Introduction to Lie algebra cohomology with a view towards BRST cohomology

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Abstract

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Introduction

This is a very short overview on Lie algebra cohomology. For a longer and more complete survey, the reader is refereed to the excellent [FeFu00].

Lie algebra cohomology is inspired by de Rham cohomology on the corresponding Lie group (see Section 2.1), and has therefore both algebraic and geometric flavour. It gives insight about the algebraic structure of the Lie algebra via the low degree interpretations of the cohomology spaces (see Section 3). On the other hand, it gives geometric information, for example about a corresponding Lie group (see Cartan's Theorem, here Corollary 2.4). We restricted ourselves to finite dimensional Lie algebras (with exception of the Lie algebra of vector fields on the circle $\mathrm{Vect}(\mathbb{S}^1)$). Therefore, we did not include Gelfand-Fuchs theory, nor current algebras, nor did we even comment on the computations of the continuous cohomology of Lie algebras of vector fields, which are also very geometric in flavour (for all these subjects, see [Fu86], [FeFu00]). Another source of links to geometry would have been the relation of Lie algebra cohomology to the Riemann-Roch Theorem, which we also ignored completely (for this, see [FeTs89], [NeTs95]).

Deformation theory of Lie algebras is the study of the variety of Lie algebra laws, i.e. the moduli space of Lie algebras, and of how a given Lie algebra deforms into another. Deformation theory is closely linked to cohomology, the space of (inequivalent) infinitesimal deformations of $\mathfrak{g}$ being $H^2(\mathfrak{g},\mathfrak{g})$ and the obstructions to prolongation of a given infinitesimal deformation being expressed as Massey products in $H^3(\mathfrak{g},\mathfrak{g})$. Unfortunately, we do not have the space to explain this here.

The content of the Sections is the following: Section 1 is about basic examples of Lie algebras and modules. Section 2 gives the definition of Lie algebra cohomology, together with its geometric motivation and the derived functor approach. Unfortunately, we completely left out *homology*, and therefore did not talk about the Loday-Quillen Theorem or K-Theory. In Section 3, we give the standard low degree interpretations of the cohomology spaces. What is less standard is the part on the Virasoro algebra and the glimpse about the interpretation of H^3 by crossed modules. In Section 4, we talk about some standard computational methods. These are kind of too sophisticated for an introduction to Lie algebra cohomology, but the reader who wants to really compute cohomology spaces will have to draw on these. Section 5 gives some very classical results on finite dimensional Lie algebras, with the intention to show that semi-simple Lie algebras have "small", while nilpotent Lie algebras have "huge" cohomology. We also cite Kostant's Theorem, leading to links with representation theory. In Section 6 we conclude with a short overview on BRST quantization, based on [KoSt87].

1 Lie algebras and modules

Let $\mathfrak{g}$ be a Lie algebra over a field k. We will always suppose k of characteristic zero. (Lie algebras over a field of characteristic $p > 0$ together with their cohomology are still a different, very rich subject.)

Examples 1.1 (a) First of all, consider finite dimensional examples – matrix Lie algebras. $\mathfrak{g} = \mathfrak{gl}(n,k)$ (reductive Lie algebra), $\mathfrak{g} = \mathfrak{sl}(n,k), \mathfrak{so}(n,k), \mathfrak{sp}(2n,k)$ (semisimple Lie algebras), $\mathfrak{g} = \mathfrak{t}(n,k)$ (upper triangular matrices – a solvable Lie algebra). $\mathfrak{t}(n,k) = \mathfrak{d}(n,k) \oplus \mathfrak{n}(n,k)$, diagonal plus strictly upper triangular matrices. $\mathfrak{n}(n,k)$ is a nilpotent Lie algebra.

(b) Let X be a (finite dimensional) manifold, and denote by $\mathrm{Vect}(X)$ the Lie algebra of vector fields on X. For example, $\mathrm{Vect}(\mathbb{S}^1)$ is the Lie algebra of vector fields on the circle $\mathbb{S}^1$. As a space

$$\mathrm{Vect}(\mathbb{S}^1) \,=\, \{f(\theta)\frac{d}{d\theta} \,|\, f \in \mathcal{C}^\infty(\mathbb{S}^1)\}.$$

The bracket reads

$$\left[f(\theta)\frac{d}{d\theta}, g(\theta)\frac{d}{d\theta}\right] \,=\, (fg' - gf')\frac{d}{d\theta}.$$

Introducing the (topological) generators $e_n := z^{n+1}\frac{d}{dz}$ with $z = e^{i\theta}$, the defining relations read

$$[e_n, e_m] \,=\, (m-n)e_{n+m}.$$

Definition 1.2 A $\mathfrak{g}$-*module* is a k-vector space M together with a map $g \otimes M \to M$, $x \otimes m \mapsto x \cdot m \in M$ such that

$$[x,y]\cdot m \,=\, x\cdot(y\cdot m) - y\cdot(x\cdot m)$$

for all $x, y \in \mathfrak{g}$ and all $m \in M$. In other words, the map $\mathfrak{g} \to \mathrm{End}(M)$, $x \mapsto x\cdot$ is a homomorphism of Lie algebras.

Example 1.3 The most important $\mathfrak{g}$-modules will be for us the *trivial module* k, i.e. the action reads $x \cdot \lambda = 0$ for all $\lambda \in k$ and all $x \in \mathfrak{g}$, and the *adjoint module* $\mathfrak{g}$, i.e. $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint action, i.e. by the bracket, and the action reads $x \cdot y = [x,y]$ for all $x, y \in \mathfrak{g}$.

Definition 1.4 The *universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ is the following associative algebra:

$$U\mathfrak{g} \,:=\, T\mathfrak{g}\,/\,(x\otimes y - y\otimes x - [x,y], \forall x,y \in \mathfrak{g}).$$

In other words, $U\mathfrak{g}$ is the quotient of the tensor algebra $T\mathfrak{g}$ on $\mathfrak{g}$ by the ideal generated by the elements $x \otimes y - y \otimes x - [x,y]$ for all $x, y \in \mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ may be regarded as included in $U\mathfrak{g}$, because no identifications take place on $\mathfrak{g}$.

Recall that each associative algebra may be regarded as a Lie algebra with the bracket obtained by antisymmetrizing the product. Then the *universal property* of $U\mathfrak{g}$ reads as follows:

Proposition 1.5 *Given an associative algebra A and a Lie algebra morphism $\varphi : \mathfrak{g} \to A$ into the underlying Lie algebra of A, there is a unique morphism of associative algebras $\Phi : U\mathfrak{g} \to A$ such that $\Phi|_{\mathfrak{g}} = \varphi$.*

(This may be formulated as saying that the functor U is a left adjoint to the functor sending an associative algebra to its underlying Lie algebra.)

2 Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module. Define the space of p-cochains on $\mathfrak{g}$ with values/coefficients in M to be

$$C^p(\mathfrak{g}, M) := \mathrm{Hom}_k(\Lambda^p\mathfrak{g}, M),$$

the space of p-linear alternating maps from $\mathfrak{g}$ to M, where for $p = 0$ we set $C^0(\mathfrak{g}, M) = M$.

Let $c \in C^p(\mathfrak{g}, M)$. Define $dc \in C^{p+1}(\mathfrak{g}, M)$ by

$$\begin{aligned} dc(x_1, \ldots, x_{p+1}) &= \sum_{1\leq i<j\leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{p+1}) \; + \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} x_i \cdot c(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}) \end{aligned}$$

The thus constructed cochain complex $(C^*(\mathfrak{g}, M), d)$ is called the Chevalley-Eilenberg complex. It computes the Lie algebra cohomology.

Denote as usual by

$$Z^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) \,|\, dc = 0\}$$

the space of p-cocycles, and by

$$B^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) \,|\, \exists c' \in C^{p-1}(\mathfrak{g}, M) : \; c = dc'\}$$

the space of p-coboundaries. Then we define the pth *cohomology space* of $\mathfrak{g}$ with values/coefficients in M as the quotient vector space

$$H^p(\mathfrak{g}, M) := Z^p(\mathfrak{g}, M) \,/\, B^p(\mathfrak{g}, M).$$

Remark 2.1 For infinite dimensional Lie algebras, the cochain spaces are very big and the cohomology is usually infinite dimensional, and thus meaningless. This can be remedied for topological Lie algebra (such as, for example, Lie algebras of vector fields) and topological $\mathfrak{g}$-modules by taking into account the topology: taking the topological tensor product (usually the π-tensor product) and restricting to *continuous* linear maps. This leads then to *continuous cohomology*, also called *Gelfand-Fuchs cohomology*, for which the interested reader is invited to consult [Fu86].

2.1 Motivation of Lie algebra cohomology by de Rham cohomology

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (and suppose for the moment M trivial, i.e. $M = \mathbb{R}$ with the trivial action.) Let $\omega \in \Omega^p(G)$ be a differential p-form on G. Then the Cartan formula for the exterior differential reads

$$\begin{aligned} d\omega(X_1, \ldots, X_{p+1}) &= \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}) \; + \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1}) \end{aligned}$$

Here X_i are vector fields on G, and the bracket is the bracket of vector fields, while the action $X_i \cdot \omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})$ denotes the Lie derivative. This formula for the exterior differential was the starting point of Lie algebra cohomology.

In order to catch the exact relation of Lie algebra cochains to differential forms, introduce the following. Here M denotes a $\mathfrak{g}$-module which also admits a smooth group action of G which differentiates to the given $\mathfrak{g}$-module structure.

Definition 2.2 A p-form $\omega \in \Omega^p(G, M)$ (differential forms with values in the vector space M) is called *equivariant* if for all $g \in G$,

$$\lambda_g^* \omega \; = \; \rho(g) \circ \omega.$$

Here λ_g is left translation on the group G and $\rho : G \times M \to M$ is the smooth group action of G on M. Denote the subspace of equivariant forms by $\Omega^p(G, M)^{\mathrm{eq}} \subset \Omega^p(G, M)$.

Proposition 2.3 (Lemma B.5 in [Ne04]) *The evaluation at* $1 \in G$

$$\mathrm{ev}_1 : \Omega^p(G, M)^{\mathrm{eq}} \to C^p(\mathfrak{g}, M)$$

defines an isomorphism of the (de Rham) complex of equivariant M-valued differential forms on G to the complex of M-valued Lie algebra cochains.

main idea of the proof: recall that $\mathfrak{g}$ may be seen as the left-invariant vector fields on G. Exactly in the same way as a left-invariant field is determined by its value at $1 \in G$, an equivariant form has the same property. □

Corollary 2.4 (Cartan's Theorem) *Let G be a connected compact Lie group. Then*

$$H^*_{\mathrm{dR}}(G) \; \cong \; H^*(\mathfrak{g}, \mathbb{R}) \; \cong \; \mathrm{Inv}_G \Lambda^p(\mathfrak{g}^*).$$

main idea of the proof: in addition to the Proposition, one uses that on a compact group, integration over the group renders differential forms invariant. Observe that equivariance means here invariance, because the module is trivial. □

A full proof of Cartan's theorem may be found in Ch. V.12 in [Bre93] and in Section 2.3.5 in [GuRo07].

2.2 Derived functor approach to cohomology

Lie algebra cohomology is the derived functor of the functor of invariants

$$M \mapsto M^{\mathfrak{g}} \,:=\, \{m \in M \,|\, \forall x \in \mathfrak{g} : x \cdot m = 0\},$$

and can thus be described as an Ext-functor:

$$H^*(\mathfrak{g}, M) \,=\, \mathrm{Ext}^*_{U\mathfrak{g}}(k, M).$$

This follows from the fact that $M^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(k, M)$ and that the Koszul complex is a resolution of the trivial $\mathfrak{g}$-module k. More precisely:

Proposition 2.5 *The augmentation map $U\mathfrak{g} \to k$ induces a quasiisomorphism $\Lambda^*\mathfrak{g} \otimes U\mathfrak{g} \to k$. This exhibits the Koszul complex*

$$\Lambda^p\mathfrak{g} \otimes U\mathfrak{g} \to \Lambda^{p-1}\mathfrak{g} \otimes U\mathfrak{g} \quad , \qquad x_1 \wedge \ldots \wedge x_p \otimes u \mapsto \sum_{i=1}^{p} (-1)^i x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_p \otimes x_i u \,+$$
$$+ \sum_{1 \leq i < j \leq p} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge \widehat{x_j} \ldots \wedge x_p \otimes u$$

as a projective resolution of the trivial $\mathfrak{g}$-module k.

Now compute the derived functor: apply the functor $\mathrm{Hom}_{U\mathfrak{g}}(-, M)$ to the resolution $U\mathfrak{g} \otimes \Lambda^*\mathfrak{g} \to k$. Simplifying

$$\mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g} \otimes \Lambda^*\mathfrak{g}, M) \,\cong\, \mathrm{Hom}_k(\Lambda^*\mathfrak{g}, M),$$

one obtains the above Chevalley-Eilenberg complex for Lie algebra cohomology.

Remark 2.6 Note that $\mathrm{Ext}^*_{U\mathfrak{g}}(k, M)$ is in general *not* the Hochschild cohomology of the associative algebra $U\mathfrak{g}$. By definition, for a $U\mathfrak{g}$-bimodule M, this is $\mathrm{Ext}^*_{U\mathfrak{g} \otimes (U\mathfrak{g})^{\mathrm{opp}}}(k, M)$. The link to Lie algebra cohomology is then

$$H^*(\mathfrak{g}, M^{\mathrm{ad}}) \,\cong\, \mathrm{Ext}^*_{U\mathfrak{g} \otimes (U\mathfrak{g})^{\mathrm{opp}}}(k, M),$$

where for a $U\mathfrak{g}$-bimodule M, M^{ad} denotes the $\mathfrak{g}$-module M with the action $x \cdot m = xm - mx$ for all $x \in \mathfrak{g}$ and all $m \in M$.

For more informations about this abstract side of cohomology, we invite the reader to consult the standard references [CaEi56], [Lo98], [We94].

3 Low degree cohomology spaces

We have seen a geometric motivation for the definition of Lie algebra cohomology. An algebraic motivation is the interpretation of the low degree cohomology spaces in terms of algebraic properties of the Lie algebra, which makes the cohomology spaces interesting invariants to compute.

These and more interpretations may be found in [Fu86] (without H^3).

3.1 Degree 0

$$H^0(\mathfrak{g}, M) \;=\; Z^0(\mathfrak{g}, M) \;=\; \{m \in M \,|\, dc = 0\} \;=\; \{m \in M \,|\, \forall x \in \mathfrak{g} : x \cdot m = 0\} \;=\; M^{\mathfrak{g}}.$$

Example: $\mathfrak{so}(3, \mathbb{R})$ acts on $\mathbb{R}^3$ as infinitesimal rotations. It is easy to see (writing down elements of $\mathfrak{so}(3, \mathbb{R})$ as antisymmetric 3×3 matrices) that $(\mathbb{R}^3)^{\mathfrak{so}(3,\mathbb{R})} = \{0\}$. This is compatible with our intuition, as the only invariant vector under rotations is the origin.

3.2 Degree 1

$$H^1(\mathfrak{g}, M) \;=\; Z^1(\mathfrak{g}, M) \,/\, B^1(\mathfrak{g}, M) \;=\; \mathrm{Der}(\mathfrak{g}, M) \,/\, \mathrm{PDer}(\mathfrak{g}, M),$$

with

$$\mathrm{Der}(\mathfrak{g}, M) \;:=\; \{f \in \mathrm{Hom}_k(\mathfrak{g}, M) \,|\, \forall x, y \in \mathfrak{g} : f([x, y]) = x \cdot f(y) - y \cdot f(x)\},$$

and

$$\mathrm{PDer}(\mathfrak{g}, M) \;:=\; \{f \in \mathrm{Hom}_k(\mathfrak{g}, M) \,|\, \exists m \in M : \forall x \in \mathfrak{g} : f(x) = x \cdot m\}.$$

One of the most important cases is here $M = \mathfrak{g}$ with the adjoint action, where

$$\mathrm{Der}(\mathfrak{g}, \mathfrak{g}) \;=\; \mathrm{Der}(\mathfrak{g}) \;=\; \{D \in \mathrm{Hom}_k(\mathfrak{g}, \mathfrak{g}) \,|\, \forall x, y \in \mathfrak{g} : D([x, y]) = [D(x), y] + [x, D(y)]\}$$

is the *Lie algebra of (all) derivations* of $\mathfrak{g}$, and where $\mathrm{PDer}(\mathfrak{g}, \mathfrak{g})$ is the ideal of *inner derivations*, i.e. $\mathrm{PDer}(\mathfrak{g}, \mathfrak{g}) = \mathrm{ad}_{\mathfrak{g}}$. The quotient space is then

$$H^1(\mathfrak{g}, \mathfrak{g}) \;=\; \mathrm{Out}(\mathfrak{g}),$$

the Lie algebra of *outer derivations* of $\mathfrak{g}$.

Another very important case is $M = k$ with the trivial action. Here

$$H^1(\mathfrak{g}, k) \;=\; \{f \in \mathrm{Hom}_k(\mathfrak{g}, k) \,|\, \forall x, y \in \mathfrak{g} : f([x, y]) = 0\} \,/\, \{0\} \;=\; (\mathfrak{g} \,/\, [\mathfrak{g}, \mathfrak{g}])^* \,.$$

Example: consider $\mathfrak{sl}(2, \mathbb{C}) = \langle e, f, h\rangle$ with the standard generators

$$e \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bracket relations are well known: $[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$. This shows easily that $H^1(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})) = \{0\}$, i.e. all derivations are inner, and that $H^1(\mathfrak{sl}(2, \mathbb{C}), k) = \{0\}$. Actually, we shall state later a theorem which implies that H^1 of $\mathfrak{sl}(2, \mathbb{C})$ with values in *any* finite dimensional module is trivial.

3.3 Degree 2

Definition 3.1 A short exact sequence of Lie algebras $0 \to M \xrightarrow{i} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0$ is called an *abelian extension* of $\mathfrak{g}$ by M in case $i(M)$ is abelian. Two such abelian extensions $\mathfrak{e}$ and $\mathfrak{e}'$ of $\mathfrak{g}$ by M are called *equivalent* in case the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\mathrm{id}_M"] & \mathfrak{e} \arrow[r, "\pi"] \arrow[d, "\varphi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"] & \mathfrak{e}' \arrow[r, "\pi'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

In this case, φ is necessarily an isomorphism. Denote by $\mathrm{Ext}(\mathfrak{g}, M)$ the set (actually an abelian group) of equivalence classes of abelian extensions of $\mathfrak{g}$ by M.

Theorem 3.2

$$H^2(\mathfrak{g}, M) \cong \mathrm{Ext}(\mathfrak{g}, M).$$

sketch of proof: in one direction, associate to a given 2-cocycle c the extension $\mathfrak{e}_c := M \oplus \mathfrak{g}$ (as vector spaces) with the bracket

$$[(a, x), (b, y)] = (x \cdot b - y \cdot a + c(x, y), [x, y]).$$

In the other direction, choose a linear section $s : \mathfrak{g} \to \mathfrak{e}$ of π. The default of s being a Lie algebra morphism gives a cocycle:

$$c(x, y) := s([x, y]) - [s(x), s(y)],$$

for all $x, y \in \mathfrak{g}$. As $\ker(\pi) = \mathrm{im}(i)$, c takes its values in $i(M) \cong M$. □

A full proof of this theorem together with the same theorem for Lie groups and the relations between the two may be found in [Ne04].

In each abelian extension $\mathfrak{e}$ of $\mathfrak{g}$ by M, the bracket on $\mathfrak{e}$ induces an action of $\mathfrak{g}$ on M by lifting elements of $\mathfrak{g}$ to $\mathfrak{e}$ and bracketing them with elements of $i(M)$. (If the $\mathfrak{g}$-module M is given, we require that this action coincides with the given one.)

Definition 3.3 An abelian extension where the action of $\mathfrak{g}$ on M is trivial is called a *central extension*. Indeed, in this case $i(M)$ is central in $\mathfrak{e}$.

The most important reason why central extensions are important to physics, is contained in the next corollary. Recall that a *projective representation* of a group G on a vector space V is a map $\rho : G \to Gl(V)$ such that

$$\rho(x)\rho(y) = c(x, y)\rho(xy).$$

Actually, this is the same as a group homomorphism $G \to PGl(V)$.

Theorem 3.4 *(a) The map $c : G \times G \to k^*$ which is associated to a projective representation $\rho : G \to Gl(V)$ is a group cocycle.*

(b) The two cocycles c and c' which are associated to different lifts $G \to Gl(V)$ of the same projective representation $G \to PGl(V)$ are cohomologuous.

(c) If the cocycle c is a coboundary, the projective representation is equivalent to a linear representation.

Corollary 3.5 *Each projective representation of G corresponds to a central extension $\widehat{G}$ of G and a linear representation of $\widehat{G}$.*

An analoguous theorem holds true for Lie algebras and central extensions of Lie algebras. This material together with full proofs may be found in Section 2.6.2 of [GuRo07].

Remark 3.6 Projective representations abound in quantum physics due to the correspondence between physical states and wave functions: two wave functions ψ_1 and ψ_2 represent the same physical state in case $\psi_1 = \lambda\psi_2$ for some $\lambda \neq 0$. In the same vein, a typical **example** of a central extension is the *Heisenberg algebra*, i.e. the 3-dimensional Lie algebra generated by x, y and z with the (only non-trivial) bracket $[x, y] = z$. It is therefore a 1-dimensional central extension of the abelian 2-dimensional Lie algebra in x and y.

3.4 On the central extension of $\mathrm{Vect}(\mathbb{S}^1)$

The Lie algebra $\mathrm{Vect}(\mathbb{S}^1)$ possesses a (unique, up to equivalence) 1-dimensional central extension, called the *Virasoro algebra*. This may be seen by direct computations (taking into account that we are here computing Gelfand-Fuchs cohomology and that we only need the degree 0 subcomplex (due to Theorem (4.3), i.e. the relevant cochain space is just

$$\bigoplus_{p=0}^{\infty} \mathbb{C}\epsilon^p \wedge \epsilon^{-p},$$

where ϵ^i denotes the generator which is dual to e_i. See the computation in Section 4.2.4 of [GuRo07].) The standard Virasoro cocycle reads:

$$c(e_n, e_m) \;=\; \frac{1}{12}(n^3 - n)\delta_{n+m,0}$$

The term $n^3\delta_{n+m,0}$ is the important part, while $n\delta_{n+m,0}$ is actually a coboundary and the factor $\frac{1}{12}$ is a normalization factor.

Another cocycle generating (a non-zero multiple of) the same cohomology class is the *Gelfand-Fuchs cocycle* which reads as

$$\omega(f, g) \;=\; \int_{\mathbb{S}^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}.$$

This expression of the cocycle comes up in Gelfand-Fuchs' computation of the continuous cohomology of $\mathrm{Vect}(\mathbb{S}^1)$. It takes into account the knowledge of the cohomology of the Lie algebra of formal vector fields on $\mathbb{R}$, together with the Gelfand-Fuchs spectral sequences. All this is explained in [Fu86].

Actually, the Gelfand-Fuchs cocycle is the fiber integral over the Godbillon-Vey cocycle, but this is different story (see for example [Wa06]).

We will take here a different point of view: using the above corollary, the Virasoro algebra appears naturally when one tries to define a representation of $\mathrm{Vect}(\mathbb{S}^1)$ on the standard fermionic Fock space. This is the point of view of the central extension of $\mathrm{Vect}(\mathbb{S}^1)$ as an *anomaly* which comes up in second quantization.

Let E be a graded vector space, $E = \langle f_i \,|\, i \in \mathbb{Z}\rangle$. Define the *space of semi-infinite forms* $\Lambda^{\frac{\infty}{2}}E$ to be

$$\Lambda^{\frac{\infty}{2}}E \;:=\; \{f_{i_0} \wedge f_{i_1} \wedge f_{i_2} \wedge \ldots \,|\, (i_n) \text{ decreasing}, \;\; \exists N : i_{n+1} = i_n - 1 \;\; \forall n \geq N\}.$$

Other names of this space are *fermionic Fock space* or *Dirac's sea/ocean.* A semi-infinite form $v \in \Lambda^{\frac{\infty}{2}}E$ has *charge* m in case there exists N such that for all $n \geq N$, $i_n = m - n$. Define the *energy* of a v of charge m to be

$$En(v) \;=\; \sum_{n=0}^{\infty} i_n + n - m.$$

Note that this is a finite sum. For an arbitrary v, the notion of energy extends by linearity, because each v is sum of elements of fixed charge.

The name "Dirac's ocean" is justified by *Dirac's formula* for the energy: let v be of charge 0. Then we have

$$En(v) \;=\; \sum_{i_n>0} i_n \;-\; \sum_{j_k\leq 0,\; j_k\neq i_n \forall n} j_k.$$

Read this formula as stating that the energy of v is given as the sum over all positive indices/modes minus all negative indices/modes which do not appear. This illustrates the idea that the absence of a mode is a mode of negative energy.

Coming back to $\mathrm{Vect}(\mathbb{S}^1)$, we choose $E = \mathrm{Vect}(\mathbb{S}^1)$, denoting its elements still f_j. We want to extend the adjoint action of $\mathrm{Vect}(\mathbb{S}^1)$ on E (i.e. $e_i \cdot f_j = (j-i)f_{i+j}$) to the space of semi-infinite forms $\Lambda^{\frac{\infty}{2}}E$. This works rather nicely for indices not involving 0; one simply extends by derivation (like the Lie derivative). Then one gets

$$e_i \cdot (e_j \cdot v) - e_j \cdot (e_i \cdot v) \;=\; [e_i, e_j] \cdot v,$$

but only if $i \neq 0$, $j \neq 0$ and $i + j \neq 0$. Trying to extend this to e_0 using $[e_{-1}, e_1] = 2e_0$ and $[e_{-2}, e_2] = 4e_0$ leads to different results. The upshot of this discussion is that the adjoint representation of $\mathrm{Vect}(\mathbb{S}^1)$ on itself extends to a *projective representation* and that the corresponding central extension of $\mathrm{Vect}(\mathbb{S}^1)$ is the Virasoro algebra. Details for this computation may be found in [GuRo07] §7.3, p. 495 (take $\mu = 0$ and $\lambda = -1$ for adjoint coefficients).

This result is due to Feigin [Fe84] and is crucial in the later treatment of BRST cohomology.

3.5 Degree 3

For degree 3, the picture is very similar to degree 2 (and permits a glimpse on the interpretation of arbitrary degree cohomology spaces). Here, $H^3(\mathfrak{g}, M)$ is isomorphic to the set/abelian group of equivalence classes of *crossed modules* with cokernel $\mathfrak{g}$ and kernel M.

Definition 3.7 A *crossed module* of Lie algebras is a homomorphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations, denoted by $m \mapsto n \cdot m$, such that for all $m, m' \in \mathfrak{m}$ and all $n \in \mathfrak{n}$

(a) $\mu(n \cdot m) \;=\; [n, \mu(m)]$

(b) $\mu(m) \cdot m' \;=\; [m, m']$.

The requirements (a) and (b) imply that the cokernel of μ, denoted $\mathfrak{g}$, is a Lie algebra, that the kernel of μ, denoted M, is a central ideal, and that $\mathfrak{g}$ acts on M. Therefore a crossed module corresponds to a 4-term exact sequence of Lie algebras

$$0 \to M \to \mathfrak{m} \to \mathfrak{n} \to \mathfrak{g} \to 0,$$

and one may associate to this a 3-cocycle by iterating the procedure of Section 3.3.

References for the degree 3 case are [Ne06] and [Wa06]. For physicists, it might be interesting to note that degree 3 cohomology is linked to Lie 2-algebras and to the string group, see for example [Sc10].

4 Main computational methods

In order to compute cohomology spaces, it may help to know how these space decompose in case $\mathfrak{g}$ decomposes, or M decomposes.

Proposition 4.1 (coefficient sequence: M decomposes) *Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of $\mathfrak{g}$-modules. Then there is a long exact sequence in cohomology

$$\ldots \to H^i(\mathfrak{g}, M'') \to H^{i+1}(\mathfrak{g}, M') \to H^{i+1}(\mathfrak{g}, M) \to H^{i+1}(\mathfrak{g}, M'') \to \ldots.$$

sketch of proof: this is the long exact sequence which holds for all Ext-functors, see for example [CaEi56], [We94]. In our case, one may show that the short exact coefficient sequence induces a short exact sequence of complexes

$$0 \to C^*(\mathfrak{g}, M') \to C^*(\mathfrak{g}, M) \to C^*(\mathfrak{g}, M'') \to 0,$$

and then apply the usual construction of the connecting homomorphism. $\square$

Theorem 4.2 (Hochschild-Serre spectral sequence: $\mathfrak{g}$ decomposes) *Let*

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0$$

be a short exact sequence of Lie algebras and M be a $\mathfrak{g}$-module. There exists a spectral sequence

$$\{E_r^{p,q}, d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\}$$

such that

(i) $E_1^{p,q} = H^q(\mathfrak{h}, C^q(\mathfrak{q}, M))$,

(ii) $E_2^{p,q} = H^p(\mathfrak{q}, H^q(\mathfrak{h}, M))$,

(iii) $\{E_r^{p,q}\} \Rightarrow H^*(\mathfrak{g}, M)$.

This theorem admits intermediate results for subalgebras which are not ideals. It has been used by Hochschild and Serre to compute cohomology using a reductive subalgebra. The original reference is [HoSe52], but see also [Fu86]. For explanations about spectral sequences, the interested reader may consult [We94].

Theorem 4.3 (inner gradings: $\mathfrak{g}$ decomposes) *Suppose $\mathfrak{g}$ possesses a grading element e_0, i.e.*

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_l = \{x \in \mathfrak{g} \,|\, [e_0, x] = lx\}.$$

Then all cochain spaces are graded and the inclusion $C_0^*(\mathfrak{g}) \subset C^*(\mathfrak{g})$ induces an isomorphism in cohomology.

sketch of proof: for an inner grading, one has formulae like in Cartan calculus:

$$i_{e_0} \circ d + d \circ i_{e_0} = l \,\mathrm{id}$$

holds on the subcomplex $C_l^*(\mathfrak{g})$ of cochains of degree l. Therefore one may construct a contracting chain homotopy for all subcomplexes with $l \neq 0$. $\square$

The theorem admits generalizations to more grading elements and to non-trivial modules. For this and a full proof, consult [Fu86] (theorem 1.5.2).

5 Some results for finite dimensional Lie algebras

5.1 $\mathfrak{g}$ complex semi-simple

Proposition 5.1 (Whitehead's Lemmas) *Let $\mathfrak{g}$ be complex semi-simple, and M be a finite-dimensional $\mathfrak{g}$-module. Then*

$$H^1(\mathfrak{g}, M) \,=\, \{0\}, \text{ and } H^2(\mathfrak{g}, M) \,=\, \{0\}.$$

The proof of this proposition may be found in Section 3.12 of [Va84].

Proposition 5.2 *Let $\mathfrak{g}$ be complex semi-simple. Then*

$$H^3(\mathfrak{g}, M) \,\cong\, \mathbb{C}^l,$$

where l is the number of simple factors contained in $\mathfrak{g}$. For each simple factor with Killing form $\langle,\rangle$, the 3-cocycle $\langle[,],\rangle$ generates the corresponding factor in $\mathbb{C}^l$.

Theorem 5.3 *Let $\mathfrak{g}$ be complex semi-simple. Then $H^*(\mathfrak{g}, \mathbb{C})$ is an exterior algebra in odd generators.*

sketch of proof: let G^c be a compact form of a connected Lie group G corresponding to $\mathfrak{g}$. Then we have by Cartan's Theorem

$$H^*(\mathfrak{g}, \mathbb{C}) \,\cong\, H^*_{\mathrm{dR}}(G^c, \mathbb{C}).$$

The latter is known to be a graded Hopf algebra, and by the Milnor-Moore Theorem, it would be infinite dimensional, if it contained even generators. $\square$

For a full proof, see, for example, Example 3, II.12, p. 143 in [FHT01].

Theorem 5.4 *Let $\mathfrak{g}$ be complex semi-simple (or even reductive). Let M be a finite dimensional, semi-simple $\mathfrak{g}$-module such that $M^{\mathfrak{g}} = \{0\}$. Then*

$$H^n(\mathfrak{g}, M) \,=\, \{0\} \qquad \forall n \geq 0.$$

For a proof, see [HoSe52].

5.2 $\mathfrak{g}$ nilpotent

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Denote by $b_i(\mathfrak{g})$ the integer $\dim H^i(\mathfrak{g}, k)$.

Theorem 5.5 (Dixmier, 1955)

$$b_i(\mathfrak{g}) \geq 2, \quad \forall\; 1 \leq i \leq n-1.$$

This is only one of the first bounds on the cohomology of nilpotent Lie algebras. It shows that where simple Lie algebras have "few" cohomology, nilpotent Lie algebras have "huge" cohomology. One motivation for finding more and more precise bounds for the cohomology of nilpotent Lie algebras is the

Toral rank conjecture (Halperin)

$$\sum b_i(\mathfrak{g}) \geq 2^d,$$

where d is the dimension of the center of $\mathfrak{g}$.

For more information about this branch of mathematics, we refer to [DeSi88].

5.3 Another famous result

Theorem 5.6 (Kostant) *Let $\mathfrak{g}$ be a semi-simple Lie algebra with root decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let V be an irreducible, finite-dimensional, highest weight $\mathfrak{g}$-module of highest weight λ.*

Then $H^k(\mathfrak{n}_+, V)$ splits (as a $\mathfrak{g}$-module) into the direct sum of 1-dimensional modules of multiplicity one. The corresponding weights are exactly all elements of the form $w(\lambda+\delta)-\delta$, where w is any Weyl group element of length k (and δ denotes the half sum of the positive roots).

For a proof and applications, see [Ko61]. Actually, this theorem permits an algebraic proof of the Borel-Weil-Bott theorem which realizes representations of semi-simple Lie groups as spaces of sections of bundles over complex manifolds. This line of thoughts still inspires many authors and has been generalized to infinite dimensional Lie algebras, see for example [Ku02] or [Ne01].

6 BRST cohomology

BRST = quantization procedure of a classical system with constraints by introducing odd variables ("ghosts")

Standard references are [Fe84], [FGZ86] and [KoSt87]. Here we followed very closely [KoSt87]. For some more recent development in this direction, see [Se99].

6.1 Review of symplectic reduction

Let X be a symplectic manifold. Let G act on X such that there is a *moment map* $\varphi : X \to \mathfrak{g}^*$. The map φ gives rise to a map $\delta : \mathfrak{g} \to \mathrm{Fun}(X)$ setting

$$\delta(\xi)(x) \,:=\, \varphi(x)(\xi)$$

for all $\xi \in \mathfrak{g}$ and all $x \in X$.

The *Marsden-Weinstein (symplectic) reduction* proceeds then as follows: first take the inverse image under the moment map of a regular value, then divide out the group action.

$$C \,:=\, \varphi^{-1}(0), \qquad B \,:=\, C\,/\,G,$$

provided 0 is a regular value.

6.2 Reformulation in cohomological terms

Suppose that C is a submanifold (for example, if 0 is a regular value) and suppose further that C corresponds to an ideal I in $\mathrm{Fun}(X)$ via

$$\mathrm{Fun}(C) \,=\, \mathrm{Fun}(X)\,/\,I.$$

By definition of C, for all $\xi \in \mathfrak{g}$, $\delta(\xi)$ vanishes on C, and therefore $\delta(\xi) \in I$. Suppose I is generated by these elements, i.e.

$$I \,=\, \mathrm{Fun}(X)\delta(\mathfrak{g}).$$

Then we obtain

$$\mathrm{Fun}(C) \,=\, \mathrm{Fun}(X)\,/\,\delta(\mathfrak{g})\mathrm{Fun}(X),$$

i.e. $\mathrm{Fun}(C)$ is the space of *coinvariants* under $\mathfrak{g}$, acting via δ.

Next $B = C\,/\,G$, and we therefore get

$$\mathrm{Fun}(B) \;=\; \mathrm{Fun}(C)^G,$$

i.e. $\mathrm{Fun}(B)$ is the space of *invariants* under the G-action.

Now consider $\Lambda\mathfrak{g} \otimes \mathrm{Fun}(X)$. This is a superalgebra (as tensor product of a superalgebras and an ordinary algebra). Define a superderivation δ (extending the above map δ) on it by

$$\begin{cases} \delta(\xi \otimes 1) & = & 1 \otimes \delta(\xi) \\ \delta(1 \otimes f) & = & 0 \end{cases}$$

This defines in fact the Koszul differential on $\Lambda\mathfrak{g} \otimes \mathrm{Fun}(X)$, and we obtain the Koszul complex. By construction, we have

$$H^0_\delta(\Lambda\mathfrak{g} \otimes \mathrm{Fun}(X)) \;=\; \mathrm{Fun}(X)\,/\,\delta(\mathfrak{g})\mathrm{Fun}(X).$$

Adding the Lie algebra cohomology, we obtain

$$\begin{tikzcd} \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes \mathrm{Fun}(X) \arrow[r, "\delta"] \arrow[d, "d"] & \Lambda^p\mathfrak{g}^* \otimes \Lambda^{q+1}\mathfrak{g} \otimes \mathrm{Fun}(X) \\ \Lambda^{p+1}\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes \mathrm{Fun}(X) & \end{tikzcd}$$

We have $\delta^2 = 0$, $d^2 = 0$, $\delta \circ d = d \circ \delta$, and therefore a *double complex*. The total differential

$$D \;=\; d + (-1)^p 2\delta$$

is called the (classical) BRST differential.

The above superalgebra also carries a super Poisson bracket - it comes from identifying it with the associated graded algebra of a filtered Clifford algebra. (Details can be found in [KoSt87] p. 65.) This is also the meaning of fermionic quantization here ! Actually, D can be expressed as the super Poisson bracket with some element Θ:

$$D \;=\; \{\Theta, -\}.$$

The element Θ arises as follows. Define the 3-form Ω by

$$\Omega(x,y,z) \;:=\; -\frac{1}{2}(x,[y,z]).$$

This can be defined for any quadratic Lie algebra $\mathfrak{a}$, $(,)$ denoting the quadratic form. We have seen in 5.1 that for a semi-simple Lie algebra $\mathfrak{a}$ and $(,)$ its Killing form, these elements give generators of H^3.

In our context, we will later take $\mathfrak{a} = \mathfrak{g} \ltimes \mathfrak{g}^*$. Disposing of a *non-degenerate* quadratic form, one can identify chains and cochains on $\mathfrak{a}$, and any element gives a 1-cochain. Now one easily verifies

$$dx(y,z) \;=\; (2i_x\Omega)(y,z)$$

for all $x,y,z \in \mathfrak{a}$, therefore in our context the Lie algebra differential d may be identifies with the Poisson bracket by Ω:

$$d \;=\; \{\Omega, -\}$$

The element Θ is therefore simply

$$\Theta = \Omega \otimes 1 + \delta.$$

6.3 BRST quantization

Suppose the quantization of $\mathrm{Fun}(X)$ as operators on some graded vector space $\mathbb{T}$ is known. The BRST procedure prescribes how to extend this to $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathrm{Fun}(X)$ such that $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ is quantized as the Clifford algebra for the quadratic vector space

$$\mathfrak{a} := \mathfrak{g} \ltimes \mathfrak{g}^*$$

and such that the above element Θ becomes quantized to an operator $\mathbf{Q} : \mathbb{T}^k \to \mathbb{T}^{k+1}$ with $\mathbf{Q}^2 = 0$.

More precisely, instead of $\mathbb{T}$, one takes the tensor product of $\mathbb{T}$ with the either a standard Clifford module (for finite dimensional $\mathfrak{g}$) or the fermionic Fock space of Section 3.4 (for infinite dimensional $\mathfrak{g}$). Both the quantization of $\mathrm{Fun}(X)$ and the Clifford algebra of $\mathfrak{g} \oplus \mathfrak{g}^*$ then acts on their respective tensor factor.

In terms of the filtration on the Clifford algebra $C(\mathfrak{a})$, the element $Q \in C_1^3(\mathfrak{a})$ is chosen such that $\mathrm{gr}_3 Q = \frac{1}{2}\Omega$, i.e. their classes modulo $C_1^1(\mathfrak{a})$ coincide. (Here the upper index is the length filtration (induced from the graduation of the tensor algebra), the lower index is the $\mathbb{Z}_2$-graduation (induced from the $\mathbb{Z}_2$-graduation of the tensor algebra).) This is the part corresponding to Ω. The quantum analogue of δ is called τ. As before, we put

$$\mathbf{Q} := Q \otimes \mathrm{id} + \tau.$$

The *BRST space*, i.e. the space of *true physical states*, is then just $H^0_{\mathbf{Q}}(\mathbb{T})$, and the elements of $\mathrm{Fun}(B)$ act by construction on it as operators. This works well for a finite dimensional Lie algebra (content of Sections 2 to 6 in [KoSt87]).

For an infinite dimensional Lie algebra $\mathfrak{g}$, the identity $Q^2 = 0$ may fail to be true. This is due to an anomaly arising in quantization – the mechanism was explained in a nutshell in Section 3.4. Kostant and Sternberg introduce the necessary machinery to tackle the infinite dimensional case, to express the anomaly/cocycle, and find a way around $Q^2 \neq 0$. Namely, adding the negative of the class of the central extension to the differential, the anomaly vanishes (see p. 88 of *loc. cit.*).

References

[Bre93] G. Bredon, Topology and geometry. Graduate Texts in Mathematics, **139**. Springer-Verlag, New York, 1993

[CaEi56] H. Cartan, S. Eilenberg, Homological Algebra. Princeton University Press 1956

[DeSi88] C. Deninger, W. Singhof, On the cohomology of nilpotent Lie algebras. Bull. Soc. Math. France **116** (1988), no. 1, 3–14

[Di55] J. Dixmier, Cohomologie des algèbres de Lie nilpotentes. Acta Sci. Math. Szeged **16** (1955) 246–250

[Fe84] B. L. Feigin, Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. Uspekhi Mat. Nauk **39**, no. 2 (1984) 195–196

[FeTs89] B. L. Feigin, B. L. Tsygan, Riemann-Roch theorem and Lie algebra cohomology. I. Proceedings of the Winter School on Geometry and Physics (Srni, 1988). Rend. Circ. Mat. Palermo (2) Suppl. No. **21** (1989) 15–52

[FeFu00] B. L. Feigin, D. B. Fuchs, Cohomologies of Lie groups and Lie algebras. Lie groups and Lie algebras, II, 125–223, Encyclopaedia Math. Sci. **21** Springer, Berlin, 2000.

[FHT01] Y. Félix, S. Halperin, J.-C. Thomas, Rational homotopy theory. Graduate Texts in Mathematics **205** Springer-Verlag, New York, 2001

[FGZ86] I. B. Frenkel, H. Garland, G. J. Zuckerman, Semi-infinite cohomology and string theory. Proc. Nat. Acad. Sci. USA, Vol. **83** (1986) 8442–8446

[Fu86] D. B. Fuchs, Cohomology of infinite dimensional Lie algebras. Consultants Bureau, New York 1986

[GuRo07] L. Guieu, C. Roger, L'Algèbre et le Groupe de Virasoro. Publications CRM, Montréal 2007

[HoSe52] G. Hochschild, J.-P. Serre, Cohomology of Lie algebras. Ann. of Math. (2) **57** (1953) 591–603

[Ko61] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem. Ann. of Math. (2) **74** (1961) 329–387

[KoSt87] B. Kostant, S. Sternberg, Symplectic Reduction, BRS Cohomology, and Infinite-Dimensional Clifford algebras. Ann. Physics **176** (1987) 49–113

[Ku02] S. Kumar, Kac-Moody groups, their flag varieties and representation theory. Progress in Mathematics **204** Birkhäuser Boston, Inc., Boston, MA, 2002

[Lo98] J.-L. Loday, Cyclic homology. Grundlehren der Mathematischen Wissenschaften **301** Springer-Verlag, Berlin, 1998

[Ne01] K.-H. Neeb, Borel-Weil theory for loop groups. Infinite dimensional Khler manifolds (Oberwolfach, 1995), 179–229

[Ne04] K.-H. Neeb, Abelian extensions of infinite-dimensional Lie groups. Travaux mathématiques. Fasc. XV, 69–194, Trav. Math., XV, Univ. Luxemb., Luxembourg, 2004.

[Ne06] K.-H. Neeb, Non-abelian extensions of topological Lie algebras. Comm. Algebra 34 (2006), no. 3, 991–1041

[NeTs95] R. Nest, B. L. Tsygan, Algebraic index theorem. Comm. Math. Phys. **172** (1995) no. 2, 223–262.

[Sc10] C. Schommer-Pries, Central Extensions of Smooth 2-Groups and a Finite-Dimensional String 2-Group. `arXiv:0911.2483`

[Se99] A. Sevostyanov, Reduction of Quantum Systems with Arbitrary First Class Constraints and Hecke Algebras. Commun. Math. Phys. **204** (1999) 137–146

[Va84] V. S. Varadarajan, Lie groups, Lie algebras, and their representations. Graduate Texts in Mathematics **102** Springer-Verlag, New York, 1984

[Wa06] F. Wagemann, On Lie algebra crossed modules. Comm. Algebra **34** (2006), no. 5, 1699–1722

[We94] C. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics **38** Cambridge University Press, Cambridge, 1994.