On some anisotropic reaction-diffusion systems with $L^1$-data modeling the propagation of an epidemic disease

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Abstract

In this paper, we prove the existence of weak solutions for a reaction-diffusion system with general anisotropic diffusivities and transport effects, supplemented with either mixed boundary conditions or no-flux boundary conditions. Initial conditions and external forcing terms are in $L^1$; this does not allow us to use classical variational formulations. Our motivation is a mathematical model describing the spatial propagation in heterogeneous environments of Feline Immunodeficiency Virus, a feline retrovirus.

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1. Introduction

This paper is devoted to the mathematical analysis of a system of nonlinear partial differential equations arising in a population dynamics model describing the spread of an
epidemic disease through a heterogeneous habitat. Our motivation is a model describing
the spatial spread of Feline immunodeficiency virus (FIV) through a population of
domestic cats in heterogeneous domains.

Let us first consider a simple unstructured population dynamics problem, where \( p \)
is the density of population, given by the logistic equation

\[
p'(t) = (b - m - kp(t))p(t), \quad t \geq 0, \quad p(0) = p_0 > 0,
\]

herein, \( b \) is the (constant) natural birth rate, \( m \) is natural death rate and \( k > 0 \) is a
positive constant yielding a density dependent death rate \( \delta(p) = m + kp \). For \( b - m > 0 \),
\( K = (b - m)/k \) is the carrying capacity and \( p(t) \to K \) as \( t \to +\infty \).

More details concerning the propagation of FIV may be found in [7] and refer-
ences therein. We denote by \( u = (u, v, w) \) the respective densities of susceptible, exposed and infected individuals at time \( t \), \( p(t) = u(t) + v(t) + w(t) \)
being the total population density. Let \( \sigma(u, v, w) \) be the incidence function, i.e. the
recruitment of newly infected cats, \( \alpha \) the additional disease induced death rate in the
infected class, and \( 1/p \) the length of latency period or duration of the exposed stage.
When no spatial consideration is involved the dynamics of the propagation of FIV
within a population of cats is governed by the following system of ordinary differential
equations

\[
\begin{align*}
\frac{du}{dt} &= -\sigma(u, v, w) - (m + kp(t))u(t) + b(u(t) + v(t) + w(t)),
\frac{dv}{dt} &= +\sigma(u, v, w) - (m + kp(t))v(t) - \rho v(t),
\frac{dw}{dt} &= -(m + kp(t))w(t) + \rho v(t) - \alpha w(t).
\end{align*}
\]

Note that there is no vertical transmission, this is all offsprings are susceptible at
birth.

The incidence term can take one of the following forms: mass action where \( \sigma(u, v, w) = \sigma_1 vw \) for some \( \sigma_1 > 0 \), proportionate mixing where \( \sigma(u, v, w) = \sigma_2 uv/p \) for some
\( \sigma_2 > 0 \), or a more general form \( \sigma(u, v, w) = \sigma_3 v^p/(1 + p^v)(uv/p) \) for some \( \sigma_3 > 0 \) and
\( v > 0 \).

Let us consider a spatial and bounded open domain \( \Omega \) in \( \mathbb{R}^N \), \( N > 1 \), with Lipchitz
boundary \( \partial \Omega \) such that \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( |\Gamma_0| > 0 \); \( \eta \) is the outward unit
normal to \( \Omega \) on \( \partial \Omega \). We set \( Q_T = (0, T) \times \Omega \) for \( T > 0 \). Our state variables \( (u, v, w) \) are
the space and time dependent densities \( (u(t,x), v(t,x), w(t,x)) \), \( x \in \Omega, t > 0 \) of suscep-
tible, exposed and infected individuals, from which the time dependent densities are
calculated upon integrating over the spatial domain \( \Omega \)

\[
\int_{\Omega} u(t,x) \, dx, \quad \int_{\Omega} v(t,x) \, dx, \quad \int_{\Omega} w(t,x) \, dx.
\]

We assume the spatial habitat to be heterogeneous. We are led to consider spatially
natural birth rate, \( b(t,x) \), natural death rate, \( m(t,x) \), disease induced death rate in the
infected class, \( \alpha(t,x) \), and rate at which infected cats enter the third stage, \( \rho(t,x) \).
When we have an external supply of individuals, $f \geq 0$, $g \geq 0$ and $h \geq 0$ the dynamics of the cat-(FIV) system is governed by the set of semilinear equations in $(0, T) \times \Omega$,
\[
\partial_t u(t, x) - \text{div}(A_1(t, x, \nabla u) + u(t, x) K_1(t, x)) + r_1(t, x, u, v, w) = - \sigma(t, x, u, v, w) - m(t, x) u(t, x) + b(t, x)(u(t, x) + v(t, x) + w(t, x)) + f(t, x),
\]
\[
\partial_t v(t, x) - \text{div}(A_2(t, x, \nabla v) + v(t, x) K_2(t, x)) + r_2(t, x, u, v, w) = \sigma(t, x, u, v, w) - (m(t, x) + \rho(t, x)) v(t, x) + g(t, x),
\]
\[
\partial_t w(t, x) - \text{div}(A_3(t, x, \nabla w) + w(t, x) K_3(t, x)) + r_3(t, x, u, v, w) = \rho(t, x) v(t, x) - (m(t, x) + \alpha(t, x)) w(t, x) + h(t, x),
\]
where $A_i$, $K_i$ and $r_i$ $(i = 1, 2, 3)$ represent the diffusivity field, the transport vector and the density dependent mortality rate in each class.

In this work we distinguish two types of boundary conditions: either

- **mixed boundary conditions** on $(0, T) \times \partial \Omega$

\[
\begin{cases}
(A_1(t, x, \nabla u) + u(t, x) K_1(t, x)) \cdot \eta(x) = 0 & \text{on } (0, T) \times \Gamma_1, \\
(A_2(t, x, \nabla v) + v(t, x) K_2(t, x)) \cdot \eta(x) = 0 & \text{on } (0, T) \times \Gamma_1, \\
(A_3(t, x, \nabla w) + w(t, x) K_3(t, x)) \cdot \eta(x) = 0 & \text{on } (0, T) \times \Gamma_1, \\
K_i(t, x) \cdot \eta(x) \geq 0, & \text{for } i = 1, 2, 3 \text{ on } (0, T) \times \Gamma_1, \\
u(t, x) = v(t, x) = w(t, x) = 0 & \text{on } (0, T) \times \Gamma_0,
\end{cases}
\]  

(1.2)

- **no-flux boundary conditions** on $(0, T) \times \partial \Omega$

\[
\begin{cases}
(A_1(t, x, \nabla u) + u(t, x) K_1(t, x)) \cdot \eta(x) = 0, \\
(A_2(t, x, \nabla v) + v(t, x) K_2(t, x)) \cdot \eta(x) = 0, \\
(A_3(t, x, \nabla w) + w(t, x) K_3(t, x)) \cdot \eta(x) = 0, \\
K_i(t, x) \cdot \eta(x) \geq 0, & \text{for } i = 1, 2, 3.
\end{cases}
\]  

(1.3)

A set of initial conditions at $t = 0$ in $\Omega$ is prescribed:
\[
u(0, x) = v_0(x), \quad w(0, x) = w_0(x).
\]  

(1.4)

**Assumptions.** To contrast with previous works on reaction-diffusion systems motivated by population dynamics problems, [2,3], in this work we shall consider anisotropic systems. We confine ourselves to a model where the diffusivity $A_i$, $i = 1, 2, 3$, is a
Carathéodory function in \((0,T) \times \Omega \times \mathbb{R}^N\) whose components are \(a_{l,i}\) for \(l = 1,\ldots,N\) and \(i = 1,2,3\), satisfying for \(\zeta \in \mathbb{R}^N\):

\[
\begin{aligned}
\left\{
\begin{array}{l}
\text{there exists } p_l > 1, \\
a_{l,i}(t,x,\zeta) = \beta_{l,i}(t,x)|\zeta_i|^{p_l-2}\zeta_i,
\end{array}
\right.
\tag{1.5}
\end{aligned}
\]

herein the nonnegative function \(\beta_{l,i}\) is bounded on \(Q_T\). We assume there exists a real positive constant \(a\), such that for \(i = 1,2,3\) and for any \(\zeta \in \mathbb{R}^N\):

\[
A_i(t,x,\zeta) \cdot \zeta \geq a \sum_{l=1}^N |\zeta|^{p_l}, \quad \text{a.e. } (t,x) \in Q_T.
\tag{1.6}
\]

The transport vector \(K_i\) \((i = 1,2,3)\) is bounded on \(Q_T\) and satisfies

\[
K_i \in (L^\infty(Q_T))^N \quad \text{and} \quad \text{div}(K_i) \in L^\infty(Q_T) \quad \text{for } i = 1,2,3.
\tag{1.7}
\]

The functions \(m, b, \rho \) and \(\sigma\) are defined on \(Q_T\) with values in \(\mathbb{R}_+\) and satisfy

\[
m,b,\rho \quad \text{and} \quad \sigma \in L^\infty(Q_T).
\tag{1.8}
\]

The density dependent mortality rates have the following form

\[
\begin{aligned}
r_1(t,x,u,v,w) &= k_1(t,x)u|u + v + w|^{p_u-1}, \\
r_2(t,x,u,v,w) &= k_2(t,x)v|u + v + w|^{p_v-1}, \\
r_3(t,x,u,v,w) &= k_3(t,x)w|u + v + w|^{p_w-1},
\end{aligned}
\tag{1.9}
\]

where \(p_\theta\), for \(\theta = u,v,w\), satisfies

\[
p_\theta \geq \max_{1 \leq l \leq N} \left( \frac{p_l}{p_l - 1}, p_l \right) > 1,
\tag{1.10}
\]

and the function \(k_i\), \(i = 1,2,3\), defined on \(Q_T\) with values in \(\mathbb{R}_+\) satisfies

\[
k_i \in L^\infty(Q_T) \quad \text{and} \quad k_i(t,x) \geq k_0 > 0 \text{ a.e. } (t,x) \in Q_T \quad \text{for } i = 1,2,3.
\tag{1.11}
\]

Last, \(\sigma : Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+\) is measurable on \(Q_T\), continuous with respect to \(u, v\) and \(w\), a.e. in \(Q_T\) and satisfies a growth condition

\[
\begin{aligned}
\left\{
\begin{array}{l}
\text{there exists two bounded functions } L, M : \mathbb{R}^N \times (0,\infty) \rightarrow (0,\infty), \\
\text{and } s',s'', \in \mathbb{R}_+ \text{ such that}
\end{array}
\right.
\end{aligned}
\tag{1.12}
\]

\[
1 \leq s < \max_{1 \leq l \leq N} \left( \frac{p_l}{\bar{p}} \left( \frac{N}{\bar{p}} - \frac{N}{N + 1} \right), p_u, p_v, p_w \right) \text{ and}
\]

\[
|\sigma(t,x,u,v,w)| \leq L(t,x)(|u|^{s'}|v|^s + |u|^{s''}|w|^s) + M(t,x) \quad \text{a.e. } (t,x) \in Q_T,
\]
where \( \frac{1}{p} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{p_l} \), and a nonnegativity condition

\[
\begin{align*}
\sigma(t,x,0,v,w) &= 0 \quad \text{if } v \geq 0 \text{ and } w \geq 0, \\
\sigma(t,x,u,0,w) &= 0 \quad \text{if } u \geq 0 \text{ and } w \geq 0, \\
\sigma(t,x,u,v,0) &= 0 \quad \text{if } u \geq 0 \text{ and } v \geq 0.
\end{align*}
\]

(1.13)

In the logistic case \((p_u = p_v = p_w = 2)\) and for isotropic equations \((p_l = 2 \text{ for } l = 1 \ldots N)\), existence results are established in [10] with \(L^\infty\)-data.

Assuming \(\sigma = 0\), \(r_i = 0\) for \(i = 1,2,3\), \(p_l = p > 1\) for \(l = 1 \ldots N\) and under Dirichlet boundary conditions, existence results for the corresponding nonlinear elliptic and parabolic equations with right hand side measure data are established in [4,5,14]. We recall that for the case of a single equation, e.g. \(\sigma = 0\), \(r_i = 0\) for \(i = 1,2,3\), with no advection terms and under Dirichlet boundary conditions, existence results for weak solution with \(L^\infty\)-data have been obtained for anisotropic elliptic equations by [1] and [16]. For the corresponding anisotropic parabolic equations, the existence of a solution is established in [13]. In [6] the authors obtained an existence result for the elliptic equations with \(L^1\) data. For the corresponding anisotropic parabolic equations with measure data, the existence of a solution is established in [9].

In this paper, we extend the results of [9] for anisotropic parabolic equations to anisotropic reaction-diffusion-advection systems and obtain the appropriate function space for solutions. The anisotropic parabolic system (1.1) is a generalization of the isotropic parabolic system studied in [2], modeling Feline Leukemia Virus.

The plan of the paper is as follows. Section 2 is devoted to statement of main results corresponding to the case of mixed boundary conditions and to the case no-flux boundary conditions. Some preliminary results are given in Section 3. Main results are proved in Section 4 (for boundary condition (1.2)) and Section 5 (for boundary condition (1.3)).

2. Main results

We denote

\( L^1_+(\Omega) = \{ u \in L^1(\Omega), \ u \geq 0 \ \text{a.e. in } \Omega \}, \)

\( D_+(\Omega) \) the space of nonnegative functions in \( C_0^\infty(\Omega) \),

\( W^{1,p}_{F_0}(\Omega) = \{ u \in W^{1,p}(\Omega) \mid u = 0 \ \text{on } F_0 \} \).

Set \( W^{1,p^1}_{F_0}(\Omega) = \{ u \in W^{1,1}(\Omega) \mid (\partial u/\partial x_l) \in L^p(\Omega) \} \) the anisotropic Sobolev space, with

\[
\| u \|_{W^{1,p^1}(\Omega)} = \| u \|_{W^{1,1}(\Omega)} + \left\| \frac{\partial u}{\partial x_l} \right\|_{L^p(\Omega)},
\]
and let

\[ W_{1;0}^{1,p,i}(\Omega) = \{ u \in W^{1,p,i}(\Omega) | u = 0 \text{ on } \Gamma_0 \}. \]

We denote \( C^1_c([0,T] \times \Omega) \) the set of all \( C^1 \)-functions with compact support in \([0,T) \times \Omega\).

### 2.1. The case of mixed boundary conditions

In this section we give the definition of a weak solution for nonlinear parabolic systems of type (1.1)–(1.2)–(1.4). Then, we supply our existence result.

We recall \( \frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \).

**Definition 2.1.** Let \( 1 \leq q_i < (p_i/\bar{p})(\bar{p} - N/(N + 1)), \ l = 1 \ldots N \). A weak solution of (1.1)–(1.2)–(1.4), is a triple \( (u,v,w) \) of nonnegative functions, with \( \theta = u,v,w \) belonging to \( \bigcap_{i=1}^{N} L^{q_i}(0,T; W_{1;0}^{1,q_i,i}(\Omega)) \cap L^{p_i}(0,T; L^{p_i}(\Omega)) \cap C([0,T]; L^{l}(\Omega)), \ l = 1, \ldots, N, \) such that \( \sigma(\cdot,u,v,w) \) and \( r_i(\cdot,u,v,w) \), for \( i = 1,2,3 \), belong to \( L^1(\Omega_T) \), and satisfying

\[
\begin{align*}
&- \int_0^T \int_\Omega u \varphi_t \, dx \, dt + \int_0^T \int_\Omega (A_1(t,x,\nabla u) + uK_1(t,x)) \cdot \nabla \varphi \, dx \, dt \\
&+ \int_0^T \int_\Omega m u \varphi \, dx \, dt - \int_0^T \int_\Omega b(u + v + w) \varphi \, dx \, dt \\
&+ \int_0^T \int_\Omega \sigma(t,x,u,v,w) \varphi \, dx \, dt + \int_0^T \int_\Omega r_1(t,x,u,v,w) \varphi \, dx \, dt \\
&= \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_\Omega \varphi(0,x)u_0(x) \, dx, \quad (2.1)
\end{align*}
\]

\[
- \int_0^T \int_\Omega v \psi \, dx \, dt + \int_0^T \int_\Omega (A_2(t,x,\nabla v) + vK_2(t,x)) \cdot \nabla \psi \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega (m + \rho) v \psi \, dx \, dt - \int_0^T \int_\Omega \sigma(t,x,u,v,w) \psi \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega r_2(t,x,u,v,w) \psi \, dx \, dt = \int_0^T \int_\Omega g \psi \, dx \, dt + \int_\Omega \psi(0,x)v_0(x) \, dx, \quad (2.2)
\]

\[
- \int_0^T \int_\Omega w \chi \, dx \, dt + \int_0^T \int_\Omega (A_3(t,x,\nabla w) + wK_3(t,x)) \cdot \nabla \chi \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega (m + \zeta) w \chi \, dx \, dt - \int_0^T \int_\Omega \rho v \chi \, dx \, dt + \int_0^T \int_\Omega r_3(t,x,u,v,w) \chi \, dx \, dt
\]

\[
= \int_0^T \int_\Omega h \chi \, dx \, dt + \int_\Omega \chi(0,x)w_0(x) \, dx, \quad (2.3)
\]

for all \( \varphi, \psi, \chi \in C^1_c([0,T) \times \Omega) \).
Theorem 2.1. Assume that (1.5)–(1.13) hold and \(2 - \frac{1}{N+1} < p_l < \frac{p(N+1)}{N} \) \((l = 1 \ldots N)\). Let \(u_0, v_0, w_0 \in L^1_+(\Omega)\) and \(f, g, h \in L^1_+(Q_T)\). Then the system (1.1)–(1.2)–(1.4) has a weak solution.

Remark 2.1. To the expense of extra technicalities one may assume distinct anisotropic conditions for the three state variables, i.e. we can modify condition (1.5) into:

\[
\begin{align*}
\text{for } i = 1, 2, 3 \text{ there exists } p_{l,i} > 1, \\
A_{l,i}(t,x,\xi) &= \beta_{l,i}(t,x)\xi_i |\xi_i|^{p_{l,i}-2}\xi_i.
\end{align*}
\]

2.2. The case of no-flux boundary

In this section we consider the system (1.1) with no-flux boundary conditions (1.3) and initial conditions (1.4).

Definition 2.2. Let \(1 \leq q_l < (p_l/\tilde{p})(\tilde{p} - N/(N + 1)), l = 1 \ldots N\). A weak solution of (1.1)–(1.2)–(1.4), is a triple \((u, v, w)\) of nonnegative functions, with \(\theta = u, v, w\) belonging to \(\bigcap_{l=1}^N L^{q_l}(0,T; W^{1,q_l;1}(\Omega)) \cap L^{p_l}(0,T; L^{p_l}(\Omega)) \cap C([0,T]; L^1(\Omega)), l = 1 \ldots N\), such that \(\sigma(\cdot, \cdot, u, v, w)\) and \(r_i(\cdot, \cdot, u, v, w), \) for \(i = 1, 2, 3, \) belong to \(L^1(Q_T),\) and satisfying (2.1)–(2.2)–(2.3).

Theorem 2.2. Assume that (1.5)–(1.13) hold and \(2 - 1/(N+1) < p_l < \tilde{p}(N+1)/N \) \((l = 1 \ldots N)\). Let \(u_0, v_0, w_0 \in L^1_+(\Omega)\) and \(f, g, h \in L^1_+(Q_T)\). Then the system (1.1)–(1.3)–(1.4) has a weak solution.

3. Preliminary results

We establish the following result

Lemma 3.1. Let \((u_\epsilon)_{0 < \epsilon \leq 1}\) in \(\bigcap_{l=1}^N L^{p_l}(0,T; W^{1,p_l;1}(\Omega)) \cap L^\infty(0,T; L^1(\Omega))\) satisfy: there exists \(\beta > 0\) independent of \(\epsilon,\) such that

\[
\begin{align*}
\sup_{t \in (0,T)} \int_\Omega |u_\epsilon(t,x)| \, dx &\leq \beta, \\
\int_{Q_T} |u_\epsilon|^{p_l} \, dx \, dt &\leq \beta,
\end{align*}
\]

and

\[
\sup_{\gamma > 0} \sum_{l=1}^N \int_{B_\gamma} \left| \frac{\partial u_\epsilon}{\partial x_l} \right|^{p_l} \, dx \, dt \leq \beta,
\]

where \(B_\gamma = \{(t,x) \in Q_T, \gamma \leq |u_\epsilon| \leq \gamma + 1\}.\)
Let \( \bar{p} \leq N + N/(N + 1) \), then for every \( 1 \leq q_l < (p_l/\bar{p})(\bar{p} - N/(N + 1)) \), \( l = 1 \ldots N \), there exists a positive constant \( c \) depending on \( Q_T \), \( N \), \( p_l \), \( q_l \), \( \beta \), such that

\[
\left\| \frac{\partial u_e}{\partial x_l} \right\|_{L^{q_l}(Q_T)} \leq c, \tag{3.4}
\]

\[
\| u_e \|_{L^q(Q_T)} \leq c, \tag{3.5}
\]

where \( q \) satisfies \( \frac{1}{q} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{q_l} \).

**Remark 3.1.** A similar result is found in [9] in the case of Dirichlet boundary conditions, more precisely, when the Sobolev Space \( W^{1,p_l,l}(\Omega) \) is replaced by \( W^{1,p_l,l}_{\partial\Omega}(\Omega) \) and assumption (3.3) replaced by

\[
\sum_{l=1}^{N} \int_{Q_T} \frac{\left| \frac{\partial u_e}{\partial x_l} \right|^{p_l}}{(1 + |u_e|)^p} \, dx \, dt \leq \beta,
\]

in that case (3.2) is a consequence of a result of [17] for homogeneous Dirichlet boundary condition.

**Proof.** Let \( q_l < p_l \) and \( \gamma_0 \in \mathbb{N}^* \); one has

\[
\int_{0}^{T} \int_{\Omega} \left| \frac{\partial u_e}{\partial x_l} \right|^{q_l} \, dx \, dt = \sum_{\gamma=0}^{\gamma_0-1} \int_{B_{\gamma}} \left| \frac{\partial u_e}{\partial x_l} \right|^{q_l} \, dx \, dt + \sum_{\gamma=\gamma_0}^{\infty} \int_{B_{\gamma}} \left| \frac{\partial u_e}{\partial x_l} \right|^{q_l} \, dx \, dt
\]

\[
\leq c_{\gamma_0} + \sum_{\gamma=\gamma_0}^{\infty} \left( \int_{B_{\gamma}} \left| \frac{\partial u_e}{\partial x_l} \right|^{p_l} \, dx \, dt \right)^{q_l/p_l} \left( \text{meas}(B_{\gamma}) \right)^{1-q_l/p_l}
\]

\[
\leq c_{\gamma_0} + c_1 \sum_{\gamma=\gamma_0}^{\infty} \left( \text{meas}(B_{\gamma}) \right)^{1-q_l/p_l}. \tag{3.6}
\]

Let \( r > 0 \); then

\[
\frac{1}{r^p} \int_{B_{\gamma}} |u_e(t,x)|^r \, dx \, dt \geq \text{meas}(B_{\gamma}), \tag{3.7}
\]

so that

\[
\int_{0}^{T} \int_{\Omega} \left| \frac{\partial u_e}{\partial x_l} \right|^{q_l} \, dx \, dt \leq c_2 + c_3 \sum_{\gamma=\gamma_0}^{\infty} \frac{1}{r^p} \left( \int_{B_{\gamma}} |u_e|^r \, dx \, dt \right)^{p_l-q_l/p_l}
\]

\[
\leq c_2 + c_3 \left( \sum_{\gamma=\gamma_0}^{\infty} \frac{1}{r^p} \left( \int_{B_{\gamma}} |u_e|^r \, dx \, dt \right)^{p_l-q_l/p_l} \right). \tag{3.8}
\]
Note that \( \bar{q} < N \), and \( \bar{q}^* > 1 \) with \( \bar{q}^* = N \bar{q}/(N - \bar{q}) \); applying an interpolation argument yields
\[
\|u_t(\cdot,t)\|_{L^q(\Omega)} \leq \|u_t(\cdot,t)\|^a_{L^q(\Omega)} \|u_t(\cdot,t)\|^a_{L^{q^*}(\Omega)} \leq c_4 \|u_t(\cdot,t)\|^a_{L^{q^*}(\Omega)},
\]
where \( a = \bar{q}^*(1-r)/r(1 - \bar{q}^*) \). Thus
\[
\|u_t\|_{L^q(0,T;L^q(\Omega))} \leq c_4 \int_0^T \|u_t(t,.)\| \left( \frac{\bar{q}^*(1-r)}{r(1 - \bar{q}^*)} \right) dt.
\]
Choose now \( r \) such that \( \bar{q}^*(1-r)/(1 - \bar{q}^*) = \bar{q} \), that is, \( r = \bar{q}(N+1)/N \); one gets
\[
\|u_t\|_{L^q(0,T;L^q(\Omega))} \leq c_4 \|u_t\|^\bar{q}_{L^\bar{q}(0,T;L^{\bar{q}^*}(\Omega))}.
\]

By the anisotropic Sobolev inequality of Theorem 4.3 in [1], see also [17], one finds
\[
\left( \int_\Omega |u_t(t,x)|^{\bar{q}^*} dx \right)^{1/\bar{q}^*} \leq c_5 \prod_{l=1}^N \left( \int_\Omega \left| \frac{\hat{\partial}_l u_t(t,x)}{\hat{\partial}_j} \right|^{q_l} dx \right)^{1/q_l}
\]
\[
+ \left( \int_\Omega |u_t(t,x)|^{q_l} dx \right)^{1/q_l} \right)^{1/N}.
\]
Since \( q_l < p_l \), it follows from (3.2)
\[
\left( \int_\Omega |u_t(t,x)|^{\bar{q}^*} dx \right)^{1/\bar{q}^*} \leq c_6 \prod_{l=1}^N \left( \int_\Omega \left| \frac{\hat{\partial}_l u_t(t,x)}{\hat{\partial}_j} \right|^{q_l} dx \right)^{1/q_l} \right)^{1/N} + c_7.
\]

Using Hölder inequality, and the estimates (3.8), (3.9) one has
\[
\|u_t\|_{L^{\bar{q}}(0,T;L^{\bar{q}^*}(\Omega))} \leq c_8 \prod_{l=1}^N \left( \int_\Omega \int_0^T \left| \frac{\hat{\partial}_l u_t(t,x)}{\hat{\partial}_j} \right|^{q_l} dx dt \right)^{1/q_l}
\]
\[
\leq c_{10} + c_{11} \prod_{l=1}^N \|u_t\|_{L^{\bar{q}}(0,T;L^{\bar{q}^*}(\Omega))} \left( \sum_{\gamma = \gamma_0}^\infty \frac{1}{(p_l-q_l)\bar{q}(N+1)} \right)^{\frac{1}{N p_l}}
\]
\[
\leq c_{12} + c_{13} \|u_t\|_{L^{\bar{q}}(0,T;L^{\bar{q}^*}(\Omega))} \left( \sum_{\gamma = \gamma_0}^\infty \frac{1}{(p_l-q_l)\bar{q}(N+1)} \right)^{\frac{1}{N p_l}}.
\]

Select \( \theta, 0 < \theta < \frac{1}{\bar{p}}(\bar{p} - \frac{N}{N+1}) = 1 - \frac{1}{\bar{p}} \frac{N}{N+1} \), and let \( q_l = \theta p_l, l = 1, \ldots, N \). Then
\[
\frac{(p_l - q_l)\bar{q}(N+1)}{q_l N} = (1 - \theta) \bar{p} \frac{N+1}{N} > 1 \quad \text{for } l = 1, \ldots, N,
\]
which ensures that the series that appears in (3.12) is convergent, we deduce from
(3.12) and \( \sum_{l=1}^{N} \tilde{q}(p_l - q_l)/Nq_l p_l = 1 - \tilde{q}/\tilde{p} < 1 \), that
\[
\|u_e\|_{L^q(0,T;L^p(\Omega))} \leq c_{14},
\]
thus from (3.8) and (3.9) we obtain
\[
\|\frac{\partial u_e}{\partial x_l}\|_{L^q(\Omega_T)} \leq c_{15}.
\]

4. Proof of Theorem 2.1

The proof is organised as follows. First, an extension of the nonlinear function \( \sigma \) is made in order to ensure the nonnegativity of solutions; next a regularisation of the data is done. In Section 4.2, a priori estimates for solutions are given; Section 4.3 is devoted to the strong convergence of nonlinear terms.

**Extension.** Let \( \hat{\sigma} \) be a measurable function on \( QT \), continuous with respect to \( u, v \) and \( w \), a.e. in \( QT \), defined by
\[
\hat{\sigma}(t,x,u,v,w) = \begin{cases} 
\sigma(t,x,u,v,w) & \text{if } u \geq 0, v \geq 0, w \geq 0, \\
\sigma(t,x,u,v,0) & \text{if } u \geq 0, v \geq 0, w < 0, \\
\sigma(t,x,u,0,w) & \text{if } u \geq 0, v < 0, w \geq 0, \\
\sigma(t,x,0,v,w) & \text{if } u < 0, v \geq 0, w \geq 0, \\
\sigma(t,x,0,0,w) & \text{if } u < 0, v < 0, w \geq 0, \\
\sigma(t,x,u,0,0) & \text{if } u \geq 0, v < 0, w < 0, \\
\sigma(t,x,0,0,0) & \text{if } u < 0, v \geq 0, w < 0, \\
\sigma(t,x,0,0,0) & \text{if } u < 0, v < 0, w < 0.
\end{cases}
\]

Then, we are concerned with system (1.1)–(1.2)–(1.4) where \( \sigma \) is replaced by \( \hat{\sigma} \).

**Regularisation.** Let us introduce the following smooth approximations of the data \( u_0, v_0, w_0 \) and \( f, g, h \); let \( Z_e = f_e, g_e, h_e \) and \( Z_{0,e} = u_{0,e}, v_{0,e}, w_{0,e} \) be such that

\[
\begin{align*}
Z_e &\in D_+((0,T) \times \Omega) \quad \text{and} \quad Z_{0,e} \in D_+(\Omega), \\
\|Z_e\|_{L^q(\Omega_T)} &\leq \|Z\|_{L^q(\Omega_T)}, \quad Z_e \to Z \text{ in } L^1(\Omega_T), \text{ as } e \to 0, \quad (4.1) \\
\|Z_{0,e}\|_{L^q(\Omega)} &\leq \|Z_0\|_{L^q(\Omega)}, \quad Z_{0,e} \to Z_0 \text{ in } L^1(\Omega), \text{ as } e \to 0,
\end{align*}
\]

here \( Z = f, g, h \) and \( Z_0 = u_0, v_0, w_0 \). Then, classical results, see e.g [12,11], provide the existence of a sequence \( u_{e}, v_{e}, w_{e} \in \bigcap_{l=1}^{N} L^{p_l}(0,T; W^{1,p_l}_{1; \Omega}) \cap L^{p_{\max}}(\Omega) \cap C([0,T]; L^2(\Omega)), p_{\max} = \max(p_u, p_v, p_w), \) with \( \hat{\sigma}_t u_e, \hat{\sigma}_t v_e, \hat{\sigma}_t w_e \in \sum_{l=1}^{N} L^{p_l}(0,T; (W^{1,p_l}_{1; \Omega})'), \) of
solutions to (1.1)–(1.2)–(1.4) where \( u_0, v_0, w_0 \) and \( f, g, h \) are replaced by \( u_{0,\varepsilon}, v_{0,\varepsilon}, w_{0,\varepsilon} \) and \( f_\varepsilon, g_\varepsilon, h_\varepsilon \) respectively, and \( \sigma \) is replaced by \( \tilde{\sigma} \). \( \sum_{l=1}^{N} L^{p_l}(0, T; (W^{1,p_1}_0(\Omega))' \cap L^\infty(Q_T) \) denotes the dual space of \( \bigcap_{l=1}^{N} L^{p_l}(0, T; W^{1,p_1}_0(\Omega)) \) with \( p'_l = p_l/(p_l - 1) \).

In order to control the zero-order terms that appear in (2.1)–(2.3), we consider \( \lambda > 0 \) satisfying

\[
\begin{cases}
\lambda - b(t,x) \geq 0 & \text{a.e. } (t,x) \in Q_T, \\
\lambda - b(t,x) - \text{div}(K_1(t,x)) \geq 0 & \text{a.e. } (t,x) \in Q_T, \\
\lambda - \text{div}(K_i(t,x)) \geq 0 & \text{a.e. } (t,x) \in Q_T \text{ for } i = 1, 2, 3.
\end{cases}
\]

We will often write \( \tilde{\sigma}(t,x,\cdot,\cdot,\cdot) = \tilde{\sigma}(\cdot,\cdot,\cdot) \) and \( r_i(t,x,\cdot,\cdot,\cdot) = r_i(\cdot,\cdot,\cdot) \) for \( i = 1, 2, 3 \) when no confusion can arise.

We set \( u_\varepsilon = e^{\lambda t}\tilde{u}_\varepsilon, v_\varepsilon = e^{\lambda t}\tilde{v}_\varepsilon \) and \( w_\varepsilon = e^{\lambda t}\tilde{w}_\varepsilon \); then \( \tilde{u}_\varepsilon, \tilde{v}_\varepsilon \) and \( \tilde{w}_\varepsilon \) satisfy

\[
\int_0^T \langle \partial_t \tilde{u}_\varepsilon, \phi \rangle \ dt + \int_0^T \int_\Omega \tilde{A}_1(t,x,\nabla \tilde{u}_\varepsilon) \cdot \nabla \phi \ dx \ dt + \int_0^T \int_\Omega \tilde{u}_\varepsilon K_1 \cdot \nabla \phi \ dx \ dt
\]

\[
+ \int_0^T \int_\Omega e^{-\lambda t} \tilde{\sigma}(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon, e^{\lambda t}\tilde{w}_\varepsilon) \phi \ dx \ dt + \int_0^T \int_\Omega (\lambda + m - b) \tilde{u}_\varepsilon \phi \ dx \ dt
\]

\[
- \int_0^T \int_\Omega b(\tilde{v}_\varepsilon + \tilde{w}_\varepsilon) \phi \ dx \ dt + \int_0^T \int_\Omega r_{1,\lambda}(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon) \phi \ dx \ dt
\]

\[
= \int_0^T \int_\Omega e^{-\lambda t} f_\varepsilon \phi \ dx \ dt,
\]

\[
\int_0^T \langle \partial_t \tilde{v}_\varepsilon, \psi \rangle \ dt + \int_0^T \int_\Omega \tilde{A}_2(t,x,\nabla \tilde{v}_\varepsilon) \cdot \nabla \psi \ dx \ dt + \int_0^T \int_\Omega \tilde{v}_\varepsilon K_2 \cdot \nabla \psi \ dx \ dt
\]

\[
+ \int_0^T \int_\Omega (\lambda + m + \rho) \tilde{v}_\varepsilon \psi \ dx \ dt - \int_0^T \int_\Omega e^{-\lambda t} \tilde{\sigma}(e^{\lambda t}\tilde{u}_\varepsilon, e^{\lambda t}\tilde{v}_\varepsilon, e^{\lambda t}\tilde{w}_\varepsilon) \psi \ dx \ dt
\]

\[
+ \int_0^T \int_\Omega r_{2,\lambda}(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon) \psi \ dx \ dt = \int_0^T \int_\Omega e^{-\lambda t} g_\varepsilon \psi \ dx \ dt,
\]

\[
\int_0^T \langle \partial_t \tilde{w}_\varepsilon, \chi \rangle \ dt + \int_0^T \int_\Omega \tilde{A}_3(t,x,\nabla \tilde{w}_\varepsilon) \cdot \nabla \chi \ dx \ dt + \int_0^T \int_\Omega \tilde{w}_\varepsilon K_3 \cdot \nabla \chi \ dx \ dt
\]

\[
+ \int_0^T \int_\Omega (\lambda + m + \alpha) \tilde{w}_\varepsilon \chi \ dx \ dt - \int_0^T \int_\Omega \rho \tilde{v}_\varepsilon \chi \ dx \ dt
\]

\[
+ \int_0^T \int_\Omega r_{3,\lambda}(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon) \chi \ dx \ dt = \int_0^T \int_\Omega e^{-\lambda t} h_\varepsilon \chi \ dx \ dt,
\]

for all \( \phi, \psi, \chi \in \bigcap_{l=1}^{N} L^{p_l}(0, T; W^{1,p_1}_0(\Omega)) \cap L^\infty(Q_T) \).
Herein
\[ \tilde{A}_i(t, x, \zeta) = e^{-i\lambda t} A_i(t, x, e^{i\lambda t} \zeta), \quad \text{for } \zeta \in \mathbb{R}^N, \]
and
\[ r_{i, \lambda}(t, x, \tilde{u}_e, \tilde{v}_e, \tilde{w}_e) = e^{(p_1 - 1)i\lambda t} r_i(t, x, \tilde{u}_e, \tilde{v}_e, \tilde{w}_e), \quad p_1 = p_u, \quad p_2 = p_v, \quad p_3 = p_w. \]

4.1. Nonnegativity

Lemma 4.1. The solution \((\tilde{u}_e, \tilde{v}_e, \tilde{w}_e)\) is nonnegative.

Proof. This proof is based on the choice of suitable test functions, namely truncation functions. For any positive real number \(\gamma \in \mathbb{R}^+\), a truncation function \(T_\gamma\) is defined as
\[ T_\gamma(z) = \min(\gamma, \max(z, -\gamma)). \tag{4.6} \]

We also set
\[ \begin{aligned}
S_\gamma(z) &= \int_0^z T_\gamma(\tau) \, d\tau, \\
\phi_\gamma &= T_{\gamma+1} - T_\gamma, \\
\Psi_\gamma(z) &= \int_0^z \phi_\gamma(\tau) \, d\tau.
\end{aligned} \tag{4.7} \]

We note that \(T_\gamma\) and \(\phi_\gamma\) are Lipschitz continuous functions, satisfying \(0 \leq |\phi_\gamma(z)| \leq 1\) and \(|\Psi_\gamma(z)| \leq |z|\) for \(\gamma > 0\) and \(z \in \mathbb{R}\).

Let us substitute \(\psi = -T_\gamma(\tilde{v}_e^-)\) in (4.4), where \(\tilde{v}_e^- = \sup(0, -\tilde{v}_e)\). Using the definition of \(\tilde{\sigma}\) and the fact that \(r_{2, \lambda}\) has the same sign as \(\tilde{v}_e^-\), one has
\[ \int_0^T \int_\Omega e^{-i\lambda t} \tilde{\sigma}(e^{i\lambda t} \tilde{u}_e, e^{i\lambda t} \tilde{v}_e^-) T_\gamma(\tilde{v}_e^-) \, dx \, dt \geq 0, \]
and
\[ \int_0^T \int_\Omega r_{2, \lambda}(\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) T_\gamma(\tilde{v}_e^-) \, dx \, dt \leq 0. \]

Note that, by the choice of \(\lambda\) in (4.2) and \(K_2 \cdot \eta \geq 0\) on \((0, T) \times \Gamma_1\), one gets as in [2]
\[ \int_0^T \int_\Omega (\lambda + m + \rho) \tilde{v}_e^- T_\gamma(\tilde{v}_e^-) \, dx \, dt + \int_0^T \int_\Omega \tilde{v}_e^- K_2 \cdot \nabla T_\gamma(\tilde{v}_e^-) \, dx \, dt \geq 0. \tag{4.8} \]

Since \(g_e\) is nonnegative and from estimate (4.8) one obtains
\[ \frac{d}{dt} \int_\Omega S_\gamma(\tilde{v}_e^-(t, x)) \, dx + a \sum_{i=1}^N \int_{\{\tilde{v}_e^- \leq \gamma\}} \frac{\partial T_\gamma(\tilde{v}_e^-(t, x))}{\partial x_i} \, dx \leq 0, \]
which yields
\[ \frac{d}{dt} \int_\Omega S_\gamma(\tilde{v}_e^-(t, x)) \, dx \leq 0. \]

Since the data \(v_{0e}\) is nonnegative, we deduce that \(\tilde{v}_e^- = 0\).
Next, substitute $\chi = T_\gamma(\tilde{W}_e^-)$ in (4.5). Since $h_e$ and $\tilde{v}_e$ are nonnegative, by the choice of $\lambda$ in (4.2) and $K_2 \cdot \eta \geq 0$ on $(0,T) \times \Gamma_1$, one gets
$$\frac{d}{dt} \int_\Omega S_\gamma(\tilde{W}_e^- (t,x)) \, dx \leq 0,$$
Since the data $w_{0e}$ is nonnegative, we deduce that $\tilde{W}_e^- = 0$.
Along the same lines we show $\tilde{U}_e(t,x) \geq 0$ a.e. $(t,x) \in Q_T$. □

4.2. A priori estimates

Proposition 4.1. Assume that (1.5)–(1.13) hold. Then, there exists $c_1 > 0$, $c_2 > 0$, not depending on $\varepsilon$, such that the sequence $(\tilde{U}_e, \tilde{V}_e, \tilde{W}_e)_{0 < \varepsilon \leq 1}$ satisfies
$$\|\tilde{U}_e + \tilde{V}_e + \tilde{W}_e\|_{L^\infty(0,T;L^1(\Omega))} \leq c_1,$$  
(4.9)

$$\|r_{i,j}(\tilde{U}_e, \tilde{V}_e, \tilde{W}_e)\|_{L^1(Q_T)} + \|\sigma(\varepsilon^{i,j} \tilde{U}_e, \varepsilon^{i,j} \tilde{V}_e, \varepsilon^{i,j} \tilde{W}_e)\|_{L^1(Q_T)} \leq c_2 \quad \text{for } i = 1, 2, 3,$$
(4.10)

and
$$\|\tilde{U}_e\|_{L^p(\Omega)} + \|\tilde{V}_e\|_{L^p(\Omega)} + \|\tilde{W}_e\|_{L^p(\Omega)} \leq c_3.$$
(4.11)

Let $\bar{p} \leq N + N/(N + 1)$; then for every $1 \leq q_l < (p_l/\bar{p})(\bar{p} - N/(N + 1))$, $l = 1, \ldots, N$, there exists a positive constant $c$ depending on $Q_T$, $N$, $p_l$, $q_l$, $\|f\|_{L^1(Q_T)}$, $\|u_0\|_{L^1(\Omega)}$, such that
$$\left\| \frac{\partial \tilde{Z}_e}{\partial x_l} \right\|_{L^q(\Omega)} \leq c$$
(4.12)

and
$$\|\tilde{Z}_e\|_{L^q(\Omega)} \leq c,$$
(4.13)

where $\tilde{z}_e = \tilde{U}_e, \tilde{V}_e, \tilde{W}_e$, and $\bar{q}$ satisfies $1/\bar{q} = 1/N \sum_{l=1}^N 1/q_l$.

Proof. The proof relies on the choice of suitable truncation functions.

Proof of estimate (4.9). Using Hölder inequality one has
$$\int_0^T \int_\Omega \tilde{U}_e K \cdot \nabla \left( \frac{1}{\gamma \gamma} T_\gamma(\tilde{U}_e) \right) \, dx \, dt$$
$$\leq C(\gamma) + \frac{a}{2\gamma} \sum_{l=1}^N \int_{\{1/\varepsilon \leq \gamma\}} \left| \frac{\partial \tilde{U}_e}{\partial x_l} \right|^{p_l} \, dx \, dt,$$
(4.14)

where $C(\gamma)$ tends to 0 when $\gamma \to 0$.

We choose $\phi = (1/\gamma \gamma) T_\gamma(\tilde{U}_e)$, $\psi = (1/\gamma \gamma) T_\gamma(\tilde{V}_e)$ and $\chi = (1/\gamma \gamma) T_\gamma(\tilde{W}_e)$ as a test functions in (4.3), (4.4) and (4.5). Using estimate (4.14), the nonnegativity of the solutions and the
choice of $\lambda$ in (4.2), and letting $\gamma$ go to zero, it follows upon adding up the resulting equations

$$\|(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon)(t, \cdot)\|_{L^1(\Omega)} \leq C + \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)} + \int_0^T \|(f + g + h)(\tau, \cdot)\|_{L^1(\Omega)} \, d\tau,$$

for $0 \leq t < T$. Then, we deduce (4.9).

**Proof of estimate (4.10).** Substituting $\phi = (1/\gamma)T_\gamma(\tilde{u}_\varepsilon)$ in Eq. (4.3) and using estimate (4.14), one has

$$\int_0^T \left< \partial_t \tilde{u}_\varepsilon + \frac{1}{\gamma} T_\gamma(\tilde{u}_\varepsilon), \right> \, dt + \int_0^T \int_\Omega r_{1,2}(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon) \frac{1}{\gamma} T_\gamma(\tilde{u}_\varepsilon) \, dx \, dt$$

$$- \int_0^T \int_\Omega b(\tilde{v}_\varepsilon + \tilde{w}_\varepsilon) \frac{1}{\gamma} T_\gamma(\tilde{u}_\varepsilon) \, dx \, dt$$

$$+ \int_0^T \int_\Omega e^{-\beta t} \sigma(\varepsilon^2 \tilde{u}_\varepsilon, e^{\beta t} \tilde{v}_\varepsilon, e^{\beta t} \tilde{w}_\varepsilon) \frac{1}{\gamma} T_\gamma(\tilde{u}_\varepsilon) \, dx \, dt$$

$$+ \int_0^T \int_\Omega (\lambda + m - b)\tilde{u}_\varepsilon \frac{1}{\gamma} T_\gamma(\tilde{u}_\varepsilon) \, dx \, dt \leq C + \|f\|_{L^1(\Omega T)}.$$

Since the solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon)$ is nonnegative, by the choice of $\lambda$ in (4.2) and from estimates (4.9), we deduce upon letting $\gamma$ go to zero,

$$\|\sigma(\varepsilon^2 \tilde{u}_\varepsilon, e^{\beta t} \tilde{v}_\varepsilon, e^{\beta t} \tilde{w}_\varepsilon)\|_{L^1(\Omega T)} \leq e^{\beta T} (C + \|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega T)} + \|b\|_{L^\infty(\Omega T)}e^{1 T}).$$

Now we prove the second estimate in (4.10). We choose $\phi = (1/\gamma)T_\gamma(\tilde{u}_\varepsilon)$, $\psi = (1/\gamma)T_\gamma(\tilde{v}_\varepsilon)$ and $\chi = (1/\gamma)T_\gamma(\tilde{w}_\varepsilon)$ as a test function in (4.3), (4.4) and (4.5), respectively. After letting $\gamma$ go to zero, one gets

$$\int_0^T \int_\Omega (k_1 \tilde{u}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1 + k_2 \tilde{v}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1 + k_3 \tilde{w}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1) \, dx \, dt$$

$$\leq C + \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)} + \|f + g + h\|_{L^1(\Omega T)}.$$

Then, using condition (1.11) for $k_i$, $i = 1, 2, 3$, yields

$$k_0 \left( \int_0^T \int_\Omega (\tilde{u}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1 + \tilde{v}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1 + \tilde{w}_\varepsilon |\tilde{u}_\varepsilon + \tilde{v}_\varepsilon + \tilde{w}_\varepsilon|^p - 1) \, dx \, dt \right)$$

$$\leq C' + \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)} + \|f + g + h\|_{L^1(\Omega T)}.$$

and consequently the proof of estimate (4.11) is complete.
Proof of estimate (4.12) and (4.13). Substituting $\phi = \varphi_\varepsilon(\tilde{u}_\varepsilon)$ in Eq. (4.3), one finds
\begin{align*}
&\int_0^T \int_\Omega \tilde{\varepsilon}_l \tilde{u}_\varepsilon \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt + \int_0^T \int_\Omega (\tilde{A}_1(t,x,\nabla \tilde{u}_\varepsilon) + \tilde{u}_\varepsilon K_1) \cdot \nabla \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (\lambda + m - b) \tilde{u}_\varepsilon \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt + \int_0^T \int_\Omega r_{l,\varepsilon}(t,x,\tilde{u}_\varepsilon) \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt \\
&\quad + \int_0^T \int_\Omega \sigma(\epsilon^{2t} \tilde{u}_\varepsilon, \epsilon^{2t} \tilde{v}_\varepsilon, \epsilon^{2t} \tilde{w}_\varepsilon) \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt + \int_0^T \int_\Omega b(\tilde{v}_\varepsilon + \tilde{w}_\varepsilon) \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt \\
&\quad = \int_0^T \int_\Omega e^{-\lambda t} f_\varepsilon \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt. \quad (4.15)
\end{align*}

Using the estimate (4.10) and Hölder inequality, one obtains
\begin{align*}
&\int_0^T \int_\Omega \tilde{u}_\varepsilon K_1 \cdot \nabla \phi_\varepsilon(\tilde{u}_\varepsilon) \, dx \, dt \leq C + \frac{\tilde{a}}{2} \sum_{l=1}^N \int_\Omega \int_{B_1} \frac{\partial \tilde{u}_\varepsilon}{\partial x_l} \right|_{x_l}^{p_l} \, dx \, dt.
\end{align*}

The choice of $\lambda$ in (4.2), the nonnegativity of $\tilde{u}_\varepsilon$, $\tilde{v}_\varepsilon$ and $\tilde{w}_\varepsilon$, the coercivity of $\tilde{A}_1$ allow to deduce from (4.15)
\begin{align*}
\frac{\tilde{a}}{2} \sum_{l=1}^N \int_\Omega \int_{B_1} \frac{\partial \tilde{u}_\varepsilon}{\partial x_l} \right|_{x_l}^{p_l} \, dx \, dt &\leq C_1(C_2 + ||f||_{L^1(Q_T)} + ||\tilde{u}_0||_{L^1(\Omega)} + ||b||_{L^\infty(\Omega)}c_1 T),
\end{align*}
which implies that there exists a positive constant $C_3$ such that
\begin{align*}
\sum_{l=1}^N \int_\Omega \int_{B_1} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_l} \right|_{x_l}^{p_l} \, dx \, dt &\leq C_3. \quad (4.16)
\end{align*}

In the same way, one gets that the above estimate remains valid for $\tilde{v}_\varepsilon$ and $\tilde{w}_\varepsilon$. Then, estimates (4.12) and (4.13) are a direct consequence of Lemma 3.1. \hfill $\square$

4.3. Strong convergence in $L^1(Q_T)$

Let $q_0 = \min_{1 \leq i \leq N} \{ q_i \}$; then $(\tilde{u}_\varepsilon)_{0 < \varepsilon \leq 1}, (\tilde{v}_\varepsilon)_{0 < \varepsilon \leq 1}$ and $(\tilde{w}_\varepsilon)_{0 < \varepsilon \leq 1}$ are bounded in $L^{q_0}(0,T; W^{1,q_0}_0(\Omega))$. This implies that $(\tilde{c}_l \tilde{u}_\varepsilon)_{0 < \varepsilon \leq 1}, (\tilde{c}_l \tilde{v}_\varepsilon)_{0 < \varepsilon \leq 1}, (\tilde{c}_l \tilde{w}_\varepsilon)_{0 < \varepsilon \leq 1}$ are bounded in $L^1(0,T; W^{1,q_0}_0(\Omega)) + L^1(Q_T)$. Therefore, possibly at the cost of extracting subsequences denoted $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon)_{0 < \varepsilon \leq 1}$, see e.g. [15, Corollary 4], we can assume that as $\varepsilon \to 0$
\begin{align*}
\begin{cases}
\tilde{u}_\varepsilon \to \tilde{u} \quad \text{strongly in } L^{q_0}(Q_T) \text{ and a.e. in } Q_T, \\
\tilde{v}_\varepsilon \to \tilde{v} \quad \text{strongly in } L^{q_0}(Q_T) \text{ and a.e. in } Q_T, \\
\tilde{w}_\varepsilon \to \tilde{w} \quad \text{strongly in } L^{q_0}(Q_T) \text{ and a.e. in } Q_T,
\end{cases} \quad (4.17)
\end{align*}
and

\[
\begin{cases}
\sigma(e^{ij} \tilde{u}_e, e^{ij} \tilde{v}_e, e^{ij} \tilde{w}_e) \to \sigma(e^{ij} \tilde{u}, e^{ij} \tilde{v}, e^{ij} \tilde{w}) & \text{a.e. in } QT, \\
r_{1,i}(t,x,\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) \to r_{1,i}(t,x,\tilde{u}, \tilde{v}, \tilde{w}) & \text{a.e. in } QT, \\
r_{2,i}(t,x,\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) \to r_{2,i}(t,x,\tilde{u}, \tilde{v}, \tilde{w}) & \text{a.e. in } QT, \\
r_{3,i}(t,x,\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) \to r_{3,i}(t,x,\tilde{u}, \tilde{v}, \tilde{w}) & \text{a.e. in } QT,
\end{cases}
\]

(4.18)

Now, we are interested in the strong convergence of the nonlinear terms \( \sigma, r_{1,i}, r_{2,i} \) and \( r_{3,i} \). A proof of the following result is found in [2, Proposition 3].

**Proposition 4.2.** The sequences \( (\sigma(e^{ij} \tilde{u}_e, e^{ij} \tilde{v}_e, e^{ij} \tilde{w}_e))_{0 < \varepsilon < 1} \) and \( (r_{1,i}(\tilde{u}_e, \tilde{v}_e, \tilde{w}_e))_{0 < \varepsilon < 1} \), satisfy

\[
\lim_{\gamma \to \infty} \sup_{0 < \varepsilon < 1} \left( \int_{\{|\tilde{u}_e| \geq \gamma\}} (r_{1,i}(\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) + \sigma(e^{ij} \tilde{u}_e, e^{ij} \tilde{v}_e, e^{ij} \tilde{w}_e) \, dx \, dt) = 0 \right),
\]

(4.19)

and for \( i = 1, 2, 3 \), almost everywhere in \( QT \) and strongly in \( L^1(Q_T) \)

\[
\sigma(e^{ij} \tilde{u}_e, e^{ij} \tilde{v}_e, e^{ij} \tilde{w}_e) \to \sigma(e^{ij} \tilde{u}, e^{ij} \tilde{v}, e^{ij} \tilde{w}),
\]

(4.20)

\[
r_{i,i}(\tilde{u}_e, \tilde{v}_e, \tilde{w}_e) \to r_{i,i}(\tilde{u}, \tilde{v}, \tilde{w}).
\]

(4.21)

### 4.4. End of the proof of Theorem 2.1

We complete the properties of the sequences \( \tilde{u}_e, \tilde{v}_e \) and \( \tilde{w}_e \) with the following two results.

**Lemma 4.2.** The sequences \( (\tilde{A}_1(t,x,\nabla \tilde{u}_e))_{0 < \varepsilon < 1}, (\tilde{A}_2(t,x,\nabla \tilde{v}_e))_{0 < \varepsilon < 1} \) and \( (\tilde{A}_3(t,x,\nabla \tilde{w}_e))_{0 < \varepsilon < 1} \) converge to \( \tilde{A}_1(t,x,\nabla \tilde{u}), \tilde{A}_2(t,x,\nabla \tilde{v}) \) and \( \tilde{A}_3(t,x,\nabla \tilde{w}) \) almost everywhere in \( QT \) and strongly in \( L^1(Q_T) \).

**Proof.** It suffices to show that \((\nabla \tilde{u}_e)_{0 < \varepsilon < 1}\) is a Cauchy sequence in measure, i.e. \( \forall \mu > 0 \)

\[
\text{meas}\{(t,x); |(\nabla \tilde{u}_e' - \nabla \tilde{u}_e)(t,x)| \geq \mu\} \to 0
\]

as \( \varepsilon, \varepsilon' \to 0 \). Let \( \gamma > 0 \) and \( \delta > 0 \), we have:

\[
\begin{align*}
\{(t,x); |(\nabla \tilde{u}_e(t,x) - \nabla \tilde{u}_e')(t,x)| \geq \mu\} & \\
\subset \{|\nabla \tilde{u}_e \geq \gamma\} \cup \{|\nabla \tilde{u}_e' \geq \gamma\} \cup \{|\tilde{u}_e - \tilde{u}_e' \geq \delta\} \\
\cup \{|\nabla \tilde{u}_e - \nabla \tilde{u}_e' \geq \gamma, |\nabla \tilde{u}_e| \leq \gamma, |\tilde{u}_e - \tilde{u}_e'| \leq \delta\}.
\end{align*}
\]

Let us denote the four sets of the right hand side as \( L_1 \) to \( L_4 \).
By choosing γ large enough, meas(L_1) and meas(L_2) are arbitrarily small (see the proof of Lemma 4 in [2]). Since \( \tilde{u}_\varepsilon \) is a Cauchy sequence in \( L^1(Q_T) \), then, for \( \delta > 0 \) fixed, meas(L_3) tends to 0 when \( \varepsilon, \varepsilon' \to 0 \).

It is now sufficient to bound meas(L_4). Thanks to the monotonicity of \( A_1 \), we have for all \( \xi_1, \xi_2 \in \mathbb{R}^N \)

\[
[A_1(t,x,\xi_1) - A_1(t,x,\xi_2)][\xi_1 - \xi_2] > 0, \quad \xi_1 \neq \xi_2.
\]

Since the set of \((\xi_1, \xi_2)\) such that \( |\xi_1| \leq \gamma, |\xi_2| \leq \gamma \) and \( |\xi_1 - \xi_2| \geq \mu \), is a compact set and \( A_1 \) is continuous with respect to \( \xi \) for almost all \((t,x) \in Q_T\), the quantity \([A_1(t,x,\xi_1) - A_1(t,x,\xi_2)][\xi_1 - \xi_2]\) reaches its minimum value on this compact set that we will denote \( q(t,x) \); one may verify \( q(t,x) > 0 \) a.e. in \( Q_T \). Since \( q(t,x) > 0 \) a.e. in \( Q_T \), for all \( \beta > 0 \) there exists \( \beta' > 0 \) such that

\[
\int_{L_4} q(t,x) \, dx \, dt < \beta' \implies \text{meas}(L_4) \leq \beta.
\]

Hence, to show \( \text{meas}(L_4) \leq \beta \), it is sufficient to show that for \( \delta \) small enough one has

\[
\int_{L_4} q(t,x) \, dx \, dt < \beta'.
\]

By definition of \( q(t,x) \) and \( L_4 \), one has

\[
\int_{L_4} q(t,x) \, dx \, dt \leq \int_{L_4} [A_1(t,x,\nabla \tilde{u}_\varepsilon) - A_1(t,x,\nabla \tilde{u}_{\varepsilon'})][\nabla \tilde{u}_\varepsilon - \nabla \tilde{u}_{\varepsilon'}]1_{\{|\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}| \leq \delta\}},
\]

moreover \( \nabla T_\delta(\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}) = (\nabla \tilde{u}_\varepsilon - \nabla \tilde{u}_{\varepsilon'})1_{\{|\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}| \leq \delta\}} \), hence one gets

\[
\int_{L_4} q(t,x) \, dx \, dt \leq \int_{L_4} [A_1(t,x,\nabla \tilde{u}_\varepsilon) - A_1(t,x,\nabla \tilde{u}_{\varepsilon'})]\nabla T_\delta(\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}).
\]

Let \( \varphi = T_\delta(\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}) \); since \( T_\delta(s) \leq \delta \) and \( 0 \leq S_\delta(s) \leq \delta |s| \), one finds

\[
\int_0^T \int_\Omega [A_1(t,x,\nabla \tilde{u}_\varepsilon) - A_1(t,x,\nabla \tilde{u}_{\varepsilon'})]\nabla T_\delta(\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'})
\]

\[
\leq 2\delta \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(Q_T)} + \|\sigma(e^{\varepsilon \tilde{u}_\varepsilon}, e^{\varepsilon \tilde{v}_\varepsilon}, e^{2\varepsilon \tilde{w}_\varepsilon})\|_{L^1(Q_T)}
\]

\[
+ \|b\|_{L^\infty(Q_T)}(\|\tilde{u}_\varepsilon\|_{L^1(Q_T)} + \|\tilde{v}_\varepsilon\|_{L^1(Q_T)} + \|\tilde{w}_\varepsilon\|_{L^1(Q_T)})
\]

\[
+ \|r_{1,\varepsilon}(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \tilde{w}_\varepsilon)\|_{L^1(Q_T)} + C \|\nabla \tilde{u}_\varepsilon\|_{L^1(Q_T)}
\]

(4.23)

goes to 0 as \( \delta \) goes to 0, uniformly in \( \varepsilon \) and \( \varepsilon' \).

For \( \delta \) small enough, one has

\[
\int_{L_4} q(t,x) \, dx \, dt \leq \int_0^T \int_\Omega [A_1(t,x,\nabla \tilde{u}_\varepsilon) - A_1(t,x,\nabla \tilde{u}_{\varepsilon'})]\nabla T_\delta(\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}) \leq \beta',
\]

and using (4.22), we deduce \( \text{meas}(L_4) \leq \beta \). Thus, we have the convergence of \((\nabla \tilde{u}_\varepsilon)_{0 < \varepsilon \leq 1}\) to \((\nabla \tilde{u})\) in measure.
Let $E$ be a measurable set in $Q_T$ and $s_l < q_l$ for $l = 1 \ldots N$. Since $q_l < p_l/\tilde{p} (\tilde{p} - N/(N+1))$, and by the Hölder inequality,

$$
\int \int_E |a_{l,1}(t,x, \nabla \tilde{u}_\varepsilon) - a_{l,1}(t,x, \nabla \tilde{u})|^\frac{s_l}{q_l} \, dx \, dt \\
\leq \left( \int \int_E |a_{l,1}(t,x, \nabla \tilde{u}_\varepsilon) - a_{l,1}(t,x, \nabla \tilde{u})^{\frac{q_l}{p_l}} \, dx \, dt \right)^{\frac{s_l}{q_l}} \left( \text{meas}(E) \right)^{1 - \frac{s_l}{q_l}} \\
\leq C \text{meas}(E).
$$

Thus, by a theorem of Vitali, see e.g. [8], the sequence $|a_{l,1}(t,x, \nabla \tilde{u}_\varepsilon) - a_{l,1}(t,x, \nabla \tilde{u})|^{p_l}$ is strongly convergent in $L^1(Q_T)$ to 0. Then $(a_{l,1}(t,x, \nabla \tilde{u}_\varepsilon))_{0 < \varepsilon < 1}$ is strongly convergent to $a_{l,1}(t,x, \nabla \tilde{u})$ in $L^{s_l}(Q_T)$ for $s_l < (p_l/\tilde{p})(\tilde{p} - N/(N+1))$.

Along the same lines we show $(a_{l,2}(t,x, \nabla \tilde{v}_\varepsilon))_{0 < \varepsilon < 1}$ and $(a_{l,3}(t,x, \nabla \tilde{w}_\varepsilon))_{0 < \varepsilon < 1}$ are convergent to $a_{l,2}(t,x, \nabla \tilde{v})$ and $a_{l,3}(t,x, \nabla \tilde{w})$, strongly in $L^{s_l}(Q_T)$ for $s_l < (p_l/\tilde{p})(\tilde{p} - N/(N+1))$ ($l = 1 \ldots N$). This ends the proof of Lemma 4.2. \qed

The proof of the second result is similar to the proof of Lemma 8 in [2].

**Lemma 4.3.** The sequences $(\tilde{u}_\varepsilon)_{0 < \varepsilon < 1}, (\tilde{v}_\varepsilon)_{0 < \varepsilon < 1}$ and $(\tilde{w}_\varepsilon)_{0 < \varepsilon < 1}$ are Cauchy sequences in $C([0,T];L^1(\Omega))$.

Finally, by passing to the limit $\varepsilon \to 0$ in the following weak formulation (see (4.3)–(4.4)–(4.5))

\begin{align*}
&- \int_0^T \int_\Omega u_\varepsilon \phi_\varepsilon \, dx \, dt + \int_0^T \int_\Omega (A_1(t,x, \nabla u_\varepsilon) + u_\varepsilon K_1(t,x)) \cdot \nabla \phi \, dx \, dt \\
&\quad + \int_0^T \int_\Omega \mu u_\varepsilon \phi_\varepsilon \, dx \, dt - \int_0^T \int_\Omega b(u_\varepsilon + v_\varepsilon + w_\varepsilon) \phi \, dx \, dt \\
&\quad + \int_0^T \int_\Omega \sigma(t,x,u_\varepsilon,v_\varepsilon,w_\varepsilon) \phi \, dx \, dt + \int_0^T \int_\Omega r_1(t,x,u_\varepsilon,v_\varepsilon,w_\varepsilon) \phi \, dx \, dt \\
&\quad = \int_0^T \int_\Omega f_\varepsilon \phi_\varepsilon \, dx \, dt + \int_\Omega \phi(0,x)u_{0\varepsilon}(x) \, dx, \\
&- \int_0^T \int_\Omega v_\varepsilon \psi_\varepsilon \, dx \, dt + \int_0^T \int_\Omega (A_2(t,x, \nabla v_\varepsilon) + v_\varepsilon K_2(t,x)) \cdot \nabla \psi \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (m + \rho) v_\varepsilon \psi_\varepsilon \, dx \, dt - \int_0^T \int_\Omega \sigma(t,x,u_\varepsilon,v_\varepsilon,w_\varepsilon) \psi \, dx \, dt \\
&\quad + \int_0^T \int_\Omega r_2(t,x,u_\varepsilon,v_\varepsilon,w_\varepsilon) \psi \, dx \, dt = \int_0^T \int_\Omega g_\varepsilon \psi \, dx \, dt + \int_\Omega \psi(0,x)v_{0\varepsilon}(x) \, dx,
\end{align*}
\[
- \int_0^T \int_{\Omega} w \chi_t \, dx \, dt + \int_0^T \int_{\Omega} \left( A_3(t,x, \nabla w) + w K_3(t,x) \right) \cdot \nabla \chi \, dx \, dt \\
+ \int_0^T \int_{\Omega} (m + \chi) w \chi_t \, dx \, dt - \int_0^T \int_{\Omega} \rho v \chi \, dx \, dt \\
+ \int_0^T \int_{\Omega} r_3(t,x,u,v,w) \chi \, dx \, dt = \int_0^T \int_{\Omega} h \chi \, dx \, dt + \int_{\Omega} \chi(0,x) w_0(x) \, dx,
\]

with \( \phi, \psi, \chi \in C^1([0,T) \times \Omega) \), we obtain in this way that the limit \((u,v,w)\) is a solution of system (1.1)–(1.2)–(1.4) in the sense of Definition 3.1.

5. Proof of Theorem 2.2

Let \( u,v,w \in \bigcap_{l=1}^N L^{p_l}(0,T; W^{1,p_l} \Omega) \cap L^{p_{max}}(Q_T) \cap C([0,T]; L^2 \Omega) \), \( p_{max} = \max(p_u, p_v, p_w) \) be the classical solutions (see e.g. [12], [11]) of (1.1)–(1.3)–(1.4), satisfies (4.3)–(4.4)–(4.5) for all \( \phi, \psi, \chi \in \bigcap_{l=1}^N L^{p_l}(0,T; W^{1,p_l} \Omega) \cap L^{\infty} \Omega \).

Using the same technique as in the mixed boundary case and small modifications in the proofs of Proposition 4.1 we get estimates (4.9) and estimates (4.10).

The main idea in the proof of Theorem 2.2 consists in deriving a \( L^{q_l}(0,T; W^{1,q_l} \Omega) \) estimate on the solutions, depending only on the \( L^1 \) norm of the right-hand sides and initial data for \( l = 1 \ldots N \). The statement of compactness properties is given in Lemma 3.1.

References