname{sgn}(\sigma)$ stands for the signature of a permutation σ.

Theorem 2.1.4. [Cartan-Eilenberg] *Suppose that $\mathfrak{g}$ is projective over $\mathbb{K}$. Then, the antisymmetrization map $F_* : CK_*(\mathfrak{g}) \to B_*(U\mathfrak{g})$ defined above is a morphism of projective resolutions of the $U\mathfrak{g}$-bimodule $U\mathfrak{g}$ over the identity map $\mathrm{Id}_{U\mathfrak{g}} : U\mathfrak{g} \to U\mathfrak{g}$.*

Denoting by $U\mathfrak{g}^{op}$ the opposite algebra of $U\mathfrak{g}$, this implies that

Corollary 2.1.5. *For any $U\mathfrak{g}$-bimodule M, the map $\mathrm{Id}_M \otimes F_* : M \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}^{op}} CK_*(\mathfrak{g}) \to M \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}^{op}} B_*(U\mathfrak{g})$ is an homotopy equivalence of chain complexes.*

2.2 Building a quasi-inverse to F_*

Definition 2.2.1. *Let $G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g})$ be the unique $U\mathfrak{g}$-bimodule map defined by induction on the homological degree via*

$$G_0 := \mathrm{Id} : B_0(U\mathfrak{g}) = U\mathfrak{g}^{\otimes 2} \to CK_0(\mathfrak{g}) = U\mathfrak{g}^{\otimes 2}$$

and

$$G_n(1 < x_1|\cdots|x_n > 1) := hG_{n-1}d^B(1 < x_1|\cdots|x_n > 1) \quad , \quad n > 0 \tag{2}$$

for all x_1, ..., x_n in $U\mathfrak{g}$.

Proposition 2.2.2. *The map $G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g})$ defined above is a morphism of resolutions of $U\mathfrak{g}$ over the identity map $\mathrm{Id}_{U\mathfrak{g}} : U\mathfrak{g} \to U\mathfrak{g}$.*

Thanks to the explicit formula defining h, one can get rid of the induction in definition 2.2.1:

Theorem 2.2.3. *The morphism of resolutions $G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g})$ defined above satisfies*

$$G_n(<x_1|\cdots|x_n>) = \int_{[0,1]^n} dt_1\cdots dt_n \Gamma_n(x_1^{(1)},\cdots,x_n^{(1)}) \otimes B_n^1(x_1^{(2)},\cdots,x_n^{(2)}) \wedge \cdots \wedge B_n^n(x_1^{(n+1)},\cdots,x_n^{(n+1)}) \otimes S\Gamma_n(x_1^{(n+2)},\cdots,x_n^{(n+2)}) x_1^{(n+3)}\cdots x_n^{(n+3)} \quad (3)$$

for all x_1, ..., x_n in $U\mathfrak{g}$. Here, $\Gamma_n : U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}[t_1,\cdots,t_n]$ and $B_n^i : U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}[t_1,\cdots,t_n]$, $1 \le i \le n$ are the operators defined by

$$\Gamma_n(y_1,\cdots,y_n) := \phi_{t_1}(y_1\phi_{t_2}(y_2\phi_{t_3}(y_3\cdots\phi_{t_n}(y_n)\cdots)))$$

and

$$B_n^i := \Gamma_n(y_1^{(1)},\cdots,y_n^{(1)})\frac{d\Gamma_n}{dt_i}(y_1^{(2)},\cdots,y_n^{(2)})$$

for all y_1, ..., y_n in $U\mathfrak{g}$.

Proof. Define $\tilde{G}_n$ to be the $\mathbb{K}[t_1,\cdots,t_n]$-linear map equal to the integrand under $\int_{[0,1]^n} dt_1\cdots dt_n$ of the right-hand side of (3). We have to proove that for all n, $G_n = \int_{[0,1]^n} dt_1\cdots, dt_n\tilde{G}_n$. Since both are bimodule maps that coincide in degree zero, we only have to check that the $\int_{[0,1]^n} dt_1\cdots dt_n\tilde{G}_n$'s satisfy the induction relation (2) on elementary tensors of the form $< x_1|\cdots|x_n >$.

Lemma 2.2.4. *For every $n \ge 0$ and y_1, ..., y_n, y in $U\mathfrak{g}$,*

$$h\tilde{G}_n(< y_1|\cdots|y_n > y) = 0$$

where it's undersood that h has been extended to $C_n(\mathfrak{g})[t_1,\cdots,t_n]$ by $\mathbb{K}[t_1,\cdots,t_n]$-linearity.

Proof. Proof of lemma 2.2.4. Let $h_t : C_*(\mathfrak{g})[t_1,\cdots,t_n] \to C_*(\mathfrak{g})[t,t_1,\cdots,t_n]$ be the $\mathbb{K}[t_1,\cdots,t_n]$ linear map defined by

$$h_t(a\otimes g_1\wedge\cdots\wedge g_n\otimes b) := \phi_t(a^{(1)})\otimes \mathrm{pr}(a^{(2)})\wedge A_t(a^{(3)},g_1)\wedge\cdots\wedge A_t(a^{(n+2)},g_n)\otimes\phi_{1-t}(a^{(n+3)})b$$

for all a, b in $U\mathfrak{g}$ and g_1, ..., g_n in $\mathfrak{g}$, so that

$$h\tilde{G}_n = \int_0^1 dt\, h_t\tilde{G}_n$$

on $C_*(\mathfrak{g})[t_1,\cdots,t_n]$. We have

$$\begin{aligned}
h_t\tilde{G}_n(<y_1|\cdots|y_n> y) =& h_t\left(\Gamma_n(y_1^{(1)},\cdots,y_n^{(1)})\otimes B_n^1(y_1^{(2)},\cdots,y_n^{(2)})\wedge\cdots\wedge B_n^n(y_1^{(n+1)},\cdots,y_n^{(n+1)})\otimes S\Gamma_n(y_1^{(n+2)},\cdots,y_n^{(n+2)})y_1^{(n+3)}\cdots y_n^{(n+3)}\right)\\
=& \phi_t\Gamma_n(y_1^{(1)},\cdots,y_n^{(1)})\otimes\mathrm{pr}\left(\Gamma_n(y_1^{(2)},\cdots,y_n^{(2)})\right)\wedge A_t\left(\Gamma_n(y_1^{(3)},\cdots,y_n^{(3)}),B_n^1(y_1^{(4)},\cdots,y_n^{(4)})\right)\wedge\cdots\\
&\cdots\wedge A_t\left(\Gamma_n(y_1^{(2n+1)},\cdots,y_n^{(2n+1)}),B_n^n(y_1^{(2n+2)},\cdots,y_n^{(2n+2)})\right)\otimes\phi_{1-t}(\Gamma_n(y_1^{(2n+3)},\cdots,y_n^{(2n+3)}))S\Gamma_n(y_1^{(2n+4)},\cdots,y_n^{(2n+4)})y_1^{(2n+5)}\cdots y_n^{(2n+5)}
\end{aligned}$$

But for all z_1, ..., z_n in $U\mathfrak{g}$, the identities of proposition 1.2.9 imply that

$$\begin{aligned}
A_t(\Gamma_n(z_1^{(1)},\cdots,z_n^{(1)}),B_n^1(z_1^{(2)},\cdots,z_n^{(2)}) =& \phi_{-t}(\Gamma_n(z_1^{(1)},\cdots,z_n^{(1)}))\phi_t(\frac{\partial\Gamma_n}{\partial t_1}(z_1^{(2)},\cdots,z_n^{(2)}))\\
=& t_1\mathrm{pr}(\Gamma_n(z_1,\cdots,z_n))
\end{aligned}$$

Thus, we see that by cocommutativity of the coproduct of $U\mathfrak{g}$, $h_t\tilde{G}_n(<y_1|\cdots|y_n> y)$ is invariant under the transposition that exchanges its first and second wedge factors, which implies that it must be zero.

□

We are now ready to proove that the $\int_{[0,1]^n} dt_1\cdots dt_n\tilde{G}_n$'s satisfy the induction relation (2). Indeed, the preceeding lemma implies that all terms but the first of $d^B(< x_1,\cdots,x_n >) = x_1 < x_2|\cdots|x_n > +\cdots$ are sent to zero under $h_t\tilde{G}_{n-1}$. Writing $\tilde{G}'_{n-1}$, Γ'_{n-1} and B'^i_{n-1} for the operators $\tilde{G}_{n-1}$, Γ_{n-1} and B^i_{n-1} where the variables t_1, ..., t_{n-1} have been changed to t_2, ..., t_n, one gets

$$\begin{aligned}
h_{t_1}\tilde{G}'_{n-1}d^B(<x_1|\cdots|x_n>) =& h_{t_1}\tilde{G}'_{n-1}(x_1 < x_2|\cdots|x_n >)\\
=&\ \phi_{t_1}(x_1^{(1)}\Gamma'_{n-1}(x_2^{(1)},\cdots,x_n^{(1)}))\otimes\mathrm{pr}\left(x_1^{(2)}\Gamma'_{n-1}(x_2^{(2)},\cdots,x_n^{(2)})\right)\wedge A_{t_1}\left(x_1^{(3)}\Gamma'_{n-1}(x_2^{(3)},\cdots,x_n^{(3)}),B'^1_{n-1}(x_2^{(4)},\cdots,x_n^{(4)})\right)\wedge\cdots\\
&\cdots\wedge A_{t_1}\left(x_1^{(n+2)}\Gamma'_{n-1}(x_2^{(2n+1)},\cdots,x_n^{(2n+1)}),B'^{n-1}_{n-1}(x_2^{(2n+2)},\cdots,x_n^{(2n+2)})\right)\otimes\\
&\otimes\phi_{1-t_1}(x_1^{(n+3)}\Gamma'_{n-1}(x_2^{(2n+3)},\cdots,x_n^{(2n+3)}))S\Gamma'_{n-1}(x_2^{(2n+4)},\cdots,x_n^{(2n+4)})x_2^{(2n+5)}\cdots x_n^{(2n+5)}
\end{aligned}$$

Since, for all z_1, ..., z_n in $U\mathfrak{g}$ and i in $\{1,\cdots,n-1\}$ the following identities hold

- $\phi_{t_1}(z_1\Gamma'_{n-1}(z_2,\cdots,z_n)) = \Gamma_n(z_1,\cdots,z_n)$,
- $\mathrm{pr}\left(z_1\Gamma'_{n-1}(z_2,\cdots,z_n)\right) = B_n^1(z_1,\cdots,z_n)$,
- $A_{t_1}\left(z_1\Gamma'_{n-1}(z_2^{(1)},\cdots,z_n^{(1)}), B'^{i}_{n-1}(z_2^{(2)},\cdots,z_n^{(2)})\right) = B_n^{i+1}(z_1,\cdots,z_n)$,
- $\phi_{1-t_1}(z_1\Gamma'_{n-1}(z_2^{(1)},\cdots,z_n^{(1)}))S\Gamma'_{n-1}(z_2^{(2)},\cdots,z_n^{(2)}) = S\Gamma_n(z_1^{(1)},z_2,\cdots,z_n)z_1^{(2)}$,

this leads to

$$h_{t_1}\tilde{G}'_{n-1}d^B(<x_1|\cdots|x_n>) = \tilde{G}_n(<x_1|\cdots|x_n>)$$

Thus, by an obvious change of variables, we get that

$$\begin{aligned}\int_{[0,1]^n} dt_1\cdots dt_n\,\tilde{G}_n(<x_1|\cdots|x_n>) &= \int_{[0,1]^n} dt_1\cdots dt_n\,h_{t_1}\tilde{G}'_{n-1}d^B(<x_1|\cdots|x_n>)\\ &= h\int_{[0,1]^{n-1}} dt_1\cdots dt_{n-1}\tilde{G}_{n-1}d^B(<x_1|\cdots|x_n>)\end{aligned}$$

which prooves that the right-hand side of (3) satifies the induction relation (2) and concludes the proof of theorem 2.2.3.

□

As a consequence of theorem 2.2.3 and proposition 2.2.2, we have the following

Corollary 2.2.5. *For any $U\mathfrak{g}$-bimodule M, the pair of maps*

$$\begin{tikzcd} M\otimes_{U\mathfrak{g}\otimes U\mathfrak{g}^{op}} B_*(U\mathfrak{g}) \arrow[r, two heads, "\mathrm{Id}_M\otimes G_*"'] & M\otimes_{U\mathfrak{g}\otimes U\mathfrak{g}^{op}} CK_*(\mathfrak{g}) \arrow[l, bend right, "\mathrm{Id}_M\otimes F_*"'] \end{tikzcd}$$

is a deformation retract of chain complexes.

Note that the preceeding corollary means that $G_*\circ F_* = \mathrm{Id}_{CK_*(\mathfrak{g})}$, which follows easily from the properties of the eulerian idempotent and the B_n^i's, and that there exists a graded map of $U\mathfrak{g}$-bimodules $H_* : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g})$ of degree $+1$ such that

$$H_*\circ d^B + d^B\circ H_* = F_*\circ G_* - \mathrm{Id}_{B_*(U\mathfrak{g})},$$

with the convention $B_{-1}(U\mathfrak{g}) := \{0\}$.

If the existence of H_* is a consequence of the fundamental lemma of calculus of derived functors, one may ask for an explicit formula for it, in view of further applications. It turns out that once again, the answer relies on the knowledge of some explicit contracting homotopy. Let's first recall the following standart result:

Definition-Proposition 2.2.6.

1. *The degree $+1$ graded $\mathbb{K}$-linear map $h^B : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g})$ defined in degree n by*

$$h^B(a<x_1|\cdots|x_n>b) := 1<a|x_1|\cdots|x_n>b$$

for all a, b, x_1, ..., x_n in $U\mathfrak{g}$, is a contracting homotopy of the bar resolution $B_*(U\mathfrak{g})$.

2. *Moreover, the graded map $\tilde{h} : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g})$ defined from h^B by*

$$\tilde{h} := h^B\circ d^B\circ h^B,$$

is still a contracting homotopy of $B_*(U\mathfrak{g})$, which satisfies in addition the gauge condition

$$\tilde{h}^2 = 0$$

Definition-Proposition 2.2.7. *Let $H_* : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g})$ be the unique graded endomorphism of $U\mathfrak{g}$-bimodule of degree $+1$ defined by induction on the degree n via*

$$H_0 := 0 : B_0(U\mathfrak{g}) \to B_1(U\mathfrak{g})$$

and

$$H_{n+1}(<x_1|\cdots|x_{n+1}>) := \tilde{h}\circ(F_{n+1}G_{n+1} - \mathrm{Id}_{B_{n+1}(U\mathfrak{g})} - H_nd^B)(<x_1|\cdots|x_{n+1}>) \quad , \quad n\geq 0 \tag{4}$$

for all x_1, ..., x_n in $U\mathfrak{g}$. Then H_* is a homotopy between F_*G_* and $\mathrm{Id}_{B_*(U\mathfrak{g})}$.

Proof. Let's proove that H_* satisfies

$$H_{n-1}d^B + d^B H_n = F_n G_n - \mathrm{Id}_{B_n(U\mathfrak{g})} \quad , n \geq 0 \tag{5}$$

by induction on n. For $n = 0$ we have

$$F_0 G_0 - \mathrm{Id}_{B_0(U\mathfrak{g})} = 0 = d^B H_0.$$

Assuming that (5) is true for all $0 \leq n \leq k$, using that $d^B\tilde{h} + \tilde{h}d^B = \mathrm{Id}$ in strictly positive degrees, we get

$$\begin{aligned} d^B H_{k+1}(< x_1|\cdots|x_{k+1} >) =& d^B\tilde{h}(F_{k+1}G_{k+1} - \mathrm{Id}_{B_{k+1}(U\mathfrak{g})} - H_k d^B)(< x_1|\cdots|x_{k+1} >) \\ =&(F_{k+1}G_{k+1} - \mathrm{Id}_{B_{k+1}(U\mathfrak{g})} - H_k d^B)(< x_1|\cdots|x_{k+1} >) \\ &- \tilde{h}d^B(F_{k+1}G_{k+1} - \mathrm{Id}_{B_{k+1}(U\mathfrak{g})} - H_k d^B)(< x_1|\cdots|x_{k+1} >) \end{aligned}$$

But, because F_*G_* is an endomorphism of chain complex and thanks to the induction hypothesis:

$$d^B(F_{k+1}G_{k+1} - \mathrm{Id}_{B_{k+1}(U\mathfrak{g})} - H_k d^B) = (F_k G_k - \mathrm{Id}_{B_k(U\mathfrak{g})} - d^B H_k)d^B = H_{k-1}(d^B)^2 = 0$$

Thus

$$d^B H_{k+1}(< x_1|\cdots|x_{k+1} >) = (F_{k+1}G_{k+1} - \mathrm{Id}_{B_{k+1}(U\mathfrak{g})} - H_k d^B)(< x_1|\cdots|x_{k+1} >)$$

which prooves that (5) is true for $n = k+1$, when applied to tensors of the form $< x_1|\cdots|x_{k+1} >$. As both sides of (5) are morphisms of bimodules, this implies that they have to coincide on the whole $B_{k+1}(U\mathfrak{g})$.

□

One could ask why, in the preceeding proposition, we have used h^B instead of $\tilde{h}$ to define the homotopy H_*, since the proof doesn't involve the gauge condition $\tilde{h}^2 = 0$. This choice of particular contraction is in fact motivated by the following result:

Proposition 2.2.8. *Denote by $C_* : B_*(U\mathfrak{g}) \to B_*(U\mathfrak{g})$ the endomorphism of graded bimodule $F_*G_* - \mathrm{Id}_{B_*(U\mathfrak{g})}$. The homotopy H_* defined in 2.2.7 satisfies*

$$H_n(< x_1|\cdots|x_n >) = \sum_{i=1}^{n-1}(-1)^{i+1}\tilde{h}(x_1(\tilde{h}(x_2\cdots\tilde{h}(x_{i-1}\tilde{h}C_{n-i+1}(< x_i|\cdots|x_n >))\cdots))) \tag{6}$$

for all x_1, ..., x_n in $U\mathfrak{g}$.

Proof. Let $\tilde{H}_* : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g})$ be the degree $+1$ endomorphism of bimodule defined by the right hand side of (6) on tensors of the form $< x_1|\cdots|x_n >$.

As $d^B(< x_1|\cdots|x_n >) = x_1 < x_2|\cdots|x_n > + R$, where R is a sum of tensors of the form $< y_2|\cdots|y_n > y_{n+1}$ on which $\tilde{h}\circ\tilde{H}_{n-1}$ vanishes because $\tilde{h}^2 = 0$, we see that

$$\tilde{h}(C_n - \tilde{H}_{n-1}d^B)(< x_1|\cdots|x_n >) = \tilde{h}C_n(< x_1|\cdots|x_n >) - \tilde{h}(x_1\tilde{H}_{n-1}(< x_2|\cdots|x_n >)) = \tilde{H}_n(< x_1|\cdots|x_n >)$$

which prooves that, as H_*, $\tilde{H}_*$ satisfies the induction relation (4). Since $\tilde{H}_0 = H_0 = 0$, they have to coincide on the whole $B_*(U\mathfrak{g})$. □

Corollary 2.2.9. *For all x in $U\mathfrak{g}$,*

$$H_1(< x >) = \int_0^1 dt \ < \phi_t(x^{(1)})|\mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) \quad - < 1|x >$$

Proof. Let x be an element of $U\mathfrak{g}$. Then

$$\begin{aligned} H_1(< x >) =& \tilde{h}C_1(< x >) \\ =& h^B d^B h^B\left(\int_0^1 dt\,\phi_t(x^{(1)}) < \mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) \quad - < x >\right) \\ =& h^B\Bigg(\int_0^1 dt\,\phi_t(x^{(1)}) < \mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) - < \phi_t(x^{(1)})\mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) + < \phi_t(x^{(1)}) > \mathrm{pr}(x^{(2)})\phi_{1-t}(x^{(3)}) \\ & - < 1 > x\Bigg) \\ =& h^B\left(\int_0^1 dt\,\phi_t(x^{(1)}) < \mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) - \int_0^1 dt\,\frac{d}{dt}\left(< \phi_t(x^{(1)}) > \phi_{1-t}(x^{(2)})\right) \quad - < 1 > x\right) \\ =& \int_0^1 dt \ < \phi_t(x^{(1)})|\mathrm{pr}(x^{(2)}) > \phi_{1-t}(x^{(3)}) \quad - < 1|x > \end{aligned}$$

□

Remark 2.2.10. *In fact, $h^b C_1 = \tilde{h} C_1$ so choosing h^b instead of $\tilde{h}$ in the definition of H_* would have led to the same result in degree 1. This doesn't seem to be true any longer in higher degrees, and it's not clear whether a compact formula like (6) could be obtained without the gauge condition.*