# ALGEBRAIC COMBINATORICS ON TRACE MONOIDS: EXTENDING NUMBER THEORY TO WALKS ON GRAPHS

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**Abstract.** Trace monoids provide a powerful tool to study graphs, viewing walks as words whose letters, the edges of the graph, obey a specific commutation rule. A particular class of traces emerges from this framework, the hikes, whose alphabet is the set of simple cycles on the graph. We show that hikes characterize undirected graphs uniquely, up to isomorphism, and satisfy remarkable algebraic properties such as the existence and unicity of a prime factorization. Because of this, the set of hikes partially ordered by divisibility hosts a plethora of relations in direct correspondence with those found in number theory. Some applications of these results are presented, including an immanantal extension to MacMahon's master theorem and a derivation of the Ihara zeta function from an abelianization procedure.

**Keywords:** Digraph; poset; trace monoid; walks; weighted adjacency matrix; incidence algebra; MacMahon master theorem; Ihara zeta function.
**MSC:** 05C22, 05C38, 06A11, 05E99

**1. Introduction.** Several school of thoughts have emerged from the literature in graph theory, concerned with studying walks (also known as paths) on graphs as algebraic objects. Among the numerous structures proposed over the years are those based on walk concatenation [3], later refined by nesting [13] or the cycle space [9]. A promising approach by trace monoids consists in viewing the directed edges of a graph as letters forming an alphabet and walks as words on this alphabet. A crucial idea in this approach, proposed by [5], is to define a specific commutation rule on the alphabet: two edges commute if and only if they initiate from different vertices. This construction yields a semi-commutative monoid which allows for a great flexibility in the walk structure while preserving the ability to distinguish between different walks composed of the same edges. A remarkable consequence of this construction is the existence of a stable subset of traces, formed by collections of cycles: the hikes. More specifically, hikes constitute a simplified trace monoid that carries most of the relevant information pertaining to the graph structure and, in the case of undirected graphs, all the information. We show that the simple cycles form the alphabet of the trace monoid of hikes, while its independence relation is characterized by vertex-disjointness.

Of fundamental importance for the trace-monoid of the hikes is the hitherto underappreciated prime-property satisfied by the simple cycles. Recall that an element of a monoid is prime if and only if, whenever it is factor of the product of two elements, then it is a factor of at least one of the two. The importance of the prime property lies in that because of it, the partially ordered set $P_G$ formed by the hikes ordered by divisibility is host to a plethora of algebraic relations in direct extension to number theory. This includes identities involving many more objects beyond the well-studied zeta and Möbius functions [5, 18], such as the von Mangoldt and Liouville functions. In this respect hikes are natural objects to consider, as most of their algebraic properties follow from analytical transformations of the weighted adjacency matrix. The study of the algebraic structures associated with hikes is the main subject of the present work. These structures provide an extended semi-commutative framework to number theory from which both well-known and novel relations in general combinatorics are derived as particular consequences.

The article is organized as follows. In Section 2, we present the theoretical setting required for the construction of hikes as elements of a specific trace monoid. We discuss some immediate consequences, such as the unicity of the prime decomposition, the hike analogous of the fundamental theorem of arithmetic. Section 3 is devoted to the study of

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algebraic relations between formal series on hikes. In particular, we introduce the walk von Mangoldt function $\Lambda$ in §3.1 and establish various results relating it with the zeta function of $P_G$ and the length of hikes. Further consequences of $\Lambda$ concerning totally multiplicative functions over the hikes are presented in §3.2. In particular, we obtain a closed form formula for the inverse of such functions and show that this result yields an immanantal extension to MacMahon's master theorem as a special case. This is then illustrated in §3.3 via the walk Liouville function. We establish the relation between these results and their number-theoretic counterparts in §3.4, by showing that there exists a class of graphs on which $P_G$ is isomorphic to the poset of integers ordered by divisibility.

In Section 4 we elucidate the connection between $P_G$ and the Ihara zeta function $\zeta_I$ of the graph $G$. This connection suggests that $P_G$ holds more information than $\zeta_I$, something we confirm in §4.2 by showing that $P_G$ determines undirected graphs uniquely, up to isomorphism.

Future perspectives and possible extensions of our work are discussed in the conclusion.

## 2. General setting.

Let $G = (V, E)$ be a directed graph with finite vertex set $V = \{v_1, \dots, v_N\}$ and edge set $E$, which may contain loops. Let $\mathsf{W} = (w_{ij})_{i,j=1,\dots,N}$ represent the weighted adjacency matrix of the graph, built by attributing a formal variable $w_{ij}$ to every pair $(v_i, v_j) \in V^2$ and setting $w_{ij} = 0$ whenever there is no edge from $v_i$ to $v_j$. In this setting, an edge is identified with a non-zero variable $w_{ij}$.

A walk, or path, of length $\ell$ from $v_i$ to $v_j$ on $G$ is a sequence $p = w_{ii_1}w_{i_1i_2}\cdots w_{i_{\ell-1}j}$ of $\ell$ contiguous edges. The walk $p$ is *open* if $i \neq j$ and *closed* (a cycle) otherwise. A walk $p$ is *self-avoiding*, or *simple*, if it does not cross the same vertex twice, that is, if the indices $i, i_1, \dots, i_{\ell-1}, j$ are mutually different (with the possible exception $i = j$ if $p$ is closed).

### 2.1. Partially commutative structure on the edges.

We endow the edges $w_{ij}$ with a partially commutative structure which allows the permutation of two edges only if they do not start from the same vertex. In this section, we discuss in details the motivations and implications of this structure to study walks and cycles on a graph. Most of the results are consequences of [5].

**Commutation rule:** Two different edges $w_{ij}$ and $w_{i'j'}$ commute if, and only if, $i \neq i'$.

The finite sequences of edges form a free partially commutative monoid $\mathcal{M}$, also called *trace monoid*, with alphabet $\Sigma_{\mathcal{M}} := \{w_{ij} : w_{ij} \neq 0\}$ and independence relation

$$I_{\mathcal{M}} = \{(w_{ij}, w_{kl}) : i \neq k\}.$$

This particular trace monoid is considered in [5]. A trace $t$, i.e. an element of $\mathcal{M}$, can be viewed as equivalence class in the free monoid generated by the edges. The equivalence relation is then defined as follows: two sequences of edges $s, s'$ are equivalent if $s$ can be obtained from $s'$ upon permuting edges with different starting points. Different elements of an equivalence class $t \in \mathcal{M}$ will be referred to as *representations* of a trace.

In this setting, a walk (a sequence of contiguous edges) may have non-contiguous representations. For instance, the walk $w_{12}w_{23}$ from $v_1$ to $v_3$ can be rewritten as $w_{23}w_{12}$ since $w_{23}$ and $w_{12}$ start from different vertices. In fact, an open walk always has a unique contiguous representation, as any allowed permutations of edges would break the contiguity. Surprisingly, the unicity of the contiguous representation no longer holds for closed walks. This consequence is an important feature of the partially commutative structure on the edges: two closed walks starting from different vertices define the same object if they can be obtained from one another by permuting edges with different starting points.

To illustrate this statement, consider the example pictured in Figure 1. There is

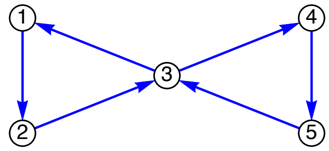



FIGURE 1. *The closed walks $c_1 = w_{34}w_{45}w_{53}w_{31}w_{12}w_{23}$ and $c_2 = w_{31}w_{12}w_{23}w_{34}w_{45}w_{53}$ are different although composed of the same edges. Both are achievable starting from $v_3$ but only $c_1$ is achievable from $v_1$.*

one closed walk starting from $v_1$ that covers every vertex exactly once, namely $c_1 := w_{12}w_{23}w_{34}w_{45}w_{53}w_{31}$. Since the only non-commuting edges are $w_{31}$ and $w_{34}$, the cycle can be rewritten as starting from $v_3$ by $c_1 = w_{34}w_{45}w_{53}w_{31}w_{12}w_{23}$. On the other hand, there are two closed walks starting from $v_3$ covering every edge once, one is $c_1$ and the other is $c_2 := w_{31}w_{12}w_{23}w_{34}w_{45}w_{53}$. One cannot go from $c_1$ to $c_2$ without permuting $w_{31}$ and $w_{34}$, thus $c_1 \neq c_2$. Here, the cycles $w_{12}w_{23}w_{31}$ and $w_{31}w_{12}w_{23}$ are equal since the permutations of the edges to go from one to the other are allowed. More generally, the starting vertex of a simple cycle never influences its value. This is no longer true if the cycle is not self-avoiding, as illustrated in Figure 1.

### 2.2. Multiplication and factorization of hikes.

A closed walk can be characterized as a contiguous sequence of edges comprising the same number of ingoing and outgoing edges for each vertex. A closed hike is obtained upon relaxing the connectedness condition:

DEFINITION 2.1. *A closed hike (or simply hike) is a trace $h = w_{i_1j_1}\cdots w_{i_\ell j_\ell} \in \mathcal{M}$ whose edges $w_{i_kj_k}$ satisfy for all $i = 1,\dots,N$,*

$$\sum_{k=1}^{\ell} \mathbb{1}\{i_k = i\} = \sum_{k=1}^{\ell} \mathbb{1}\{j_k = i\}, \tag{2.1}$$

*where $\mathbb{1}\{.\}$ stands for the indicator function.*

REMARK 2.1. Closed hikes correspond to the partially commutative extension of the homonymous objects introduced in [10]. While open hikes could be defined similarly as particular traces in $\mathcal{M}$, we choose to focus on closed hikes due to their more natural algebraic structure. For now on, closed hikes will be simply referred to as hikes.

We denote by $\mathcal{H}$ the set of hikes, which is a subset of $\mathcal{M}$. By convention, the trivial walk 1 viewed as the empty sequence is considered to be a hike. We emphasize that, since hikes are elements of $\mathcal{M}$, they obey the partially commutative structure on the edges: two hikes $h$ and $h'$ are equal if, and only if, $h'$ can be obtained from $h$ by permuting edges in $h$ with different starting point. In particular, while every cycle is a hike, a hike is a cycle only if it has a contiguous representation. In this case we say that $h$ is connected. Moreover, a hike is self-avoiding if and only if all its edges commute.

The multiplication of two hikes $h, h'$, simply defined as the concatenation, yields a hike and shall be denoted by $h.h'$ or simply $hh'$ in the sequel. We define hike division as the reverse operation: $d \in \mathcal{H}$ left divides $h \in \mathcal{H}$, which we write $d|h$, if there exists $h' \in \mathcal{H}$ such that $h = d.h'$. We shall use the standard division notation

$$h = d.h' \iff h' = \frac{h}{d}.$$

Here the choice of left-division, rather than right-division, is only a matter of convention. Remark that because the multiplication of hikes is not commutative, $d|h$ does not necessarily implies that $h/d$ divides $h$.

Theorem 2.2. *Every non-trivial hike $h$ has a representation as a product of simple cycles $h = c_1 \cdots c_k$. This decomposition is unique up to permutations of consecutive vertex-disjoint simple cycles.*

*Proof.* We start by showing that a non-trivial hike has at least one simple cycle as a divisor. For $v_i$ in $V(h)$, let $j(i)$ denote the end vertex of the first edge starting from $i$ in $h$. By (2.1), we know that $h$ contains an edge starting from $j(i)$. Thus, the path $w_{ij(i)}w_{j(i)j(j(i))}\cdots$ eventually returns to a previously visited vertex, resulting in a simple cycle $c$. Since the edges composing this cycle start from the first occurrences of vertices in $h$, $c$ divides $h$.

The simple cycles dividing $h$ are vertex-disjoint. Indeed, if two simple cycles $c, c'$ have a vertex in common, one can find different edges $e \in c$, $e' \in c'$ starting from the same vertex, so that $c$ and $c'$ cannot both divide $h$. Thus, the maximal self-avoiding divisor $s(h)$, defined as the product of the simple cycles dividing $h$, is unique and we have

$$h = s(h).h'$$

for some hike $h'$. If $h' = 1$, then $h = s(h)$ and the result holds. Otherwise, the process can be reiterated on $h'$ until all edges $w_{ij}$ in $h$ have been made part of a simple cycle. □

The representation of hikes as products of simple cycles is actually a *prime factorization*. Rigorously, an element $p$ of a monoid is prime if and only if, whenever $p$ is a factor of $a.b$, then $p$ is a factor of $a$ or $b$ or both. Prime hikes are directly identified using Theorem 2.2:

Corollary 2.3. *A hike $h$ is prime if and only if it is a simple cycle.*

Theorem 2.2 thus indicates that the prime factorization of hikes always exists and is unique. We emphasize that, because of the lack of commutativity, the prime factors of $h$, i.e. the elements of the prime-decomposition, are different from its prime divisors. Switching two different consecutive cycles in the prime-decomposition $h = c_1 \cdots c_k$ changes the value of $h$ as soon as $V(c_i) \cap V(c_{i+1}) \neq \emptyset$. This property highlights that $\mathcal{H}$ forms a sub-monoid of $\mathcal{M}$, whose alphabet is the set of prime hikes $\Sigma_{\mathcal{H}} := \{c_1, \ldots, c_k\}$ and with independence relation defined by

$$I_{\mathcal{H}} = \big\{(c_i, c_j) : V(c_i) \cap V(c_j) = \emptyset\big\}. \tag{2.2}$$

Self-avoiding hikes are the independence cliques in the commutation subgraph of $\mathcal{H}$ (see [1, 5] for more details). The maximal self-avoiding divisor of a hike $h$, defined as the product $s(h)$ of its prime divisors, is the first clique in the Cartier-Foata decomposition of $h$. This decomposition can be built recursively as follows. If $h$ is self-avoiding, then $s(h) = h$ and $h$ is its own Cartier-Foata decomposition. Otherwise, consider a collection of self-avoiding hikes $s_k$, initiated by $s_1 = s(h)$, and setting

$$s_{k+1} = s\Big(\frac{h}{s_1 \cdots s_k}\Big)$$

until all edges of $h$ are made part of a clique $s_k$.

The prime-decomposition of a hike $h = w_{i_1j_1} \cdots w_{i_\ell j_\ell}$ can be obtained simply by considering the simple cycles as they are formed in the edge sequence. This construction is somewhat similar to Lawler's loop-erasing procedure [15] which divides a closed walk into a finite sequence of simple cycles. By considering hikes, we argue that this decomposition remains natural when relaxing the connectedness condition, the other important point to our claim being that two consecutive cycles in the sequence can be permuted if they are vertex-disjoint.

In the sequel, $\ell(h)$ represents the length of a closed hike $h$ while the number of elements in its prime-decomposition is denoted by $\Omega(h)$. If $h$ is self-avoiding, $\Omega(h)$ is equal to its number of connected components. By convention, the trivial hike 1 is not prime and thus $\Omega(1) = 0$.

### 2.3. Hikes incidence algebra.

The hikes on a (di)graph $G$, ordered by division, form a locally finite partially ordered set, or *poset*, which we denote $P_G$. The reduced incidence algebra on this poset is the set $\mathcal{F}$ of real-valued functions on $\mathcal{H}$ endowed with the Dirichlet convolution

$$f * g(h) = \sum_{d|h} f(h)g\Big(\frac{h}{d}\Big)\ ,\ h \in \mathcal{H}.$$

Here, the sum is taken over all left-divisors $d$ of $h$, including $h$ itself and the trivial hike 1. One verifies easily that the Dirichlet convolution is associative and distributive over addition. However, it is not commutative since $d$ can divide $h$ without it being the case for $h/d$. The reduced incidence algebra is isomorphic to the algebra of formal series

$$\sum_{h\in\mathcal{H}} f(h)h\ ,\ f \in \mathcal{F}$$

endowed with hike multiplication. Indeed, for $f, g : \mathcal{H} \to \mathbb{R}$, we have

$$\Big(\sum_{h\in\mathcal{H}} f(h)h\Big).\Big(\sum_{h\in\mathcal{H}} g(h)h\Big) = \sum_{h\in\mathcal{H}} f * g(h)h.$$

Important functions of the reduced incidence algebra include the identity $\delta(.)$ equal to one for $h = 1$ and zero otherwise, the zeta function $\zeta(h) = 1\ ,\ \forall h \in \mathcal{H}$ or the Möbius function, the inverse of $\zeta$ through the Dirichlet convolution. We refer to [18] for a more comprehensive study of the zeta and Möbius functions of arbitrary posets. It is one of the main results of the present work that many more number-theoretic functions beyond $\zeta$ and $\mu$ have generalized analogs in the reduced incidence algebra $(\mathcal{F}, *)$ and that these analogs satisfy the same relations as their number-theoretic counterparts, see §3.

The next theorem gives the expression of the Möbius function on $\mathcal{H}$. This result is discussed in Remark 3.6 in [5]. Nevertheless, we provide an elementary proof for sake of completeness.

Theorem 2.4. *The Möbius function on $\mathcal{H}$ is given by*

$$\mu(h) := \begin{cases} 1 & \text{if } h = 1 \\ (-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

*Proof.* The Möbius function on $\mathcal{H}$ is the inverse of $\zeta$, i.e. the unique function such that $\mu(1) = 1$ and

$$\forall h \neq 1,\ \mu * \zeta(h) = \sum_{d|h} \mu(d)\zeta\Big(\frac{h}{d}\Big) = \sum_{d|h} \mu(d) = 0. \tag{2.4}$$

We need to verify that $\mu$ as defined in (2.3) satisfies this relation. Let $h \neq 1$ and $s(h)$ denote the largest self-avoiding divisor of $h$, i.e. the product of all its prime divisors. Since the self-avoiding divisors of $h$ are the divisors of $s(h)$ and $\mu(d) = 0$ whenever $d$ is not self-avoiding, it follows that $\mu*\zeta(h) = \mu*\zeta(s(h))$. So, it suffices to show the result for $h$ self-avoiding. We proceed by induction. If $h = c$ is a simple cycle, then we verify easily the relation

$$\mu * \zeta(c) = \mu(1) + \mu(c) = 1 - 1 = 0.$$

Now, let $c_1, \ldots, c_k, c_{k+1}$ be vertex-disjoint cycles and assume that (2.4) holds for $h = c_1 \ldots c_k$. We have

$$\sum_{d|h.c_{k+1}} \mu(d) = \sum_{d|h} \mu(d) + \sum_{d|h} \mu(d.c_{k+1}) = \sum_{d|h} \mu(d) - \sum_{d|h} \mu(d) = 0,$$

ending the proof. ◻

Theorem 2.4 confirms the characterization of $\mathcal{H}$ as the trace monoid generated by the alphabet of simple cycles $\Sigma_{\mathcal{H}} = \{c_1, \dots, c_k\}$ with independence relation defined in Equation (2.2) (see Chapter 2.5 in [19]), i.e. $\mathcal{H} = \Sigma_{\mathcal{H}}^*/I_{\mathcal{H}}$ where $\Sigma_{\mathcal{H}}^*$ is the Kleene star of $\Sigma_{\mathcal{H}}$. The formal series associated to the Möbius function for $\mathcal{H}$ then appears in the identity

$$\det(\mathsf{I} - \mathsf{W}) = \sum_{h \in \mathcal{H}} \mu(h) h, \tag{2.5}$$

a proof of which can be found in Theorem 1 of [17] on noting that for self-avoiding hikes, the concatenation of edges coincides with the ordinary multiplication. Theorem 2.4 thus provides a determinant formula for the Möbius function of $\mathcal{H}$ and the series associated to the zeta function is obtained via the formal inversion

$$\det(\mathsf{I} - \mathsf{W})^{-1} = \frac{1}{\sum_{h \in \mathcal{H}} \mu(h) h} = \sum_{h \in \mathcal{H}} \zeta(h) h = \sum_{h \in \mathcal{H}} h.$$

Remark 2.2 (Coprimality). The Möbius function is multiplicative on vertex-disjoint hikes,

$$V(h) \cap V(h') = \emptyset \implies \mu(hh') = \mu(h)\mu(h'). \tag{2.6}$$

This identity is reminiscent of the multiplicative property of the number-theoretic Möbius function $\mu_{\mathbb{N}}$ for which $\mu_{\mathbb{N}}(nm) = \mu_{\mathbb{N}}(n)\mu_{\mathbb{N}}(m)$ whenever $n$ and $m$ are coprime integers. The fact that (2.6) only holds for vertex-disjoint hikes suggests a more general notion of coprimality on $\mathcal{H}$: two hikes are coprime if they share no vertex in common. In particular, coprime hikes have different prime factors, but contrary to natural integers, this condition is in general not sufficient. The two notions of coprimality coincide on a class of graphs where $\mu_{\mathbb{N}}$ is recovered from $\mu$, see §3.4.

A determinantal expression for the Möbius function of certain trace monoids $\mathcal{M}$ that is similar to Eq. (2.5) was obtained in [6]. Nevertheless, these two results have different domains of validity and arise from different constructions. In [6], the expression of the Mobius function as $\det(\mathsf{I} - \mathsf{X})$ involves a matrix $\mathsf{X}$ whose entries are polynomials in the letters of the trace monoid. Furthermore, this formula holds if and only if the independence relation admits a transitive orientation (see [8]).

The situation is different for Eq. (2.5) since the weighted adjacency matrix $\mathsf{W}$ involves the edges of $G$, which are *subdivisions* of the simple cycles of $G$ and thus subdivisions of the letters of $\mathcal{H}$. In this setting, a transitive orientation is not necessary anymore for Eq. (2.5) to hold, since $\mathcal{H}$ is not necessarily transitively orientable (see Example 2.1 below). This means that the determinant formula Eq. (2.5) holds in situations where the result of [6] does not apply. On the other hand, the reverse is also true: there exist transitively orientable trace monoids which do not constitute hike trace monoids. For instance, one can show with little work that no digraph has the cycle on six vertices $C_6$ as hike commutation graph, yet one can easily construct a transitively orientable trace monoid with commutation graph $C_6$. Thus, our result and those of [6, 8] seem to be complementary. A complete characterization of hike trace monoids is beyond the scope of this work.

Example 2.1 (Hike trace monoid with no transitive orientation and a determinantal Möbius function). *Let $G$ be the cycle graph on 5 vertices illustrated in Figure 2. There are seven simple cycles on $G$: $a = w_{13}w_{31}$, $b = w_{24}w_{42}$, $c = w_{25}w_{52}$, $d = w_{14}w_{41}$, $e = w_{35}w_{53}$, $f = w_{13}w_{35}w_{52}w_{24}w_{41}$ and $g = w_{14}w_{42}w_{25}w_{53}w_{31}$.*


FIGURE 2. *The bidirected cycle graph on 5 vertices (left) and its hike commutation graph (right).*

*Therefore, the hikes on $G$ form the trace monoid $\mathcal{H}$ on seven letters $\Sigma_{\mathcal{H}} = \{a, b, c, d, e, f, g\}$ with independence relation*

$$I_{\mathcal{H}} = \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c), (d, e), (e, d), (e, a), (a, e)\}.$$

*The commutation graph of $\mathcal{H}$, presented on Figure 2 (right), is not a comparability graph since it contains a cycle of length 5 as an induced subgraph (see also Example 11.ii of [8]). Consequently, $I_{\mathcal{H}}$ does not admit a transitive orientation, yet Eq. (2.5) indicates that*

$$\sum_{h\in\mathcal{H}} \mu(h)h = 1 - a - b - c - d - e - f - g + ac + ad + bd + be + ce,$$

$$= \det(\mathsf{I} - \mathsf{W}) = \det \begin{pmatrix} 1 & 0 & -w_{13} & -w_{14} & 0 \\ 0 & 1 & 0 & -w_{24} & -w_{25} \\ -w_{31} & 0 & 1 & 0 & -w_{35} \\ -w_{41} & -w_{42} & 0 & 1 & 0 \\ 0 & -w_{52} & -w_{53} & 0 & 1 \end{pmatrix},$$

*that is, the Möbius function of $\mathcal{H}$ admits a determinantal form.*

## 3. Algebraic relations between series on hikes.

In this section we show that a plethora of number theoretic relations find natural extensions on the trace monoid of hikes. These provide powerful algebraic tools in a novel graph theoretic context and yield further insights into well established results. For example, we find in §3.2 that MacMahon's master theorem and the Dirichlet inverse of totally multiplicative functions over the integers both originate from the same general result about series of hikes. Throughout this section, $G$ designates a (di)graph and $P_G$ is the poset of hikes on $G$ ordered by divisibility.

DEFINITION 3.1. *We denote $\mathcal{S}f(s)$ the formal series $\mathcal{S}f(s) := \sum_{h\in\mathcal{H}} e^{-s\ell(h)} f(h)h$ associated to the function $f \in \mathcal{F}$. In particular we define $\zeta(s) := \mathcal{S}1(s)$.*

Recall that because of the lack of commutativity between hikes, Dirichlet convolution typically acts non-commutatively on functions on hikes $g * f \neq f * g$ and thus hike-series also multiply non-commutatively, i.e. $\mathcal{S}f.\mathcal{S}g = \mathcal{S}(f * g) \neq \mathcal{S}g.\mathcal{S}f = \mathcal{S}(g * f)$. For convenience, we write $\frac{\mathcal{S}f}{\mathcal{S}g}$ for the *right* multiplication with the inverse $\mathcal{S}f.(\mathcal{S}g)^{-1}$.

We begin with two simple relations counting the left divisors and left prime divisors of a hike:

PROPOSITION 3.2. *Let $\tau(h)$ be the number of left divisors of $h \in \mathcal{H}$. Then*

$$\mathcal{S}\tau(s) = \zeta^2(s).$$

*Let $1_p$ be the indicator function on primes and $\omega(h)$ the number of prime divisors of $h$. Then*

$$\mathcal{S}\omega(s) = \mathcal{S}1_p(s) \, . \, \zeta(s).$$

*Proof.* The results follow immediately from combinatorial arguments on the reduced incidence algebra of $P_G$. First, we have $\tau(h) = \sum_{d|h} 1 = (1 * 1)(h) = \zeta^2(h)$. For the second result, observe that $(\mathcal{S}1_p \,.\, \zeta)(h) = (1_p * 1)(h) = \sum_{d|h} 1_p(d)$ counts the distinct left prime divisors of $h$. □

While the relations satisfied by the functions $\tau$ and $\omega$ stem from straightforward combinatorial arguments, more advanced algebraic concepts also have natural extensions on the monoid of hikes. We begin with the von Mangoldt function on hikes.

### 3.1. Walk von Mangoldt function.

Definition 3.3 (Walk von Mangoldt function). *The walk von Mangoldt function $\Lambda : \mathcal{H} \to \mathbb{N}$ is defined as the number of connected representations of a hike that is, $\Lambda(h)$ is the number of walks in the equivalence class of the trace $h$.*

Equivalently, $\Lambda(h)$ is the number of possible contiguous rearrangements of the edges in $h$, obtained without permuting two edges with the same starting point. For a non-trivial hike $h$, the walk von Mangoldt function $\Lambda(h)$ is the coefficient of $h$ in the trace of the resolvent $\mathsf{R} := (\mathsf{I} - \mathsf{W})^{-1}$,

$$\Lambda(h) = (\operatorname{Tr} \mathsf{R})(h). \tag{3.1}$$

This follows immediately from the definition of $\Lambda$ together with the observations that the trace $\operatorname{Tr}(\mathsf{W}^\ell)$ generates all closed walks of length $\ell$ and $\mathsf{R} = \sum_{\ell \geq 0} \mathsf{W}^\ell$. By convention we set the value on the trivial hike to $\Lambda(1) = 0$.

In Section 3.4 we show that the simple Definition 3.3 is enough to recover the number theoretic von Mangoldt function on a special class of graphs. In this section we prove that the walk von Mangoldt function introduced above satisfies the same relations as its number-theoretic counterpart:

Proposition 3.4. *Let $G$ be a graph with $\zeta$ the zeta function of $P_G$. Then*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_h e^{-s\ell(h)} \Lambda(h) h, \quad \text{and} \quad \log \zeta(s) = \sum_{h:\, \ell(h) \neq 0} e^{-s\ell(h)} \frac{\Lambda(h)}{\ell(h)}\, h, \tag{3.2}$$

*where $\zeta'(s) = d\zeta(s)/ds$. Furthermore, the walk von Mangoldt function is the Möbius inverse of the walk length*

$$\Lambda = \ell * \mu. \tag{3.3}$$

*Proof.* We begin by proving Eqs. (3.2) of Proposition 3.4. Observe that

$$\operatorname{Tr}[e^{-sk} \mathsf{W}^k] = \sum_{w:\, \ell(w) = k} e^{-sk}\, w = \sum_{h:\, \ell(h) = k} e^{-s\ell(h)}\, \Lambda(h) h,$$

where $w$ is a walk. Then, since $\mathsf{R}(s) := (e^s \mathsf{I} - \mathsf{W})^{-1}$, we have $e^s \operatorname{Tr} \mathsf{R}(s) = N + \sum_h e^{-s\ell(h)} \Lambda(h) h$, where $N$ is the number of vertices of $G$. Now recall the relation between the trace of the resolvent and $\chi(s) := \det(e^s \mathsf{I} - \mathsf{W})$, the characteristic polynomial of $\mathsf{W}$ [4],

$$\operatorname{Tr} \mathsf{R}(s) = e^{-s} \frac{\chi'(s)}{\chi(s)},$$

with $\chi'(s) := d\chi(s)/ds$. Noting that $\zeta(s) = e^{sN} / \det(e^s \mathsf{I} - \mathsf{W}) = e^{sN} \chi(s)^{-1}$ thus leads to

$$-\sum_{h \in \mathcal{H}} e^{-s\ell(h)}\, \Lambda(h)\, h = N - e^s \operatorname{Tr} \mathsf{R}(s) = \frac{\zeta'(s)}{\zeta(s)}, \tag{3.4}$$

where $\zeta'(s) := d\zeta(s)/ds$. This is Eq. (3.2). To obtain the logarithm of $\zeta$, we may integrate Eq. (3.4),

$$-\int ds \sum_{h\in\mathcal{H}} e^{-s\ell(h)}\,\Lambda(h)\,h = \log\zeta(s),$$

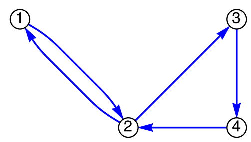
which gives $\log\zeta(s) = \sum_{h:\,\ell(h)\neq 0} e^{-s\ell(h)}\,\frac{\Lambda(h)}{\ell(h)}\,h$.

Eq. (3.4) also relates the walk von Mangoldt function to the length of individual walks. Indeed

$$-\sum_{h\in\mathcal{H}} e^{-s\ell(h)}\,\Lambda(h)\,h = \frac{\zeta'(s)}{\zeta(s)} \Rightarrow -e^{-s\ell(h)}\Lambda(h) = \Big(\zeta'(s)\zeta^{-1}(s)\Big)[h],$$
$$\Rightarrow \Lambda(h) = \sum_{d|h}\ell(d)\mu\Big(\frac{h}{d}\Big),$$

that is $\Lambda = \ell * \mu$. □

EXAMPLE 3.1. *To illustrate the relation $\Lambda = \ell * \mu$, consider the following graph on 4 vertices:*




*Let $p_1$ be the backtrack and $p_2$ the triangle and let us calculate $\Lambda(p_1p_2)$ and $\Lambda(p_2p_1)$ from $\ell * \mu$. Since the left divisors of $p_1p_2$ are 1, $p_1$ and $p_1p_2$, we have*

$$\Lambda(p_1p_2) = \ell(1)\mu(p_1p_2) + \ell(p_1)\mu(p_2) + \ell(p_1p_2)\mu(1),$$
$$= 0\times 0 + 2\times(-1) + 5\times 1 = 3.$$

*We proceed similarly for $\Lambda(p_2p_1)$:*

$$\Lambda(p_2p_1) = \ell(1)\mu(p_2p_1) + \ell(p_2)\mu(p_1) + \ell(p_2p_1)\mu(1),$$
$$= 0\times 0 + 3\times(-1) + 5\times 1 = 2.$$

*Let us now compare these results with a direct calculation of $\Lambda$, by way of counting all the walks in the equivalence classes $p_1p_2$ and $p_2p_1$. We find*

$$w_{21}w_{12}w_{23}w_{34}w_{42} \simeq p_1p_2, \qquad w_{12}w_{23}w_{34}w_{42}w_{21} \simeq p_2p_1,$$
$$w_{42}w_{21}w_{12}w_{23}w_{34} \simeq p_1p_2, \qquad w_{23}w_{34}w_{42}w_{21}w_{12} \simeq p_2p_1.$$
$$w_{34}w_{42}w_{21}w_{12}w_{23} \simeq p_1p_2,$$

*This confirms that $\Lambda(p_1p_2) = 3$ and $\Lambda(p_2p_1) = 2$, as expected.*

Within the framework presented here, the relation of Eq. (3.2) between the zeta function and the walk von Mangoldt function gives rise to a generalized Riemann-von Mangoldt explicit formula. This formula reduces to counting the walks on $G$ from the spectrum of its ordinary adjacency matrix $\mathsf{A}$. Remarkably, we show in Section 3.4 that the number-theoretic Riemann-von Mangoldt explicit formula can be interpreted in this way as well. This suggests that counting walks provides non-trivial information on the primes. In the spirit of the number-theoretic approach, one should be able to extract this information from a form of $\log\zeta$ that only involves the primes. In the case of the integers, total commutativity implies that this form stems from the (relatively simple) Euler product. The situation is much more complicated on arbitrary graphs, where the logarithm of the zeta function can be shown to be a branched continued fraction over the primes.[1]

[1]This result will be presented in a future work and stems from [13].

### 3.2. Totally multiplicative functions on hikes.

A consequence of the Möbius inversion between $\Lambda$ and $\ell$, $\Lambda = \ell * \mu$, concerns totally multiplicative function on hikes $f \in \mathcal{F}$. We say that $f$ is totally multiplicative if and only if $f(hh') = f(h)f(h')$ for all $h, h' \in \mathcal{H}$.

COROLLARY 3.5. *Let $f$ be a totally multiplicative function on hikes. Define $F(s) := \mathcal{S}f(s) = \sum_{h\in\mathcal{H}} e^{-s\ell(h)} f(h)h$ and $F'(s) := dF(s)/ds$. Then*

$$\frac{F'(s)}{F(s)} = -\sum_{h\in\mathcal{H}} e^{-s\ell(h)} \Lambda(h) f(h)h.$$

This corollary stems from a fundamental property of totally multiplicative functions over the hikes:

LEMMA 3.6. *Let $f$ be a totally multiplicative function and let $F(s) := \mathcal{S}f(s)$. Then*

$$F(s) = \frac{1}{\sum_{h\in\mathcal{H}} e^{-s\ell(h)} \mu(h) f(h)h}. \tag{3.5}$$

*Proof.* This follows from a direct calculation:

$$\begin{aligned} F(s)\sum_{h} e^{-s\ell(h)}\mu(h)f(h)h &= \sum_{h'} e^{-s\ell(h')} f(h')h' \sum_{h''} e^{-s\ell(h'')}\mu(h'')f(h'')h'', \\ &= \sum_{h} e^{-s\ell(h)} f(h)h \sum_{d|h} \mu(d), \\ &= 1, \end{aligned}$$

where $\sum_{d|h} \mu(d) = \delta(h)$ and $f(1) = 1$ since $f$ is totally multiplicative. Lemma 3.6 is reminiscent of the inverse of totally multiplicative functions in number theory, $f^{-1}(n) = \mu_{\mathbb{N}}(n)f(n)$, $n \geq 0$, with $\mu_{\mathbb{N}}$ the number-theoretic Möbius function. The relation between these two results is explained in Section 3.4.

Now let $F'(s) := dF(s)/ds$. Then using Eq. (3.5) for $F(s)$ we obtain $F'(s).F^{-1}(s)$ as

$$\begin{aligned} \frac{F'(s)}{F(s)} &= -\sum_{h'\in\mathcal{H}} e^{-s\ell(h')}\ell(h')f(h')h' \sum_{h''\in\mathcal{H}} e^{-s\ell(h'')}\mu(h'')f(h'')h'', \\ &= -\sum_{h\in\mathcal{H}}\sum_{d|h} e^{-s\ell(h)}\ell(d)\mu\Big(\frac{h}{d}\Big)f(h)h = -\sum_{h\in\mathcal{H}} e^{-s\ell(h)}\Lambda(h)f(h)h, \end{aligned}$$

where the last equality follows from Proposition 3.4, that is $\Lambda = \ell * \mu$. □

Lemma 3.6 giving the formal series of totally multiplicative functions on hikes constitute an important extension to MacMahon's master theorem. To see this, consider first a weighted version of the graph $G$ where all edges pointing to a vertex $i$ are given a formal weight $t_i$. The adjacency matrix of this weighted graph is $\mathsf{TA}$, with $\mathsf{T}$ the diagonal matrix where $\mathsf{T}_{ii} = t_i$. Now observe that a totally multiplicative function on hikes is completely determined by its value on the primes (since $f(hh') = f(h)f(h')$ regardless of the commutativity of $h$ and $h'$). We may therefore consider the totally multiplicative function which associates any prime $p$ with its weight,

$$f(p) = \text{weight}(p) = t_{i_2} \cdots t_{i_{\ell(p)}} t_{i_1}. \tag{3.6}$$

where $\{i_1, \cdots, i_{\ell(p)}\}$ is the set of vertices visited by $p$. Then Lemma 3.6 yields

$$\mathcal{S}f(1) := \sum_{h\in\mathcal{H}} f(h)h = \frac{1}{\sum_h \mu(h)f(h)h} = \frac{1}{\det(\mathsf{I} - \mathsf{TA})}, \tag{3.7}$$

where the last equality follows on noting that $f$ is equivalent to a map sending all primes to 1 on the graph with adjacency matrix $\mathsf{TA}$. This is the non-commutative generalization of MacMahon's theorem discovered by Cartier and Foata [5]. MacMahon's original result [16] is then recovered upon letting all $t_i$ variables commute.

In general, totally multiplicative functions on hikes do not have to take on the extremely restricted form of Eq. (3.6). In these cases Lemma 3.6 goes beyond even the non-commutative generalization of MacMahon's theorem. We present an explicit example illustrating this observation in the next section, where Lemma 3.6 yields a *permanent* in relation with a simple totally multiplicative function. More generally, Lemma 3.6 is capable of producing any matrix immanant:

Definition 3.7. *Let $\mathsf{M} = m_{ij}$ be a $N \times N$ matrix and $\chi_\lambda$ an irreducible character of the symmetric group $S_N$. Then the immanant of $\mathsf{M}$ associated with $\chi_\lambda$ is*

$$\text{Imm}_\lambda \mathsf{M} := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)}.$$

Now let $G$ be a (di)graph on $N$ vertices, $\mathsf{W}$ its weighted adjacency matrix and fix $\chi_\lambda$ an irreducible character of the symmetric group $S_N$.

Corollary 3.8. *Let $f \in \mathcal{F}$ be the totally multiplicative function on hikes which for any prime $p$ takes on the value $f(p) := -\chi_\lambda(p)$. Then*

$$\mathcal{S}f(s) = \frac{1}{1 - \chi_\lambda(1) + e^{-sN}\text{Imm}_\lambda(e^s\mathsf{I} - \mathsf{W})}.$$

*Proof.* Since $f$ is totally multiplicative for any non-trivial self-avoiding hike $h$, $f(h) = (-1)^{\Omega(h)}\chi_\lambda(h)$. Now $\mu(h) = (-1)^{\Omega(h)}$ for a self-avoiding hike $h$, and thus for all non-trivial self-avoiding hikes $f(h)\mu(h) = \chi_\lambda(h)$. The case of the trivial hike is peculiar because $f$ being totally multiplicative we necessary have $f(1) = 1$, while in general $\chi_\lambda(1) \neq 1$. Thus, Lemma 3.6, indicates that

$$\begin{aligned}
\mathcal{S}f(s)^{-1} &= 1 + \sum_{\substack{h \neq 1 \\ h \text{ self avoiding}}} e^{-s\ell(h)}\chi_\lambda(h)h, \\
&= 1 - \chi_\lambda(1) + \sum_{h \text{ self avoiding}} e^{-s\ell(h)}\chi_\lambda(h)h, \\
&= 1 - \chi_\lambda(1) + e^{-sN}\text{Imm}_\lambda(e^s\mathsf{I} - \mathsf{W}).
\end{aligned}$$

This gives the corollary. □

Corollary 3.8 is an immanantal extension to MacMahon's master theorem and holds for all weighted adjacency matrices $\mathsf{W}$, including $\mathsf{W} = \mathsf{TA}$. It should be noted however that Lemma 3.6 does not reduce to this immanantal extension since totally multiplicative functions may not be of the form $f(p) = -\chi_\lambda(p)$ for $p$ prime and thus may not give rise to an immanant.

### 3.3. Walk Liouville function.

The Liouville function of number theory is defined as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of the positive integer $n$. We define the walk Liouville function similarly:

Definition 3.9 (Walk Liouville function). *The walk Liouville function $\lambda(h) : \mathcal{H} \to \{-1, 1\}$ is defined by $\lambda(h) := (-1)^{\Omega(h)}$, where $\Omega(h)$ is the number of prime factors of hike $h$.*

The series $\mathcal{S}\lambda(s) := \sum_{h\in\mathcal{H}} e^{-s\ell(h)}\lambda(h)h$ associated to the walk Liouville function has a remarkably simple expression showing that calculating it is #P-complete on arbitrary graphs:

PROPOSITION 3.10. *Let $G$ be a graph on $n$ vertices with weighted adjacency matrix $\mathsf{W}$. Then the formal series $\mathcal{S}\lambda$ satisfies*

$$\mathcal{S}\lambda(s) = e^{sN}\frac{1}{\mathrm{perm}(e^s\mathsf{I}+\mathsf{W})},$$

*where* perm *designates the permanent.*

*Proof.* Observe that since $\Omega(h)$ is totally additive, the walk Liouville function is totally multiplicative. In addition, $f(p) = -1$ for all primes $p$, and $f$ is therefore the totally multiplicative function associated with the trivial character $\chi_1$ in Corollary 3.8. This gives the proposition. ◻

Since $\mathcal{S}\mu(s) = \sum_{h\in\mathcal{H}} e^{-s\ell(h)}\mu(h)h = e^{-sN}\det(e^s\mathsf{I}-\mathsf{W})$ we can express the formal series for the absolute value of $\mu$ as $\mathcal{S}|\mu|(s) = e^{-sN}\sum_{h\in\mathcal{H}} e^{-s\ell(h)}|\mu(h)|h$, which is immediately seen to be the permanent $e^{-sN}\mathrm{perm}(e^s\mathsf{I}+\mathsf{W})$. Hence, by Proposition 3.10 the walk Liouville function is the inverse of $\mathcal{S}|\mu|(s)$. We show in Section 3.4 that this implies the number theoretic result concerning the Liouville function as the Dirichlet inverse of the absolute value of the Möbius function.

### 3.4. Relation to number theory.

The unique factorization of hikes into products of hikes satisfying the prime property is evidently reminiscent of the fundamental theorem of arithmetic. The difference between these two results stems from the non-commutativity of the product operation between hikes. Unsurprisingly then, on a graph where all prime cycles commute, the prime factorization of hikes identifies with that of the integers and the poset $P_G$ becomes isomorphic to the poset of integers ordered by divisibility $P_{\mathbb{N}}$.

DEFINITION 3.11. *Let $\mathcal{G}_{\mathbb{N}}$ be the class of all directed graphs with no isolated vertices and comprising infinitely many simple cycles, all of which are vertex disjoint.*

Since all simple cycles of any $G \in \mathcal{G}_{\mathbb{N}}$ are vertex-disjoint, the trace monoid $\mathcal{H}$ formed by the hikes is free and Abelian.

THEOREM 3.12. *Let $G \in \mathcal{G}_{\mathbb{N}}$. Then $P_G$ is isomorphic to $P_{\mathbb{N}}$. Furthermore, the reduced incidence algebra of $P_G$, $(\mathcal{F}, *)$, is isomorphic to the algebra of Dirichlet series equipped with ordinary multiplication. The zeta and Möbius functions are sent by this isomorphism to*

$$\zeta \longrightarrow \zeta_R(s) = \sum_n \frac{1}{n^s},$$

$$\mu \longrightarrow \zeta_R(s)^{-1} = \sum_n \frac{\mu_{\mathbb{N}}(n)}{n^s},$$

*with $\zeta_R(s)$ the Riemann zeta function and $\mu_{\mathbb{N}}(n)$ the number-theoretic Möbius function.*

*Proof.* Let $\varphi : \mathcal{H} \longrightarrow \mathbb{N}$ be a map such that $h \in \mathcal{H}$ is a prime hike if and only if $\varphi(h)$ is a prime integer. Since $\mathcal{H}$ is free Abelian, concatenation acts commutatively and $\varphi$ is an isomorphism between $(\mathcal{H}, .)$ and $(\mathbb{N}, \times)$. Consequently, $G \in \mathcal{G}_{\mathbb{N}} \Rightarrow P_G \simeq P_{\mathbb{N}}$ and the reduced incidence algebras of $P_G$ and $P_{\mathbb{N}}$ are isomorphic as well. Finally, the reduced incidence algebra of $P_{\mathbb{N}}$ is known to be isomorphic to the algebra of Dirichlet series [18]. ◻

In this context, all results obtained in Section 3 yield valid number theoretic results on any $G \in \mathcal{G}_{\mathbb{N}}$. For example, Proposition 3.2 entails the following well-known equalities between Dirichlet series at the formal level[2]

$$\mathcal{D}\tau = \zeta_R^2(s), \quad \text{and} \quad \mathcal{D}\omega = \zeta_p(s)\,\zeta_R(s),$$

where $\mathcal{D}f := \sum_n f(n)/n^s$ designates the Dirichlet series associated with any function $f$ over the integers, $\tau(n)$ is the number of divisors of $n$ and $\zeta_p(s) = \sum_{p\text{ prime}} p^{-s}$ is the prime zeta function. Remark that on $G \in \mathcal{G}_{\mathbb{N}}$ all primes are vertex-disjoint and thus commute. It follows that all distinct prime factors of a hike are divisors of this hike and $\omega(h)$ is equal to the number of distinct prime factors of $h$.

The walk von Mangoldt function also yields its number theoretic counterpart. Indeed on $G \in \mathcal{G}_{\mathbb{N}}$ all closed walks are of the form $p^k$ with $p$ a prime cycle and $k$ an integer. Then Eq. (3.1) yields $\Lambda(p^k) = \ell(p)$, i.e.

$$\Lambda(h) = \begin{cases} \ell(p), & \text{if } h = p^k,\ p \text{ prime} \\ 0, & \text{otherwise.} \end{cases}$$

This is the number-theoretic von Mangoldt function provided we identify the length of a hike with the logarithm of an integer, i.e. provided that for every prime hike on $G \in \mathcal{G}_{\mathbb{N}}$, the isomorphism from $P_G$ to $P_{\mathbb{N}}$ sends $\ell(p)$ to $\log \varphi(p)$. Assuming this, the Riemann-von Mangoldt explicit formula can be interpreted as counting the closed walks on any $G \in \mathcal{G}_{\mathbb{N}}$. Indeed such walks are of the form $p^k$ with $p$ prime and therefore Proposition 3.4 gives $\log \zeta(s) = \sum_p \sum_k e^{-sk\log p} \frac{1}{k} p^k = \sum_w e^{-s\ell(w)} w$, hence $\log \zeta_R(s) = \varphi(\sum_w e^{-s\ell(w)} w)$.

## 4. Relation to the Ihara zeta and the characterisation of graphs.

The Ihara zeta function plays an important role in algebraic graph theory and network analysis as it was shown to relate to *some* properties of the graph [20]. In this section, we elucidate the relation between the zeta function of the poset of hikes ordered by divisibility and the Ihara zeta function. We then show that the poset $P_G$ and its zeta function $\zeta(s)$ determine undirected graphs uniquely, up to isomorphism.

### 4.1. Ihara zeta function.

The basic objects underlying the Ihara zeta function are certain equivalence classes defined over the closed walks of a graph, called primitive orbits [20]. We begin by recalling basic definitions pertaining to the primitive orbits.

Two closed walks are said to be equivalent if one can be obtained from the other upon changing its starting point and deleting its immediate backtracks, e.g. $w_{12}w_{23}w_{34}w_{43}w_{31} \simeq w_{23}w_{31}w_{12}$. The resulting equivalence classes on the set of all walks are called *backtrackless orbits*.[3] An orbit is *primitive* if and only if it is not a perfect power of another orbit, i.e. $p_o \not\simeq p_o'^k$, $k > 1$. The Ihara zeta function is then defined in analogy with the Euler product form of the Riemann zeta function as

$$\zeta_I(u) := \prod_{\tilde{p}_o \in \tilde{C}_G} \frac{1}{1 - u^{\ell(\tilde{p}_o)}},$$

where $\tilde{C}_G$ is the set of backtrackless primitive orbits on $G$. In the following it will be convenient to consider orbits for which immediate backtracks have been retained. In this case, two walks represent the same orbit if and only if one can be obtained from the other upon changing its starting point. In this situation $w_{12}w_{23}\underline{w_{34}w_{43}}w_{31}$ and $w_{12}w_{23}w_{31}$ define different (primitive) orbits. We denote by $C_G$ the set of primitive orbits including those with immediate backtracks.

[2] That is, irrespectively of questions of convergence.

[3] Backtrackless orbits are necessarily connected and may still have one or more backtracks as prime factors.

Primitive orbits *do not obey the prime property*, that is a primitive orbit may be a factor of the product of two walks $w.w'$ without being a factor of $w$ or $w'$ and the factorization of walks into products of primitive orbits is not unique. We further note that counting primitive orbits is indeed much easier than counting prime cycles:[4]

PROPOSITION 4.1. *Let $G$ be a graph and let $\pi_{C_G}(\ell)$ be the number of primitive orbits of length $\ell$ on $G$ with immediate backtracks retained. Then*

$$\pi_{C_G}(\ell) = \frac{1}{\ell}\sum_{n|\ell}\mu_{\mathbb{N}}(\ell/n)\,\mathrm{Tr}\,\mathsf{A}^n, \tag{4.1}$$

*where $\mu_{\mathbb{N}}$ is the number theoretic Möbius function and $\mathsf{A}$ is the adjacency matrix of $G$.*

REMARK 4.1. A similar result already exists for backtrackless primitive orbits, in this case $\mathsf{A}$ is replaced by the edge-adjacency matrix, see [20].

Before we prove Proposition 4.1, it is instructive to relate the zeta function $\zeta(s)$ of $P_G$ to the Ihara zeta function. We start with Eq (3.2),

$$\log\zeta(s) = \sum_{h:\,\ell(h)\neq 0} e^{-s\ell(h)}\,\frac{\Lambda(h)}{\ell(h)}\,h.$$

Observe that $\Lambda(h)$ is non-zero only if $h$ is connected. Furthermore, a connected hike either defines a primitive orbit or is a power of one, $h = p_o^k$, $k \geq 1$, where we write $p_o$ for a primitive orbit in order to avoid confusion with primes. Then we can recast Eq (3.2) as

$$\log\zeta(s) = \sum_{p_o\in C_G}\sum_{k>0} e^{-s\ell(p_o^k)}\,\frac{\Lambda(p_o^k)}{\ell(p_o^k)}\,p_o^k, \tag{4.2}$$

with $C_G$ the set of primitive orbits on $G$ (including those with immediate backtracks). There are $\ell(p_o)$ walks in the equivalence class $p_o^k$ since two walks are equivalent if and only if one can be obtained from the other upon changing its starting point. Then $\Lambda(p_o^k) = \ell(p_o)$ and Eq. (4.2) gives

$$\log\zeta(s) = \sum_{p_o\in C_G}\sum_{k>0}\frac{1}{k}e^{-sk\ell(p_o)}p_o^k, \tag{4.3}$$

Exponentiating the series above necessitates some precautions: being hikes, primitive orbits do not commute $p_o p_o' \neq p_o' p_o$ as soon as $p_o$ and $p_o'$ share at least one vertex. We will present the result of this exponentiation in a future work as it is sufficient for the purpose of relating $\zeta$ with $\zeta_I$ to bypass this difficulty by eliminating all formal variables. This is equivalent to substituting $\mathsf{W}$ with $\mathsf{A}$ in Eq. (4.3). This procedure immediately yields

$$\zeta_{\mathsf{A}}(s) := \frac{e^{sN}}{\det(e^s\mathsf{I}-\mathsf{A})} = \prod_{p_o\in C_G}\frac{1}{1-e^{-s\ell(p_o)}},$$

this being an Abelianization of $\zeta(s)$. Defining $u := e^{-s}$ now gives

$$\zeta_{\mathsf{A}}(u) = \frac{1}{\det(\mathsf{I}-u\mathsf{A})} = \prod_{p_o\in C_G}\frac{1}{1-u^{\ell(p_o)}}. \tag{4.4}$$

We separate the product above into a product over primitive orbits with no immediate backtracks, yielding the Ihara zeta function, and the product $\zeta_b(u) := \prod_{p_b\in C_G}(1-u^{\ell(p_b)})^{-1}$, involving primitive orbits $p_b$ with at least one immediate backtrack. This yields

$$\zeta_{\mathsf{A}}(u) = \zeta_I(u)\zeta_b(u),$$

[4]In contrast, just determining the existence of a prime cycle of length $n$ on a graph on $n$ vertices is known to be NP-complete, being the Hamiltonian cycle problem.

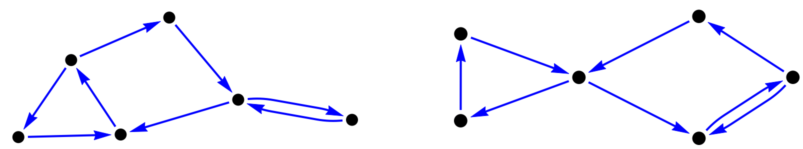


FIGURE 3. *Two non-isomorphic directed graphs with isomorphic posets of hikes ordered by divisibility and thus identical zeta functions. Note that this implies that these graphs are cospectral.*

which indicates that the Ihara zeta function originates from the unlabeled, Abelianized version $\zeta_{\mathsf{A}}$ of the zeta function $\zeta(s)$. Thus, we may expect $\zeta(s)$ or $P_G$ to hold more information on the graph $G$ than the Ihara zeta function does. In the next section we prove that this is indeed the case and that $P_G$ and $\zeta(s)$ determine undirected graphs uniquely.

*Proof.* [Proof of Proposition 4.1] Starting from Eq. (4.4) we have $\det(\mathsf{I}-u\mathsf{A}) = \prod_{j=1}^{\infty}(1-u^j)^{\pi_{C_G}(j)}$. Taking the logarithm on both sides yields

$$\sum_{i=1}^{\infty}\frac{1}{i}\,u^i\,\mathrm{Tr}\,\mathsf{A}^i = \sum_{j=1}^{\infty}\pi_{C_G}(j)\sum_{k=1}^{\infty}\frac{u^{kj}}{k},$$

and the result follows upon equating the coefficients of $z^\ell$ on both sides. Remark that the product expansion of $\zeta_{\mathsf{A}}$ over the primitive orbits also yields a Lambert series for the ordinary resolvent $\mathsf{R}(u) = (u\mathsf{I}-\mathsf{A})^{-1}$ of $G$,

$$u^{-1}\,\mathrm{Tr}\,\mathsf{R}(u^{-1}) = N + \sum_{\ell\geq 0}\ell\,\pi_{C_G}(\ell)\frac{u^\ell}{1-u^\ell}, \tag{4.5}$$

where $\pi_{C_G}(\ell)$ is the number of primitive orbits of length $\ell$ on $G$. This follows from Eq. (3.2) together with Eq. (4.4) for $\zeta_{\mathsf{A}}(u)$. □

### 4.2. The poset $P_G$ determines undirected graphs.

THEOREM 4.2. *Let $G$ be an undirected graph, $\mathsf{W}$ its weighted adjacency matrix and $\zeta(s)$ the zeta function of $P_G$. Then both $P_G$ and $\zeta(s)$ determine $G$ uniquely, up to isomorphism.*

The requirement that $G$ be undirected is essential: we show a counterexample to the theorem involving directed graphs on Figure (3). This leads to the curious observation that *there exist pairs of non-isomorphic directed graphs with exactly the same sets of closed hikes and in particular the same sets of closed walks.* Such pairs are cospectral and, by Lemma 3.6, all their immanantal polynomials are identical. It is a basic, rarely questioned, tenet of network analysis that walks accurately reflect the properties of the network on which they take place. Following this tenet most techniques used to distinguish networks are walk-based, see e.g. [7, 11, 12] and references therein. Yet, the failure of Theorem 4.2 on directed graphs shows that this tenet is incorrect, at least for closed walks, even if the graphs considered are strongly connected. Consequences of this observation in network analysis as well as methods to generate pairs of non-isomorphic digraphs with identical walk sets will be discussed in a future work.

*Proof.* [Proof of Theorem 4.2] We prove the theorem by showing that when $G$ is undirected, $P_G$ determines the line graph $L(G)$ of $G$. To this end, we prove that there exists an isomorphism between two posets $P_G$ and $P_H$ that preserves the length of individual hikes if and only if the line graphs of $G$ and $H$ are isomorphic. We will conclude the proof by relating $\zeta(s)$ and $P_G$.

Central to our proof is a graph encoding the relations of commutations between primes, which we call the $\gamma$-dual:

Definition 4.3. *Let $G$ be a graph and $\Gamma_G$ be the set of prime cycles on $G$. The $\gamma$-dual $\gamma_G$ of $G$ is the graph defined as*

*i) every prime $p \in \Gamma_G$ corresponds to a vertex $v(p)$ on $\gamma_G$;*

*ii) two vertices $v(p)$ and $v(p')$ of $\gamma_G$ share an edge if and only if $[p, p'] \neq 0$.*

The $\gamma$-dual of $G$ is the complement of the commutation graph of $\mathcal{H}$ defined in [1, 5]. Posets of hikes ordered by divisibility and $\gamma$-duals are essentially the same objects.

Lemma 4.4. *Let $G$ and $H$ be two graphs, $P_G$ and $P_H$ their posets of hikes ordered by divisibility and $\gamma_G$ and $\gamma_H$ their $\gamma$-duals. Then*

$$P_G \simeq P_H \iff \gamma_G \simeq \gamma_H.$$

*Proof.* We prove the forward direction. Two posets $P_1$ and $P_2$ are isomorphic, denoted $P_1 \simeq P_2$, if and only if there exists an order preserving bijective map $\Phi$ from $P_1$ to $P_2$, that is for all $x, y \in P_1$, $x \leq_{P_1} y \iff \Phi(x) \leq_{P_2} \Phi(y)$. Then $P_G \simeq P_H$ implies that there exists an order-preserving bijective map $\Phi : P_G \to P_H$. In particular:

i) $\Phi$ maps primes to primes: $p \in \Gamma_G \iff \Phi(p) \in \Gamma_H$. Posets of hikes ordered by divisibility are graded, the gradation being $\Omega(h)$ the number of prime factors of $h \in \mathcal{H}$. Since $\Phi$ is order preserving, it must be rank preserving $\Omega(h) = \Omega(\Phi(h))$ and thus maps primes to primes. Consequently $\Phi$ is a bijection between the prime sets $\Gamma_G$ and $\Gamma_H$.

ii) $\Phi$ preserves the commutation relations between primes: $p_1, p_2 \in \Gamma_G$, $[p_1, p_2] \neq 0 \iff [\Phi(p_1), \Phi(p_2)] \neq 0$. Suppose the opposite, i.e. there exists at least one pair of non-commuting primes $p_1, p_2 \in \Gamma_G$ with $\Phi(p_1p_2) = \Phi(p_2p_1)$. Since $\Phi$ is order preserving $p_1 \leq p_1p_2$ implies $\Phi(p_1) \leq_{P_H} \Phi(p_1p_2)$ and similarly $\Phi(p_2) \leq_{P_H} \Phi(p_2p_1)$. Then $\Phi(p_1p_2) = \Phi(p_2p_1)$ entails that both $\Phi(p_1)$ and $\Phi(p_2)$ lie under $\Phi(p_1p_2)$ in $P_H$, that is $\Phi^{-1}\big(\Phi(p_i)\big)$, $i = 1, 2$, are under $p_1p_2$ in $P_G$. $\Phi$ being a bijection between $\Gamma_G$ and $\Gamma_H$, $\Phi^{-1}\big(\Phi(p_i)\big) = p_i$ and both $p_1$ and $p_2$ are left prime factors of $p_1p_2$, a contradiction. Similarly, supposing that there exists at least one pair of commuting primes $p_1, p_2 \in \Gamma_G$ with $\Phi(p_1p_2) \neq \Phi(p_2p_1)$ leads to a contradiction.

From point i) it follows that $\Phi$ is a bijection between the sets of vertices of $\gamma_G$ and $\gamma_H$; and by point ii) this bijection preserves adjacency. Thus $\gamma_G$ and $\gamma_H$ are isomorphic graphs.

The backward direction works similarly, $\gamma_G \simeq \gamma_H$ implies that there exists an adjacency-preserving bijective mapping $\phi$ between the vertices of $\gamma_G$ and $\gamma_H$. We verify easily that $\phi$ induces an order preserving isomorphism between $P_G$ and $P_H$. ◻

We now turn to a specific class of isomorphisms between posets of hikes: those that preserve the length of individual hikes. That is, for two graphs $G$ and $H$ we say that $P_G$ and $P_H$ are *length isomorphic*, denoted $P_G \simeq_\ell P_H$, if and only if there exists an order preserving bijective map $\Phi$ from $P_G$ to $P_H$ and such that for all hikes $h \in \mathcal{H}$, $\ell(h) = \ell(\Phi(h))$. Since the length is totally additive, it is completely determined by its value on the primes. Thus, we may reflect the additional information carried by length-preserving isomorphisms on labeling the vertices of $\gamma_G$ with the length of the corresponding primes. That is, for $p \in \Gamma_G$, $v(p)$ is given the label $\ell(p)$. We denote $\ell\gamma_G$ the resulting labelled $\gamma$-dual.

By Lemma 4.4 a length-preserving isomorphism between $P_G$ and $P_H$ induces an isomorphism between labelled $\gamma$-dual graphs, i.e.

$$P_G \simeq_\ell P_H \iff \ell\gamma_G \simeq \ell\gamma_H.$$

This isomorphism in turn induces an isomorphism between the line graphs $L(G)$ and $L(H)$ since these are the induced subgraphs of $\ell\gamma_G$ and $\ell\gamma_H$ comprising all vertices with labels "1" or "2". It is known [14] that $L(G) \simeq L(H) \Rightarrow G \simeq H$ as long as $G, H \neq K_3$, $K_{1,3}$. This last condition can be dispensed with on noting that while $L(K_3)$ and $L(K_{1,3})$ are isomorphic, the labeled $\gamma$-duals $\ell\gamma_{K_3}$ and $\ell\gamma_{K_{1,3}}$ are not. Thus, $P_G \simeq_\ell P_H \Rightarrow \ell\gamma_G \simeq \ell\gamma_H \Rightarrow L(G) \simeq L(G) \Rightarrow G \simeq H$.

We complete the proof of the theorem by showing that $P_G$ can be recovered from $\zeta(s)$ alone. The zeta function determines the primes, their lengths and the relations of coprimality since

$$\zeta(s)^{-1} = \sum_{h\text{: self avoiding}} e^{-s\ell(h)}(-1)^{\Omega(h)} h. \tag{4.6}$$

This (trivial) observation becomes false if one replaces $\mathsf{W}$ with $\mathsf{A}$ in $\zeta(s)$ since we then lose the labels and the ability to identify the primes that comes with them. □

**5. Conclusion.** Our results demonstrate that an "algebraic theory of hikes" can be developed in close parallel to number theory. Although hikes only form a semi-commutative monoid, an equivalent to the fundamental theorem of arithmetic holds on it and implies a plethora of relations between formal series, with consequences in both general combinatorics and number theory. For example, we found that MacMahon's master theorem and the number-theoretic inverse of a totally multiplicative function $f$ over the integers $f(n)^{-1} = f(n)\mu_{\mathbb{N}}(n)$ [2], both originate from the same general result concerning hikes.

We believe that our approach also offers a novel perspective on outstanding open problems of enumerative combinatorics on graphs. Most notably, proving asymptotic estimates for the number of self-avoiding paths on infinite regular lattices corresponds to establishing the prime number theorem for hikes. In this respect, an "algebraic theory of hikes" would find itself in the situation of number theory in the mid 19th-century. Accordingly, partial progress towards asymptotic prime-counting may be possible via a better understanding of the relation between the zeta function of $P_G$ and the primes.

Some results pertaining to this relation have been left out of the present study because of length considerations and will be presented in future works. In particular, i) there exists an exact relation between $\zeta$ and the ordinary generating function of the primes; ii) $\zeta$ admits an infinite product expansion giving rise to a functional equation on at least some types of graphs; and iii) its logarithm is a branched continued fraction involving only the primes. Furthermore, $\zeta$-based systematic procedures for enumerating certain types of hikes are readily available. Observe indeed that Eq. (3.2) indicates that the set of hikes with non-zero coefficient in $\log\zeta$, called the support of $\log\zeta$, is the set of connected hikes (i.e. the walks). In fact, the logarithm of $\zeta$ is one of the simplest member of an infinite family of hypergeometric functions of $\zeta$, whose supports are sets of hikes obeying precise connectivity constraints. For example, the support of $2(\zeta - \log\zeta - 1)\zeta^{-1}$ is the set of hikes $h = p_1 \cdots p_n$ for which there exists a prime $p_i$ such that $h = p_1 \cdots p_{i-1}p_{i+1} \cdots p_n$ is non-connected. In every case, exploiting the spectral decomposition of $\zeta$ provides explicit Riemann-von Mangoldt formulas counting the hikes of the support from the spectrum of $\mathsf{A}$.

A more straightforward extension of our work concerns open hikes, defined similarly by relaxing the connectedness condition of open walks. Commutative versions of open hikes are studied in [10], where they are shown to appear naturally from manipulations of the weighted adjacency matrix, e.g. in Lemma 2.2. In our partially commutative framework, a characterization of open hikes would be the following: $h$ is an open hike from $v_i$ to $v_j$ if, and only if, $hw_{ji}$ is a closed hike (although this definition requires that $w_{ji}$ be given a non-zero value even if $v_j$ is not incident to $v_i$ in the digraph). Due to this simple connection, the algebraic structure on open hikes can be understood from the properties of closed hikes. In particular, the prime factorization of an open hike always exists and is unique, up to permutations of consecutive vertex-disjoint simple cycles. This factorization involves one self-avoiding path, the remainder of Lawler's loop-erasing procedure. Nevertheless, the slightly more complex algebraic structure of open hikes seems to exhibit less connections with number theory, diverting us from our original motivations.

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