TWISTED CUBICS ON CUBIC FOURFOLDS

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ABSTRACT. We construct a new twenty-dimensional family of projective eight-dimensional holomorphically symplectic manifolds: the compactified moduli space $M_3(Y)$ of twisted cubics on a smooth cubic fourfold Y that does not contain a plane is shown to be smooth and to admit a contraction $M_3(Y) \to Z(Y)$ to a projective eight-dimensional symplectic manifold $Z(Y)$. The construction is based on results on linear determinantal representations of singular cubic surfaces.

INTRODUCTION

According to Beauville and Donagi [4], the Fano variety $M_1(Y)$ of lines on a smooth cubic fourfold $Y \subset \mathbb{P}^5_{\mathbb{C}}$ is a smooth four-dimensional holomorphically symplectic variety which is deformation equivalent to the second Hilbert scheme of a K3-surface. The symplectic structure can be constructed as follows: let $C \subset M_1(Y) \times Y$ denote the universal family of lines and let pr_i be the projection onto the i-th factor of the ambient space. For any generator $\alpha \in H^{3,1}(Y) \cong \mathbb{C}$ one gets a holomorphic two-form $\omega_1 := \mathrm{pr}_{1*}\mathrm{pr}_2^*\alpha$ on $M_1(Y)$.

More generally, one may consider moduli spaces of smooth rational curves of arbitrary degree d on Y. For $d \geq 2$ such spaces are no longer compact, and depending on the purpose one might consider compactifications in the Chow variety or the Hilbert scheme of Y or in the moduli space of stable maps to Y. To be specific we let $M_d(Y)$ denote the compactification in the Hilbert scheme $\mathrm{Hilb}^{dn+1}(Y)$. The moduli spaces $M_d(Y)$ and their rationality properties have been studied by de Jong and Starr [8]. They showed that any desingularisation of $M_d(Y)$ carries a canonical 2-form ω_d which at a generic point of $M_d(Y)$ is non-degenerate if d is odd and ≥ 5 and has 1-dimensional radical if d is even and ≥ 6. For the remaining small values of d, de Jong and Starr found that the radical of the form has dimension 3, 2 and 3 at a generic point if $d = 2$, 3 or 4, respectively.

The geometric reason for the degeneration of ω_2 can be seen as follows: Any rational curve C of degree 2 on Y spans a two dimensional linear space $E \subset \mathbb{P}^5$ which in turn cuts out a plane curve of degree 3 from Y. As this curve contains C, it must have a line L as residual component. Mapping $[C]$ to $[L]$ defines a natural morphism $M_2(Y) \to M_1(Y)$, the fibre over a point $[L] \in M_1(Y)$ being isomorphic to the three dimensional space of planes in $\mathbb{P}^5$ that contain the line L.

The geometry of $M_3(Y)$ is much more interesting. We show first:

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Theorem A — *Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface that does not contain a plane. Then the moduli space $M_3(Y)$ of generalised twisted cubic curves on Y is a smooth and irreducible projective variety of dimension 10.*

Let ω_3 denote the holomorphic 2-form defined by de Jong and Starr. The purpose of this paper is to produce a contraction $M_3(Y) \to Z$ to an 8-dimensional symplectic manifold Z. More precisely, we will prove:

Theorem B — *Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface that does not contain a plane. Then there is a smooth eight dimensional holomorphically symplectic variety Z and morphisms $u : M_3(Y) \to Z$ and $j : Y \to Z$ with the following properties:*

(1) *The symplectic structure ω on Z satisfies $u^*\omega = \omega_3$.*
(2) *The morphism j is a closed embedding of Y as a Lagrangian submanifold in Z.*
(3) *The morphism u factors as follows:*

$$\begin{tikzcd} M_3(Y) \arrow[rr, "u"] \arrow[dr, "a"'] & & Z \\ & Z', \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where $a : M_3(Y) \to Z'$ is a $\mathbb{P}^2$-fibre bundle and $\sigma : Z' \to Z$ is the blow-up of Z along Y.

(4) *The topological Euler number of Z is $e(Z) = 25650$.*

Since 25650 is also the Euler number of $\mathrm{Hilb}^4(K3)$, it seems likely that Z is deformation equivalent to the fourth Hilbert scheme of a K3 surface.

The manifold Z does of course depend on Y and should systematically be denoted by $Z(Y)$. In order to increase the readability of the paper we have decided to stick with Z. Nevertheless, the construction works well for any flat family $\mathcal{Y} \to T$ of smooth cubic fourfolds without planes and yields a family $\mathcal{Z} \to T$ of symplectic manifolds.

The two-step contraction $u : M_3(Y) \to Z$ has an interesting interpretation in terms of matrix factorisations. Let $P = \mathbb{C}[x_0, \dots, x_5]$ and let $R = P/f$, where f is the equation of a smooth cubic hypersurface $Y \subset \mathbb{P}^5$. The ideal $I \subset R$ of a generalised twisted cubic $C \subset Y$ is generated by two linear forms and three quadratic forms. As Eisenbud [12] has shown, the minimal free resolution

$$0 \longleftarrow I \longleftarrow R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \dots$$

becomes 2-periodic for an appropriate choice of bases for the free R-modules R_i. Going back in the resolution, information about I gets lost at each step before stabilisation sets in. One can show that this stepwise loss of information corresponds exactly to the two phases

$$M_3(Y) \to Z' \quad \text{and} \quad Z' \to Z$$

of the contraction of $M_3(Y)$. Thus periodicity begins one step earlier for curves that are arithmetically Cohen-Macaulay (aCM) than for those that are not (non-CM). Consequently, Z truly parameterises isomorphism classes of Cohen-Macaulay approximations in the sense of Auslander and Buchweitz [2]. We intend to return to these questions in a subsequent paper.

Structure of the paper. In Section §1 we introduce the basic objects of the discussion: generalised twisted cubics and their moduli space. The focus lies on describing the possible degenerations of a smooth twisted cubic space curve and understanding the fundamental difference between curves that are arithmetically CM and those that are not. Any generalised twisted cubic C spans a 3-dimensional projective space $\langle C\rangle$ and defines a cubic surface $S = Y \cap \langle C\rangle$. In Section §2 we describe the moduli spaces of generalised twisted cubics on possibly singular cubic surfaces S. Such curves are related to linear determinantal representations of S. In Section §3 we study this relation in the universal situation of integral cubic surfaces in a fixed $\mathbb{P}^3$. This is the technical heart of the paper. The main tool are methods from geometric invariant theory. The results obtained in this section will be applied in Section §4 to the family of cubic surfaces cut out from Y by arbitrary 3-dimensional projective subspaces in $\mathbb{P}^5$. With these preparations we can finally prove all parts of the main theorems.

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§1. HILBERT SCHEMES OF GENERALISED TWISTED CUBICS

A rational normal curve of degree 3, or twisted cubic for short, is a smooth curve $C \subset \mathbb{P}^3$ that is projectively equivalent to the image of $\mathbb{P}^1$ under the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ of degree 3. The set of all twisted cubics is a 12-dimensional orbit under the action of PGL_4. Piene and Schlessinger [29] showed that its closure H_0 is a smooth 12-dimensional component of $\mathrm{Hilb}^{3n+1}(\mathbb{P}^3)$ and that the full Hilbert scheme is in fact scheme theoretically the union of H_0 and a 15-dimensional smooth variety H_1 that intersect transversely along a smooth divisor $J_0 \subset H_0$. The second component H_1 parameterises plane cubic curves together with an additional and possibly embedded point; it will play no further rôle in our discussion.

We will refer to any subscheme $C \subset \mathbb{P}^3$ that belongs to a point in H_0 as a *generalised twisted cubic* and to H_0 as the Hilbert scheme of generalised twisted cubics on $\mathbb{P}^3$.

There is an essential difference between curves parameterised by $H_0 \setminus J_0$ and those parameterised by J_0. This difference is crucial for almost all arguments in this article and enters all aspects of the construction. We therefore recall the following facts from the articles of Ellingsrud, Piene, Schlessinger and Strømme [29, 14, 13] in some detail.

(1) Curves C with $[C] \in H_0 \setminus J_0$ are arithmetically Cohen-Macaulay (aCM), i.e. their affine cone in $\mathbb{C}^4$ is Cohen-Macaulay at the origin. The homogeneous ideal of such a curve is generated by a net of quadrics (q_0, q_1, q_2) that arise as minors of a 3×2-matrix A_0 with linear entries. There is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \xrightarrow{A_0} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3} \xrightarrow{\Lambda^2 A_0^t} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{1.1}$$

Up to projective equivalence there are exactly 8 isomorphism types of aCM-curves represented by the following matrices:

$$A^{(1)} = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ 0 & x_3 \end{pmatrix}, \quad A^{(4)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_1 \\ 0 & x_3 \end{pmatrix}$$

$$A^{(5)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_3 \end{pmatrix}, \quad A^{(6)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_3 \end{pmatrix}, \quad A^{(7)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \end{pmatrix}, \quad A^{(8)} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \end{pmatrix}.$$

The dimension of the corresponding strata in H_0 are 12, 11, 10, 9, 9, 8, 7 and 4 in the given order. $A^{(1)}$ defines a smooth twisted cubic, $A^{(2)}$ the union of a smooth plane conic and a line, and $A^{(3)}$ a chain of three lines. These three types are local complete intersections. $A^{(4)}$ defines the union of three collinear but not coplanar lines. The matrices in the second row define non-reduced curves that contain a line with multiplicity ≥ 2, but are always purely 1-dimensional.

(2) Curves C with $[C] \in J_0$ are not Cohen-Macaulay (non-CM). The homogeneous ideal of such a curve C is generated by three quadrics, which in appropriate coordinates can be written as x_0^2, x_0x_1, x_0x_2, and a cubic polynomial $h(x_1, x_2, x_3) = x_1^2 a(x_1, x_2, x_3) + x_1x_2b(x_1, x_2, x_3) + x_2^2 c(x_1, x_2, x_3)$. The latter defines a cubic curve in the plane $\{x_0 = 0\}$ with a singularity at the point $[0 : 0 : 0 : 1]$. Note that the three quadratic generators still arise as minors of a 3×2-matrix, namely $A_0 = \left(\begin{smallmatrix} 0 & -x_0 & x_1 \\ x_0 & 0 & -x_2 \end{smallmatrix}\right)^t$. There is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(-3)^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0.$$

Up to projective equivalence there are 9 isomorphism types of non-CM curves: The generic 11-dimensional orbit is represented by a nodal curve with polynomial $h = x_1^3 + x_2^3 + x_1x_2x_3$, and the 6-dimensional unique closed orbit by a line with a planar triple structure defined by $h = x_1^3$.

In each case, the linear span of C is the ambient space $\mathbb{P}^3$. Because of this it is easy to see that for any $m \geq 3$ the Hilbert scheme $\mathrm{Hilb}^{3n+1}(\mathbb{P}^m)$ contains a smooth component $\mathrm{Hilb}^{gtc}(\mathbb{P}^m)$ that parameterises generalised twisted cubics and that fibres locally trivially over the Grassmannian variety of 3-spaces in $\mathbb{P}^m$. The morphism

$$s : \mathrm{Hilb}^{gtc}(\mathbb{P}^m) \to \mathrm{Grass}(\mathbb{C}^{m+1}, 4)$$

maps a generalised twisted cubic in $\mathbb{P}^m$ to the projective 3-space $\langle C \rangle$ spanned by C. Conversely, if $[p] \in \mathrm{Grass}(\mathbb{C}^{m+1}, 4)$ is a point represented by an epimorphism $p : \mathbb{C}^{m+1} \to W$ onto a four-dimensional vector space W, or equivalently, by a threedimensional space $\mathbb{P}(W) \subset \mathbb{P}^m$, then the fibre $s^{-1}([p])$ is the Hilbert scheme of generalised twisted cubics in $\mathbb{P}(W)$. Clearly, $\dim \mathrm{Hilb}^{gtc}(\mathbb{P}^m) = 4m$. For any projective scheme $X \subset \mathbb{P}^m$ let $\mathrm{Hilb}^{gtc}(X) := \mathrm{Hilb}^{3n+1}(X) \cap \mathrm{Hilb}^{gtc}(\mathbb{P}^m)$ denote the *Hilbert scheme of generalised twisted cubics on* X.

Let $\mathcal{C} \subset \mathrm{Hilb}^{gtc}(\mathbb{P}^5) \times \mathbb{P}^5$ denote the universal family of generalised twisted cubics and let pr_1 and pr_2 be the projections to $\mathrm{Hilb}^{gtc}(\mathbb{P}^5)$ and $\mathbb{P}^5$, respectively. It follows from [13], Cor. 2.4., that the sheaf $\mathcal{A} := \mathrm{pr}_{1*}(\mathcal{O}_{\mathcal{C}} \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^5}(3))$ is locally free of rank 10 and that the natural restriction homomorphism $\varepsilon : S^3\mathbb{C}^6 \otimes \mathcal{O}_{\mathrm{Hilb}^{gtc}(\mathbb{P}^5)} \to \mathcal{A}$ is surjective. Let $f \in S^3\mathbb{C}^6$ be a non-zero homogeneous polynomial of degree 3 and $Y = \{f = 0\}$ the corresponding cubic hypersurfaces. Then the Hilbert scheme

$$M_3(Y) := \mathrm{Hilb}^{gtc}(Y) \tag{1.2}$$

of generalised twisted cubic curves on Y is scheme theoretically isomorphic to the vanishing locus of the section $\varepsilon(f) \in H^0(\mathrm{Hilb}^{gtc}(\mathbb{P}^5), \mathcal{A})$. In particular, any irreducible component of $M_3(Y)$ is at least 10-dimensional.

A simple dimension count shows that the set of cubic polynomials in six variables that vanish along a plane is 55 dimensional and hence a divisor in the 56-dimensional space of all cubic polynomials. We will from now on impose the condition that Y is smooth and does not contain a plane. As we will show in Section 4.3 this implies that $M_3(Y)$ is smooth as well.

To simplify the notation we put $\mathbb{G} := \mathrm{Grass}(\mathbb{C}^6, 4)$. Closed points in $\mathbb{G}$ parameterise epimorphisms $p : \mathbb{C}^6 \to W$ or equivalently 3-dimensional linear subspaces $\mathbb{P}(W) \subset \mathbb{P}^5$. Since a smooth cubic hypersurface cannot contain a 3-space, the intersection $S = \mathbb{P}(W) \cap Y$ is a cubic surface in $\mathbb{P}(W)$, and since Y does not even contain a plane, the surface S is reduced and irreducible, i.e. integral.

By construction, $M_3(Y) = \mathrm{Hilb}^{gtc}(Y)$ comes equipped with a morphism

$$s : \mathrm{Hilb}^{gtc}(Y) \to \mathbb{G}, \quad [C \subset Y] \mapsto [\langle C \rangle \subset \mathbb{P}^5],$$

with fibres

$$s^{-1}([p]) = \mathrm{Hilb}^{gtc}(S), \qquad S = Y \cap \mathbb{P}(W).$$

§2. TWISTED CUBICS ON CUBIC SURFACES

Since the morphism $s : \mathrm{Hilb}^{gtc}(Y) \to \mathbb{G}$ constructed at the end of the previous paragraph has fibres of the form $\mathrm{Hilb}^{gtc}(S)$, where S is an integral cubic surface, we will study these Hilbert schemes for arbitrary integral cubic surfaces abstractly and quite independently of Y.

Cubic surfaces form a classical subject of algebraic geometry. The classification of the different types of singularities was given by Schläfli [31] in 1864. A classical source of information on cubic surfaces is the book of Henderson [19]. For treatments in modern terminology see the papers of Looijenga [26] and Bruce and Wall [7]. We refer to the book of Dolgachev [10], Ch. 9, and the seminar notes of Demazure [9] for further references and all facts not proved here. A cubic surface $S \subset \mathbb{P}^3$ belongs to one of the following four classes:

(1) S has at most rational double point singularities,
(2) S has a simple-elliptic singularity,
(3) S is integral but not normal, or
(4) S is not integral, i.e. its defining polynomial is reducible.

Let $\mathbb{B} := \mathbb{P}(S^3\mathbb{C}^{4*})$ denote the 19-dimensional moduli space of *embedded* cubic surfaces, and let $\mathbb{B}^{\mathrm{int}} \subset \mathbb{B}$ denote the open subset of integral surfaces. It is stratified by locally closed subsets $\mathbb{B}(\Sigma)$, where Σ is a string describing the common singularity type of the surfaces $[S] \in \mathbb{B}(\Sigma)$. For example, $\mathbb{B}(A_1 + 2A_2)$ will denote the 5-codimensional stratum of surfaces with one A_1 and two A_2-singularities, whereas the 7-codimensional stratum $\mathbb{B}(\tilde{E}_6)$ parameterises surfaces with a simple-elliptic singularity. For most singularity types, the stratum $\mathbb{B}(\Sigma)$ is a single PGL_4-orbit with the exception of $\Sigma = \emptyset, A_1, 2A_1, 3A_1, A_2, A_1 + A_2$ and $\tilde{E}_6$. In these cases, the isomorphism type is not determined by the singularity type. The moduli problem for *isomorphism types* of cubic surfaces is treated by Beauville in [3] in terms of geometric invariant theory.

2.1. **Cubic surfaces with rational double points.** Let $S \subset \mathbb{P}^3$ be a cubic surface with at most rational double point singularities and let $\sigma : \tilde{S} \to S$ be its minimal resolution. The canonical divisors of S and $\tilde{S}$ are $K = -H$, if H denotes a hyperplane section, and $\tilde{K} = -\sigma^* H$, since σ is crepant. In fact, σ is defined by the complete anti-canonical linear system $|-\tilde{K}|$. The smooth surface $\tilde{S}$ is an almost (or weak) Del Pezzo surface. The orthogonal complement $\Lambda := \tilde{K}^\perp \subset H^2(\tilde{S};\mathbb{Z})$ of the canonical divisor is a negative definite root lattice of type E_6. The components $E_1, \ldots, E_m$ of the exceptional divisor of σ are -2-curves whose classes $\alpha_1, \ldots, \alpha_m$ form a subset Δ_0 in the root system $R \subset \Lambda$ that is a root basis for a subsystem $R_0 \subset R$. Let $\Lambda_0 \subset \Lambda$ denote the corresponding sub-lattice. Configurations $\Lambda_0 \subset \Lambda$ are classified by subdiagrams of the extended Dynkin diagram $\tilde{E}_6$ (cf. [6] exc. 4.4, p. 126, or [32], Thm. 2B.). That all lattice theoretically admissible configurations also arise geometrically was shown in [26]. (As Looijenga pointed out to us, the equivalent statement is not true for the other simple elliptic singularities.) The connected components of the Dynkin diagram of R_0 are in bijection with the singularities of S. This limits the possible combinations of singularity types of S to the following list: A_1, $2A_1$, A_2, $3A_1$, $A_1 + A_2$, A_3, $4A_1$, $2A_1 + A_2$, $A_1 + A_3$, $2A_2$, A_4, D_4, $2A_1 + A_3$, $A_1 + 2A_2$, A_5, D_5, $A_1 + A_5$, $3A_2$, E_6.

It is classically known that there is a close connection between roots in the E_6-lattice of the resolution $\tilde{S}$, twisted cubics on S and representations of the cubic equation of S as a linear determinant, and we will further exploit this connection in Section §3. We refer to the book of Dolgachev [10] for further information. We could, however, not find a reference for the rôle of the Weyl group in this context and therefore include a detailed discussion here. We also take the occasion (cf. Table 1 in Sec. 3.1) to correct Table 9.2. in [10], where this action was overlooked.

Let $W(R_0)$ denote the subgroup of the Weyl group $W(R)$ that is generated by the reflections s_i in the effective roots α_i, $i = 1, \ldots, m$. The root system R decomposes into finitely many orbits with respect to this action. The orbits contained in R_0 are exactly the irreducible components of R_0 and are therefore in bijection with the singularities of S. It is a well-known property of root systems that every $W(R_0)$-orbit of $\Lambda_0 \otimes \mathbb{Q}$ meets the closed Weyl chamber $\overline{\mathcal{C}} = \{\beta \mid \beta.\alpha_i \leq 0\}$ (and the opposite chamber $-\bar{\mathcal{C}}$) exactly once (cf. [21] Thm. 1.12). If we apply this to the orthogonal projection of any root α to $\Lambda_0 \otimes \mathbb{Q}$ we find in every $W(R_0)$-orbit $B \subset R$ unique roots α_B^+ and α_B^- that are characterised by the property $\pm\alpha_B^\pm.\alpha_i \leq 0$ for $i = 1, \ldots, m$. We will refer to α_B^+ and α_B^- as the *maximal* resp. *minimal* root of the orbit. Note that $-\alpha_B^+$ equals α_B^- only if $B = -B$, i.e. if B is a subset of R_0. If R_p is the irreducible subsystem of R_0 that corresponds to a singularity $p \in S$, then $\alpha_{R_p}^+$ is the longest root in the root system R_p with respect to the root basis given by exceptional curves in the fibre of p. It also equals the cohomology class of the fundamental cycle Z_p as defined by Artin [1].

Theorem 2.1 — *Let S be a cubic surface with at most rational double point singularities. Then*

$$\mathrm{Hilb}^{gtc}(S)_{\mathrm{red}} \cong \coprod_{B\in R/W(R_0)} |\mathcal{O}_{\tilde{S}}(\alpha_B^- - \tilde{K})| \;\cong\; (R/W(R_0)) \times \mathbb{P}^2.$$

Moreover, an orbit B corresponds to families of non-CM or aCM-curves depending on whether B contains effective roots or not. The generic curve in a linear system of aCM curves is smooth.

Some components of $\mathrm{Hilb}^{gtc}(S)$ can be non-reduced, as can be easily seen from the fact that the morphism $\mathrm{Hilb}^{gtc}(Y) \to \mathbb{G}$ is ramified along the divisor in $\mathbb{G}$ that corresponds to singular surfaces. For the purpose of this article there is no need to discuss this question in any detail.

We will prove the theorem in several steps.

Proposition 2.2 — *1. Let $C \subset S$ be a generalised twisted cubic, and let $\tilde{C} = \sigma^{-1}(C) \subset \tilde{S}$ denote the scheme theoretic inverse image. Then $\tilde{C}$ is an effective divisor such that the class of $\tilde{C} + \tilde{K}$ is a root in R. This root is the maximal root in its orbit. Moreover, $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$.*
2. Conversely, let α be a maximal root and let $\tilde{C} \in |\alpha - \tilde{K}|$. Then $C := \sigma(\tilde{C}) \subset S$ is a subscheme with Hilbert polynomial $3n + 1$.

Proof. Ad 1: Let $I \subset \mathcal{O}_S$ and $\tilde{I} \subset \mathcal{O}_{\tilde{S}}$ denote the ideal sheaves of C and $\tilde{C}$, respectively, so that $\sigma^*I \twoheadrightarrow \tilde{I}$ and $I \subset \sigma_*\tilde{I}$. For any singular point $p \in S$, there is an open neighbourhood U and an epimorphism $\mathcal{O}_U^n \twoheadrightarrow I|_U$. This induces surjective maps $\mathcal{O}_V^n \to \sigma^*I|_V \to \tilde{I}|_V$ on a neighbourhood $V = \sigma^{-1}(U)$ of the fibre $\sigma^{-1}(p)$. As σ has at most 1-dimensional fibres, all second or higher direct images of coherent sheaves on $\tilde{S}$ vanish, and pushing down the epimorphism $\mathcal{O}_V^n \to \tilde{I}|_V$ along σ yields an epimorphism $(R^1\sigma_*\mathcal{O}_{\tilde{S}})^n|_U \to R^1\sigma_*\tilde{I}|_U$. Since S has rational singularities, $R^1\sigma_*\mathcal{O}_{\tilde{S}} = 0$ and so $R^1\sigma_*\tilde{I} = 0$. This implies that in the following commutative diagram both rows are exact, that α is injective and that β is surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_*\tilde{I} & \longrightarrow & \sigma_*\mathcal{O}_{\tilde{S}} & \longrightarrow & \sigma_*\mathcal{O}_{\tilde{C}} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\uparrow & & \| & & {\scriptstyle\beta}\uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \end{array} \tag{2.1}$$

The homomorphism β is generically an isomorphism. If C has no embedded points, β is an isomorphism everywhere. In this case $\tilde{C}$ cannot have embedded points either as they would show up as embedded points in $\sigma_*\mathcal{O}_{\tilde{C}}$. Hence $\tilde{C}$ is an effective divisor.

If on the other hand C has an embedded point at p then C is a non-CM curve, and it follows from the global structure of such curves that p is a singular point of S, say with ideal sheaf $\mathfrak{m}$, and that I is of the form $\mathfrak{m} \cdot \mathcal{O}_S(-H)$ for a hyperplane section H through p. Let Z_p denote the fundamental cycle supported on the exceptional fibre $\sigma^{-1}(p)$. By Artin's Theorem 4 in [1], $\sigma^*\mathfrak{m} \cdot \mathcal{O}_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(-Z_p)$ and $\sigma_*\mathcal{O}_{\tilde{S}}(-Z_p) = \mathfrak{m}$, so that $\tilde{I} = \mathcal{O}_{\tilde{S}}(-Z_p - \sigma^*H)$ and $I = \sigma_*\tilde{I}$.

Thus $\tilde{C}$ is always an effective divisor and $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$. Since $R^i\sigma_*\mathcal{O}_{\tilde{S}} = 0$ and $R^i\sigma_*\tilde{I} = 0$ for $i > 0$ one also gets $R^i\sigma_*\mathcal{O}_{\tilde{C}} = 0$ for $i > 0$, and $\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) = 1$.

Since $\tilde{C}.(-\tilde{K}) = C.H = 3$, an application of the Riemann-Roch-formula gives $(\tilde{C})^2 = 1$ and hence $(\tilde{C} + \tilde{K}).\tilde{K} = 0$ and $(\tilde{C} + \tilde{K})^2 = -2$. This shows that $\alpha := \tilde{C} + \tilde{K}$ is a root in the lattice Λ. Since the ideal sheaf $\tilde{I} = \mathcal{O}_{\tilde{S}}(-\tilde{C}) = \mathcal{O}_{\tilde{S}}(-\alpha + \tilde{K})$ is generated by global sections in a neighbourhood of every effective (-2)-curve E one gets $\alpha.E = -\deg(\tilde{I}|_E) \leq 0$. This shows that α is the maximal root of its orbit.

Ad 2: Taking direct images of $0 \to \mathcal{O}_{\tilde{S}}(-\tilde{C}) \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{C}} \to 0$ one gets an exact sequence $0 \to I_C \to \mathcal{O}_S \to \pi_*\mathcal{O}_{\tilde{C}} \to R^1\sigma_*\mathcal{O}_{\tilde{S}}(-\tilde{C}) \to 0$, where I_C is the ideal sheaf of C, and all other higher direct image sheaves vanish. As α is maximal, the restriction of $\mathcal{O}_{\tilde{S}}(-\tilde{C})$ to any exceptional curve has non-negative degree. Let Z denote the sum of the fundamental cycles of all exceptional fibres. According to [1], Lemma 5, one has $H^1(Z, \mathcal{O}_{\tilde{S}}(-\tilde{C} - mZ)) = 0$ for all $m \geq 0$, and the Theorem on Formal Functions [15], Prop. III.4.2.1, now yields $R^1\sigma_*(\mathcal{O}_{\tilde{S}}(-\tilde{C})) = 0$ and thus $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$. It follows that

$$\begin{aligned}
\chi(\mathcal{O}_C(nH)) &= \chi(\mathcal{O}_{\tilde{C}}(-n\tilde{K})) = \chi(\mathcal{O}_{\tilde{S}}(-n\tilde{K})) - \chi(\mathcal{O}_{\tilde{S}}(-\tilde{C} - n\tilde{K}))\\
&= \tfrac{1}{2}\Big(n(n+1)\tilde{K}^2 - (-\tilde{C} - n\tilde{K})(-\tilde{C} - (n+1)\tilde{K})\Big)\\
&= \tfrac{1}{2}\Big(-\tilde{C}^2 + (2n+1)\tilde{C}(-\tilde{K})\Big) = 3n + 1.
\end{aligned}$$

□

The intersection product of an irreducible curve $D \subset \tilde{S}$ with $-\tilde{K}$ can only take the following values: Either $(-\tilde{K}) \cdot D = 0$, in which case D is an exceptional (-2)-curve, or $(-\tilde{K}) \cdot D = 1$, which implies that the image of D in S is a line, so that D itself must be a smooth rational curve with $D^2 = -1$, or, finally, $(-\tilde{K}) \cdot D \geq 2$ and $D^2 \geq 0$.

Lemma 2.3 — *If α is a minimal root, then $(\alpha - \tilde{K}) \cdot F \geq 0$ for every effective divisor F with $F \cdot (-\tilde{K}) \leq 1$.*

Proof. F is the sum of (-2)-curves and at most one (-1)-curve. As α is minimal it intersects each (-2)-curve non-negatively. It suffices to treat the case that F is a (-1)-curve. But then $u = \frac{1}{3}\tilde{K} + F$ lies in $\Lambda \otimes \mathbb{Q}$ with $u^2 = -\frac{4}{3}$. Now $(\alpha - \tilde{K}).F = \alpha \cdot u + 1$, so by Cauchy-Schwarz we get $(\alpha - \tilde{K}).F \geq 1 - \sqrt{2}\sqrt{\frac{4}{3}} > -\frac{2}{3}$. But the left hand side is an integer. □

Lemma 2.4 — *Let α be a minimal root. Then the linear system $|\alpha - \tilde{K}|$ is two-dimensional and base point free. In particular, the generic element in $|\alpha - \tilde{K}|$ is a smooth rational curve.*

Proof. Let $L_\alpha = \mathcal{O}_{\tilde{S}}(\alpha - \tilde{K})$. Since $(2\tilde{K} - \alpha) \cdot (-\tilde{K}) = -6 < 0$, the divisor $2\tilde{K} - \alpha$ cannot be effective. This shows that $h^2(L_\alpha) = h^0(\mathcal{O}(2\tilde{K} - \alpha)) = 0$. Any irreducible curve D with $0 > \deg L(\alpha)|_D = (\alpha - \tilde{K})D$ must be a fixed component of the linear system $|\alpha - \tilde{K}|$ satisfying $D^2 < 0$ and hence $D(-\tilde{K}) \leq 1$. But this contradicts Lemma 2.3. Hence L_α is nef and even big, and a fortiori $L_\alpha(-\tilde{K})$ is as well. The Kawamata-Viehweg Vanishing Theorem now implies that $h^1(L_\alpha) = 0$, and Riemann-Roch gives $h^0(L_\alpha) = 3$.

Suppose that F is the fixed component of $|\alpha - \tilde{K}|$ and M a residual irreducible curve. Then M is effective and nef, and $M - \tilde{K}$ is big and nef. This implies that $h^i(\mathcal{O}(M)) = 0$ for $i > 0$ and $\chi(\mathcal{O}(M)) = h^0(\mathcal{O}(M)) = h^0(L_\alpha) = 3$. Now Riemann-Roch gives $M^2 = 4 - M(-\tilde{K}) = 1 + F(-\tilde{K}) \geq 1$. As M cannot be a (-1) or (-2) curve, we have $M(-\tilde{K}) \geq 2$ and $F(-\tilde{K}) \leq 1$. By Lemma 2.3 we get $1 = (\alpha - \tilde{K})^2 = (\alpha - \tilde{K})F + FM + M^2 \geq M^2$. This shows in turn $M^2 = 1$, $FM = 0$, $F^2 = 0$ and $F(-\tilde{K}) = 0$. Since Λ is negative definite, $F = 0$. This shows that $|\alpha - \tilde{K}|$ has no fixed component.

Since $(\alpha - \tilde{K})^2 = 1$, there is at most one base point p. If there were such a point, consider the blow-up $\widehat{S} \to \tilde{S}$ at p with exceptional divisor E. The linear system $-\widehat{K} = -\tilde{K} - E$ is effective, big and nef, and since $|\alpha - \tilde{K} - E|$ has not fixed components either, another application of the Kawamata-Viehweg Vanishing Theorem gives the contradiction $\mathbb{C} = H^0(E, \mathcal{O}(\alpha - \tilde{K})|_E) \hookrightarrow H^1(\widehat{S}, \mathcal{O}(\alpha - \tilde{K} - E)) = 0$.

The smoothness of a generic curve in the linear system follows from Bertini's theorem. □

Proposition 2.5 — *Let $\alpha \in R \setminus R_0$, and let α^+ and α^- denote the maximal and the minimal root, resp., of its orbit.*

(1) *The linear system $|\alpha - \tilde{K}|$ is independent of the choice of α in its $W(R_0)$-orbit. More precisely, the differences $e_+ = \alpha^+ - \alpha$ and $e_- = \alpha - \alpha^-$ are sums of (-2)-curves, and the multiplication by these effective classes gives isomorphisms*

$$|\alpha^- - \tilde{K}| \xrightarrow{e_-} |\alpha - \tilde{K}| \xrightarrow{e_+} |\alpha^+ - \tilde{K}|.$$

In particular, $\dim |\alpha - \tilde{K}| = 2$. The linear system $|\alpha^- - \tilde{K}|$ is base point free.

(2) *For every curve $\tilde{C} \in |\alpha^- - \tilde{K}|$ one has $C := \sigma(\tilde{C}) = \sigma(\tilde{C} + e_-)$, and C is a generalised twisted cubic.*

(3) *The image $C = \sigma(\tilde{C})$ of a generic curve $\tilde{C} \in |\alpha - \tilde{K}|$ is smooth.*

Proof. As before, let $L_\alpha = \mathcal{O}_{\tilde{S}}(\alpha - \tilde{K})$.

Assume first that $\alpha^- \neq \alpha^+$, and let β be any root from the orbit of α, different from α^-. Then there is an effective root α_i such that $\beta.\alpha_i \leq -1$. In fact, $\beta.\alpha_i = -1$, since $\beta.\alpha_i = -2$ implies $\beta = \alpha_i$ contradicting the assumption that no root of the orbit of α is effective. Let $\beta' = \beta - \alpha_i = s_i(\beta)$ be the root obtained by reflecting β in α_i. Now multiplication with the equation of the exceptional (-2)-curve E_i gives an exact sequence $0 \to L_{\beta'} \to L_\beta \to L_\beta|_{E_i} \to 0$. Since $L_\beta|_{E_i} = \mathcal{O}_{E_i}(-1)$ has no cohomology, one gets $h^i(L_{\beta'}) = h^i(L_\beta)$ for all i. In particular, $|L_{\beta'}| \to |L_\beta|$ is an isomorphism. If $\tilde{C} \in |L_{\beta'}|$, there is an exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-\tilde{C} - E_i) \to \mathcal{O}_{\tilde{S}}(-C) \to \mathcal{O}_{E_i}(-1) \to 0,$$

so that the ideal sheaves $\sigma_*(\mathcal{O}_{\tilde{S}}(-\tilde{C} - E_i)) = \sigma_*(\mathcal{O}_{\tilde{S}}(-C)) \subset \mathcal{O}_S$ define the same image curve $\sigma(\tilde{C} + E_i) = \sigma(\tilde{C})$. Replacing β by β' subtracts a fixed component from the linear system $|L_\beta|$. Iterations of this step lead in finitely many steps to the minimal root α_-. The argument can be reversed to move in the opposite direction from β to α^+.

Hence all roots in the $W(R_0)$-orbit of α define isomorphic linear systems and the same family of subschemes in S. Of course, if $\alpha^- = \alpha^+$, this is true as well.

Taking $\alpha = \alpha^+$, it follows from Proposition 2.2 that these subschemes are generalised twisted cubics. Taking $\alpha = \alpha^-$, it follows from Lemma 2.4 that the linear system is two-dimensional and that the generic curve $\tilde{C} \in |L_{\alpha^-}|$ is smooth. If $p \in S$ is any singular point and $R_p \subset R_0 \subset R$ the corresponding root system, the pre-image $\sigma^{-1}(p)$ equals the effective divisor corresponding to the maximal root $\alpha^+_{R_p}$. As $\alpha^-.\alpha^+_{R_p}$ can only take the values 0 or 1, the curve $C := \sigma(\tilde{C})$ has multiplicity 0 or 1 at p. Hence p is a smooth point of C or no point of C at all. As σ is birational off the singular locus of S, the scheme C is a smooth curve. $\square$

The situation for effective roots is slightly different:

Proposition 2.6 — *Let $p \in S$ be a singular point, let $R_p \subset R_0 \subset R$ denote the corresponding irreducible root system with maximal root α^+ and minimal root $\alpha^- = -\alpha^+$. Let $\alpha \in R_p$ be an effective root.*

(1) *The difference $e := \alpha^+ - \alpha$ is effective. Multiplication with the effective classes e, α, and e, resp., induces the following isomorphisms*

$$\mathbb{P}^2 \cong |\alpha^- - \tilde{K}| \xrightarrow{\cong} |-\alpha - \tilde{K}| \subsetneq \mathbb{P}^3 \cong |-\tilde{K}| \xrightarrow{\cong} |\alpha - \tilde{K}| \xrightarrow{\cong} |\alpha^+ - \tilde{K}|.$$

(2) *For every curve $\tilde{C} \in |\alpha^- - \tilde{K}|$, the image $C = \sigma(\tilde{C} + 2Z_p)$ is a generalised twisted cubic in S with an embedded point at p, and every non aCM-curve $C \subset S$ with an embedded point at p arises in this way.*

Proof. As long as $\beta \in R_p$ is a non-effective root the first part of the proof of the previous proposition still holds and shows that $\beta - \alpha^-$ is effective, represented, say, by a curve E', that multiplication with E' defines an isomorphism $|\alpha^- - \tilde{K}| \to |\beta - \tilde{K}|$ and that for every curve $\tilde{C} \in |\alpha^- - \tilde{K}|$ the divisors $\tilde{C}$ and $\tilde{C} + E$ have the same scheme theoretic image in S. The same method shows that for every effective root $\beta \in R_p$ the linear systems $|\beta - \tilde{K}|$ and $|\alpha^- - \tilde{K}|$ are isomorphic and give the same family of subschemes in S.

Multiplication by the fundamental cycle Z_p (of class α^+) defines an embedding of the two-dimensional linear system $|-\alpha^- - \tilde{K}|$ into the three-dimensional linear system $|-\tilde{K}|$ of hyperplane sections with respect to the contraction $\sigma : \tilde{S} \to S \subset \mathbb{P}^3$. The image of the embedding is the linear subsystem of hyperplane sections through p. Let $\tilde{C}$ be any curve in the linear system $|\alpha^- - \tilde{K}|$. Its image $C_0 = \sigma(\tilde{C})$ is a hyperplane section $C_0 = H \cap S$ for a hyperplane H through p. Then $\tilde{C}$ and $\tilde{C} + Z_p$ have the same image C, but $\sigma(C + 2Z_p)$ has an additional embedded point at p. By Proposition 2.2, the image is a generalised twisted cubic, necessarily of non-CM type. $\square$

The Propositions 2.5 and 2.6 together imply Theorem 2.1.

2.2. **Cubic surfaces with a simple-elliptic singularity.** Simple-elliptic singularities were introduced and studied in general by Saito in [30] and further studied by Looijenga [26]. A cubic surface with a simple-elliptic singularity is a cone over a smooth plane cubic curve $E \subset \mathbb{P}^2 \subset \mathbb{P}^3$ with a vertex $p \in \mathbb{P}^3 \setminus \mathbb{P}^2$. The type of such a simple-elliptic singularity is denoted by $\tilde{E}_6$.

In appropriate coordinates $x_0, \ldots, x_3$ the surface S is given by the vanishing of $g = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3$ for some parameter $\lambda \in \mathbb{C}$, $\lambda^3 \neq 1$. The same equation defines a smooth elliptic curve E in the plane $\{x_0 = 0\}$, and S is the cone over E with vertex $p = [1 : 0 : 0 : 0]$. The parameter λ determines the j-invariant of the curve E. The Jacobian ideal of g in the local ring $\mathcal{O}_{S,p}$ is generated by the quadrics $x_1^2 - \lambda x_2 x_3$, $x_2^2 - \lambda x_1 x_3$, $x_3^2 - \lambda x_1 x_2$. The monomials $1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3$ form a basis of $\mathcal{O}_{S,p}/J(g)$ and hence of the tangent space to the deformation space of the singularity. Since the total degree of all monomials is ≤ 3, all deformations are realised by deformations of g in the space of cubic polynomials. This shows that $\mathbb{B}$ is the base of a versal deformation for the singularity of S. Note that although the Milnor ring $\mathcal{O}_{S,p}/J(g)$ is 8-dimensional the stratum $\mathbb{B}(\tilde{E}_6)$ has codimension 7 since the parameter corresponding to the monomial $x_1x_2x_3$ only changes the isomorphism type of the elliptic curve.

Proposition 2.7 — *Let $S \subset \mathbb{P}^3$ be the cone over a plane elliptic curve E with vertex p. Then*

$$\mathrm{Hilb}^{gtc}(S)_{\mathrm{red}} \cong \mathrm{Sym}_3(E) = E^3/S_3,$$

the third symmetric product of E. If $q = [q_1 + q_2 + q_3] \in \mathrm{Sym}_3(E)$ is not a collinear triple, the corresponding generalised twisted cubic is the union of the three lines connecting p with each q_i. If $q = E \cap H$ for a hyperplane H through p, the generalised twisted cubic is $H \cap S$ with an embedded point at p. The addition map $\mathrm{Sym}_3(E) \to E$ is a $\mathbb{P}^2$-bundle, and the non-CM curves in $\mathrm{Hilb}^{gtc}(S)$ form the fibre over the zero element $0 \in E$.

Proof. The only irreducible rational curves on S are lines connecting the vertex p with a point $q \in E$. Let C be the union of three such lines over possibly coinciding points $q_1, q_2, q_3 \in E$. The Hilbert polynomial of C is $3n + 1$ unless the points are collinear: the Hilbert polynomial then drops by one to $3n$. In this case, one has to augment C by an embedded point at p. □

2.3. **Non-normal integral cubic surfaces.** Assume that the cubic surface S is irreducible and reduced, but not normal. Then S is projectively equivalent to one of four surfaces given by the following explicit equations:

$$\begin{aligned} X_6 &= \{t_0^2 t_2 + t_1^2 t_3 = 0\}, & X_7 &= \{t_0 t_1 t_2 + t_0^2 t_3 + t_1^3 = 0\}, \\ X_8 &= \{t_1^3 + t_2^3 + t_1 t_2 t_3 = 0\}, & X_9 &= \{t_1^3 + t_2^2 t_3 = 0\}. \end{aligned}$$

The labelling is chosen in such a way that in each case the stratum $\mathbb{B}(X_n)$ is a single PGL_4-orbit of codimension n in $\mathbb{B}$. Moreover, each X_m lies in the closure the orbit of X_{m-1}.

In fact, the mutual relation between these strata can be made explicit: Both $\mathbb{B}(X_9)$ and $\overline{\mathbb{B}(X_6)}$ are smooth. A slice F in $\overline{\mathbb{B}(X_6)}$ to $\mathbb{B}(X_9)$ through the point X_9 is three-dimensional. One such slice, or more precisely, the family of non-normal surfaces parameterised by it, is

$$\tilde{f} = t_1^3 + t_2^2 t_3 + a t_1^2 t_3 + b t_0 t_1 t_2 + c t_0 t_1^2, \quad (a, b, c) \in \mathbb{C}^3.$$

The discriminant of this family is $\Delta = ab^2 + c^2$. One obtains the following stratification: $\tilde{f}_{a,b,c}$ defines a surface isomorphic to

$$\begin{cases} X_9, \\ X_8, \\ X_7, \\ X_6, \end{cases} \text{if} \begin{cases} a = b = c = 0, \\ a \neq 0,\ b = c = 0, \\ \Delta = 0, b \neq 0, \\ \Delta \neq 0. \end{cases}$$

In particular, there are three different types of X_6 surfaces over the real numbers corresponding to the components of the complement of the Whitney-umbrella $\{\Delta = 0\}$.

We will now describe $\mathrm{Hilb}^{gtc}(X_8)$; the other cases can be treated similarly. The surface $S = X_8$ is a cone in $\mathbb{P}^3$ over a plane nodal cubic. Its normalisation $\tilde{S}$ is a cone in $\mathbb{P}^4$ over a smooth twisted cubic B in a hyperplane $U \subset \mathbb{P}^3$. Let v denote the vertex of $\tilde{S}$. The normalisation morphism $\nu : \tilde{S} \to S$ is the restriction to $\tilde{S}$ of a central projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ with centre in a point c on a secant line L of B. Finally, let $\hat{S} \to \tilde{S}$ denote the minimal resolution of the singularity of $\tilde{S}$. The exceptional curve E is a rational curve with self intersection -3, and $\hat{S}$ is isomorphic to Hirzebruch surface $\mathbb{F}_3$. Lines in $\tilde{S}$ through the vertex v correspond to fibres F of the ruling $\hat{S} \to \mathbb{P}^1$, and both E and B are sections to this fibration. Any generalised twisted cubic on S when considered as a cycle, arises as the image of a divisor on $\hat{S}$ of degree 3 with respect to $E + 3F$. Now, the only irreducible curves of degree ≤ 3 on $\hat{S}$ belong to the linear systems $|E|$, $|F|$, $|E + 3F|$ (cf. [18]). As E is contracted to a point in $\tilde{S}$, it suffices to consider the curves in $|E + 3F| =: P \cong \mathbb{P}^4$. Note that P is the dual projective space to the $\mathbb{P}^4$ containing $\tilde{S}$. The images in $\tilde{S}$ of the the curves in the linear system $|E + 3F|$ are exactly the hyperplane sections. Let $T \subset \mathbb{P}^4$ denote the plane through the line L and the vertex v, and let $T^\perp \subset P$ denote the dual line. The plane T intersects $\tilde{S}$ in two lines F_0 and F_∞ which are glued to a single line F' in S by the normalisation map. So far we have identified the underlying cycles of a generalised twisted cubics on S as images of hyperplane sections of $\tilde{S}$: they are parameterised by P. In order to get the scheme structures as well, we need to blow-up P along $T^\perp$. The fibres of the corresponding fibration $P' := \mathrm{Bl}_{T^\perp}(P) \to T^*$ have the following description: If $[M] \in T^*$ is represented by a line $M \subset T$, the fibre over $[M]$ is the $\mathbb{P}^2$ of all hyperplanes in $\mathbb{P}^4$ that contain T. It is clear that the families of hyperplanes through the lines F_0 and F_∞ parameterise the same curves in S. Identifying $[F_0]$ and $[F_\infty]$ in T^* and the corresponding fibres in P' we obtain non-normal varieties $T^\dagger := T^*/\sim$ and $P^\dagger/\sim$ with a natural $\mathbb{P}^2$-fibration $P^\dagger \to T^\dagger$. It is not difficult to explicitly describe the family of curves parameterised by $P^\dagger$: We may choose coordinates $z_0, \dots, z_4$ for $\mathbb{P}^4$ in such a way that $\tilde{S}$

is the vanishing locus of the minors of the matrix $\left(\begin{smallmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{smallmatrix}\right)$ and $c = [0:1:0:0:-1]$. Let the central projection be given by $x_i = z_i$ for $i = 0, 2, 3$ and $x_1 = z_1 + z_4$, so that $S = \{g = 0\}$ with $g = x_1x_2x_3 - x_2^3 - x_3^3$. For a generic choice of $[a] \in P$, the hyperplane $\{a_0z_0 + \ldots + a_4z_4 = 0\}$ produces a curve in $\tilde{S}$ defined by the equation $g = 0$ and the vanishing of the minors of

$$\begin{pmatrix} a_0x_0 + a_4x_1 + a_2x_2 + a_3x_3 & x_2 & \frac{1}{2}(a_4 - a_1)x_3 \\ \frac{1}{2}(a_4 - a_1)x_2 & x_3 & -a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 \end{pmatrix}.$$

This fails to give a curve only if a_0, a_1 and a_4 vanish simultaneously, i.e. along $T^\perp \subset P$, and is corrected by the blowing-up of P along $T^\perp$. The identification in P' that produces $P^\dagger$ is in these coordinates given by $[0:0:a_2:a_3:a_4] \mapsto [0:2a_2:a_3:\frac{1}{2}a_4:0]$, and it is easy to see that corresponding matrices yield equal subschemes in S. We infer:

Proposition 2.8 — $\mathrm{Hilb}^{gtc}(X_8)_{\mathrm{red}}$ *is isomorphic to the four-dimensional non-normal projective variety* $P^\dagger$. □

Similar calculations can be done for the other non-normal surfaces. In fact, for the proof of the main theorems we only need the dimension estimate $\dim(\mathrm{Hilb}^{gtc}(X_m)) \leq 4$ for $m = 6, 7, 8, 9$, and this result can be obtained much simpler without studying the Hilbert schemes themselves using Corollary 3.11.

§3. Moduli of Linear Determinantal Representations

This section is the technical heart of the paper. There is a close relation between generalised twisted cubics on a cubic surface and linear determinantal representations of that surface as we will explain first. This motivates the construction of various moduli spaces using Geometric Invariant Theory as a basic tool.

Fix a three-dimensional projective space $\mathbb{P}(W)$. We will first recall a construction of Ellingsrud, Piene and Strømme [14] of the Hilbert scheme H_0 of twisted cubics in $\mathbb{P}(W)$ in terms of determinantal nets of quadrics. We will then adapt their method to construct a moduli space of determinantal representations of cubic surfaces in $\mathbb{P}(W)$, and establish the relation between these two moduli spaces. The main intermediate result is the construction of a $\mathbb{P}^2$-fibration for the Hilbert scheme of generalised twisted cubics for the universal family of integral cubic surfaces (Theorem 3.13).

Every step in the construction will be equivariant for the action of $\mathrm{GL}(W)$ and will therefore carry over to the relative situation for the projective bundle $a : \mathbb{P}(\mathcal{W}) \to \mathbb{G}$ where $\mathcal{O}_{\mathbb{G}}^6 \to \mathcal{W}$ is the tautological quotient of rank 4 over the Grassmannian variety $\mathbb{G} = \mathrm{Grass}(\mathbb{C}^6, 4)$. The ground is then prepared for passing to the particular case of the family of cubic surfaces over $\mathbb{G}$ defined by the cubic fourfold $Y \subset \mathbb{P}^5$.

Beauville's article [5] gives a thorough foundation to the topic of determinantal and pfaffian hypersurfaces with numerous references to both classical and modern treatments of the subject.

3.1. **Linear determinantal representations.** Let $S = \{g = 0\} \subset \mathbb{P}^3 = \mathbb{P}(W)$ be an integral cubic surface and let $C \subset S$ be a generalised twisted cubic. We saw earlier that the homogeneous ideal I_C of C is generated by the minors of a 3×2-matrix A_0 with coefficients in $W \cong \mathbb{C}^4$ if C is an aCM-curve. As the cubic polynomial $g \in S^3W$ that defines S must be contained in I_C, it is a linear combination of said minors and hence can be written as the determinant of a 3×3-matrix

$$A = \left(A_0 \middle| \begin{smallmatrix} * \\ * \\ * \end{smallmatrix} \right). \tag{3.1}$$

As any two such representations of g differ by a relation among the minors of A_0, it follows from the resolution (1.1) that the third column is uniquely determined by A_0 up to linear combinations of the first two columns. Such a matrix A with entries in W and $\det(A) = g$ is called a *linear determinantal representation* of S or g. Conversely, given a linear determinantal representation A of g, any choice of a two-dimensional subspace in the space generated by the column vectors of A gives a 3×2-matrix A_0'. We will see in Section 3.4) that A_0' is always sufficiently non-degenerate to define a generalised twisted cubic. In this way every generalised twisted cubic of aCM-type sits in a natural $\mathbb{P}^2$-family of such curves on S regardless of the singularity structure of S or C.

If on the other hand C is not CM the situation is similar but slightly different: the ideal I_C is cut out by g and the minors of a matrix $A_0^t = \left(\begin{smallmatrix} 0 & -x_0 & x_1 \\ x_0 & 0 & -x_2 \end{smallmatrix}\right)$. This matrix may be completed to a skew-symmetric matrix as follows:

$$A = \left(\begin{smallmatrix} 0 & x_0 \\ -x_0 & 0 \\ x_1 & -x_2 \end{smallmatrix} \middle| \begin{smallmatrix} -x_1 \\ x_2 \\ 0 \end{smallmatrix} \right). \tag{3.2}$$

Any A_0' with linearly independent vectors from the space of column vectors of A defines a non-CM curve on S as before. In fact, the $\mathbb{P}^2$-family is in this case much easier to see geometrically: Let $p = \{x_0 = x_1 = x_2 = 0\}$ denote the point defined by the entries of A, necessarily a singular point of S. Then curves in the $\mathbb{P}^2$-family simply correspond to hyperplane sections through the point p.

The $\mathbb{P}^2$-families of generalised twisted cubics that arise in this way from 3×3-matrices provide a natural explanation for the appearance of the $\mathbb{P}^2$-components of $\mathrm{Hilb}^{gtc}(S)$, if S has at most rational double points, and for the $\mathbb{P}^2$-fibration $\mathrm{Hilb}^{gtc}(S) \cong \mathrm{Sym}_3(E) \to E$, if S has a simple-elliptic singularity. We will exploit this idea further by constructing moduli spaces of determinantal representations in the next section.

We end this section by making the connection between the structure of $\mathrm{Hilb}^{gtc}(S)$ and the set of essentially different determinantal representations of S if S is of ADE-type. Here two matrices A and A' are said to give equivalent linear determinantal representations if A can be transformed into A' by row and column operations.

Let S be a cubic surface with at most rational double points. According to the previous discussion, essentially different determinantal representations correspond bijectively to families of generalised twisted cubics of aCM-type on S. We have seen in Theorem 2.1 that these are in natural bijection with $W(R_0)$ orbits on $R \setminus R_0$.

R_0	Type	#	R_0	Type	#	R_0	Type	#
$\emptyset$	I	72	$4A_1$	XVI	13	A_1+2A_2	XVII	6
A_1	II	50	$2A_1+A_2$	XIII	12	A_1+A_4	XIV	4
$2A_1$	IV	34	A_1+A_3	X	10	A_5	XI	4
A_2	III	30	$2A_2$	IX	12	D_5	XV	2
$3A_1$	VIII	22	A_4	VII	8	A_1+A_5	XIX	1
A_1+A_2	VI	20	D_4	XII	6	$3A_2$	XXI	2
A_3	V	16	$2A_1+A_3$	XVIII	5	E_6	XX	0

Table 1: Numbers of inequivalent linear determinantal representations of cubic surfaces of given singularity type.

This leads to the data in Table 1: For a surface with at most rational double points the first column gives the Dynkin type of R_0 or equivalently, the configuration of singularities of S, the second column the type notation used by Dolgachev ([10], Ch. 9) and the third column the number of $W(R_0)$-orbits on $R \setminus R_0$. The table can easily be computed with any all purpose computer algebra system.

Here are two examples:

Example 3.1. ($3A_2$ singularities) — Let $p_0, p_1, p_2 \in \mathbb{P}^2$ denote the points corresponding to the standard basis in $\mathbb{C}^3$. Consider the linear system of cubics through all three points that are tangent at p_i to the line p_ip_{i+1} (indices taken mod 3). A basis for this linear system is $z_0 = x_0x_1^2$, $z_1 = x_1x_2^2$, $z_2 = x_2x_0^2$ and $z_3 = x_0x_1x_2$. The image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is the cubic surface S with the equation $f = z_0z_1z_2 - z_3^3 = 0$. It has three A_2-singularities at the points $q_0 = [1:0:0:0]$, $q_1 = [0:1:0:0]$ and $q_2 = [0:0:1:0]$. The reduced Hilbert scheme $\mathrm{Hilb}^{gtc}(S)_{\mathrm{red}}$ consists of five copies of $\mathbb{P}^2$. Three of them are given by the linear systems $|\mathcal{O}_S(-q_i)|$, $i = 0, 1, 2$, and correspond to non-CM curves with an embedded point at q_i. The remaining two components correspond to the 2 orbits listed in the table above. Representatives of these orbits are obtained by taking the strict transforms L and Q of a general line L' and a general quadric Q' through p_0, p_1 and p_2. To be explicit, take $L' = \{x_0+x_1+x_2 = 0\}$ and its Cremona transform $Q' = \{x_0x_1+x_1x_2+x_2x_0 = 0\}$. The corresponding ideals then are $I_L = (z_0(z_2+z_3)+z_3^2, z_1(z_0+z_3)+z_3^2, z_2(z_1+z_3)+z_3^2)$ and $I_Q = (z_0(z_1+z_3)+z_3^2, z_1(z_2+z_3)+z_3^2, z_2(z_0+z_3)+z_3^2)$ and differ only by the choice of a cyclic order of the variables z_0, z_1 and z_2. Both L and Q are smooth twisted cubics that pass through all three singularities. They lead to the following two essentially different determinantal representations of the polynomial f:

$$f = \det\begin{pmatrix} 0 & -z_3 & z_0 \\ z_1 & 0 & -z_3 \\ -z_3 & z_2 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & -z_3 & z_0 \\ z_2 & 0 & -z_3 \\ -z_3 & z_1 & 0 \end{pmatrix}$$

Example 3.2. ($4A_1$ singularities) — Let $\ell_0, \ell_1, \ell_2, \ell_3$ be linear forms in three variables that define four lines in $\mathbb{P}^2$ in general position (i.e. no three pass through one point) and such that $\sum_i \ell_i = 0$. The linear system of cubics through the six intersection points has a basis consisting of monomials $z_i = \prod_{j\neq i} \ell_j$ for $i = 0, \dots, 3$. The image of the induced rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is a cubic surface S with the equation

$$f = z_1z_2z_3 + z_0z_2z_3 + z_0z_1z_3 + z_0z_1z_2$$

and with four A_1-singularities that result from the contraction of the four lines. An explicit calculation shows that there are 17 root orbits of different lengths. They correspond to families of twisted cubics on S as follows: the transform H of a general line in $\mathbb{P}^2$ gives a twisted cubic on S passing through all four singularities. It corresponds to the unique orbit of length 16 and yields the following determinantal representation.

$$f = \det\begin{pmatrix} 0 & z_0+z_3 & z_0 \\ z_1+z_2 & 0 & z_1 \\ z_2 & z_3 & 0 \end{pmatrix}$$

Despite the apparent asymmetry the matrix is in fact symmetric with respect to all variables up to row and column operations. Now there are 16 possible choices of non-collinear triples out of the 6 intersection points of the four lines. For each triple take a general smooth conic through these points. There are four triples that form the *vertices of a triangle* of lines. These yield plane curves in S that pass twice through the singularity corresponding to the line not in the triangle: the associated generalised twisted cubics are non-CM and do not lead to linear determinantal representations. They account for four orbits of effective roots of length 2. The remaining 12 triples of points yield families of twisted cubics that pass through any two out of the four singularities. These families account for the remaining 12 inequivalent linear determinantal representations and correspond to root orbits of length 4.

3.2. **Kronecker modules I: twisted cubics.** Let the group $\mathrm{GL}_3 \times \mathrm{GL}_2$ act on $U_0 := \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3 \otimes W)$, with $W \cong \mathbb{C}^4$, by

$$(g,h)\cdot A_0 = (g \otimes \mathrm{id}_W)A_0h^{-1}. \tag{3.3}$$

We will think of homomorphisms $A_0 \in U_0$ as 3×2-matrices with values in W and write simply $A_0 \mapsto gA_0h^{-1}$ for the action. The diagonal subgroup $\Delta_0 = \{(tI_3, tI_2) \mid t \in \mathbb{C}^*\}$ acts trivially, so that the action factors through the reductive group $G_0 = \mathrm{GL}_3 \times \mathrm{GL}_2 / \Delta_0$. We are interested in the invariant theoretic quotient $U_0^{ss} /\!\!/ G_0$. For an introduction to geometric invariant theory see any of the standard texts by Mumford and Fogarty [27] or Newstead [28]. In the given context, the conditions for A_0 to be semistable resp. stable were worked out by Ellingsrud, Piene and Strømme. The general case for arbitrary W and arbitrary ranks of the general linear groups was treated by Drezet [11] and Hulek [20]. We refer to these papers for proofs of the following lemma and of Lemma 3.4.

Lemma 3.3 — *A matrix $A_0 \in U_0$ is semistable if and only if it does not lie in the G_0-orbit of a matrix of the form*

$$\begin{pmatrix} * & * \\ 0 & * \\ 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

In this case, A_0 is automatically stable. The isotropy subgroup of any stable matrix is trivial. $\square$

Let $U_0^s = U_0^{ss} \subset U_0$ denote the open subset of stable points. Then

$$X_0 := U_0^s /\!\!/ G_0$$

is a 12-dimensional smooth projective variety, and the quotient map

$$q_0 : U_0^{ss} \to X_0$$

is a principal G_0-bundle. There is a universal family of maps $a_0 : F_0 \to E_0 \otimes W$, where F_0 and E_0 are vector bundles of rank 2 and 3, respectively, on X_0 with $\det(F_0) = \det(E_0)$. Moreover, $\Lambda^2 a_0 : E_0 \to S^2W$ is an injective bundle map and defines a closed embedding $X_0 \to \mathrm{Grass}(3, S^2W)$ into the Grassmannian of nets of quadrics on $\mathbb{P}(W)$, see [14]. Let $I_0 \subset \mathbb{P}(W) \times \mathbb{P}(W^*)$ denote the incidence variety of all pairs (p, V) consisting of a point $p = \{x_0 = x_1 = x_2\}$ on a hyperplane $V = \{x_0 = 0\}$. Sending (p, V) to the net (x_0^2, x_0x_1, x_0x_2) defines a map $I_0 \to \mathrm{Grass}(3, S^2W)$. Ellingsrud, Piene and Strømme show that I_0 is a closed immersion, that it factors through X_0, and that the Hilbert scheme H_0 of twisted cubics on $\mathbb{P}^3$ is isomorphic to the blow-up of X_0 along I_0. Finally, under the isomorphism $H_0 \cong \mathrm{Bl}_{I_0}(X_0)$, the divisor $J_0 = H_0 \cap H_1$ is identified with the exceptional divisor. We let $\pi_0 : H_0 \to X_0$ denote the contraction of J_0.

$$\begin{array}{ccc} J_0 & \longrightarrow & H_0 \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ I_0 & \longrightarrow & X_0 \end{array}$$

3.3. **Kronecker modules II: determinantal representations.** The reductive group $G = \mathrm{GL}_3 \times \mathrm{GL}_3 / \Delta$, with $\Delta = \{(tI_3, tI_3) \mid t \in \mathbb{C}^*\}$, acts on the affine space

$$U = \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3 \otimes W)$$

with the analogous action by $(g, h).A := gAh^{-1}$. In contrast to the case of 3×2-matrices the notions of stability and semistability differ here. Again, this is a special case of a more general result of Drezet and Hulek.

Lemma 3.4 — *A matrix $A \in U$ is semistable if it does not lie in the G-orbit of a matrix of the form*

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

and is stable if it does not lie in the G-orbit of a matrix of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \tag{3.6}$$

The isotropy subgroup of any stable matrix is trivial. □

Consequently, the quotient

$$X := U^{ss} /\!\!/ G$$

is an irreducible normal projective variety of dimension $\dim X = \dim U - \dim G = 19$. The stable part $X^s = U^s /\!\!/ G$ is a smooth dense open subset, and the quotient

$$q^s : U^s \to X^s$$

is a principal G-bundle. The character group of G is generated by $\chi : G \to \mathbb{C}^*$, $\chi(g,h) = \det(g)/\det(h)$, and the trivial line bundle $\mathcal{O}_U(\chi)$, endowed with the G-linearisation defined by χ, descends to the ample generator L_X of $\mathrm{Pic}(X)$.

The tautological homomorphism $a_U : \mathcal{O}_U^3 \to \mathcal{O}_U^3 \otimes W$ induces a map $\det(a_U) : \mathcal{O}_U(-\chi) \to \mathcal{O}_U \otimes S^3W$ that descends to a homomorphism $\det : L_X^{-1} \to \mathcal{O}_X \otimes S^3W$, which in turn induces a rational map $\det : X \dashrightarrow \mathbb{P}(S^3W^*)$. We need to understand the degeneracy locus of this map.

Proposition 3.5 — *Let $A \in U^{ss}$ be a semistable matrix and consider its determinant $\det(A) \in S^3W^*$.*

(1) *If A is semistable but not stable then $\det(A)$ is a non-zero reducible polynomial.*
(2) *If $\det(A) = 0$, then A is stable and is conjugate under the G-action to a skew-symmetric matrix.*

Lemma 3.6 — *Let B be a matrix with values in a polynomial ring over a field. If $\mathrm{rk}(B) \leq 1$, i.e. if all 2×2-minors of B vanish, there are vectors u and v with values in the polynomial ring such that $B = vu^t$. If all entries of B are homogeneous of the same degree then the same is true for both u and v.*

Proof. We may assume that B has no zero columns. Extracting from each column its greatest common divisor we may further assume that each column consists of coprime entries. As all columns are proportional over the function field we find for each pair of column vectors B_i and B_j coprime polynomials g_i and g_j such that $g_jB_i = g_iB_j$. As g_i and g_j are coprime, g_i must divide every entry of B_i. Hence g_i is unit, and for symmetry reasons g_j is as well. Therefore all columns of B are proportional over the ground field. The last assertion follows easily. □

Proof of Proposition 3.5.
1. Assume first that A is semistable but not stable. Replacing A by another matrix from its orbit we may assume that

$$A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}. \tag{3.7}$$

It is clear that $\det(A)$ factors into a linear and a quadric polynomial in S^*W. If $\det(A) = 0$, either the linear or the quadratic factor must vanish. If the linear factor vanishes A has a trivial row or column, which contradicts its semistability. If the quadratic polynomial vanishes, the lower right respectively upper left 2×2-block B satisfies $\det(B) = 0$. According to Lemma 3.6, appropriate row or column operations will eliminate a row or column of B. This contradicts again the semistability of A.

2. Let A be a stable matrix with $\det(A) = 0$ and let $C = \mathrm{adj}(A) \in (S^2W)^{3\times 3}$ denote its adjugate matrix. So $C_{ij} = (-1)^{i+j}\det(A^{ji})$ where A^{ji} is the matrix obtained from A by erasing the j-th row and the i-th column. If $\det(A^{ji})$ were 0, the rows or columns of A^{ji} would be $\mathbb{C}$-linearly dependent according to Lemma 3.6. Row or column operations applied to A would produce a row or a column with at least two zeros, contradicting the

stability of A. This shows that all entries of C are non-zero, and this holds even after arbitrary row and column operations on C, since such operations correspond to column resp. row operations on A. In particular, all columns and all rows of C contain $\mathbb{C}$-linearly independent entries. Since $\mathrm{adj}(C) = \det(A)A = 0$, one has $\mathrm{rk}(C) \leq 1$. By Lemma 3.6, there are homogeneous column vectors $u, v \in S^*W$ such that $C = uv^t$. Since the entries of the rows and columns of C are $\mathbb{C}$-linearly independent, u and v must have entries of degree 1, and these must be linearly independent for each vector. In an appropriate basis x_0, x_1, x_2, x_3 of W we may write $u = (x_2\, x_1\, x_0)^t$. Since the entries of u form a regular sequence their syzygy module is given by the Koszul matrix $K = \left(\begin{smallmatrix} 0 & x_0 & -x_1 \\ -x_0 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{smallmatrix}\right)$. Since $AC = 0$ implies $Au = 0$, it follows that $A = MK$ for some $M \in \mathbb{C}^{3\times 3}$. Finally, since the columns of A are $\mathbb{C}$-linearly independent because of the stability of A, the transformation matrix M must be invertible, and $A \sim_G K$ as claimed. □

The proposition allows for a simple stability criterion in terms of the determinant:

Corollary 3.7 — *For any $A \in U$ the following holds:*

(1) *If $\det(A) \neq 0$, then A is semistable.*
(2) *If $\det(A)$ is irreducible, then A is stable.*
(3) *If A is stable, then either $\det(A) \neq 0$ or A is in the G-orbit of a skew-symmetric matrix.*

□

We continue the discussion of the rational map $\det : X \dashrightarrow \mathbb{P}(S^3W^*)$. The following commutative diagram is inserted here as an optical guide through the following arguments. The notation will be introduced step by step.

$$
\begin{tikzcd}
\mathrm{Hom}'(\mathbb{C}^3, W) \arrow[r, "\cong"] \arrow[dr, "/\!\!/ \mathrm{GL}_3"'] & T^{ss} \arrow[r] \arrow[d, "/\!\!/ \Gamma"] & U^{ss} \subset \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3 \otimes W) \arrow[d, "/\!\!/ G"] & \\
& \mathbb{P}(W) \arrow[r] & X \arrow[r, dashed, "\det"] & \mathbb{P}(S^3W^*) \\
& \mathbb{P}(N') = J \arrow[r] \arrow[u, "\sigma"] & H \arrow[u, "\sigma"] \arrow[ur, "\delta"'] &
\end{tikzcd}
\tag{3.8}
$$

Consider the splitting $U = V \oplus T$ into the subspaces $V = \{a \in U \mid a^t = a\}$ of symmetric and $T = \{a \in U \mid a^t = -a\}$ of skew-symmetric matrices. According to Proposition 3.5, the smooth closed subset

$$T^{ss} := T \cap U^{ss} \subset U^{ss}$$

is in fact contained in the open subset U^s of stable points, and its G-orbit $G.T^{ss}$ is the vanishing locus of the determinant $\det(a_U) : \mathcal{O}_{U^{ss}}(-\chi) \to \mathcal{O}_{U^{ss}} \otimes S^3W$. An element $A \in T^{ss}$ is mapped back to T^{ss} by $[g, h] \in G$ if and only if $(gAh^{-1})^t = -gAh^{-1}$. This is equivalent to saying that $[h^tg, g^th]$ is a stabiliser of A. Hence $h = \lambda(g^t)^{-1}$ for some $\lambda \in \mathbb{C}^*$. In fact, changing h and g by an appropriate scalar, we get $[g, h] = [\gamma, (\gamma^t)^{-1}]$ for some $\gamma \in \mathrm{GL}_3$, well-defined up to a sign ± 1. We conclude that

$$T^{ss} /\!\!/ \Gamma = G.T^{ss} /\!\!/ G \subset U^{ss} /\!\!/ G = X,$$

where $\Gamma := \mathrm{GL}_3 / \pm I$ acts freely on T^{ss} via $\gamma.A = \gamma A \gamma^t$. Any deformation $a \in U$ of $A \in T^{ss}$ can be split into its symmetric and its skew-symmetric part. The skew-symmetric part gives a tangent vector to T^{ss} at A. Among the symmetric deformations those of the form $uA - Au^t$, $u \in \mathfrak{gl}_3 \cong \mathrm{Lie}(\Gamma)$, are tangent to the G-orbit of A. The bundle homomorphism

$$\rho : \mathfrak{gl}_3 \otimes \mathcal{O}_{T^{ss}} \to V \otimes \mathcal{O}_{T^{ss}}, \quad (A, u) \mapsto (A, uA - Au^t), \tag{3.9}$$

is equivariant with respect to the natural action of $\gamma \in \Gamma$ given by $\gamma.u = \gamma u \gamma^{-1}$ and $\gamma.a = \gamma a \gamma^t$ and has constant rank 8. The cokernel of ρ therefore has rank 16 and is isomorphic to the restriction to T^{ss} of the normal bundle of $G.T^{ss}$ in U^{ss}. It descends to the normal bundle of $T^{ss} /\!\!/ \Gamma$ in X.

We can look at T^{ss} in a different way that will lead to an isomorphism $T^{ss} /\!\!/ \Gamma \cong \mathbb{P}(W)$ and to an identification of its normal bundle: Let $\mathrm{Hom}'(\mathbb{C}^3, W)$ denote the open subset of injective homomorphisms $v : \mathbb{C}^3 \to W$. The group GL_3 acts naturally on $\mathbb{C}^3$, and we consider the induced action on $\mathrm{Hom}'(\mathbb{C}^3, W)$ given by $g.v := v \circ g^{-1}$. The projection $\mathrm{Hom}'(\mathbb{C}^3, W) \to \mathbb{P}(W)$ is a principal fibre bundle with respect to this action. The isomorphism

$$\tau : \mathrm{Hom}'(\mathbb{C}^3, W) \to T^{ss}, \quad v \mapsto \begin{pmatrix} 0 & v(e_3) & -v(e_2) \\ -v(e_3) & 0 & v(e_1) \\ v(e_2) & -v(e_1) & 0 \end{pmatrix}$$

is equivariant for the group isomorphism

$$\mathrm{GL}_3 \to \Gamma = \mathrm{GL}_3 / \pm I_3, \quad h \mapsto \frac{h}{\sqrt{\det(h)}}.$$

We conclude that $\mathbb{P}(W) = \mathrm{Hom}'(\mathbb{C}^3, W) /\!\!/ \mathrm{GL}_3 \cong T^{ss} /\!\!/ \Gamma$. The pull-back of the bundle homomorphism ρ in (3.9) to $\mathrm{Hom}'(\mathbb{C}^3, W)$ via τ is a homomorphism

$$\hat{\rho} : \mathrm{Hom}'(\mathbb{C}^3, W) \times \mathfrak{gl}_3 \longrightarrow \mathrm{Hom}'(\mathbb{C}^3, W) \times V, \quad (v, u) \mapsto (v, u\tau(v) - \tau(v)u^t),$$

that is GL_3-equivariant with respect to the adjoint representations on $\mathfrak{gl}_3$ and the representation

$$h.a = \frac{h}{\sqrt{\det(h)}} a \frac{h^t}{\sqrt{\det}(h)} = \frac{1}{\det(h)} hah^t$$

on V. The trivial bundle $\mathrm{Hom}'(\mathbb{C}^3, W) \times \mathbb{C}^3$ descends to the kernel K in the tautological sequence $0 \to K \to W \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(1) \to 0$ on $\mathbb{P}(W)$. Accordingly, the homomorphism $\hat{\rho}$ descends to a bundle homomorphism

$$\tilde{\rho} : \mathrm{End}(K) \to S^2 K \otimes W \otimes \det(K)^{-1}$$

on $\mathbb{P}(W)$. Rewriting the first sheaf as $\mathrm{End}(K) = K \otimes K^* = K \otimes \Lambda^2 K \otimes \det(K)^{-1}$, this bundle map is explicitly given by $w \otimes w' \wedge w'' \otimes \mu \mapsto (ww' \otimes w'' - ww'' \otimes w') \otimes \mu$. In particular, the cokernel of $\tilde{\rho}$ is isomorphic to $N \otimes \det(K)^{-1}$, where

$$N := \mathrm{im}(S^2 K \otimes_{\mathbb{C}} W \to \mathcal{O}_{\mathbb{P}(W)} \otimes_{\mathbb{C}} S^3 W)$$

is the image of the natural multiplication map. From this we conclude:

Proposition 3.8 — *The morphism* $i : \mathbb{P}(W) \cong T^{ss} /\!\!/ \Gamma \hookrightarrow X$ *constructed above is an isomorphism onto the indeterminacy locus of the rational map* $\det : X \dashrightarrow \mathbb{P}(S^3 W^*)$*. The normal bundle of* $\mathbb{P}(W)$ *in* X *is isomorphic to* $N \otimes \det(K)^{-1}$*, and* $i^*(L_X) \cong \det(K)^{-1} \cong \det(W)^{-1} \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$.

Proof. Only the last statement has not yet been shown. In fact, the composite character $\chi' : \mathrm{GL}_3 \xrightarrow{\cong} \Gamma \hookrightarrow G \xrightarrow{\chi} \mathbb{C}^*$ is given by $\chi'(h) = \det(\frac{h}{\sqrt{h}})^2 = \det(h)^{-1}$. This implies $i^* L_X \cong \det(K)^{-1}$. It follows from the exactness of the tautological sequence $0 \to K \to \mathcal{O}_{\mathbb{P}(W)} \otimes W \to \mathcal{O}_{\mathbb{P}(W)}(1) \to 0$ that $\det(K)^{-1} \cong \det(W)^{-1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(W)}(1)$. $\square$

The one-dimensional vector space $\det(W)$ appears in the proposition in order to keep all statements equivariant for the natural action of $\mathrm{GL}(W)$. Let

$$\begin{array}{ccc} J & \longrightarrow & H \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathbb{P}(W) & \longrightarrow & X \end{array}$$

denote the blow-up of X along $\mathbb{P}(W)$ with exceptional divisor J. According to the previous proposition $J = \mathbb{P}(N')$, where $N' := (N \otimes \det(K)^{-1})^*$. Note that the fibre of $\sigma : J \to \mathbb{P}(W)$ over a point p is exactly the $\mathbb{P}^{15}$-family of cubic surfaces that are singular at p. The Picard group of H is generated by $\sigma^* L_X$ and $\mathcal{O}_H(J)$.

Proposition 3.9 — *The rational map* $\det : X \dashrightarrow \mathbb{P}(S^3 W^*)$ *extends to a well-defined morphism*

$$\delta : H \to \mathbb{P}(S^3 W^*).$$

Moreover, there are bundle isomorphisms

$$\mathcal{O}_H(J)|_J \cong \mathcal{O}_{N'}(-1) \quad \textit{and} \quad \delta^* \mathcal{O}_{\mathbb{P}(S^3 W^*)}(1) \cong \sigma^* L_X \otimes \mathcal{O}_H(-J).$$

In view of this proposition we may call H the *universal linear determinantal representation.*

Proof. Let $p \in \mathbb{P}(W)$ be defined by the vanishing of the linear forms $x_0, x_1, x_2 \in W$. Its image in X is represented by the skew-symmetric matrix $A = \begin{pmatrix} 0 & x_0 & -x_1 \\ -x_0 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{pmatrix} \in T^{ss}$. The 16-dimensional vector space

$$N_0 := \{a \in U \mid a = a^t\} / \{uA - Au^t \mid u \in \mathfrak{gl}_3\}$$

represents a slice transversal to the G-orbit through A, as we have seen before. The differential of $\det : U \to S^3W$ restricted to $A + N_0$ at A equals

$$(D_A \det)(a) = \mathrm{tr}(a\,\mathrm{adj}(A)) = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}^t a \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. \tag{3.10}$$

An explicit calculation now shows that $D_K \det : N_0 \to S^3W$ is injective. This implies that $\det : X \setminus \mathbb{P}(W) = H \setminus J \to \mathbb{P}(S^3W^*)$ extends to a morphism $\delta : H \to \mathbb{P}(S^3W^*)$. The restriction $\delta|_J : J = \mathbb{P}(N') \to \mathbb{P}(S^3W^*)$ is induced by the bundle epimorphisms $\mathcal{O}_{\mathbb{P}(N')} \otimes_{\mathbb{C}} S^3W^* \twoheadrightarrow \sigma^*N^* \twoheadrightarrow \mathcal{O}_{N'}(1) \otimes \sigma^*\det(K)^{-1}$, so that

$$\delta^*\mathcal{O}_{\mathbb{P}(S^3W^*)}(1)|_{\mathbb{P}(N')} = \mathcal{O}_{N'}(1) \otimes \sigma^*\det(K)^{-1}.$$

There are integers m, m' such that $\delta^*\mathcal{O}_{\mathbb{P}(S^3W^*)}(1) = \sigma^*L_X^m \otimes \mathcal{O}_H(J)^{m'}$. The restriction to J becomes

$$\delta^*\mathcal{O}_{\mathbb{P}(S^3W^*)}(1)|_J = \sigma^*(L_X^m|_{\mathbb{P}(W)}) \otimes \mathcal{O}_H(J)|_J^{m'} = \det(K)^{-m} \otimes \mathcal{O}_{N'}(-m').$$

Comparison of the two expressions for $\delta^*\mathcal{O}_{\mathbb{P}(S^3W^*)}(1)|_J$ shows $m = 1$ and $m' = -1$. □

Corollary 3.10 — *The line bundle $\mathcal{O}_H(J)$ is ample relative $\delta : H \to \mathbb{P}(S^3W^*)$.*

Proof. Let $F \subset H$ be a subvariety of a fibre of δ. Then

$$\mathcal{O}_F \cong \delta^*\mathcal{O}_{\mathbb{P}(S^3W^*)}(1)|_F \cong \sigma^*L_X|F \otimes \mathcal{O}_H(-J)|_F,$$

so that $\mathcal{O}_H(J)|_F \cong \sigma^*L_X|_F$. Since δ is an embedding on fibres of σ, the variety F projects isomorphically into X. Hence $\sigma^*L_X|_F$ is ample. □

Corollary 3.11 — *For any cubic surface $S \subset \mathbb{P}(W)$ the δ-fibre over the corresponding point $[S] \in \mathbb{P}(S^3W^*)$ is finite if S has at most ADE-singularities and satisfies the estimate*

$$\dim \delta^{-1}([S]) \leq \dim \mathrm{Sing}(S) + 1,$$

otherwise.

Proof. The case of surfaces with ADE-singularitites was treated in Section §2. Otherwise, a point in J encodes a point $p \in \mathbb{P}(W)$ together with a cubic surface S that is singular at p. Hence $J \cap \delta^{-1}([S])$ is isomorphic to the singular locus of S through projection to $\mathbb{P}(W)$. Since J is an effective Cartier divisor that is ample relative δ, the intersection with every irreducible component of $\delta^{-1}([S])$ of positive dimension is non-empty and of codimension ≤ 1 in this component. This implies the asserted inequality. □

3.4. The $\mathbb{P}^2$-fibration for the universal family of cubic surfaces.

Let

$$R \subset H_0 \times \mathbb{P}(S^3W^*)$$

denote the incidence variety of all points $([C], [S])$ such that the generalised twisted cubic C is contained in the cubic surface S. Of the two projections $\alpha : R \to H_0$ and $\beta : R \to$

$\mathbb{P}(S^3W^*)$ the first is a $\mathbb{P}^9$-bundle by [13], Cor. 2.4, so that R is smooth and of dimension 21. We have arrived at the following set-up:

$$\begin{tikzcd}[column sep=small] & R \arrow[dl, "\alpha"', "\mathbb{P}^9"] \arrow[dr, "\beta"] & & H \arrow[dl, "\delta"'] \arrow[dr, "\sigma"] & \\ H_0 & & \mathbb{P}(S^3W^*) & & X \arrow[ll, dashed, "\det"'] \end{tikzcd} \tag{3.11}$$

Consider the open subset $\mathbb{P}(S^3W^*)^{\mathrm{int}} \subset \mathbb{P}(S^3W^*)$ of integral surfaces and the corresponding open subsets

$$H^{\mathrm{int}} = \delta^{-1}(\mathbb{P}(S^3W^*)^{\mathrm{int}}) \quad \text{and} \quad R^{\mathrm{int}} = \beta^{-1}(\mathbb{P}(S^3W)^{\mathrm{int}}).$$

By part (1) of Proposition 3.5, one has $H^{\mathrm{int}} \subset H^s \subset H$, where $H^s = \sigma^{-1}(X^s)$.

For any matrix $A \in U$ let $\mathrm{res}(A) \in U_0$ denote the submatrix consisting of its first two columns. A comparison of the Lemmas 3.4 and 3.3 shows immediately, that res restricts to a map $\mathrm{res} : U^s \to U_0^s$. Let $P' \subset \mathrm{GL}_3$ denote the parabolic subgroup of elements that stabilise the subspace $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$. The parabolic subgroup $P = (\mathrm{GL}_3 \times P')/\mathbb{C}^* \subset G$ has a natural projection $\gamma : P \to G_0$ through its Levi factor, and $\mathrm{res} : U^s \to U_0^s$ is equivariant with respect to this group homomorphism, i.e. $\gamma(p).\,\mathrm{res}(A) = \mathrm{res}(p.A)$ for all $A \in U^s$ and $p \in P$.

Since $q^s : U^s \to X^s$ is a principal G-bundle, it factors through maps

$$U^s \xrightarrow{q_P} U^s/P \xrightarrow{a_P} U^s /\!\!/ G = X^s, \tag{3.12}$$

where a_P is an étale locally trivial fibre bundle with fibres isomorphic to $G/P \cong \mathbb{P}^2$. As res is γ-equivariant it descends to a morphism $\overline{\mathrm{res}} : U^s/P \to X_0 = U_0^s/G_0$. This provides us with morphisms

$$X_0 \xleftarrow{\overline{\mathrm{res}}} U^s/P \xrightarrow{a_P} X^s. \tag{3.13}$$

Let $\sigma_Q : Q \to U^s/P$ denote the blow-up along $a_P^{-1}(I)$. By the universal property of the blow up, there is a natural morphism $a_Q : Q \to H^s$, which is again a $\mathbb{P}^2$-bundle.

$$\begin{array}{ccc} Q & \xrightarrow{\sigma_Q} & U^s/P \\ a_Q\big\downarrow & & \big\downarrow a_P \\ H^s & \xrightarrow{\sigma} & X^s \end{array}$$

Let $Q^{\mathrm{int}} = a_Q^{-1}(H^{\mathrm{int}})$.

Proposition 3.12 — *$R^{\mathrm{int}} \cong Q^{\mathrm{int}}$ as schemes over $X_0 \times \mathbb{P}(S^3W^*)^{\mathrm{int}}$*

Proof. Q^{int} parameterises via the composite morphism $Q^{\mathrm{int}} \to H^{\mathrm{int}} \to \mathbb{P}(S^3W^*)$ a family of cubic surfaces $S_q = \{g_q = 0\}$, $q \in Q^{\mathrm{int}}$, and via the composite morphism $Q^{\mathrm{int}} \to U^s/P \to X_0$ a family of determinantal nets of quadrics $(Q_q^{(1)}, Q_q^{(2)}, Q_q^{(3)})$, $q \in Q^{\mathrm{int}}$, in such a way that either the ideal $I_q := (Q_q^{(1)}, Q_q^{(2)}, Q_q^{(3)})$ defines an aCM generalised twisted cubic on the surface S_q, or I_q is the ideal of a hyperplane with an embedded point on S_q. But in both cases the ideal $I_q' := I_q + (g_q)$ defines a generalised twisted cubic C_q on S_q. As the base scheme Q^{int} of this family is reduced and the Hilbert polynomial of the family of curves C_q is constant, this family is flat. Since R is the moduli space of pairs $(C \subset S)$ of a generalised twisted cubic on a cubic surface, there is classifying morphism $\psi : Q^{\mathrm{int}} \to R$ whose image is obviously contained in R^{int}. As both Q^{int} and R^{int} are smooth it suffices to show that ψ is bijective.

Let $([A], g)$ be a point in Q^{int}. We need to show that A can be reconstructed up to the action of P from $([A_0], g)$ where $A_0 = \mathrm{res}(A)$. If A_0 defines an aCM-curve, it follows from the presentation (1.1) that any extension of A_0 to a matrix B with $\det(B) = g$ and $\mathrm{res}(B) = A_0$ is unique up to adding multiples of the first two columns to the last. But this is exactly the way that P acts on the columns of A. If on the other hand A_0 (together with g) defines a non-CM curve, the point $[A_0]$ belongs to I_0, and the determinant of any B with $\mathrm{res}(B) = A_0$ will split off a linear factor. As $[B]$ is required to lie in Q^{int} this is only possible when $\det(B) = 0$ according to part (1) of Proposition 3.5. By part (2) of the same proposition it follows again that B is in the P-orbit of A. This proves the injectivity of ψ.

Assume finally that a point $n \in R^{\mathrm{int}}$ be given. It determines and is determined by a pair $([A_0], f)$. If $[A_0] \in I_0$, the existence of a stable matrix A with $\mathrm{res}(A) = A_0$ is clear. If $[A_0] \notin I_0$, there is a unique matrix $A \in U$ up to column transformations with $\mathrm{res}(A) = A_0$ and $\det(A) = g$. Since g is non-zero and irreducible, A is stable. This shows that ψ is surjective as well. $\square$

We can summarise the results of this section as follows:

Theorem 3.13 — *Let R^{int} denote the moduli space of pairs (C, S) of an integral cubic surface S and a generalised twisted cubic $C \subset S$ in a fixed three-dimensional projective space $\mathbb{P}(W)$.*

(1) *The projection $R^{\mathrm{int}} \to H_0$ to the first component is a surjective smooth morphism whose fibres are open subsets in $\mathbb{P}^9$. In particular, R^{int} is smooth.*
(2) *The projection $R^{\mathrm{int}} \to \mathbb{P}(S^3W^*)^{\mathrm{int}}$ is projective and factors as follows:*

$$R^{\mathrm{int}} \xrightarrow{a_R} H^{\mathrm{int}} \xrightarrow{\delta} \mathbb{P}(S^3W^*)^{\mathrm{int}}, \tag{3.14}$$

where a_R is a $\mathbb{P}^2$-bundle and δ is generically finite.

$\square$

§4. TWISTED CUBICS ON Y

In the previous Section §3 we have discussed the geometry of generalised twisted cubics on cubic surfaces for the universal family of cubic surfaces in a fixed 3-dimensional projective space $\mathbb{P}(W)$, the main result being the construction of maps

$$H_0 \longleftarrow R^{\mathrm{int}} \longrightarrow H \xrightarrow{\delta} \mathbb{P}(S^3W^*).$$

The cubic fourfold Y has played no rôle in the discussion so far. The intersections of Y with all 3-spaces in $\mathbb{P}^5$ form a family of cubic surfaces parameterised by the Grassmannian $\mathbb{G} = \mathrm{Grass}(\mathbb{C}^6, 4)$. All schemes discussed in the previous section come with a natural

$\mathrm{GL}(W)$-action, and all morphisms are $\mathrm{GL}(W)$-equivariant. This allows us to generalise all results to this relative situation over the Grassmannian.

In this section we will construct the morphisms $\mathrm{Hilb}^{gtc}(Y) \to Z' \to Z$ and prove that Z is an 8-dimensional connected symplectic manifold.

4.1. **The family over the Grassmannian.** Let $\mathbb{G} := \mathrm{Grass}(\mathbb{C}^6, 4)$ denote as before the Grassmannian of three-dimensional linear subspaces in $\mathbb{P}^5$, let $\mathcal{O}_{\mathbb{G}}^6 \to \mathcal{W}$ denote the universal quotient bundle of rank 4. The projectivisation $\mathbb{P}(\mathcal{W})$ is a partial flag variety and comes with two natural projections $a : \mathbb{P}(\mathcal{W}) \to \mathbb{G}$ and $q : \mathbb{P}(\mathcal{W}) \to \mathbb{P}^5$. Let

$$0 \to K \to a^*\mathcal{W} \to \mathcal{O}_a(1) \to 0 \tag{4.1}$$

denote the tautological exact sequence. Then $\det(K)^{-1} = \mathcal{O}_a(1) \otimes a^*\det(\mathcal{W})^{-1}$. Furthermore, let $\mathbb{S} := \mathbb{P}(S^3\mathcal{W}^*)$ denote the space of cubic surfaces in the fibres of a, let $\mathbb{S}^{\mathrm{int}} \subset \mathbb{S}$ denote the open subset corresponding to integral surfaces, and let $c : \mathbb{S} \to \mathbb{G}$ denote the natural projection.

We will build up the following commutative diagram of morphisms step by step:

$$\begin{tikzcd}
\mathbb{P}(\mathcal{N}') \arrow[r, hook, "j"] \arrow[d, "\sigma"] & \mathbb{H} \arrow[d, "\tilde{\sigma}"] \arrow[dr, "\delta"] & \\
\mathbb{P}(\mathcal{W}) \arrow[r, hook, "i"] \arrow[d, "q"] \arrow[dr, "a"] & \mathbb{X} \arrow[r, dashed, "\det"] \arrow[d, "b"] & \mathbb{S} \arrow[dl, "c"] \\
\mathbb{P}^5 & \mathbb{G} &
\end{tikzcd} \tag{4.2}$$

Generalising the results of Section 3.3 to the relative case we consider the vector bundle $\mathcal{H}om(\mathbb{C}^3, \mathbb{C}^3 \otimes \mathcal{W})$ on $\mathbb{G}$ and the quotient $\mathbb{X}$ of its open subset of semistable points by the group $G = (\mathrm{GL}_3 \times \mathrm{GL}_3)/\mathbb{C}^*$. The natural projection $b : \mathbb{X} \to \mathbb{G}$ is a projective morphism and a Zariski locally trivial fibre bundle with fibres isomorphic to X. There is a canonical embedding $i : \mathbb{P}(\mathcal{W}) \to \mathbb{X}$ of $\mathbb{G}$-schemes such that the normal bundle of $\mathbb{P}(\mathcal{W})$ in $\mathbb{X}$ is given by

$$\nu_{\mathbb{P}(\mathcal{W})/\mathbb{X}} \cong \mathcal{N} \otimes \det(K)^{-1} \cong \mathcal{N} \otimes \mathcal{O}_a(1) \otimes a^*\det(\mathcal{W})^{-1}, \tag{4.3}$$

where $\mathcal{N}$ is the image of the natural multiplication map $S^2K \otimes a^*\mathcal{W} \to a^*S^3\mathcal{W}$. Let $\tilde{\sigma} : \mathbb{H} \to \mathbb{X}$ denote the blow-up of $\mathbb{X}$ along $\mathbb{P}(\mathcal{W})$. The exceptional divisor of $\tilde{\sigma}$ can be identified with $\mathbb{P}(\mathcal{N}')$, where $\mathcal{N}' := \nu^*_{\mathbb{P}(\mathcal{W})/\mathbb{X}}$, and we let $\sigma : \mathbb{P}(\mathcal{N}') \to \mathbb{P}(\mathcal{W})$ and $j : \mathbb{P}(\mathcal{N}') \to \mathbb{H}$ denote the canonical projection and inclusion, respectively. As we have seen in previous sections, the rational map $\det : \mathbb{X} \dashrightarrow \mathbb{S}$ extends to a well-defined morphism $\delta : \mathbb{H} \to \mathbb{S}$.

Finally, let $\mathbb{H}_0 \to \mathbb{G}$ denote the relative Hilbert scheme of generalised twisted cubics in the fibres of $a : \mathbb{P}(\mathcal{W}) \to \mathbb{G}$, and let $\mathcal{R}^{\mathrm{int}}$ denote the moduli space of pairs (C, S) where S is an integral cubic surface in a fibre of a and C is a generalised twisted cubic in

S. Generalising Theorem 3.13 to the relative situation over the Grassmannian we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{H} \arrow[d] & \mathbb{H}^{\mathrm{int}} \arrow[l] \arrow[d] & \mathcal{R}^{\mathrm{int}} \arrow[l, "a"'] \arrow[d] \\ \mathbb{S} \arrow[dr] & \mathbb{S}^{\mathrm{int}} \arrow[l] & \mathbb{H}_0 \arrow[dl] \\ & \mathbb{G} & \end{tikzcd} \tag{4.4}$$

where a is a $\mathbb{P}^2$-bundle.

Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface defined by a polynomial $f \in S^3\mathbb{C}^6$ and assume that Y does not contain a plane. Then f defines a nowhere vanishing section in $S^3\mathcal{W}$ and hence a section $\gamma_f : \mathbb{G} \to \mathbb{S}$ to the bundle projection c. For a point $[\mathbb{P}(W)] \in \mathbb{G}$, its image $[S] = \gamma_f([\mathbb{P}(W)])$ is the surface $S = \mathbb{P}(W) \cap Y$. Since Y does not contain a plane, γ_f takes values in the open subset $\mathbb{S}^{\mathrm{int}} \subset \mathbb{S}$ of integral surfaces.

We define a projective scheme Z' with a Cartier divisor $D \subset Z'$ by the following pull-back diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{N}') & \hookrightarrow & \mathbb{H} & \longrightarrow & \mathbb{S} \\ \cup & & \cup & & \cup \\ D & \hookrightarrow & Z' & \longrightarrow & \gamma_f(\mathbb{G}) \end{array}$$

As $\gamma_f(\mathbb{G})$ is contained in $\mathbb{S}^{\mathrm{int}}$, the scheme Z' is in fact contained in the open subset $\mathbb{H}^{\mathrm{int}} \subset \mathbb{H}$.

Proposition 4.1 — $a^{-1}(Z') \cong \mathrm{Hilb}^{gtc}(Y)$, *and* $a^{-1}(D)$ *is the closed subset of non-CM curves.*

Proof. The natural projection $\mathrm{Hilb}^{gtc}(Y) \to \mathbb{G}$ lifts both to a closed immersion

$$\mathrm{Hilb}^{gtc}(Y) \to \mathbb{H}_0$$

and to a morphism $\mathrm{Hilb}^{gtc}(Y) \to \mathbb{S}^{\mathrm{int}}$, sending a curve C with span $\langle C \rangle = \mathbb{P}(W)$ to the point $[C] \in \mathrm{Hilb}^{gtc}(\mathbb{P}(W)) \subset \mathbb{H}_0$ and the point $[\mathbb{P}(W) \cap Y]$, respectively. By the definition of $\mathcal{R}^{\mathrm{int}}$, these two maps induce a closed immersion $\mathrm{Hilb}^{gtc} \to \mathcal{R}^{\mathrm{int}}$, whose image equals $a^{-1}(Z')$ by Theorem 3.13. The second assertion follows similarly. □

We have proved the first part of Theorem B: the existence of a natural $\mathbb{P}^2$-fibration

$$\mathrm{Hilb}^{gtc}(Y) \xrightarrow{a} Z'$$

relative to $\mathbb{G}$.

Proposition 4.2 — *Let* Y *be a smooth cubic fourfold. Then the closure of the set of points* $[\mathbb{P}(W)] \in \mathbb{G}$ *such that* $S = \mathbb{P}(W) \cap Y$ *is a non-normal integral surface is at most 4-dimensional.*

Proof. Let $L \subset Y = \{f = 0\}$ be a line, and let $U \subset \mathbb{C}^6$ denote the four-dimensional space of linear forms that vanish on L, so that $L = \mathbb{P}(V)$ for $V = \mathbb{C}^6/U$. By assumption, the cubic polynomial $f \in S^3\mathbb{C}^6$ vanishes on L and hence is contained in the kernel of $S^3\mathbb{C}^6 \to S^3V$. Its leading term is a polynomial $\bar{f} \in U \otimes S^2V = \mathrm{Hom}(U^*, S^2V)$. That Y is smooth along L is equivalent to saying that the four quadrics in the image of $\bar{f} : U^* \to S^2V$ must not have a common zero on L. Hence $\bar{f}$ has at least rank 2. On the other hand, if L is the line of singularities of a non-normal surface $Y \cap \mathbb{P}(W)$, then $\bar{f}$ has at most rank 2, and $W^* \subset \mathbb{C}^{6*}$ is determined as the preimage of $\ker(\bar{f})$ under the projection $\mathbb{C}^{6*} \to U^*$. In particular, every line $L \subset Y$ is the singular locus of at most one non-normal integral surface of the form $S = Y \cap \mathbb{P}(W)$. As the space of lines on a smooth cubic fourfold is four-dimensional, the assertion follows. □

Since non-normal surfaces form a stratum of codimension 6 in $\mathbb{P}(S^3\mathbb{C}^4)$, the 'non-normal' locus in $\mathbb{G}$ is in fact only 2-dimensional for a generic fourfold Y.

Proposition 4.3 — *Let Y be a smooth cubic fourfold not containing a plane. Then the closure of the set of points $[\mathbb{P}(W)] \in \mathbb{G}$ such that $S = \mathbb{P}(W) \cap Y$ has a simple-elliptic singularity is at most 4-dimensional.*

Proof. Let $p \in Y = \{f = 0\}$ be a point. Any 3-space $\mathbb{P}(W)$ with the property that $S = Y \cap \mathbb{P}(W)$ is a cone with vertex p must be contained in the tangent space to Y at p. Then one may choose coordinates $x_0, \ldots, x_5$ in a way that $x_0, \ldots, x_4$ vanish at p, that $x_0 = 0$ defines the tangent space and that f takes the form $f = x_5^2x_0 + x_5q(x_1, \ldots, x_4) + c(x_0, \ldots, x_4)$ for a quadric polynomial q and a cubic polynomial c. If q vanishes identically, we may choose a line L in $\{x_0 = 0 = c\} \subset \mathbb{P}^4$. As the plane spanned by L and p would be contained in Y this case is excluded. A 3-space through p intersects Y in a cone if and only if it is the span of p and a plane in the quadric surface $\{x_0 = 0 = q\}$. Clearly, for any point $p \in Y$ there at most two such planes. Thus the family of such 3-spaces is at most 4-dimensional. □

Again, the expected dimension of the 'simple-elliptic' locus is much smaller. We may restate the argument in a coordinate free form as follows: Let $f \in S^3\mathbb{C}^6$ denote the cubic polynomial that defines a smooth fourfold $Y \subset \mathbb{P}^5$ as before. The restriction to Y of the Jacobi map $Jf : \mathcal{O}_Y(-2) \to \mathcal{O}_Y^6$ takes values in $\Omega_{\mathbb{P}^5}(1)|_Y$. Since Y is smooth, this map vanishes nowhere, giving rise to a short exact sequence $0 \to \Omega_Y(1) \to \mathcal{F} \to \mathcal{O}_Y(1) \to 0$ with $\mathcal{F} = \mathcal{O}_Y^6/\mathcal{O}_Y(-2)$. By construction, the image of f under the canonical map $S^3\mathbb{C}^6 \to H^0(Y, S^3\mathcal{F})$ takes values in the subbundle $\mathcal{F} \cdot S^2(\Omega_Y(1))$ with leading term $\tilde{f} \in H^0(Y, S^2(\Omega_Y(1)) \otimes \mathcal{O}_Y(1)) = \mathrm{Hom}_Y(\mathcal{O}_Y(-3), S^2\Omega_Y)$. Considering $\tilde{f}$ considered as a symmetric map $\mathcal{O}_Y(-3) \otimes \Omega_Y^* \to \Omega_Y$ we may ask for the locus where its rank is ≤ 2. Standard intersection theoretic methods [17] allow to calculate the expected cycle class as $35h^3$, where h is the class of a hyperplane section in Y. This implies:

Corollary 4.4 — *Let Y be smooth cubic fourfold not containing a plane. Then there is a 3-space $\mathbb{P}(W) \subset \mathbb{P}^5$ such that $Y \cap \mathbb{P}(W)$ has a simple-elliptic singularity.*

4.2. **The divisor $D \subset Z'$.** A closed point $[C]$ in $D \subset Z'$ corresponds to a family of non-CM curves on a surface $S = \mathbb{P}(W) \cap Y$ for some three-dimensional linear subspace $\mathbb{P}(W) \subset \mathbb{P}^5$. In fact, such a family is obtained by intersecting S with all planes in $\mathbb{P}(W)$ through a fixed singular point $p \in S$ (and adding the unique non-reduced structure at p).

On the other hand, if $p \in Y$ is any point, a three-dimensional linear space $\mathbb{P}(W)$ through p intersects Y in such a way that p becomes a singular point of $S = \mathbb{P}(W) \cap Y$ if and only if $\mathbb{P}(W)$ is contained in the projective tangent space of Y at p. This defines a bijective morphism $j : \mathbb{P}(T_Y) \to D \subset Z'$. In fact:

Proposition 4.5 — *Let $\pi : \mathbb{P}(T_Y) \to Y$ denote the projectivisation of the tangent bundle of Y. The morphism $j : \mathbb{P}(T_Y) \to D$ is an isomorphism, and $j^*\mathcal{O}_{Z'}(D) = \mathcal{O}_\pi(-1)$.*

Proof. Let $0 \to U \to \pi^* T_Y \to \mathcal{O}_\pi(1) \to 0$ denote the tautological bundle sequence on $\mathbb{P}(T_Y)$. Starting from the Euler sequence on $\mathbb{P}^5$ we obtain the following pull-back diagram of short exact sequences of sheaves on $\mathbb{P}(T_Y)$.

$$\begin{array}{ccccccccc}
0 & \to & \pi^*\mathcal{O}_Y(-1) & \to & \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}(T_Y)} & \to & \pi^*(T_{\mathbb{P}^5}|_Y \otimes \mathcal{O}_Y(-1)) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \pi^*\mathcal{O}_Y(-1) & \to & V' & \to & \pi^*(T_Y \otimes \mathcal{O}_Y(-1)) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \pi^*\mathcal{O}_Y(-1) & \to & V & \to & U \otimes \pi^*\mathcal{O}_Y(-1) & \to & 0
\end{array}$$

The bundle inclusions $\pi^*\mathcal{O}_Y(-1) \subset V \subset \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}(T_Y)}$ induce a closed immersion $u : \mathbb{P}(T_Y) \to \mathbb{P}(\mathcal{W})$ with $V^* = u^*a^*\mathcal{W}$ and $u^*\mathcal{O}_a(1) = \pi^*\mathcal{O}_Y(1)$. Moreover, the composite map $\mathcal{O}_{\mathbb{P}(T_Y)} \xrightarrow{f} S^3\mathcal{O}^6_{\mathbb{P}(T_Y)} \to u^*a^*S^3\mathcal{W}$ takes values in the subbundle $u^*\mathcal{N}$ (cf. (4.3)), inducing a bundle monomorphism

$$u^*(\mathcal{O}_a(1) \otimes a^*\det(\mathcal{W})^{-1}) \to u^*(N')$$

and hence a morphism $v : \mathbb{P}(T_Y) \to \mathbb{P}(\mathcal{N}')$ with $\sigma \circ v = u$ and

$$v^*\mathcal{O}_\sigma(-1) = u^*(\mathcal{O}_a(1) \otimes a^*\det(\mathcal{W})^{-1}) = \pi^*\mathcal{O}_Y(1) \otimes (a \circ u)^*\det(\mathcal{W})^{-1}. \tag{4.5}$$

Adding u and v to diagram (4.2) we get

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{N}') \arrow[r, hook, "j"] \arrow[d, "\sigma"] & \mathbb{H} \arrow[d, "\tilde{\sigma}"] \\
\mathbb{P}(T_Y) \arrow[ur, "v"] \arrow[r, "u"'] \arrow[d, "\pi"] & \mathbb{P}(\mathcal{W}) \arrow[r, hook, "i"] \arrow[d, "q"] \arrow[dr, "a"] & \mathbb{X} \arrow[d, "b"] \\
Y \arrow[r, hook] & \mathbb{P}^5 & \mathbb{G}
\end{tikzcd} \tag{4.6}$$

Since $(a \circ u)^*\det(\mathcal{W})^{-1} = \det(V) = \pi^*\mathcal{O}_Y(-1)^4 \otimes \det(U)$ we may simplify this as follows:

$$v^*\mathcal{O}_\sigma(-1) \cong \pi^*(\det(T_Y) \otimes \mathcal{O}_Y(-3)) \otimes \mathcal{O}_\pi(-1) \cong \mathcal{O}_\pi(-1) \tag{4.7}$$

Since u is a closed immersion, so is v. By construction, the image of v is contained in D. This shows that $\mathbb{P}(T_Y) \cong D_{\mathrm{red}}$. But the pull-back of the normal bundle $\mathcal{O}_{\mathbb{H}}(\mathbb{J})|_{\mathbb{J}} = \mathcal{O}_\sigma(-1)$ to $\mathbb{P}(T_Y)$ equals $\mathcal{O}_\pi(-1)$ according to equation (4.7) and hence is not a power of any other line bundle. This implies that $\mathbb{P}(T_Y)$ indeed is isomorphic to the scheme-theoretic intersection $D = Z' \cap \mathbb{J}$ and that $\mathcal{O}_{Z'}(D)|_D = \mathcal{O}_\pi(-1)$ with respect to the identification $D = \mathbb{P}(T_Y)$. □

Corollary 4.6 — *Z' is smooth along D.*

Proof. Since D is smooth and a complete intersection in Z', the ambient space Z' must be smooth along D as well. □

4.3. **Smoothness and Irreducibility.** Let $Y = \{f = 0\} \subset \mathbb{P}^5$ be a smooth cubic hypersurface that does not contain a plane. In this section we prove that $\mathrm{Hilb}^{gtc}(Y)$ is smooth and irreducible. Due to the $\mathbb{P}^2$-bundle map $a : \mathrm{Hilb}^{gtc}(Y) \to Z'$ both assertions are equivalent to the analogous statement about Z'.

Theorem 4.7 — *$\mathrm{Hilb}^{gtc}(Y)$ is smooth of dimension 10.*

Proof. 1. Since $\mathrm{Hilb}^{gtc}(Y)$ is the zero locus of a section in a vector bundle of rank 10 on a 20-dimensional smooth variety $\mathbb{H}_0 = \mathrm{Hilb}^{gtc}(\mathbb{P}^5)$, every irreducible component of $\mathrm{Hilb}^{gtc}(Y)$ has dimension ≥ 10. In order to proof smoothness, it suffices to show that all Zariski tangent spaces are 10-dimensional.

Due to the existence of a $\mathbb{P}^2$-fibre bundle map $a : \mathrm{Hilb}^{gtc}(Y) \to Z'$, the Hilbert scheme is smooth at a point $[C]$ if and only if Z' is smooth at $a([C])$, or equivalently, if $\mathrm{Hilb}^{gtc}(Y)$ is smooth at *any* point of the fibre $a^{-1}(a([C]))$. And due to Corollary 4.6 which takes care of the non-CM-locus, it suffices to consider aCM-curves, for which there is a functorial interpretation of tangent space: $T_{[C]}\mathrm{Hilb}^{gtc}(Y) \cong \mathrm{Hom}(I_{C/Y}, \mathcal{O}_C)$.

Thus it remains to prove that $\hom(I_{C/Y}, \mathcal{O}_C) = 10$ for any generalised twisted cubic $C \subset Y$ of aCM-type whose isomorphism type is generic within the family $a^{-1}(a([C]))$.

2. Given an aCM-curve $C \subset Y$ we may choose coordinates $x_0, \ldots, x_5$ in such a way that the ideal sheaf $I_{C/\mathbb{P}^5}$ is defined by the linear forms x_4 and x_5 and the quadratic minors of a 3×2-matrix A_0 with linear entries in the coordinates $x_0, \ldots, x_3$. The surface $S = Y \cap \{x_4 = x_5 = 0\}$ is cut out by a cubic polynomial $g \in \mathbb{C}[x_0, x_1, x_2, x_3]$. There are quadratic polynomials q_4 and q_5 such that $f = g + x_4 q_4 + x_5 q_5$ and linear forms ℓ_0, ℓ_1, ℓ_2 in $x_0, \ldots, x_3$ such that

$$g = \det(A) \qquad \text{for} \qquad A = \left(A_0 \;\middle|\; \begin{smallmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{smallmatrix} \right).$$

The ideal sheaf $I_{C/\mathbb{P}^5}$ has a presentation

$$\mathcal{O}_{\mathbb{P}^5}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^5}(-3)^6 \oplus \mathcal{O}_{\mathbb{P}^5}(-2) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^5}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^5}(-1)^2 \longrightarrow I_{C/\mathbb{P}^5} \longrightarrow 0,$$

with

$$M = \left(\begin{array}{c|c|c} A_0 & * & 0 \\ \hline 0 & * & * \end{array} \right),$$

where the entries denoted by $*$ give the tautological relations between the quadrics and the linear forms defining $I_{C/\mathbb{P}^5}$. They vanish identically when restricted to C. Therefore, $\mathcal{H}om(I_{C/\mathbb{P}^5},\mathcal{O}_C) = F \oplus \mathcal{O}_C(1)^2$ with $F = \ker(\mathcal{O}_C(2)^3 \xrightarrow{A_0^t} \mathcal{O}_C(3)^2)$. Since Y is smooth along C, the natural homomorphism $\varphi : \mathcal{H}om(I_{C/\mathbb{P}^5},\mathcal{O}_C) \to N_{Y/\mathbb{P}^5}|_C = \mathcal{O}_C(3)$ is surjective, and $\ker(\varphi) = \mathcal{H}om(I_{C/Y},\mathcal{O}_C)$. The homomorphism φ can be lifted to $\mathcal{O}_C(2)^3 \oplus \mathcal{O}_C(3)^2$ in such a way that there is an exact sequence

$$0 \longrightarrow \mathcal{H}om(I_{C/Y},\mathcal{O}_C) \longrightarrow \mathcal{O}_C(2)^3 \oplus \mathcal{O}_C(1)^2 \xrightarrow{B} \mathcal{O}_C(3)^3 \tag{4.8}$$

with

$$B = \left(\begin{array}{ccc|cc} & A_0^t & & 0 & 0 \\ \hline \ell_0 & \ell_1 & \ell_2 & q_4 & q_5 \end{array}\right)$$

Note that $\varphi|_F$ vanishes at a point of C if and only if the surface S is singular at this point. We will now analyse B for the four reduced types of aCM-curves. In the first three cases, the curve C is in fact locally a complete intersection, and $N_{C/Y} = \mathcal{H}om(I_{C/Y},\mathcal{O}_C)$ is locally free of rank 3.

3. Assume that C is a smooth twisted cubic. For an appropriate choice of coordinates we have $A_0^t = \left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right)$, and we parameterise the curve by

$$\iota : \mathbb{P}^1 \to C, \quad [s:t] \to [s^3 : s^2t : st^2 : t^3 : 0 : 0].$$

Then $\iota^* A_0^t = \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s^2 & st & t^2 \end{smallmatrix}\right)$ has kernel $\iota^* F = \mathcal{O}_{\mathbb{P}^1}(5)^2$, and

$$\mathcal{H}om(I_{C/Y},\mathcal{O}_C) \cong \ker\Big(B' : \mathcal{O}_{\mathbb{P}^1}(5)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2 \to \mathcal{O}_{\mathbb{P}^1}(9)\Big)$$

with $B' = (t\ell_0 - s\ell_1 \quad t\ell_1 - s\ell_0 \quad q_4 \quad q_5)$. The kernel of B' has rank 3 and degree 7. Writing it in the form $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $5 \geq a \geq b \geq c$, it follows that either $b \leq 3$ (and hence $c \geq -1$) or $a \geq b \geq 4$. In the first case $h^1(N_{C/Y}) = 0$ and $h^0(N_{C/Y}) = 10$, as desired. In the second case, we must have $\mathcal{O}_{\mathbb{P}^1}(5)^2 \subset N_{C/Y}$, since the kernel is saturated. But this implies that S is singular along C, which is impossible. Hence $\mathrm{Hilb}^{gtc}(Y)$ is smooth at any point $[C]$ whose corresponding curve C is smooth.

4. Assume that C is the union of a line L and a quadric Q. We may take $A_0^t = \left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ 0 & x_2 & x_3 \end{smallmatrix}\right)$, so that $L = \{x_2 = x_3 = 0\}$ and $Q = \{x_0 = x_1x_3 - x_2^2 = 0\}$. Then $A_0^t|_L = \left(\begin{smallmatrix} x_0 & x_1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ has kernel $\mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$ and

$$N_{C/Y}|_L = \ker(B' : \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \oplus \mathcal{O}_L(1)^2 \to \mathcal{O}_L(3))$$

with $B' = (x_1\ell_0 - x_0\ell_1 \quad \ell_2 \quad q_4 \quad q_5)$. Since $N_{C/X}|_L$ has rank 3 and degree 2 and is a subsheaf of $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^3$, it cannot have a direct summand of degree -2. This implies $h^1(N_{C/Y}|_L) = 0$ and hence $h^0(N_{C/Y}|_L) = 5$. We parameterise the second component of C by $\iota : \mathbb{P}^1 \to Q$, $[s:t] \to [0 : s^2 : st : t^2 : 0 : 0]$. The kernel of $\iota^* A_0^t = \left(\begin{smallmatrix} 0 & s^2 & st \\ 0 & st & t^2 \end{smallmatrix}\right)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, and

$$N_{C/Y}|_Q = \ker(B' : \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^2 \to \mathcal{O}_{\mathbb{P}^1}(6))$$

with $B' = (\ell_0 \quad t\ell_1 - s\ell_0 \quad q_4 \quad q_5)$. The sheaf $\mathcal{O}_{\mathbb{P}^1}(4)$ can lie in the kernel only if $\ell_0|_Q = 0$, i.e. if ℓ_0 is a multiple of x_0, which is impossible since x_0 must not divide $\det(A)$. If two copies of $\mathcal{O}_{\mathbb{P}^1}(3)$ were contained in the kernel they would have to lie in $\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, and since the kernel is saturated, this would imply that $\mathcal{O}_{\mathbb{P}^1}(4)$ is contained in the kernel as well, a case we just excluded. Therefore we have $N_{C/Y}|_Q = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a \geq b \geq c$ and $a \leq 3$ and $b \leq 2$. Since $a+b+c = 5$, this implies $c \geq 0$. Now $N_{C/Y}|_Q$ not only has vanishing H^1 but is in fact globally generated, so that $H^0(N_{C/Y}|_Q) \to H^0(N_{C/Y}|_{L\cap Q})$ is surjective. Hence it follows from the exact sequence

$$0 \to H^0(N_{C/Y}) \longrightarrow H^0(N_{C/Y}|_L) \oplus H^0(N_{C/Y}|_Q) \longrightarrow H^0(N_{C/Y}|_{L\cap Q})$$

that $h^0(N_{C/Y}) = 5 + 8 - 3 = 10$.

5. Assume that C is the union of three lines L_1, M and L_2 that intersect in two distinct points $p_1 = L_1 \cap M$ and $p_2 = M \cap L_2$. In appropriate coordinates C is defined by the minors of $A_0^t = \left(\begin{smallmatrix} x_0 & x_1 & 0 \\ 0 & x_2 & x_3 \end{smallmatrix}\right)$, and $L_1 = \{x_0 = x_1 = 0\}$, $M = \{x_0 = x_3 = 0\}$ and $L_2 = \{x_2 = x_3 = 0\}$. Then $A_0^t|_{L_1} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & x_2 & x_3 \end{smallmatrix}\right)$ has kernel $F|_{L_1} = \mathcal{O}_{L_1}(2) \oplus \mathcal{O}_{L_1}(1)$, so that

$$N_{C/Y}|_{L_1} = \ker(B' : \mathcal{O}_{L_1}(2) \oplus \mathcal{O}_{L_1}(1)^3 \to \mathcal{O}_{L_1}(3))$$

with $B' = (\ell_0 \quad x_3\ell_1 - x_2\ell_2 \quad q_4 \quad q_5)$. Assume first that $\ell_0|_{L_1} = 0$. Then ℓ_0 must be a linear combination of x_0 and x_1. If it were a multiple of x_0, the determinant $\det(A)$ would be divisible by x_0, contradicting the assumptions on Y. Hence $\ell_0 = \alpha x_0 + \beta x_1$ with $\beta \neq 0$. Then for any $\varepsilon \in \mathbb{C}$ the matrix $A_\varepsilon^t = \left(\begin{smallmatrix} x_0 & x_1 & 0 \\ \varepsilon\ell_0 & x_2+\varepsilon\ell_1 & x_3+\varepsilon\ell_2 \end{smallmatrix}\right)$ defines a curve C_ε in the $\mathbb{P}^2$-family of C, which for generic choice of ε is the union of a quadric and a line. Hence the isomorphism type of C is not generic in the family, and we need not further consider this case. If on the other hand $\ell_0|_{L_1} \neq 0$, then the maximal degree of a direct summand of in the kernel of B' is 1, so that $N_{C/Y}|_{L_1}$ is isomorphic to $\mathcal{O}_{L_1}(1)^2 \oplus \mathcal{O}_{L_1}$, has exactly 5 global sections and is even globally generated. By symmetry, the same is true for L_2.

Similarly, $A_0^t|_M = \left(\begin{smallmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \end{smallmatrix}\right)$ has kernel $F|_M = \mathcal{O}_M(2)^2$, and

$$N_{C/Y}|_M = \ker(B' : \mathcal{O}_M(2)^2 \oplus \mathcal{O}_M(1)^2 \to \mathcal{O}_M(3))$$

with $B' = (\ell_0 \quad \ell_2 \quad q_4 \quad q_5)$. Hence $N_{C/Y}|_M$ has degree 3, and any direct summand has degree ≤ 2. The only possibility for $N_{C/Y}$ not to be globally generated is $N_{C/Y}|_M = \mathcal{O}_M(2)^2 \oplus \mathcal{O}_M(-1)$, but even then it has vanishing H^1 and hence $h^0 = 6$. Since the restrictions of $N_{C/Y}$ to the lines L_1 and L_2 are globally generated, we conclude as in the previous step that the map

$$H^0(N_{C/Y}|_{L_1}) \oplus H^0(N_{C/Y}|_M) \oplus H^0(N_{C/Y}|_{L_2}) \longrightarrow H^0(N_{C/Y}|_{p_1}) \oplus H^0(N_{C/Y}|_{p_2})$$

is surjective, and that $h^0(N_{C/Y}) = 5 + 6 + 5 - 3 - 3 = 10$.

6. Assume that C is the union of three collinear lines L_1, L_2 and L_3 that meet in a point p but are not coplanar. We may take $A_0^t = \left(\begin{smallmatrix} x_0 & 0 & -x_2 \\ 0 & -x_1 & x_2 \end{smallmatrix}\right)$ and index the lines so that x_i and x_3 are the only non-zero coordinates on L_i. In particular, every column of A_0^t vanishes on two of the lines identically. We obtain $F = \bigoplus_{i=0}^2 F_i$ with $F_i = \ker(\mathcal{O}_C(2) \xrightarrow{x_i} \mathcal{O}_C(3) \cong \mathcal{O}_{L_{i+1}}(1) \oplus \mathcal{O}_{L_{i+2}}(1)$ with indices taken mod 3, and need to analyse the exact

sequences of the form

$$0 \longrightarrow N \longrightarrow \bigoplus_i \mathcal{O}_{L_i}(1)^2 \oplus \mathcal{O}_C(1)^2 \longrightarrow \mathcal{O}_C(3) \to 0. \tag{4.9}$$

At most one line is contained in the singular locus of S. Should this be the case we may renumber the coordinates so that that line is L_0. In any case, we may restrict sequence (4.9) to L_0 and divide out the zero-dimensional torsion. We obtain a commutative diagram or purely 1-dimensional sheaves with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N' & \to & \bigoplus_{i=1,2}\left(\mathcal{O}_{L_i}(1)^2 \oplus \mathcal{O}_{L_i}^2\right) & \to & \bigoplus_{i=1,2}\mathcal{O}_{L_i}(2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & \bigoplus_{i=0}^2 \mathcal{O}_{L_i}(1)^2 \oplus \mathcal{O}_C(1)^2 & \to & \mathcal{O}_C(3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N'' & \to & \mathcal{O}_{L_0}(1)^2 \oplus \mathcal{O}_{L_0}(1)^2 & \to & \mathcal{O}_{L_0}(3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now $N' = N'_1 \oplus N'_2$ where each summand $N'_i = \ker(\mathcal{O}_{L_i}(1)^2 \oplus \mathcal{O}_{L_i}^2 \to \mathcal{O}_{L_i}(2))$ is a vector bundle of rank 3 and degree 0 on L_i. Since S is not singular along L_i for $i = 1, 2$, the two summands $\mathcal{O}_{L_i}(1)$ cannot both be contained in N'. Necessarily, we have $N'_i \cong \mathcal{O}_{L_i}(a) \oplus \mathcal{O}_{L_i}(b) \oplus \mathcal{O}_{L_i}(c)$ with $(a, b, c) = (1, 0, -1)$, $(0, 0, 0)$. In any case, N' has vanishing H^1 and 6 global sections. On the other hand, N'' is locally free on L_0 of rank 3 and degree 1. Admissible decompositions $N'' = \mathcal{O}_{L_0}(a) \oplus \mathcal{O}_{L_0}(b) \oplus \mathcal{O}_{L_0}(c)$ are $(a, b, c) = (1, 1, -1)$ and $(1, 0, 0)$. In any case, $H^1(N'') = 0$ and $h^0(N'') = 4$. It follows that $h^0(N) = h^0(N') + h^0(N'') = 10$.

7. Assume that C is the first infinitesimal neighbourhood of a line in $\mathbb{P}^3$, defined by, say, $A_0^t = \left(\begin{smallmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & x_1 \end{smallmatrix}\right)$. We will show that the corresponding $\mathbb{P}^2$-family contains a non-reduced curve, so that this case is reduced to those treated before. The curve C necessarily forms the singular locus of S, and S must be one of the four types of non-normal surfaces. In each case there is only one determinantal representation up to equivalence and coordinate change, namely

$$A = \left(\begin{smallmatrix} x_0 & 0 & x_2 \\ x_1 & x_0 & 0 \\ 0 & x_1 & x_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 & 0 & x_1 \\ x_1 & x_0 & x_2 \\ 0 & x_1 & x_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 & 0 & x_1 \\ x_1 & x_0 & x_2 \\ 0 & x_1 & x_0 \end{smallmatrix}\right), \text{ and } \left(\begin{smallmatrix} x_0 & 0 & x_2 \\ x_1 & x_0 & 0 \\ 0 & x_1 & x_0 \end{smallmatrix}\right).$$

A reduced curve in the corresponding $\mathbb{P}^2$-family is provided for example by the matrices

$$A'_0 = \left(\begin{smallmatrix} x_0 & x_2 \\ x_1 & 0 \\ 0 & x_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 & x_1 \\ x_1 & x_2 \\ 0 & x_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 & x_1 \\ x_0+x_1 & x_2 \\ x_1 & x_0 \end{smallmatrix}\right), \text{ and } \left(\begin{smallmatrix} x_0 & x_2 \\ x_0+x_1 & 0 \\ x_1 & x_0 \end{smallmatrix}\right),$$

respectively.

8. The remaining three types of non-reduced aCM-curves (corresponding to matrices $A^{(5)}$, $A^{(6)}$ and $A^{(7)}$ in the enumeration of Section §1) are each the union of two lines L and M, of which one, say L, has a double structure. As we have already shown that any $\mathbb{P}^2$-family containing the most degenerate type also contains a non-reduced curve, it

suffices to show that there is no $\mathbb{P}^2$-family parameterising only non-reduced curves with two components. Assume that $A \in W^{3\times 3}$ defines such a family. The corresponding bundle homomorphism is the composite map

$$\Omega_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}^3 \otimes W.$$

We form $\Lambda^2\Omega_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \to \Lambda^2(\mathcal{O}_{\mathbb{P}^2}^3) \otimes S^2W$ and obtain the associated family of nets of quadrics $\mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \otimes S^2W$. To each parameter $\lambda \in \mathbb{P}^2$ in the family there are associated subspaces $B_\lambda \subset U_\lambda \subset W$, where B_λ defines the plane spanned by the lines L_λ and M_λ, and U_λ defines the line L_λ. Let $\mathcal{B} \subset \mathcal{U} \subset \mathcal{O}_{\mathbb{P}^2} \otimes_{\mathbb{C}} W$ denote the corresponding vector bundles. Then there are inclusions

$$\mathcal{B} \cdot \mathcal{U} \subset \mathcal{O}_{\mathbb{P}^2}(-1)^3 \subset \mathcal{O}_{\mathbb{P}^2} \otimes S^2W.$$

But such a configuration of vector bundles is impossible: Both inclusions $\mathcal{B} \subset \mathcal{U}$ and $\mathcal{B}\mathcal{U} \subset \mathcal{O}_{\mathbb{P}^2}(-1)^3$ would have to split, say $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(a)$, $\mathcal{U} = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ and finally $\mathcal{O}_{\mathbb{P}^2}(-1)^3 \cong \mathcal{O}_{\mathbb{P}^2}(2a) \oplus \mathcal{O}_{\mathbb{P}^2}(a+b) \oplus \mathcal{O}_{\mathbb{P}^2}(c)$, and the latter isomorphism is clearly impossible. □

Theorem 4.8 — *Z' is an 8-dimensional smooth irreducible projective variety.*

Proof. Due to the existence of the $\mathbb{P}^2$-fibration $\mathrm{Hilb}^{gtc}(Y) \to Z'$, the smoothness of $\mathrm{Hilb}^{gtc}(Y)$ implies that Z' is smooth as well and of dimension 8. The morphism $Z' \to \mathbb{G}$ is finite over the open subset of ADE-surfaces, and has fibre dimension ≤ 1 resp. ≤ 2 over the strata of simple-elliptic and non-normal surfaces, resp., due to Corollary 3.11. By Proposition 4.3 and Proposition 4.2, simple-elliptic and non-normal surfaces form strata in $\mathbb{G}$ of dimension ≤ 4. It follows that every irreducible component of Z' must dominate $\mathbb{G}$. The stratum of simple-elliptic surfaces in $\mathbb{G}$ is non-empty by Corollary 4.4. Since $\mathrm{Hilb}^{gtc}(S)$ is connected for a simple-elliptic surface, Z' must be connected as well. Being smooth, Z' is irreducible. □

Again, due to the existence of the $\mathbb{P}^2$-fibre bundle map $\mathrm{Hilb}^{gtc}(Y) \to Z'$, this theorem is equivalent to Theorem A.

4.4. **Symplecticity.** We continue to assume that $Y \subset \mathbb{P}^5$ is a smooth hypersurface that does not contain a plane.

De Jong and Starr [8] showed that any smooth projective model of the coarse moduli space associated to the stack of rational curves of degree d on a very general cubic fourfold carries a natural 2-form ω_d. In our context, ω_3 can be defined directly as follows: Let $\Omega = \sum_{i=0}^5 (-1)^i x_i dx_0 \wedge \ldots \widehat{dx_i} \ldots \wedge dx_5$. An equation f for Y determines a generator $\alpha \in H^{3,1}(Y)$ as the image of $[\Omega/f^2]$ under Griffiths's residue isomorphism

$$\mathrm{Res} : H^5(\mathbb{P}^5 \setminus Y, \mathbb{C}) \to H^4_{\mathrm{prim}}(Y).$$

The cycle $[\mathcal{C}] \in H_{22}(\mathrm{Hilb}^{gtc}(Y) \times Y; \mathbb{Z})$ of the universal curve $\mathcal{C} \subset \mathrm{Hilb}^{gtc}(Y) \times Y$ defines a correspondence

$$[\mathcal{C}]_* : H^4(Y, \mathbb{C}) \to H^2(\mathrm{Hilb}^{gtc}(Y), \mathbb{C})$$

via $[\mathcal{C}]_*(u) = \mathrm{PD}^{-1}\,\mathrm{pr}_{1*}(\mathrm{pr}_2^*(u) \cap [Z])$, where pr_1 and pr_2 denote the projections from $\mathrm{Hilb}^{gtc}(Y) \times Y$ to its factors. Since the homology class $[\mathcal{C}]$ is algebraic, the map $[\mathcal{C}]_*$ is of Hodge type $(-1,-1)$ and maps $H^{3,1}(Y) \cong \mathbb{C}$ to $H^{2,0}(\mathrm{Hilb}^{gtc}(Y))$. Let the two-form ω_3 be the image of $\alpha \in H^{3,1}(Y)$. More importantly, de Jong and Starr showed that the value of ω_3 on the tangent space $T_{[C]}\mathrm{Hilb}^{gtc}(Y) = H^0(C, N_{C/Y})$ at a *smooth* rational curve $C \subset Y$ has the following geometric interpretation:

There is a short exact sequence of normal bundles

$$0 \to N_{C/Y} \to N_{C/\mathbb{P}^5} \to N_{Y/\mathbb{P}^5}|_C \to 0. \tag{4.10}$$

To simplify the notation let $A := N_{C/Y}$, $N := N_{C/\mathbb{P}^5}$ and $F := N_{Y/\mathbb{P}^5}$. The fact, that Y is a *cubic* contributes the relation

$$\frac{\det A}{F} \cong \frac{\det N}{F^2} \cong \frac{\omega_C}{\omega_{\mathbb{P}^5} \otimes F^2} \cong \omega_C. \tag{4.11}$$

Taking the third exterior power of (4.10) and dividing by F one obtains a short exact sequence

$$0 \to \frac{\det A}{F} \to \frac{\Lambda^3 N}{F} \to \Lambda^2 A \to 0, \tag{4.12}$$

whose boundary operator defines a skew-symmetric pairing

$$\delta : \Lambda^2 H^0(A) \to H^0(C, \Lambda^2 A) \to H^1(C, \det(A) \otimes F^*) = H^1(C, \omega_C) \cong \mathbb{C}. \tag{4.13}$$

By Theorem 5.1 in [8], one has $\omega_3(u,v) = \delta(u \wedge v)$ for any two tangent vectors $u, v \in H^0(C, N_{C/Y})$, up to an irrelevant constant factor. By a rather involved calculation de Jong and Starr show that ω_3 generically has rank 8. We will need the following minimally sharper result:

Proposition 4.9 — *ω_3 has rank 8 at $[C] \in \mathrm{Hilb}^{gtc}(Y)$ whenever C is smooth.*

Proof. Consider the second exterior power of (4.10) and divide again by F:

$$0 \to \frac{\Lambda^2 A}{F} \to \frac{\Lambda^2 N}{F} \to A \to 0. \tag{4.14}$$

Note that $\Lambda^2 A/F \cong A^* \otimes \det A/F \cong A^* \otimes \omega_C$. The associated boundary operator defines a map

$$\delta' : H^0(C, A) \longrightarrow H^1(C, \Lambda^2 A \otimes F^*) \cong H^0(C, A)^*. \tag{4.15}$$

The commutative diagram

$$\begin{array}{ccccc}
H^0(A) \otimes H^0(A) & \to & H^0(A) \otimes H^1(\Lambda^2 A/F) & \to & H^0(A) \otimes H^1(A^* \otimes \omega_C) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\Lambda^2 A) & \to & H^1(\det A/F) & \cong & H^1(\omega_C)
\end{array}$$

shows that δ' is the associated linear map of the pairing δ.

Though it is less clear from δ' that the pairing on $H^0(A)$ is skew symmetric, it makes it easier to compute the radical of ω_3 at $[C]$, which is simply the kernel of δ' and hence the cokernel of the injective homomorphism $\gamma : H^0(C, \Lambda^2 A \otimes F^*) \to H^0(C, \Lambda^2 N \otimes F^*)$ induced by (4.14). Using an identification $C \cong \mathbb{P}^1$ we have isomorphisms $F \cong \mathcal{O}_{\mathbb{P}^1}(9)$ and $N \cong \mathcal{O}_{\mathbb{P}^1}(5)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2$. The bundle $\Lambda^2 N \otimes F^* \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ has exactly two sections. If we write $A = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a \geq b \geq c$ then $a + b + c = \deg(A) = 7$, and we know from step 3 in the proof of Theorem 4.7 that $c \geq -1$ and $a + b \leq 8$. Thus the maximal degree of a direct summand of $\Lambda^2 A/F$ is $a + b - 9 \leq -1$. This shows $h^0(\Lambda^2 A/F) = 0$ and $\dim \operatorname{rad} \omega_3([C]) = \dim \operatorname{coker}(\gamma) = h^0(\Lambda^2 N/F) = 2$. □

Theorem 4.10 — *Let* $a : \operatorname{Hilb}^{gtc}(Y) \to Z'$ *be the* $\mathbb{P}^2$*-fibration constructed before.*

(1) *There is a unique form* $\omega' \in H^0(Z', \Omega^2_{Z'})$ *such that* $a^*\omega' = \omega_3$.
(2) ω' *is non-degenerate on* $Z' \setminus D$.
(3) $K_{Z'} = mD$ *for some* $m > 0$.

Proof. 1. From the exact sequence $0 \to a^*\Omega_{Z'} \to \Omega_{M_3} \to \Omega_{M_3/Z'} \to 0$ one gets a filtration by locally free subsheaves $0 \subset a^*\Omega^2_{Z'} \subset U \subset \Omega^2_{M_3}$ with factors $U/a^*\Omega^2_{Z'} \cong a^*\Omega_{Z'} \otimes \Omega_{M_3/Z'}$ and $\Omega^2_{M_3}/U \cong \Omega^2_{M_3/Z'}$. This in turn yields exact sequences

$$0 \longrightarrow H^0(M_3, U) \longrightarrow H^0(M_3, \Omega^2_{M_3}) \longrightarrow H^0(M_3, \Omega^2_{M_3/Z'}) \tag{4.16}$$

and

$$0 \longrightarrow H^0(M_3, a^*\Omega^2_Z) \longrightarrow H^0(M_3, U) \longrightarrow H^0(M_3, a^*\Omega_{Z'} \otimes \Omega_{M_3/Z'}). \tag{4.17}$$

Since neither $\Omega_{\mathbb{P}^2}$ nor $\Omega^2_{\mathbb{P}^2}$ have nontrivial sections, $a_*\Omega_{M_3/Z'}$ and $a_*\Omega^2_{M_3/Z'}$ vanish. It follows that $H^0(M_3, \Omega^2_{M_3/Z'}) = H^0(Z', a_*\Omega^2_{M_3/Z'}) = 0$ and $H^0(M_3, a^*\Omega_{Z'} \otimes \Omega_{M_3/Z'}) = H^0(Z', \Omega_{Z'} \otimes a_*\Omega_{M_3/Z'}) = 0$. We are left with isomorphisms

$$H^0(Z', \Omega^2_{Z'}) \cong H^0(M_3, a^*\Omega^2_{Z'}) \cong H^0(M_3, U) \cong H^0(M_3, \Omega^2_{M_3}). \tag{4.18}$$

This shows that ω_3 descends to a unique 2-form ω' on Z'.

2. It follows from Proposition 4.9 that ω' is non-degenerate at all points $z \in Z'$ for which the fibre $a^{-1}(z)$ contains a point corresponding to a smooth rational curve. By Theorem 2.1, this is the case for all points corresponding to fibres with aCM-curves on a surface with at most ADE-singularities. The dimension argument in the proof of Theorem 4.8 shows that the locus of points in $Z' \setminus D$ that do not satisfy this condition has codimension ≥ 2. But the degeneracy locus of a 2-form is either empty or a divisor. Thus ω' is indeed non-degenerate on $Z' \setminus D$.

3. Since ω' is non-degenerate on $Z' \setminus D$, its 4th exterior power defines a non-vanishing section in the canonical line bundle of Z' over $Z' \setminus D$, showing that $K_{Z'} = mD$ for some $m \geq 0$. To see that $m > 0$, it suffices to note that Y has no non-trivial holomorphic 2-form, so that the restriction of ω' to $D = \mathbb{P}(T_Y)$ must vanish identically. Consequently ω' must be degenerate along D. □

A calculation of the topological Euler characteristic of the preimage curve in Z' of a generic line $L \subset \mathrm{Grass}(\mathbb{C}^6, 4)$ shows that $K_{Z'} \sim 3D$. We will not need this explicit number and hence omit the calculation. In fact, $m = 3$ easily follows *a posteriori* once we have shown the existence of a contraction $Z' \to Z$ to a manifold Z that maps D to Y.

4.5. **The extremal contraction.**

Theorem 4.11 — *There exists an* 8*-dimensional irreducible projective manifold* Z *and a morphism* $\Phi : Z' \to Z$ *with the following properties:*

(1) Φ *maps* $Z' \setminus D$ *isomorphically to* $Z \setminus \Phi(D)$.
(2) $\Phi|_D$ *factors through the projection* $\pi : D = \mathbb{P}(T_Y) \to Y$ *and a closed immersion* $j : Y \to Z$.
(3) *There is a unique holomorphic* 2*-form* $\omega \in H^0(Z, \Omega_Z^2)$ *such that* $\omega' = \Phi^*\omega$.
(4) ω *is symplectic.*

We will prove the theorem in several steps:

Lemma 4.12 — *The line bundle* $\mathcal{O}_{Z'}(D)$ *is ample relative to* $s : Z' \to \mathbb{G}$.

Proof. As the statement is relative over the Grassmannian, it suffices to prove the analogous statement for the divisor $J \subset H$ relative to the morphism $H \to \mathbb{P}(S^3 W^*)$. This is the content of Corollary 3.10. ☐

Let $\mathcal{W}$ denote the universal rank 4 bundle on $\mathbb{G}$. Then $\det(\mathcal{W})$ is very ample, and its pull-back $B := s^*\det(\mathcal{W})$ to Z' is a nef line bundle. The linear system of the line bundle

$$L := \mathcal{O}_{Z'}(D) \otimes B.$$

will produce the contraction $\Phi : Z' \to Z$. It follows from Proposition 4.5 that with respect to the identification $D = \mathbb{P}(T_Y)$ we have

$$\mathcal{O}(D)|_D = \mathcal{O}_\pi(-1) \qquad \text{and} \qquad L|_D \cong \pi^*\mathcal{O}_Y(1). \tag{4.19}$$

Lemma 4.13 — L *is nef, and all irreducible curves* $\Sigma \subset Z'$ *with* $\deg(L|_\Sigma) = 0$ *are contained in* D*, and more specifically, in the fibres of* $\pi : D = \mathbb{P}(T_Y) \to Y$.

Proof. Assume first, that Σ is an irreducible curve not contained in D. Since D is effective, $D.\Sigma \geq 0$. As B is nef, one has $\deg(L|_\Sigma) \geq 0$. Moreover, $\deg(L|_\Sigma) > 0$ unless $\deg(B|_\Sigma) = 0$, which is only possible when Σ is contained in the fibres of $Z' \to \mathrm{Grass}(\mathbb{C}^6, 4)$. But since D is relatively ample over the Grassmannian, one would have $D.\Sigma > 0$.

Conversely, if $\Sigma \subset D$, we have $\deg(L|_\Sigma) = \deg(\mathcal{O}_Y(1)|_{\pi}(\Sigma)) \geq 0$ by the previous lemma. This number is > 0 unless Σ lies in the fibre of $\pi : D \to Y$. ☐

Lemma 4.14 — *For all* $p, q > 0$ *the line bundle* $L^p \otimes B^q$ *is ample.*

Proof. As B is the pull-back of an ample line bundle on $\mathbb{G}$ and L is ample relative $\mathbb{G}$, it follows that $L \otimes B^\ell$ is ample for some large ℓ. Since both L and B are both nef,

$L^{1+m} \otimes B^{\ell+n}$ is ample for all $m, n \geq 0$ by Kleiman's numerical criterion for ampleness [22]. □

Lemma 4.15 — *The classes $[\Sigma]$ of curves with $\deg(L|_\Sigma) = 0$ form a $K_{Z'}$-negative extremal ray.*

Proof. According to the previous lemma, curves with $\deg(L|_\Sigma) = 0$ are contained in the fibres of a projective bundle $D = \mathbb{P}(T_Y) \to Y$. Any such curve is numerically equivalent to a multiple of a line in any of these fibres. Such classes $[\Sigma]$ generate a ray. Moreover, as $\mathcal{O}_D(D)$ is negative on the fibres of π by (4.19) and $K_{Z'} \sim mD$, the restriction of $K_{Z'}$ to this ray is strictly negative. □

Using the Contraction Theorem ([24] Thm. 3.7, or [23] Thm. 8-3-1) we conclude: There is a morphism $Z' \to Z$ with the following properties:

(1) Z is normal and projective, Φ has connected fibres, and $\Phi_*\mathcal{O}_{Z'} = \mathcal{O}_Z$.
(2) A curve $\Sigma \subset Z'$ is contracted to a point in Z' if and only if its class is contained in the extremal ray.
(3) There is an ample line bundle L' on Z such that $L \cong \Phi^* L'$.

Let $Y' \subset Z$ denote the image of D. By Lemma 4.13 and Lemma 4.15, the morphism Φ contracts exactly the fibres of $\pi : \mathbb{P}(T_Y) \to Y$. Since the fibres of π and of Φ are connected, Φ induces bijections $Z' \setminus D \to Z \setminus Y'$ and $Y \to Y'$. As both $Z' \setminus D$ and $Z \setminus Y'$ are normal, the restriction $\Phi : Z' \setminus D \to Z \setminus Y'$ is an isomorphism.

Lemma 4.16 — *For sufficiently large ℓ the natural map $H^0(Z', L^\ell) \to H^0(D, L^\ell|_D)$ is surjective.*

Proof. By Lemma 4.14,

$$L^\ell(-D) \otimes \mathcal{O}(-K_{Z'}) = L^\ell(-(m+1)D) = B^{m+1} \otimes L^{\ell-m-1}$$

is ample for $\ell > m+1$. Hence an application of the Kodaira Vanishing Theorem gives $H^1(Z', L^\ell(-D)) = 0$, so that $H^0(Z', L^\ell) \to H^0(D, L^\ell|_D)$ is surjective. □

Since $L|_D \cong \pi^*\mathcal{O}_Y(1)$ it follows from the previous lemma that $Y \to Y'$ is an isomorphism.

Proposition 4.17 — *Z is smooth.*

Proof. It remains to show that Z is smooth along Y. The system of ideal sheaves $I_n := \Phi^{-1}(I^n_{Y/Z})\mathcal{O}_{Z'}$ and $\mathcal{O}_{Z'}(-nD)$ are cofinal. Moreover, there are exact sequences

$$0 \longrightarrow \mathcal{O}_D(-nD) \longrightarrow \mathcal{O}_{(n+1)D} \longrightarrow \mathcal{O}_{nD} \longrightarrow 0$$

and

$$0 \longrightarrow S^n T_Y \longrightarrow \Phi_*\mathcal{O}_{(n+1)D} \longrightarrow \Phi_*\mathcal{O}_{nD} \longrightarrow 0,$$

since $\mathcal{O}_D(-nD) = \mathcal{O}_\pi(n)$ and thus $\Phi_*\mathcal{O}_D(-nD) = S^n T_Y$ and $R^i\Phi_*\mathcal{O}_D(-nD) = 0$ for all $i > 0$. It follows from Grothendieck's version of Zariski's Main Theorem ([15], Thm. III.4.1.5.) that the completion of Z along Y can be computed by

$$\hat{\mathcal{O}}_Z = \varprojlim \Phi_*(\mathcal{O}_{Z'}/I_n) = \varprojlim \Phi_*(\mathcal{O}_{nD}) = \hat{S}(T_Y).$$

This shows that Z is smooth along Y. □

Proposition 4.18 — *The form ω' on Z' descends to a symplectic form ω on Z.*

Proof. As $Y \subset Z$ has complex codimension 4, the pull-back of ω' via the isomorphism $Z \setminus Y \to Z' \setminus D$ extends uniquely to a holomorphic 2-form ω that is necessarily symplectic since the degeneracy locus of a 2-form is either empty or a divisor. □

This finishes the proof of Theorem 4.11.

4.6. **The topological Euler number.**

Theorem 4.19 — *The topological Euler number of Z is 25650.*

This number equals the Euler number of the Hilbert scheme $\mathrm{Hilb}^4(K3)$ of 0-dimensional subschemes of length 4 on a K3-surface [16]. This and the fact that the Beauville-Donagi moduli space of lines on Y is isomorphic to Hilb^2 of a K3-surface if Y is of Pfaffian type makes it very hard not to believe that Z is isomorphic to some $\mathrm{Hilb}^4(K3)$ for special choices of Y or is at least deformation equivalent to such a Hilbert scheme.

For this reason we will not give a detailed proof of the theorem here. Our method imitates the pioneering calculations of Ellingsrud and Strømme [13]. Note first that $e(Z') = e(Z) + e(Y)(e(\mathbb{P}^3) - 1) = e(Z) + 81$ and $e(\mathrm{Hilb}^{gtc}(Y)) = e(Z')e(\mathbb{P}^2) = 3e(Z')$. Hence the assertion is equivalent to $e(\mathrm{Hilb}^{gtc}(Y)) = 77193$. Now $\mathrm{Hilb}^{gtc}(Y)$ is the zero locus of a regular section in a certain 10-dimensional tautological vector bundle A on $\mathrm{Hilb}^{gtc}(\mathbb{P}^5)$ (cf. Section §1). It is therefore possible to explicitly express both the class of $\mathrm{Hilb}^{gtc}(Y)$ and the Chern classes of its tangent bundle in terms of tautological classes in the cohomology ring $H^*(\mathrm{Hilb}^{gtc}(\mathbb{P}^5), \mathbb{Q})$. Two options present themselves for the calculation:

1. Follow the model of Ellingsrud and Strømme and write down a presentation of the rational cohomology ring of $\mathrm{Hilb}^{gtc}(\mathbb{P}^5)$ in terms of generators and relations and calculate using Groebner base techniques. This is the option we chose. We wrote pages of code first in SINGULAR and then in SAGE [25].

2. Take a general linear $\mathbb{C}^*$ action on $\mathbb{P}^5$ and determine the induced local weights at any of the 1950 fixed points for the induced action on $\mathrm{Hilb}^{gtc}(\mathbb{P}^5)$. Fortunately there are only nine different types of fixed points. The relevant calculations can then be executed by means of the Bott-formula.

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