

Holomorphic current groups – Structure and Orbits

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Abstract

Let K be a finite-dimensional, 1-connected complex Lie group, and let $\Sigma_k = \Sigma \setminus \{p_1, \dots, p_k\}$ be a compact connected Riemann surface Σ , from which we have extracted $k \geq 1$ distinct points. We study in this article the regular Fréchet-Lie group $\mathcal{O}(\Sigma_k, K)$ of holomorphic maps from Σ_k to K and its central extension $\widehat{\mathcal{O}(\Sigma_k, K)}$. We feature especially the automorphism groups of these Lie groups as well as the coadjoint orbits of $\widehat{\mathcal{O}(\Sigma_k, K)}$ which we link to flat K -bundles on Σ_k .

Introduction

Let K be a connected Lie group. The *loop group* $LK := \mathcal{C}^\infty(\mathbb{S}^1, K)$ is an important symmetry group in physics, and its mathematical study has led to considerable research with respect to its structure, its representations and their applications. As physicists usually prefer the symmetry Lie algebra to the group, the loop Lie algebra $L\mathfrak{k} := \mathcal{C}^\infty(\mathbb{S}^1, \mathfrak{k})$ where \mathfrak{k} is the Lie algebra of K , also plays an important role. Most of the algebraic theory is based on $L\mathfrak{k}$ and its central extension $\widehat{L\mathfrak{k}}$ which is closely related to affine Kac-Moody Lie algebras. There is interesting geometry coming about when lifting the central extension $\widehat{L\mathfrak{k}}$ to group level defining the \widehat{LK} . A standard reference about the loop group is Pressley-Segal's book [PrSe86].

Searching for similar mapping groups in higher dimensions, Etingof and Frenkel [EtFr94] came up with the group $K^\Sigma := \mathcal{C}^\infty(\Sigma, K)$ for a compact connected Riemann surface Σ , and studied its central extensions, its automorphisms and coadjoint orbits. It turns out that the universal central extension has infinite dimensional center, but there is an interesting sub-extension by the Jacobian of Σ . The orbits of the coadjoint representation for 1-connected K correspond bijectively to equivalence classes of holomorphic principal K -bundles over the surface Σ . These bundles carry by construction a flat connection.

On the other hand, more classes of examples arise from mapping spaces with different regularity conditions. One important class of examples are spaces of meromorphic maps. Here the research mainly restricted to the Lie algebra side (it is difficult or even impossible to define Lie groups for these Lie algebras), namely to *Krichever-Novikov algebras*. Given a compact connected Riemann surface Σ and $k \geq 1$ distinct points p_1, \dots, p_k , the open Riemann surface

$$\Sigma_k := \Sigma \setminus \{p_1, \dots, p_k\}$$

is in fact an affine algebraic curve. Krichever-Novikov [KriNov87a], [KriNov87b], [KriNov89] for the genus zero case and Schlichenmaier, Schlichenmaier-Sheinman (see [Sch89]) for arbitrary genus define meromorphic analogues of the loop algebras as $Reg(\Sigma_k) \otimes_{\mathbb{C}} \mathfrak{k}$ for any complex Lie algebra \mathfrak{k} , where $Reg(\Sigma_k)$ is the algebra of regular functions on the algebraic variety Σ_k . They also study central extensions, automorphisms, coadjoint orbits and representations of these Lie algebras.

The present work takes its place between Etingof-Frenkel's work and the work on Krichever-Novikov algebras. Our goal is to study central extensions, automorphisms and coadjoint orbits of holomorphic analogues of Krichever-Novikov algebras. A new feature is that we have Fréchet-Lie groups to these Lie algebras here. Namely, instead of considering only *regular* functions on the affine algebraic curve Σ_k , one may consider all *holomorphic maps* on the Stein manifold Σ_k . These latter may have essential singularities in the points p_1, \dots, p_k , while the former are meromorphic, i.e. have at most poles of finite order in these points. We are thus considering here the topological group $\mathcal{O}(\Sigma_k, K)$ of holomorphic maps from Σ_k to K in the topology of uniform convergence on compact sets. In fact, $\mathcal{O}(\Sigma_k, K)$ is an infinite-dimensional Fréchet-Lie group [NeWa08b] with Lie algebra $\mathcal{O}(\Sigma_k, \mathfrak{k})$, the Lie algebra of holomorphic maps from Σ_k to the Lie algebra \mathfrak{k} .

In our study of the structure of $\mathcal{O}(\Sigma_k, K)$ and its central extension $\widehat{\mathcal{O}(\Sigma_k, K)}$ which occupies Section 2, we build on our previous work concerning the holomorphic current algebra [NeWa08b], [NeWa03] and on previous studies of the infinite-dimensional Lie theory of current groups [Ne02], [NeWa08a], [MaNe03] part of which we adapt to the holomorphic setting. A new result is the lifting of the Lie group structure to the central extension (Corollary 2.13).

In Section 3, we describe the automorphism groups of $\mathcal{O}(\Sigma_k, K)$ and of its central extension. Concrete results are the computation of the automorphism group (Corollary 3.7). We start the problem of determining the automorphism groups of the central extensions (Proposition 3.9), but leave it to further study to determine these groups explicitly for the different complex simple Lie algebras \mathfrak{k} .

Section 4 is the heart of the present article. Here we study the coadjoint orbits of $\mathcal{O}(\Sigma_k, K)$ in the smooth dual of $\widehat{\mathcal{O}(\Sigma_k, \mathfrak{k})}$ and establish relations to flat principal K -bundles on Σ_k as well as to the coadjoint orbits of the loop group LK (Propositions 4.3 and 4.8). Section 1 prepares the necessary material about flat bundles. As usual, these coadjoint orbits carry the *Kostant-Kirillov-Souriau* symplectic form, and they are thus examples of (weakly) holomorphic symplectic manifolds (see Corollary 4.12).

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1 Holomorphic 1-forms and flat principal bundles

1.1 Holomorphic 1-forms and the logarithmic derivative

In this subsection, we discuss the conditions under which a Lie algebra valued holomorphic 1-form is the logarithmic derivative of a group-valued holomorphic function. We follow closely [NeWa08b].

Let K be a (possibly infinite dimensional) regular complex Lie group. For a complex manifold M , let us denote by $\mathcal{O}(M, K)$ the group of holomorphic maps from M to K and by $\Omega^1(M, \mathfrak{k})$ the space of holomorphic 1-forms on M with values in the Lie algebra \mathfrak{k} . Let κ be the Maurer-Cartan form on K , i.e. the unique holomorphic left invariant 1-form with values in \mathfrak{k} on K corresponding to $\text{id}_{\mathfrak{k}}$ in $\text{Hom}(\mathfrak{k}, \mathfrak{k})$. For

$f \in \mathcal{O}(M, K)$, we denote by

$$\delta(f) := f^*(\kappa) =: f^{-1}df \in \Omega^1(M, \mathfrak{k})$$

the (left) logarithmic derivative of f . We obtain a map

$$\delta : \mathcal{O}(M, K) \rightarrow \Omega^1(M, \mathfrak{k})$$

satisfying the cocycle condition

$$\delta(f_1 f_2) = \text{Ad}(f_2)^{-1} \delta(f_1) + \delta(f_2).$$

From this it follows easily that if M is connected, then

$$\delta(f_1) = \delta(f_2) \Leftrightarrow (\exists k \in K) f_2 = \lambda_k \circ f_1,$$

where λ_k denotes left multiplication by $k \in K$. If K is abelian, then δ is a group homomorphism whose kernel consists of the locally constant maps $M \rightarrow K$. There is a (right) action of $\mathcal{O}(M, K)$ on $\Omega^1(M, \mathfrak{k})$ given by

$$\alpha * f := \delta(f) + \text{Ad}(f)^{-1} \alpha, \tag{1}$$

derived from the above cocycle condition. We will call this action the *gauge action*.

We call $\alpha \in \Omega^1(M, \mathfrak{k})$ *integrable* if there exists a holomorphic function $f : M \rightarrow K$ with $\delta(f) = \alpha$. We say that α is *locally integrable* if each point $m \in M$ has an open neighborhood U such that $\alpha|_U$ is integrable. In order to describe conditions for the integrability of an element $\alpha \in \Omega^1(M, \mathfrak{k})$, we define the bracket

$$[\cdot, \cdot] : \Omega^1(M, \mathfrak{k}) \times \Omega^1(M, \mathfrak{k}) \rightarrow \Omega^2(M, \mathfrak{k})$$

by

$$[\alpha, \beta]_p(v, w) := [\alpha_p(v), \beta_p(w)] - [\alpha_p(w), \beta_p(v)] \quad \text{for } v, w \in T_p(M).$$

Note that $[\alpha, \beta] = [\beta, \alpha]$. By definition, the *Maurer-Cartan space* associated to the complex manifold M and the complex Lie algebra \mathfrak{k} is the space of closed non-abelian differential 1-forms

$$Z_{\text{dR}}^1(M, \mathfrak{k}) := \{ \alpha \in \Omega^1(M, \mathfrak{k}) \mid d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \}.$$

The following theorem can be found with a full proof in [NeWa08b]. Here we denote by

$$\text{evol}_K : \mathcal{C}^\infty([0, 1], \mathfrak{k}) \rightarrow K$$

the *evolution map* which associates to the initial value problem

$$\gamma'(t) = \gamma(t) \cdot \xi$$

associated to $\xi \in \mathcal{C}^\infty([0, 1], \mathfrak{k})$ its solution $\gamma_\xi(1) \in K$ at $1 \in K$. The Lie group K is called *regular* if for all $\xi \in \mathcal{C}^\infty([0, 1], \mathfrak{k})$, the solution γ_ξ exists and the evolution map is smooth. For a complex regular Lie group K , the evolution map is holomorphic, cf *loc. cit.* Lemma 3.4.

Theorem 1.1 *Let M be a complex manifold, K be a regular complex Lie group and $\alpha \in \Omega^1(M, \mathfrak{k})$.*

(1) *α is locally integrable if and only if $\alpha \in Z_{\text{dR}}^1(M, \mathfrak{k})$.*

(2) If M is 1-connected and α is locally integrable, then α is integrable.

(3) Suppose that M is 1-connected, fix $m_0 \in M$ and let $\alpha \in Z_{\text{dR}}^1(M, \mathfrak{k})$. Using piecewise smooth representatives of homotopy classes, we obtain a well-defined group homomorphism

$$\text{per}_\alpha^{m_0} : \pi_1(M, m_0) \rightarrow K, [\gamma] \mapsto \text{evol}_K(\gamma^* \alpha),$$

and α is integrable if and only if this homomorphism is trivial.

Sometimes, we will suppress the base point in the notation and write more simply per_α for $\text{per}_\alpha^{m_0}$.

1.2 The classification of flat holomorphic principal bundles

In this subsection, we follow closely [Ne04] and [La07]. The main result (Proposition 1.6) of this subsection is the exact sequence linking solutions of the Maurer-Cartan equation, homomorphisms from the fundamental group π_1 to the Lie group K and flat principal K -bundles, which appears in [Ne04] and [La07]. The main idea is to describe flat bundles as bundles which become trivial when pulled back to the universal covering.

Let M be a connected complex manifold and K be a connected complex Lie group. Recall that the universal covering \widetilde{M} of M has a natural structure of a complex manifold. Let us briefly recall non-abelian 1-cocycles and non-abelian 1-cohomology.

Let Γ be a group. A Γ -group is a group G endowed with an action of Γ by automorphisms. We denote this action by $(\gamma, g) \mapsto \gamma \cdot g$. Given a Γ -group G , a G -valued 1-cochain of Γ is simply a map $\Gamma \rightarrow G$. A cochain $c : \Gamma \rightarrow G$ is called a *non-abelian 1-cocycle* in case for all $\alpha, \beta \in \Gamma$, we have

$$c(\alpha) (\alpha \cdot c(\beta)) c(\alpha\beta)^{-1} = 1.$$

Such a 1-cocycle is also called a *crossed homomorphism*. On the set $Z^1(\Gamma, G)$ of non-abelian 1-cocycles, there is a G -action given by

$$(g \cdot c)(\gamma) = g c(\gamma) (\gamma \cdot g)^{-1}.$$

The *non-abelian cohomology set* $H^1(\Gamma, G)$ is by definition the set of G -orbits in $Z^1(\Gamma, G)$. In case Γ acts trivially on G , we have

$$Z^1(\Gamma, G) = \text{Hom}(\Gamma, G) \quad \text{and} \quad H^1(\Gamma, G) = \text{Hom}(\Gamma, G) / G,$$

where G acts on $\text{Hom}(\Gamma, G)$ by the conjugation action in the image: $(g \cdot c)(\gamma) = g c(\gamma) g^{-1}$.

In the following, we will apply this formalism to the special case where $\Gamma = \pi_1(\widetilde{M})$ and $G = \mathcal{O}(\widetilde{M}, K)$, where \widetilde{M} is the universal covering of M and $\pi_1(M)$ acts on G via its action on \widetilde{M} .

Proposition 1.2 *Let $q_M : \widetilde{M} \rightarrow M$ be the universal covering projection. To a non-abelian 1-cocycle $\varphi : \pi_1(M) \rightarrow \mathcal{O}(\widetilde{M}, K)$, one associates the holomorphic principal K -bundle*

$$P_\varphi := (\widetilde{M} \times K) / \pi_1(M), \quad [(m, k)] := \pi_1(M) \cdot (m, k),$$

where $\pi_1(M)$ acts on the trivial bundle $\widetilde{M} \times K$ by

$$\gamma \cdot (m, k) = (\gamma \cdot m, \varphi(\gamma)(m)k).$$

Then the bundle $q_M^*P_\varphi$ is holomorphically trivial. Conversely, each holomorphic principal K -bundle $q : P \rightarrow M$ for which q_M^*P is holomorphically trivial is equivalent to some P_φ . For non-abelian 1-cocycles φ, ψ , we have

$$P_\varphi \cong P_\psi \Leftrightarrow (\exists f \in \mathcal{O}(\widetilde{M}, K) : \psi(\gamma) = f \cdot \varphi(\gamma) \cdot (\gamma \cdot f)^{-1} \text{ for all } \gamma \in \pi_1(M)).$$

Let $\text{Bun}_{\widetilde{M}}(M, K)$ denote the set of equivalence classes of holomorphic principal K -bundles whose pullback to \widetilde{M} is holomorphically trivial. Then the map

$$H^1(\pi_1(M), \mathcal{O}(\widetilde{M}, K)) \rightarrow \text{Bun}_{\widetilde{M}}(M, K), \quad [\varphi] \mapsto [P_\varphi]$$

is a bijection.

Proof. The proof can be adapted from [Ne04] and appears in [La07] Lemma 5. Let us recall here only its main features: The bundle P_φ is holomorphic, because the values of φ are holomorphic functions and K is a complex Lie group. Now

$$q_M^*P_\varphi = \{([m, k], m) \in P_\varphi \times \widetilde{M} \mid k \in K\},$$

and $m \mapsto ([m, 1], m)$ is a holomorphic global section. On the other hand, one shows that an arbitrary P may be identified with the set $q_M^*P / \pi_1(M)$ of $\pi_1(M)$ -orbits in q_M^*P . Furthermore, this kind of action by biholomorphisms on a trivial principal bundle is always given by a non-abelian 1-cocycle φ , and one has $P \cong P_\varphi$.

Then, an equivalence Φ of bundles $P_\varphi \cong P_\psi$ is necessarily of the form $\Phi(m, k) = (m, f(m)k)$ for some $f : \widetilde{M} \rightarrow K$. The compatibility with the $\pi_1(M)$ -action is equivalent to the equation $\psi(\gamma) = f \cdot \varphi(\gamma) \cdot (\gamma \cdot f)^{-1}$. But this equation just means that φ is sent to ψ acting on it by $f \in \mathcal{O}(\widetilde{M}, K)$, i.e. they belong to the same class in $H^1(\pi_1(M), \mathcal{O}(\widetilde{M}, K))$. ■

Remark 1.3 Suppose for the sake of this remark that K is connected and abelian. Note that the map $\varphi \mapsto P_\varphi$ appears in the spectral sequence arising in equivariant cohomology for the covering

$$\pi_1(M) \rightarrow \widetilde{M} \rightarrow M.$$

More precisely, the discrete group $H := \pi_1(M)$ gives rise to a classifying space BH and the total space EH of the universal H -bundle. The universal covering space \widetilde{M} can be described up to homotopy equivalence as $EH \times \widetilde{M}$ and M as $(EH \times \widetilde{M}) / H$. The fibration

$$EH \times \widetilde{M} \rightarrow (EH \times \widetilde{M}) / H$$

has typical fiber EH which is contractible, thus we obtain for the equivariant cohomology

$$H_H^*(\widetilde{M}, \underline{K}) = H^*((EH \times \widetilde{M}) / H, \underline{K}) = H^*(M, \underline{K}).$$

On the other hand, the fibration

$$(EH \times \widetilde{M}) / H \rightarrow EH / H = BH$$

gives rise to a spectral sequence in equivariant cohomology whose E_2 -term is $H^*(BH, H^*(\widetilde{M}, \underline{K}))$ and which converges to the equivariant cohomology of \widetilde{M} :

$$E_\infty^* = H_H^*(\widetilde{M}, \underline{K}) = H^*(M, \underline{K}).$$

For non-abelian K , these constructions only work for low-dimensional (Čech) cohomology groups. We therefore obtain in this framework maps

$$H^1(B(\pi_1(M)), \check{H}^0(\widetilde{M}, \underline{K})) = H^1(\pi_1(M), \mathcal{O}(\widetilde{M}, K)) \xrightarrow{\partial} \check{H}^1(M, \underline{K}) \xrightarrow{q_M^*} \check{H}^1(\widetilde{M}, \underline{K}).$$

Now translate $\check{H}^1(M, \underline{K}) = \text{Bun}(M, K)$ and $\ker(q_M^*) = \text{Bun}_{\widetilde{M}}(M, K)$. The upshot is that ∂ has values in $\ker(q_M^*)$ by exactness, and induces a bijection.

Corollary 1.4 *The bundle P_φ is trivial if and only if there exists a holomorphic function $f \in \mathcal{O}(\widetilde{M}, K)$ with*

$$\varphi(\gamma) = f \cdot (\gamma \cdot f)^{-1}.$$

For $\varphi \in \text{Hom}(\pi_1(M), K)$, this assertion is equivalent to the existence of an $\alpha \in Z_{\text{dR}}^1(M, \mathfrak{k})$ with $\text{per}_\alpha = \varphi$.

Proof. It remains to show that for $\varphi \in \text{Hom}(\pi_1(M), K)$, the existence of $f \in \mathcal{O}(\widetilde{M}, K)$ with $\varphi(\gamma) = f \cdot (\gamma \cdot f)^{-1}$ for all $\gamma \in \pi_1(M)$ is equivalent to the existence of $\alpha \in Z_{\text{dR}}^1(M, \mathfrak{k})$ with $\text{per}_\alpha = \varphi$. But the condition $\varphi(\gamma) = f \cdot (\gamma \cdot f)^{-1}$ may be rephrased as $f(\gamma \cdot m) = \varphi(\gamma)f(m)$ for all $m \in \widetilde{M}$, $\gamma \in \pi_1(M)$. By Theorem 1.1, as the universal cover is 1-connected, $q_M^*(\alpha)$ is a logarithmic derivative and the equation $f(\gamma \cdot m) = \varphi(\gamma)f(m)$ means that $\text{per}_\alpha = \varphi$. This argument works both ways. ■

Proposition 1.5 *A holomorphic K -bundle $q : P \rightarrow M$ is flat if and only if it is equivalent to a bundle of the form P_φ , where*

$$\varphi \in \text{Hom}(\pi_1(M), K) \subset Z^1(\pi_1(M), \mathcal{O}(\widetilde{M}, K)).$$

Here we identify K with the subgroup of constant functions in $\mathcal{O}(\widetilde{M}, K)$.

Proof. The idea is to pull the bundle P back to \widetilde{M} and to use that a flat bundle on a 1-connected manifold is trivial, see *loc. cit.* for example. The homomorphism φ is then constructed from the trivialization.

On the other hand, the trivial principal K -bundle on \widetilde{M} has the pullback of the Maurer-Cartan form on K as an invariant connection form. The associated connection form on M is the expected flat connection. ■

For the following proposition, denote by $H_{\text{dR}}^1(M, \mathfrak{k})$ the set of orbits of $Z_{\text{dR}}^1(M, \mathfrak{k})$ under the gauge action (1) of $\mathcal{O}(M, K)$ which has been introduced in the previous section. The proposition appears in the smooth context in [Ne04] and in the holomorphic context as Lemma 6 in [La07].

Proposition 1.6 *Let P be the map which assigns to $\alpha \in Z_{\text{dR}}^1(M, \mathfrak{k})$ its period homomorphism, and let $\text{Bun}_h^b(M, K)$ denote the set of equivalence classes of holomorphic flat principal K -bundles. Then there is an exact sequence of pointed sets*

$$1 \rightarrow K \rightarrow \mathcal{O}(M, K) \xrightarrow{\delta} Z_{\text{dR}}^1(M, \mathfrak{k}) \xrightarrow{P} \text{Hom}(\pi_1(M), K) \rightarrow \text{Bun}_h^b(M, K) \rightarrow *.$$

More precisely, exactness means here that the fibers of P are exactly the $\mathcal{O}(M, K)$ -orbits, and exactness in $\text{Hom}(\pi_1(M), K)$ means that the image of P consists of those homomorphisms for which the corresponding flat bundle is trivial.

The sequence induces an exact sequence in cohomology

$$0 \rightarrow H_{\text{dR}}^1(M, \mathfrak{k}) \xrightarrow{P} H^1(\pi_1(M), K) \rightarrow \text{Bun}_h^b(M, K) \rightarrow *.$$

Proof. Proposition 1.5 gives rise to the map $\text{Hom}(\pi_1(M), K) \rightarrow \text{Bun}_h^b(M, K)$. Exactness in $Z_{\text{dR}}^1(M, \mathfrak{k})$ can be seen as follows: Let $S : Z_{\text{dR}}^1(M, \mathfrak{k}) \rightarrow \mathcal{O}(\widetilde{M}, K)$ the map defined by $\delta(S(\alpha)) = q_M^* \alpha$ and $S(\alpha)(\widetilde{m}_0) = 1$ for some base point $\widetilde{m}_0 \in \widetilde{M}$ with $q_M(\widetilde{m}_0) = m_0$. If $\text{per}_\alpha^{m_0} = \text{per}_\beta^{m_0}$, then the function $S(\alpha)^{-1}S(\beta)$ is constant on the orbits of $\pi_1(M)$, hence of the form $q_M^* f$ for some $f \in \mathcal{O}(M, K)$ with $f(m_0) = 1$. By injectivity of S , one gets $\beta = \alpha * f$, i.e. α and β are in the same $\mathcal{O}(M, K)$ -orbit.

Exactness in $\text{Hom}(\pi_1(M), K)$ follows directly from Corollary 1.4. ■

Let Σ be a compact connected Riemann surface of genus g , and let p_1, \dots, p_k be k distinct points on Σ for $k \geq 1$. We will denote by Σ_k the open Riemann surface

$$\Sigma_k := \Sigma \setminus \{p_1, \dots, p_k\}.$$

Let us recall that Σ_k is homotopy equivalent to a bouquet of $2g + k - 1$ circles. The number $2g + k - 1$ will be denoted ℓ in the following. The first homotopy group $\pi_1(\Sigma_k)$ is thus the free group on ℓ generators. In the same way, the singular cohomology group $H^1(\Sigma_k, \mathbb{Z})$ is the free abelian group on ℓ generators (some explanation of this will be given in the next section).

Corollary 1.7 For $M = \Sigma_k$ and connected K , the monodromy map P induces a bijection

$$Z_{\text{dR}}^1(\Sigma_k, \mathfrak{k}) / \mathcal{O}(\Sigma_k, K) =: H_{\text{dR}}^1(\Sigma_k, \mathfrak{k}) = K^\ell / K,$$

where ℓ is the first Betti number of Σ_k and the action of $\mathcal{O}(\Sigma_k, K)$ on $Z_{\text{dR}}^1(\Sigma_k, \mathfrak{k})$ is the gauge action (1) and the action of K on K^ℓ is the conjugation in each argument.

Proof. Indeed, for $M = \Sigma_k$, the above exact sequence reads

$$0 \rightarrow H_{\text{dR}}^1(\Sigma_k, \mathfrak{k}) \xrightarrow{P} H^1(\pi_1(\Sigma_k), K) \rightarrow \text{Bun}_h^b(\Sigma_k, K) \rightarrow 0.$$

Now on the one hand, $\pi_1(\Sigma_k)$ is the free group $F(\ell)$ on ℓ generators, where ℓ is the first Betti number of Σ_k , and we have therefore

$$H^1(\pi_1(\Sigma_k), K) = \text{Hom}(F(\ell), K) / K = K^\ell / K.$$

where the action of K on K^ℓ is by conjugation in each argument.

On the other hand, exactness in $\text{Hom}(\pi_1(M), K)$ means that the image of P consists of those homomorphisms for which the corresponding flat bundle is trivial. But in our case, all holomorphic bundles are trivial, as it is shown in the following proposition. This shows our claim. ■

Proposition 1.8 Let K be a connected complex Banach Lie group. Then any holomorphic principal K -bundle on Σ_k is holomorphically trivial.

Proof. Let us first show that any K -principal bundle P on Σ_k is topologically trivial. The question is invariant under homotopy equivalence, so we may assume that the base of the bundle is a bouquet of circles $\bigvee_{j=1}^{\ell} \mathbb{S}^1$. Now we apply homotopy theory:

$$\text{Bun}(\mathbb{S}^1, K) = [\mathbb{S}^1, BK] = \pi_1(BK) / \text{conjugation} = \pi_0(K) / \text{conjugation},$$

the last space clearly being a one-point space, because K is connected. The bundle is thus trivial over each circle in the bouquet; the trivializations may be glued together in the distinguished point to trivialize the bundle on the whole $\bigvee_{j=1}^{\ell} \mathbb{S}^1$.

On the other hand, by [Rae77] (see the details in [Ram65] and our Theorem 2.3), any holomorphic K -principal bundle on a Stein manifold is holomorphically trivial if and only if it is topologically trivial. ■

Remark 1.9 The first step of the above proof is actually the first step in the theory of describing the obstructions to lifting a map from a (relative) CW complex (X, A) to some target space by lifting it step by step from the k skeleton of X to the $k + 1$ skeleton. It has been set up by Eilenberg in 1940, and a good reference for this is [Wh78] Ch. V.5 p. 229.

Remark 1.10 In [On67] (see also [La07], Lemma 7) it is stated that for a general complex manifold M and a general complex Lie group K , the period map P in the above sequence is surjective if and only if one of the following conditions is holds:

- $\pi_1(M)$ is a free group and K is connected,
- $\pi_1(M)$ is a free abelian group and K is a compact connected group whose cohomology is torsion free.

In particular, under these conditions every flat holomorphic principal K -bundle is holomorphically trivial.

2 The Lie group structure on the central extension

In this section, we recall topological and geometrical preliminaries about the holomorphic current group $\mathcal{O}(\Sigma_k, K)$.

2.1 The Lie group $\mathcal{O}(\Sigma_k, K)$

Let Σ be a compact connected Riemann surface of genus g , let p_1, \dots, p_k be k distinct points on Σ for $k \geq 1$ and let $\Sigma_k = \Sigma \setminus \{p_1, \dots, p_k\}$.

The open Riemann surface Σ_k is homotopy equivalent to a bouquet of $2g + k - 1$ circles. This follows, for example, from [Ha83]. Intuitively speaking, the usual model for Σ which proceeds by gluing the edges of a $4g$ -gon leads to a bouquet of circles, because taking out (at least) one point destroys the 2-cell. The number $2g + k - 1$ is denoted ℓ .

The first homotopy group $\pi_1(\Sigma_k)$ is thus the free group on ℓ generators. Let us denote these generators (which we fix once and for all) $\alpha_1, \dots, \alpha_{\ell} \in \mathcal{C}(\mathbb{S}^1, \Sigma_k)$. In the same way (for example, using either the knowledge of π_1 or a Mayer-Vietoris sequence), the singular cohomology group $H^1(\Sigma_k, \mathbb{Z})$ is the free abelian group on ℓ generators.

In view of Huber's Theorem [Hu61] and the local contractibility of Σ_k , the group $H^1(\Sigma_k, \mathbb{Z})$ is isomorphic to $\check{H}^1(\Sigma_k, \mathbb{Z}) \cong [\Sigma_k, \mathbb{S}^1]$, the set of homotopy classes of continuous maps from Σ_k to \mathbb{S}^1 . In particular, there exist continuous maps $f_1, \dots, f_\ell : \Sigma_k \rightarrow \mathbb{S}^1$ such that $[f_j \circ \alpha_i] = \delta_{ij} \in \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. As every homotopy class $[\Sigma_k, \mathbb{S}^1]$ contains a smooth function (which may be seen by using a homotopy equivalence of Σ_k with a compact surface with boundary), we may assume in the following that the f_j are smooth. As the 2-pointed Riemann sphere $\mathbb{S}^2 \setminus \{0, \infty\} = \mathbb{C}^\times$ is homotopy equivalent to the circle \mathbb{S}^1 ,

$$[\Sigma_k, \mathbb{S}^1] = [\Sigma_k, \mathbb{C}^\times],$$

and each homotopy class contains also a holomorphic function (see [Rae77] and Theorem 2.3). In this sense, we may also assume the functions f_j to be holomorphic.

Let Y be a sequentially complete locally convex space. Using the logarithmic derivatives $\delta(f_j)$, seen here as usual closed holomorphic 1-forms on Σ_k , one obtains an isomorphism

$$\Phi : H_{\text{dR}}^1(\Sigma_k, Y) \rightarrow Y^\ell, \quad [\beta] \mapsto \left(\int_{\alpha_j} \beta \right)_{j=1, \dots, \ell},$$

with explicit inverse $\Phi^{-1}(y_1, \dots, y_\ell) = \left[\sum_{j=1}^{\ell} \delta(f_j) y_j \right]$.

In the same way as the compact differentiable manifold Σ may also be regarded as a complete projective algebraic curve over \mathbb{C} , Σ_k may be regarded as an affine algebraic curve over \mathbb{C} . Therefore Σ_k carries a natural structure of an algebraic variety and of a complex manifold, and it makes sense to associate to Σ_k its structure sheaves of regular functions Reg_{Σ_k} and of holomorphic functions \mathcal{O}_{Σ_k} , and their vector spaces of global sections $\text{Reg}_{\Sigma_k}(\Sigma_k)$ and $\mathcal{O}_{\Sigma_k}(\Sigma_k)$, or $\text{Reg}(\Sigma_k)$ and $\mathcal{O}(\Sigma_k)$ in short hand notation. Both of them are in general infinite dimensional vector spaces over \mathbb{C} .

Let K be a connected complex Banach Lie group, and \mathfrak{k} be its Lie algebra. Most of the time, we will suppose that K is 1-connected, finite-dimensional and that \mathfrak{k} is *simple*, i.e. does not possess a non-trivial proper ideal.

Definition 2.1 The holomorphic current group $\mathcal{O}(\Sigma_k, K)$ is the topological group of holomorphic maps on Σ_k with values in K . The group structure is given by the pointwise multiplication. The topology is given by the compact open topology on the mapping space, or equivalently by the topology of uniform convergence on compact subsets. We will consider the holomorphic current algebra $\mathcal{O}(\Sigma_k, \mathfrak{k})$ as the Lie algebra of $\mathcal{O}(\Sigma_k, K)$. The bracket on $\mathcal{O}(\Sigma_k, \mathfrak{k})$ is given pointwise. $\mathcal{O}(\Sigma_k, K)$ together with the compact open topology is a Fréchet-Lie algebra.

In fact, $\mathcal{O}(\Sigma_k, K)$ is a Lie group, i.e. carries a Fréchet manifold structure such that multiplication and inversion maps are smooth. This is the content of the following theorem which is Theorem 3.12 in [NeWa08b]. It is also shown in *loc. cit.* that $\mathcal{O}(\Sigma_k, K)$ is a *regular Lie group*, i.e. that the evolution equation (see Section 1) always has solutions, i.e. the evolution map is well defined.

Observe also that $\mathcal{O}(\Sigma_k, K)$ carries an exponential map

$$\exp : \mathcal{O}(\Sigma_k, \mathfrak{k}) \rightarrow \mathcal{O}(\Sigma_k, K),$$

which is just given by the concatenation with the finite dimensional exponential map $\exp_K : \mathfrak{k} \rightarrow K$.

Let now K be a connected affine algebraic group over \mathbb{C} . It seems reasonable to expect that $\text{Reg}(\Sigma_k, K)$ also carries a structure of an infinite-dimensional algebraic group in case K is an algebraic group. One

approach in this direction would describe it as a group valued functor $\mathbf{rings} \rightarrow \mathbf{groups}$, sending a ring R to the group $\text{Reg}(\Sigma_k(R), K)$ of regular maps from the R -points of the affine algebraic curve Σ_k to the algebraic group K , but no results in this direction are known to us.

Theorem 2.2 *Let M be a non-compact connected complex curve without boundary. Assume further that $\pi_1(M)$ is finitely generated and that K is a complex Banach-Lie group. Then the group $\mathcal{O}_*(M, K)$ carries a regular complex Lie group structure for which*

$$\delta : \mathcal{O}_*(M, K) \rightarrow \Omega^1(M, \mathfrak{k})$$

is biholomorphic onto a complex submanifold, and $\mathcal{O}(M, K) \cong K \times \mathcal{O}_(M, K)$ carries a regular complex Lie group structure compatible with evaluations and the compact open topology.*

Of course, we apply this theorem here with $M = \Sigma_k$.

The manifold structure on the space of pointed maps $\mathcal{O}_*(M, K)$ is obtained using an infinite dimensional version of the regular value theorem (using a parametrized version of the implicit function theorem of Glöckner [Gl06]). More precisely, the image of the logarithmic derivative

$$\delta : \mathcal{O}_*(M, K) \rightarrow \Omega^1(M, \mathfrak{k})$$

in the Fréchet space of holomorphic 1-forms $\Omega^1(M, \mathfrak{k})$ with values in \mathfrak{k} is characterized as the inverse image of the trivial monodromy homomorphism under the period map, and is thus seen to be a split submanifold.

2.2 Topology of the Lie group $\mathcal{O}(\Sigma_k, K)$

Here we compute the homotopy groups of the Lie group $\mathcal{O}(\Sigma_k, K)$. It is well-known from Sullivan's rational homotopy theory that the rational cohomology algebra of the H-space $\mathcal{O}(\Sigma_k, K)$ has as its generators the duals of the generators of the $\pi_i(\mathcal{O}(\Sigma_k, K)) \otimes \mathbb{Q}$, thus these computations determine in particular the rational cohomology algebra.

All our computations are based on the Oka principle (see [Rae77], with details of the proof from [Ram65]):

Theorem 2.3 *Let M be a Stein manifold and K be a connected complex Banach Lie group. Then*

- (a) *each continuous map $f : M \rightarrow K$ is homotopic to a holomorphic map;*
- (b) *if two holomorphic maps $f_0, f_1 : M \rightarrow K$ are homotopic within continuous maps (i.e. in $\mathcal{C}^0(M, K)$), then they are homotopic within holomorphic maps (i.e. in $\mathcal{O}(\Sigma_k, K)$).*

Corollary 2.4 *Let M be a Stein manifold and K be a connected complex Banach Lie group. Then $\mathcal{O}(\Sigma_k, K)$ has the same homotopy type as $\mathcal{C}^0(M, K)$.*

As before, we also use for the computation of $\pi_i(\mathcal{O}(\Sigma_k, K))$ that Σ_k is homotopy equivalent to a bouquet of $\ell = 2g + k - 1$ circles \mathbb{S}^1 . The outcome is then:

Proposition 2.5 *Let K be a connected complex Banach Lie group. Then for $i \geq 0$,*

$$\pi_i(\mathcal{O}(\Sigma_k, K)) = \pi_i(K) \oplus \pi_{i+1}(K)^\ell.$$

Proof. The group $\mathcal{O}(\Sigma_k, K)$ has the same homotopy type as $\mathcal{C}^0(\Sigma_k, K)$. The topological group $\mathcal{C}^0(\Sigma_k, K)$ splits as a semi-direct product

$$\mathcal{C}^0(\Sigma_k, K) = \mathcal{C}_*^0(\Sigma_k, K) \rtimes K,$$

where $\mathcal{C}_*^0(\Sigma_k, K)$ denotes the space of continuous maps $f : \Sigma_k \rightarrow K$ such that $f(*) = 1$ for the base point $*$ of Σ_k .

Now compute (with S being the suspension and $[\cdot, \cdot]$ the based homotopy classes)

$$\begin{aligned} \pi_i(\mathcal{C}_*^0(\Sigma_k, K)) &= [S^i(\bigvee_{k=1}^\ell \mathbb{S}^1), K] \\ &= [\mathbb{S}^{i+1} \vee \dots \vee \mathbb{S}^{i+1}, K] \\ &= \pi_{i+1}(K)^\ell. \end{aligned}$$

This ends the proof. ■

Corollary 2.6 *Let M be a Stein manifold and K be a connected complex Banach Lie group. Then $\mathcal{O}(\Sigma_k, K)$ is 1-connected if and only if K is 2-connected. This is the case, for example, if K is finite dimensional and 1-connected.*

2.3 Central extensions of current algebras

Here we recall the second cohomology of current algebras. The earliest reference is usually attributed to Bloch [Bl80] and Kassel-Loday [KaLo82]. While these references deal with homology, for cohomology, and especially continuous cohomology, we refer to [NeWa08a].

Let A be a unital commutative associative algebra over a field \mathbb{K} of characteristic zero and \mathfrak{k} a \mathbb{K} -Lie algebra. The tensor product $\mathfrak{g} := A \otimes \mathfrak{k}$ is a Lie algebra when endowed with the *current bracket*

$$[a \otimes x, b \otimes y] = ab \otimes [x, y].$$

Let \mathfrak{g}' denote the derived algebra of \mathfrak{g} , i.e. the commutator ideal. The goal of this subsection is to present the second cohomology space $H^2(\mathfrak{g})$ in terms of data associated to A and \mathfrak{k} .

For this purpose, let $\Omega^1(A)$ be the *module of Kähler differentials* associated to A . It may be defined by its universal property, namely for the A -module $\Omega^1(A)$, there is a derivation $d_A : A \rightarrow \Omega^1(A)$ such that for any other A -module M and any other derivation $D : A \rightarrow M$, there is a unique morphism of A -modules $\alpha : \Omega^1(A) \rightarrow M$ such that $D = \alpha \circ d_A$. There are several explicit constructions of $\Omega^1(A)$, and we refer to [NeWa08a] for the most commonly known. For a Fréchet algebra A , there is a version of $\Omega^1(A)$ which takes the topology into account and is a Fréchet A -module, see *loc. cit.* for example.

For the Lie algebra \mathfrak{k} , we have the usual cycle, boundary and cohomology spaces $Z^n(\mathfrak{k})$, $B^n(\mathfrak{k})$ and $H^n(\mathfrak{k}) = Z^n(\mathfrak{k}) / B^n(\mathfrak{k})$, all with trivial coefficients. Recall further from [NeWa08a] that there is a map, called the *Cartan map*

$$\Gamma : \text{Sym}(\mathfrak{k})^\mathfrak{k} \rightarrow Z^3(\mathfrak{k}), \quad \Gamma(\kappa)(x, y, z) := \kappa([x, y], z),$$

linking the space $\text{Sym}(\mathfrak{k})^\mathfrak{k}$ of \mathfrak{k} -invariant symmetric bilinear forms to the space of 3-cocycles. Denote by $Z^3(\mathfrak{k})_\Gamma$ and $B^3(\mathfrak{k})_\Gamma$ the intersections $Z^3(\mathfrak{k})_\Gamma := Z^3(\mathfrak{k}) \cap \text{im}(\Gamma)$ and $B^3(\mathfrak{k})_\Gamma := B^3(\mathfrak{k}) \cap \text{im}(\Gamma)$ with the image of Γ .

One of the main results of [NeWa08a] is the following Theorem (Theorem 4.2):

Theorem 2.7 *The sequence*

$$0 \rightarrow H^2(\mathfrak{g}/\mathfrak{g}') \oplus (A \otimes H^2(\mathfrak{k})) \longrightarrow H^2(\mathfrak{g}) \longrightarrow \text{Lin}((\Omega^1(A), d_A(A)), (Z^3(\mathfrak{k})_\Gamma, B^3(\mathfrak{k})_\Gamma)) \rightarrow 0$$

is exact.

In the special case where \mathfrak{k} is a finite dimensional complex simple Lie algebra, the space $\text{Sym}(\mathfrak{k})^\mathfrak{k}$ is 1-dimensional, $\mathfrak{k}' = \mathfrak{k}$ and $H^2(\mathfrak{k}) = 0$, therefore we get

$$H^2(\mathfrak{g}) = (\Omega^1(A) / d_A(A))^*,$$

cf [Bl80] and [KaLo82].

The generating cocycle for the universal central extension is described as follows. Let κ be the Cartan-Killing form on \mathfrak{k} . Then the 2-cocycle ω_κ is defined by

$$\omega_\kappa(x, y) = [\kappa(x, d_A y)],$$

where $[\cdot]$ denotes the equivalence class in $\Omega^1(A) / d_A(A)$.

We will apply this to the case where the associative algebra A is the Fréchet algebra of holomorphic functions $\mathcal{O}(\Sigma_k)$ on the punctured Riemann surface Σ_k . In this case, the Fréchet space of Kähler differentials can be identified with the Fréchet space of differential 1-forms. This is the content of the main theorem in [NeWa03] (Theorem 2.1):

Theorem 2.8 *Let X be a Stein manifold. Then the map*

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \rightarrow \Omega^1(X),$$

induced by the universal property, is an isomorphism of Fréchet $\mathcal{O}(X)$ -modules.

Let \mathfrak{k} be a finite dimensional simple complex Lie algebra. The main object of the present article is the central extension $\widehat{\mathcal{O}(\Sigma_k, \mathfrak{k})}$ of $\mathcal{O}(\Sigma_k, \mathfrak{k})$ by means of the cocycle ω_κ . In the next subsection, we will describe how this central extension is integrated into a Fréchet-Lie group $\widehat{\mathcal{O}(\Sigma_k, K)}$, central extension of the Lie group $\mathcal{O}(\Sigma_k, K)$.

2.4 The Lie group structure on the central extension

In this subsection, we will adapt the Reduction Theorem of [MaNe03] to our holomorphic framework. This will imply that the Lie group structure on $\mathcal{O}(\Sigma_k, K)$ which was described in subsection 2.1 extends to the central extension.

Let M be a complex manifold, K be a complex Lie group and Y be a sequentially complete locally convex complex space. We will work with the topological group $G := \mathcal{O}(M, K)$ of holomorphic maps from M to K and its Lie algebra $\mathfrak{g} := \mathcal{O}(M, \mathfrak{k})$. Let $\kappa : \mathfrak{k} \times \mathfrak{k} \rightarrow Y$ be a continuous invariant symmetric bilinear Y -valued form on \mathfrak{k} . The cocycles which we consider on \mathfrak{g} are of the form

$$\omega_M(\xi, \eta) := [\kappa(\xi, d\eta)] \in \Omega^1(M, Y) / d\mathcal{O}(M, Y), \quad (2)$$

where $\xi, \eta \in \mathfrak{g}$, and $[\cdot]$ denotes the equivalence class in the quotient space

$$\mathfrak{z}_M(Y) := \Omega^1(M, Y) / d\mathcal{O}(M, Y).$$

A typical example for κ is the Cartan-Killing form on a simple complex Lie algebra \mathfrak{k} , where $Y = \mathbb{C}$. In the discussion that follows, cocycles of the same kind which are defined on mapping spaces (of smooth maps) with domain the circle \mathbb{S}^1 will play a role. We will denote them accordingly $\omega_{\mathbb{S}^1}$.

For the following lemma, observe that the usual loop group $\mathcal{C}^\infty(\mathbb{S}^1, K)$ is a *complex* Fréchet-Lie group in our case, because K is a complex Lie group. Recall that a map $f : X \rightarrow Y$ between complex Fréchet manifolds is called *holomorphic* in case f is smooth and its differentials are complex linear.

Lemma 2.9 *Let $\alpha : \mathbb{S}^1 \rightarrow M$ be smooth.*

The group homomorphism $\alpha_K : G \rightarrow \mathcal{C}^\infty(\mathbb{S}^1, K)$ defined by $f \mapsto f \circ \alpha$ is a holomorphic homomorphism of complex Lie groups, where G carries the Fréchet-Lie group structure described in Section 2.1 and $\mathcal{C}^\infty(\mathbb{S}^1, K)$ carries the usual (loop group) Fréchet-Lie group structure.

Proof. Observe that α_K is the restriction map, i.e. the restriction $f|_{\alpha(\mathbb{S}^1)}$ of holomorphic functions $f \in G = \mathcal{O}(M, K)$ to the “circle” $\alpha(\mathbb{S}^1)$. As such, it is first of all continuous, because if $f_i \rightarrow f$ in the compact-open topology in G , then $(f_i)^{[n]} \rightarrow f^{[n]}$ in the smooth compact-open topology. But then this is also true for all restrictions $(f_i)^{[n]}|_{\alpha(\mathbb{S}^1)} \rightarrow f^{[n]}|_{\alpha(\mathbb{S}^1)}$, and therefore α_K is continuous.

Let us prove that α_K is smooth. This actually follows from the same restriction argument as above, because the tangent map

$$T\alpha_K : T\mathcal{O}(M, K) \rightarrow T\mathcal{C}^\infty(\mathbb{S}^1, K)$$

can be identified with a two component restriction map under the isomorphism

$$T\mathcal{O}(M, K) \cong \mathcal{O}(M, K) \times \mathcal{O}(M, \mathfrak{k})$$

and in the same way

$$T\mathcal{C}^\infty(\mathbb{S}^1, K) \cong \mathcal{C}^\infty(\mathbb{S}^1, K) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathfrak{k}).$$

Iteration of this argument implies that α_K is smooth.

Now a map $f : X \rightarrow Y$ between complex Fréchet manifolds is holomorphic in case f is smooth and its differentials are complex linear. As the restriction map is complex linear on the tangent spaces, our map α_K is therefore holomorphic. ■

Remark 2.10 For $M = \Sigma_k$ and a smooth map $\alpha : \mathbb{S}^1 \rightarrow M$ whose image contains an interval which is not reduced to a point, the Lie group homomorphism $\alpha_K : G \rightarrow \mathcal{C}^\infty(\mathbb{S}^1, K)$ is injective. Indeed, in case the restriction of two holomorphic functions to the set $\alpha(\mathbb{S}^1)$ coincide, the two functions coincide, because the set $\alpha(\mathbb{S}^1)$ contains an accumulation point.

By definition, ω_M is a (Y -valued) 2-form on the tangent space of $G = \mathcal{O}(M, K)$ at $1 \in G$, and may thus be extended to a unique left-invariant (Y -valued) 2-form Ω_M on G such that $\Omega_M(1) = \omega_M$.

Choosing in a homotopy class a smooth representative permits to define a *period homomorphism*

$$\text{per}_{\omega_M} : \pi_2(G) \rightarrow \mathfrak{z}_M(Y), \quad \text{per}_{\omega_M}([\sigma]) := \int_{\sigma} \Omega_M.$$

Observe that the homotopy group $\pi_2(G)$ of the infinite dimensional Lie group G is not necessarily zero. In order to extend a given Lie group structure on $\mathcal{O}(M, K)$ to the central extension given by the cocycle ω_M , it is necessary and sufficient that the *period group* $\Pi_{M, \kappa} := \text{im}(\text{per}_{\omega_M})$, i.e. the image of the period homomorphism per_{ω_M} , is a discrete subspace in $\mathfrak{z} := \mathfrak{z}_M(Y)$, see [Ne02]. If this condition is fulfilled, the central extension is then an extension of $\mathcal{O}(M, K)$ by the abelian group $\mathfrak{z} / \Pi_{M, \kappa}$.

We now adapt the discussion of Section I in [MaNe03] to our context. For $\alpha \in \mathcal{C}^\infty(\mathbb{S}^1, M)$, denote by $\alpha_{\mathfrak{z}} : \mathfrak{z} \rightarrow \mathbb{C}$ the integration over $\alpha(\mathbb{S}^1)$.

Lemma 2.11 For each $\alpha \in \mathcal{C}^\infty(\mathbb{S}^1, M)$, we have

$$\alpha_{\mathfrak{z}} \circ \text{per}_{\omega_M} = \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K),$$

where $\pi_2(\alpha_K) : \pi_2(G) \rightarrow \pi_2(\mathcal{C}^\infty(\mathbb{S}^1, K))$ is the group homomorphism induced by the Lie group homomorphism $\alpha_K : G \rightarrow \mathcal{C}^\infty(\mathbb{S}^1, K)$.

Proof. The proof of Lemma I.7 in [MaNe03] is easily adapted: For $\alpha \in \mathcal{C}^\infty(\mathbb{S}^1, M)$, the map $\alpha_{\mathfrak{z}} : \mathfrak{z} \rightarrow \mathbb{C}$ denotes the integration over $\alpha(\mathbb{S}^1)$. Then $\alpha_{\mathfrak{z}} \circ \Omega_M$ is a left-invariant 2-form on G whose value at $1 \in G$ is $\alpha_{\mathfrak{z}} \circ \omega_M$. Further $\alpha_K^* \Omega_{\mathbb{S}^1}$ is a left-invariant 2-form on G whose value at $1 \in G$ is given by

$$\begin{aligned} (\xi, \eta) \mapsto \omega_{\mathbb{S}^1}(\xi \circ \alpha, \eta \circ \alpha) &= [\kappa(\xi \circ \alpha, d(\eta \circ \alpha))] \\ &= [\kappa(\alpha^* \xi, \alpha^* d\eta)] = \int_{\mathbb{S}^1} \kappa(\alpha^* \xi, \alpha^* d\eta) \\ &= \int_{\alpha(\mathbb{S}^1)} \kappa(\xi, d\eta) = \alpha_{\mathfrak{z}}(\omega_M(\xi, \eta)). \end{aligned}$$

This implies $\alpha_{\mathfrak{z}} \circ \Omega_M = \alpha_K^* \Omega_{\mathbb{S}^1}$, which shows the lemma. \blacksquare

Let us now restrict to Σ_k . Following the reasoning of Section I in [MaNe03], Corollary I.9 remains true for Σ_k , because holomorphic 1-forms are closed. Lemma I.10 does not involve the holomorphic context, thus we get:

Theorem 2.12 (Reduction Theorem) Identifying $H_{\text{dR}}^1(\Sigma_k, Y)$ with Y^ℓ via the isomorphism Φ , where ℓ is the first Betti number of Σ_k , we have

$$\Pi_{\Sigma_k, \kappa} \cong \Pi_{\mathbb{S}^1, \kappa}^\ell \subset Y^\ell \cong H_{\text{dR}}^1(\Sigma_k, Y) = \mathfrak{z}_{\Sigma_k}(Y).$$

In particular, $\Pi_{\Sigma_k, \kappa}$ is discrete if and only if $\Pi_{\mathbb{S}^1, \kappa}$ is discrete.

Proof. To each $f \in \mathcal{C}^\infty(\Sigma_k, \mathbb{S}^1)$, one associates the map $f_K : \mathcal{C}^\infty(\mathbb{S}^1, K) \rightarrow G = \mathcal{O}(\Sigma_k, K)$, $\eta \mapsto \eta \circ f$, which in turn induces a map $\pi_2(f_K) : \pi_2(\mathcal{C}^\infty(\mathbb{S}^1, K)) \rightarrow \pi_2(G)$. For $\alpha \in \mathcal{C}^\infty(\mathbb{S}^1, \Sigma_k)$, we obtain with Lemma I.10 of [MaNe03]

$$\begin{aligned} \alpha_{\mathfrak{z}} \circ \text{per}_{\omega_{\Sigma_k}} \circ \pi_2(f_K) &= \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K) \circ \pi_2(f_K) = \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K \circ f_K) \\ &= \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2((f \circ \alpha)_K) = \text{deg}(f \circ \alpha) \text{per}_{\omega_{\mathbb{S}^1}} \end{aligned}$$

For $f = f_j$ and $\alpha = \alpha_i$, it follows in particular that $\alpha_{i, \mathfrak{z}} \circ \text{per}_{\omega_{\Sigma_k}} \circ \pi_2(f_{j, K}) = \delta_{ij} \text{per}_{\omega_{\mathbb{S}^1}}$. Here $\alpha_{i, \mathfrak{z}}$ may be seen as the projection onto the i -th factor in

$$Y^\ell \cong H_{\text{dR}}^1(\Sigma_k, Y) = \bigoplus_{j=1}^{\ell} [\delta(f_j)] Y.$$

Hence

$$\text{per}_{\omega_{\Sigma_k}}(\text{im } \pi_2(f_{j, K})) = [\delta(f_j)] \Pi_{\mathbb{S}^1}$$

and further $\Pi_{\Sigma_k} \supset \sum_{j=1}^{\ell} [\delta_{ij}] \Pi_{\mathbb{S}^1} = \Pi_{\mathbb{S}^1}^\ell$.

For the reverse inclusion, we observe that $\alpha_{j, \mathfrak{z}} \circ \text{per}_{\omega_{\Sigma_k}} = \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K)$ implies that for each j , we have $\alpha_{j, \mathfrak{z}} \circ \text{per}_{\omega_{\Sigma_k}} \subset \Pi_{\mathbb{S}^1}$, and therefore $\Pi_{\Sigma_k} \subset \Pi_{\mathbb{S}^1}^\ell$. \blacksquare

Corollary 2.13 *Suppose now that the complex Lie group K is 1-connected and finite dimensional. Then $\Pi_{\Sigma_k, \kappa}$ is discrete, and the Fréchet-Lie group structure on $\mathcal{O}(\Sigma_k, K)$ lifts to a Lie group structure on the central extension.*

Proof. Indeed, it is shown in Theorem II.9 in [MaNe03] that $\Pi_{\mathbb{S}^1, \kappa}$ is discrete. For the second part of the corollary, observe that under the above hypotheses, $\mathcal{O}(\Sigma_k, K)$ is 1-connected by Corollary 2.6. It is explained in Section 6 of [Ne02] how to construct out of a Lie algebra 2-cocycle on the Lie algebra of a 1-connected Lie group a locally smooth group 2-cocycle with values in $Z := \mathfrak{z}_{\Sigma_k}(Y)/\Pi_{\Sigma_k, \kappa}$. The corresponding central extension is then a Lie group, as $\mathcal{O}(\Sigma_k, K)$ and the quotient Z are Lie groups. ■

As stated before, we will denote by $\widehat{\mathcal{O}(\Sigma_k, K)}$ the central group extension thus obtained (using the cocycle (2)). The Lie group $\widehat{\mathcal{O}(\Sigma_k, K)}$ is a regular Fréchet-Lie group.

Remark 2.14 (a) We believe that the same reasoning shows that Lie group structures on $\mathcal{O}(M, K)$ for higher dimensional M extend to central extensions by cocycles of the above form. It would be interesting to extract from the discussion in Section 4 in [NeWa08b] an interesting example of a Lie group $\mathcal{O}(M, K)$ with M of dimension greater than 1 (for example with K solvable), and from [NeWa08a] a class of cocycles on its Lie algebra $\mathcal{O}(M, \mathfrak{k})$, where a higher dimensional complex version of the Reduction Theorem applies.

(b) It is probably easy to adapt the results of Section IV of [MaNe03] to show that the central extension $\widehat{\mathcal{O}(\Sigma_k, K)}$ of $\mathcal{O}(\Sigma_k, K)$ via the universal cocycle (2) is in fact the universal central extension of $\mathcal{O}(\Sigma_k, K)$.

3 Automorphism group of $\widehat{\mathcal{O}(\Sigma_k, K)}$

In this section, we study the automorphisms of $\widehat{\mathcal{O}(\Sigma_k, \mathfrak{k})}$ and $\widehat{\mathcal{O}(\Sigma_k, K)}$. The discussion uses key results from [MaNe03] Section V.

Let us first of all recall the automorphism group of the open Riemann surface $\Sigma_k = \Sigma \setminus \{p_1, \dots, p_k\}$, obtained from a compact connected Riemann surface Σ by extracting $k \geq 1$ distinct points p_1, \dots, p_k .

Proposition 3.1 *$\text{Aut}(\Sigma_k)$ is a subgroup of $\text{Aut}(\Sigma)$, namely the subgroup of automorphisms fixing all the points p_1, \dots, p_k .*

Proof. Let $\varphi : \Sigma_k \rightarrow \Sigma_k$ be an automorphism, i.e. a biholomorphic bijective map. Let us show that φ extends to an automorphism $\widehat{\varphi} : \Sigma \rightarrow \Sigma$. It suffices to consider one extracted point $p \in \Sigma$, i.e. $k = 1$. Denote by D^* a punctured disc, centered in $p = 0$, around p . Also, denote by D_ϵ^* a punctured disc of radius ϵ .

The compact Riemann surface Σ , viewed as a projective curve, may be embedded into some projective space \mathbb{P}^n using $i : \Sigma \hookrightarrow \mathbb{P}^n$. Take a minimal embedding, i.e. n minimal. Denote by $[z_0 : \dots : z_n]$ the projective coordinates on \mathbb{P}^n , and observe that $u_{ij} = \frac{z_i}{z_j} : \mathbb{P}^n \rightarrow \mathbb{P}^1$ gives a regular map.

Composing φ with u_{ij} , φ gives rise to a map $\varphi_{ij} : D^* \rightarrow \mathbb{P}^1$ with a possible singularity in the point $p = 0$. We have to show that $p = 0$ is non singular for φ_{ij} .

Let us first note that φ_{ij} cannot have an essential singularity in $p = 0$. Indeed, if a holomorphic function $f : D^* \rightarrow \mathbb{C}$ has an essential singularity in 0, then for all $\epsilon > 0$, $f(D_\epsilon^*) \subset \mathbb{C}$ is dense. (If this was not the case, let $c \in \mathbb{C}$ be not hit by f . Then

$$g(z) := \frac{1}{f(z) - c}$$

is bounded on D_ϵ^* , and therefore $f(z) = \frac{1}{g(z)} + c$ meromorphic, which is a contradiction.) This density is incompatible with the bijective character of φ .

Now consider φ as the composition

$$D^* \xrightarrow{\varphi} \Sigma \hookrightarrow \mathbb{P}^n.$$

Composed with u_{ij} we obtain a finite covering

$$v_{ij} := u_{ij} \circ \varphi : D^* \rightarrow \mathbb{P}^1.$$

Let $\text{ord}_0(v_{10}), \dots, \text{ord}_0(v_{n0})$ be the orders (of poles or zeros) of the function v_{i0} in $p = 0 \in D^*$. By what we have already shown, there exists i such that $\text{ord}_0(v_{i0})$ is maximal. Define then $w_{ij} := \frac{v_{i0}}{v_{j0}}$. We have by construction $\text{ord}_0(w_{ij}) \geq 0$ for all i, j . Therefore the w_{ij} give affine coordinates (given by holomorphic functions on D^* without pole in 0) such that φ factors as

$$D^* \rightarrow U \hookrightarrow \mathbb{P}^n$$

for some affine open set U . This shows that φ_{ij} does not have a pole in $p = 0$ and can thus be extended to a holomorphic function on the entire disc D .

It is clear that this extension φ has to fix the points p_i one by one. ■

Corollary 3.2 *For genus $g > 1$ and $g = 1$, the automorphism group $\text{Aut}(\Sigma_k)$ is finite. For $g = 0$, $\text{Aut}(\Sigma_k)$ is finite as soon as $k > 2$.*

Proof. It is the content of Theorem 2.5, p. 88, in [Ko72] that $\text{Aut}(\Sigma)$ is a finite group for genus $g \geq 2$. As the automorphisms in $\text{Aut}(\Sigma_k)$ must fix the distinguished points $p_1, \dots, p_k \in \Sigma$ and $k \geq 1$, the automorphism group of Σ_k in genus $g = 1$ must be finite, because $\text{Aut}(\Sigma) \cong \mathbb{C}$ for $g = 1$. For genus $g = 0$, $\text{Aut}(\Sigma) \cong \text{PSL}(2, \mathbb{C})$ is 3 dimensional and an element of $\text{PSL}(2, \mathbb{C})$ fixing 3 points must be the identity. ■

Proposition 3.3 *Let \mathfrak{k} be a finite dimensional, semi-simple complex Lie algebra and M be a complex Stein space.*

Then

$$\text{Aut}(\mathcal{O}(M, \mathfrak{k})) \cong \text{Aut}(M) \times \mathcal{O}(M, \text{Aut}(\mathfrak{k})).$$

Proof. Let $\alpha : \mathcal{O}(M, \mathfrak{k}) \rightarrow \mathcal{O}(M, \mathfrak{k})$ be an automorphism. Now consider the composition

$$a_x : \mathfrak{k} \hookrightarrow \mathcal{O}(M, \mathfrak{k}) \xrightarrow{\alpha} \mathcal{O}(M, \mathfrak{k}) \xrightarrow{\text{ev}_x} \mathfrak{k},$$

where $\text{ev}_x : \mathcal{O}(M, \mathfrak{k}) \rightarrow \mathfrak{k}$ denotes the evaluation in $x \in M$. The map a_x is not zero, because $a_x = 0$ implies $\text{ev}_x \circ \alpha =: \alpha_x = 0$ by using $\mathcal{O}(M, \mathfrak{k}) = [\mathfrak{k}, \mathcal{O}(M, \mathfrak{k})]$ (see Lemma 3.4 below). Replacing \mathfrak{k} by a suitable subalgebra, this shows that a_x is an automorphism.

Clearly, $x \mapsto a_x$ is an element a of $\mathcal{O}(M, \text{Aut}(\mathfrak{k}))$, and replacing α by $a^{-1} \circ \alpha$, one can assume $a_x = \text{id}_{\mathfrak{k}}$ for all $x \in M$.

Let us show now that $\alpha_x = \text{ev}_y$ for some $y \in M$. α_x is a homomorphism of Lie algebras. Its kernel must be of the form $I \otimes \mathfrak{k}$ where I is a maximal ideal of A , by Lemma 3.5 below. Therefore, by the theory of Stein algebras (the ‘‘Verschärfung von Satz 2’’, p. 181, in [GrRe77] states that a closed ideal is necessarily of the form \mathfrak{m}_y , but on the other hand, an algebra homomorphism $\pi : \mathcal{O}(M) \rightarrow \mathbb{C}$ is automatically continuous, cf [GrRe77] p. 187) $I = \mathfrak{m}_y$ for some $y \in M$, where $\mathfrak{m}_y = \{f \in \mathcal{O}(M) \mid f(y) = 0\}$. It is then clear that $\alpha_x = \text{ev}_y$, and that $\alpha_x = \text{ev}_y$

We define now $\psi : M \rightarrow M$ by $x \mapsto y$ where y is such that $\alpha_x = \text{ev}_y$. With this definition, α is written as

$$\alpha(f)(x) = f(\psi(x)) = \psi^*(f)(x).$$

Now ψ must be holomorphic, as it sends holomorphic maps to holomorphic maps (this follows easily using projections as holomorphic functions), and finally, ψ must be an automorphism of M . ■

Lemma 3.4 $\mathcal{O}(M, \mathfrak{k}) = [\mathfrak{k}, \mathcal{O}(M, \mathfrak{k})]$

Proof. This follows directly from $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$. ■

The following lemma is very close to Lemma 6.1 in [Ne10]:

Lemma 3.5 *Let A be a unital associative commutative algebra, and \mathfrak{k} be a finite dimensional simple Lie algebra, both over an algebraically closed field \mathbb{K} .*

Any maximal ideal J of the current algebra $A \otimes \mathfrak{k}$ must be of the form $I \otimes \mathfrak{k}$ for some maximal ideal I of A .

Proof. Choose a Cartan subalgebra \mathfrak{h} of \mathfrak{k} , and consider the Cartan decomposition $\mathfrak{k} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{k}_{\alpha})$, where Φ denotes the root system of \mathfrak{k} and as usual,

$$\mathfrak{k}_{\alpha} = \{x \in \mathfrak{k} \mid \forall h \in \mathfrak{h} : [h, x] = \alpha(h)x\}.$$

Let \mathfrak{k} be of rank l .

Let $(h_i)_{i=1}^l$ be a basis of \mathfrak{h} , and complete it into a basis $(x_i)_{i=1}^n$ of \mathfrak{k} which is adapted to the Cartan decomposition, i.e. consisting of a basis of \mathfrak{h} and elements from \mathfrak{k}_{α} , such that $x_i = h_i$ for $i = 1, \dots, l$.

We claim that for any i , the projection onto x_i , parallel (with respect to the Killing form) to the subspace generated by the other x_j , is the sum of compositions of inner derivations.

Indeed, for a root α and an element $h \in \mathfrak{h}$, the eigenvalues of $\text{ad}(h)$ are $\beta(h)$, β a root, or 0. As the endomorphism $\text{ad}(h)$ of \mathfrak{k} is semi-simple, the minimal polynomial is written

$$\mu_{\text{ad}(h)}(X) = (X - \lambda_1) \cdots (X - \lambda_r),$$

and the projectors p_i^h onto the eigenspaces $\text{Ker}(\text{ad}(h) - \lambda_i)$ are polynomials in $\text{ad}(h)$. More precisely, denoting

$$Q_i(X) = (X - \lambda_1) \cdots (X - \lambda_{i-1})(X - \lambda_{i+1}) \cdots (X - \lambda_r),$$

Bezout’s theorem implies that there exist $R_1, \dots, R_r \in \mathbb{C}[X]$ such that

$$R_1 Q_1 + \dots + R_r Q_r = 1,$$

and the projector onto $\text{Ker}(\text{ad}(h) - \lambda_i)$ is then $R_i Q_i(\text{ad}(h))$.

Thus the projector p_α^h onto the eigenspace corresponding to the eigenvalue $\alpha(h)$ is a polynomial in $\text{ad}(h)$. Now the composition $p_i := p_\alpha^{h_1} \circ \dots \circ p_\alpha^{h_i}$ is the projection onto the x_i corresponding to the root α . Taking 0 instead of α , one obtains a polynomial in $\text{ad}(h)$ which represents the projector onto \mathfrak{h} .

Now let $(\epsilon_i)_{i=1}^n$ be the dual basis of $(x_i)_{i=1}^n$. Denote by $P_i(J)$ the image of the restriction of the linear maps $\text{id}_A \otimes (\epsilon_i \circ p_i) : A \otimes \mathfrak{k} \rightarrow A$ to J . Let $P(J)$ be the sum of the $P_i(J)$. $P(J)$ is an ideal, because J is closed under multiplication by an $a \in A$ in the first factor.

Indeed, the Casimir operator $\Omega = \sum_{i=1}^n x_i x^i$ where $x^i \in \mathfrak{k}$ for $i = 1, \dots, n$ are the elements in orthonormality to the x_i with respect to the Killing form, satisfies $\text{ad} \Omega = \lambda \text{id}_{\mathfrak{h}}$, and one can renormalize the x^i such that $\lambda = 1$. Then one easily computes that

$$ab \otimes y = \sum_{i=1}^n (a \otimes \text{ad}(x_i))(1 \otimes \text{ad}(x^i))(b \otimes y),$$

meaning that the multiplication by $a \in A$ in the first factor can be expressed by a composition of inner derivations.

Now let us prove that $J = P(J) \otimes \mathfrak{k}$. Indeed, the inclusion $J \subset P(J) \otimes \mathfrak{k}$ is trivial. On the other hand, one has $(\text{id}_A \otimes p_i)(J) \subset J$, because p_i is given by a sum of compositions of inner derivations, and therefore $P_i(J) \otimes \mathfrak{k} \subset J$, which sum up to $P(J) \otimes \mathfrak{k} \subset J$.

This ends the proof. ■

Corollary 3.6

$$\text{Aut}(\mathcal{O}(\Sigma_k, \mathfrak{k})) \cong \text{Aut}(\Sigma_k) \times \mathcal{O}(\Sigma_k, \text{Aut}(\mathfrak{k})).$$

The automorphism group of the Lie algebra determines the connected component of the identity of the automorphism group of the corresponding Lie group:

Corollary 3.7 *Let K be a finite-dimensional 1-connected complex Lie group. Then*

$$\text{Aut}^0(\mathcal{O}(\Sigma_k, K)) \cong \text{Aut}(\Sigma_k) \times \mathcal{O}(\Sigma_k, \text{Aut}^0(K)),$$

where $\text{Aut}^0(H)$ denotes the connected component of the identity in the automorphism group $\text{Aut}(H)$ of the Lie group H .

Proof. For a regular, connected and 1-connected Lie group G , every automorphism of \mathfrak{g} integrates uniquely to an automorphism of G (see [Mi83] Thm. 8.1). This is the case for the regular Fréchet-Lie group $G = \mathcal{O}(\Sigma_k, K)$ for finite dimensional 1-connected K . ■

Remark 3.8 (a) Observe that for a finite dimensional complex simple Lie algebra \mathfrak{k} , the group $\text{Aut}(\mathfrak{k})$ is the semi-direct product of the inner automorphisms $\text{Inn}(\mathfrak{k})$ of \mathfrak{k} and the graph automorphisms (see [Hu72] Ch. 16.5, p. 87). Therefore, for $g > 0$ or $k > 2$, the only continuous families of automorphisms of $\text{Aut}(\mathcal{O}(\Sigma_k, \mathfrak{k}))$ are holomorphic maps on Σ_k with values in $\text{Inn}(\mathfrak{k})$. In other words, up to finite subgroups, the automorphisms of $\mathcal{O}(\Sigma_k, \mathfrak{k})$ for $g > 0$ or $k > 2$ are holomorphic maps on Σ_k with values in $\text{Inn}(\mathfrak{k})$.

(b) Up to finite groups, there are thus no outer automorphisms of $\mathcal{O}(\Sigma_k, \mathfrak{k})$ for $g > 0$ or $k > 2$. This can be explained in the following way. The outer derivations of $\mathcal{O}(\Sigma_k, \mathfrak{k})$ are given by the Lie algebra $\text{Hol}(\Sigma_k)$ of holomorphic vector fields on Σ_k . As these do not integrate into an infinite-dimensional complex Lie group, the corresponding automorphism group is missing/finite dimensional.

Proposition 3.9 *If G is simply connected and ω a continuous 2-cocycle defining a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g} = L(G)$ by \mathfrak{z} and \widehat{G} a central extension of G by Z integrating $\widehat{\mathfrak{g}}$, then $\gamma = (\gamma_G, \gamma_Z) \in \text{Aut}(G) \times \text{Aut}(Z)$ lifts to an automorphism $\widehat{\gamma}$ of \widehat{G} (fixing Z) if and only if $[\gamma \cdot \omega] = [\omega]$, i.e. if the corresponding automorphism of \mathfrak{g} lifts to $\widehat{\mathfrak{g}}$.*

Proof. This is Proposition V.4 in [MaNe03]. ■

Let us investigate the condition $[\gamma \cdot \omega] = [\omega]$ for some class of automorphisms. The central extension $\widehat{G} = \mathcal{O}(\widehat{\Sigma_k}, K)$ of $G = \mathcal{O}(\Sigma_k, K)$ is given by the cocycle (2) in Section 2.4, K is finite dimensional and 1-connected with simple Lie algebra \mathfrak{k} whose Killing form is denoted by κ . The center is $Z = (\Omega^1(\Sigma_k) / d\mathcal{O}(\Sigma_k)) / \Pi_{\Sigma_k}$. $\text{Aut}(G)$ is described by Corollary 3.7.

Proposition 3.10 *Consider an automorphism of $\mathcal{O}(\Sigma_k, K)$ of the form $\gamma = (\gamma_G, \text{id}_Z) \in \text{Aut}(G) \times \text{Aut}(Z)$ with $\gamma_G \in \mathcal{O}(\Sigma_k, \text{Aut}(K))$. Then $[\gamma \cdot \omega_{\Sigma_k}] = [\omega_{\Sigma_k}]$.*

Proof. This follows from the fact that the Killing form is invariant under automorphisms, because of the form of the cocycle ω_{Σ_k} . ■

Remark 3.11 (a) It follows from the fact that a Lie algebra acts trivially on its cohomology that the identity component of the subgroup of inner automorphisms of the Lie group always act trivially on any Lie algebra cocycle. This is easily shown directly using the Cartan formula for the action of an element X in a Lie algebra \mathfrak{g} on a class $[\omega] \in H^n(\mathfrak{g})$:

$$L_X \omega = (d \circ i_X + i_X \circ d)\omega = d(i_X \omega),$$

because the representative ω is closed.

(b) We believe it to be an interesting open problem to determine explicitly all automorphism groups $\text{Aut}(\mathcal{O}(\widehat{\Sigma_k}, K))$ for different $k \geq 1$ and different finite-dimensional 1-connected simple Lie groups K where the extension is defined using the cocycle (2).

4 Coadjoint Orbits

In this section, we study the orbits of the coadjoint action of a central extension of $G := \mathcal{O}(\Sigma_k, K)$ on the dual space of a central extension of $\mathfrak{g} := \mathcal{O}(\Sigma_k, \mathfrak{k})$ with \mathfrak{k} complex simple. As usual, we will restrict the dual to the so-called *smooth dual*. The orbits we obtain in this way are closely related to the flat bundles described in Section 1. The main result is the description of the orbits in terms of conjugacy classes.

4.1 The smooth dual

We will define here the smooth dual of a central extension of $\mathcal{O}(\Sigma_k, \mathfrak{k})$. Actually, it would be more accurate to call it the holomorphic dual, but we will stick to the classical term. Following [EtFr94], we will not consider the universal central extension $\mathcal{O}(\widehat{\Sigma_k}, \mathfrak{k})$, but only a central extension with 1-dimensional central term.

For this, let us fix a singular cycle $\sigma \in Z_1(\Sigma_k)$ which we suppose to be a simple smooth curve and which we suppose to be one of the ℓ generators of $\pi_1(\Sigma_k)$. Recall the *compact dual* $\eta_\sigma \in \Omega_{\text{sm},c}^1(\Sigma_k)$, where $\Omega_{\text{sm},c}^1(\Sigma_k)$ is the space of smooth differential 1-forms with compact support, cf. [BoTu82] p. 51. Here η_σ is by definition a compactly supported differential 1-form which satisfies for any smooth 1-form α on Σ_k

$$\int_\sigma \alpha = \int_{\Sigma_k} \eta_\sigma \wedge \alpha.$$

We define a central extension

$$\widehat{\mathfrak{g}}_\sigma := \widehat{\mathcal{O}(\Sigma_k, \mathfrak{k})}_\sigma$$

of $\mathcal{O}(\Sigma_k, \mathfrak{k})$ with a 1-dimensional central term by the cocycle Ω_σ given by

$$\Omega_\sigma(X, Y) := \int_{\Sigma_k} \eta_\sigma \wedge \kappa(X, dY) = \int_\sigma \kappa(X, dY),$$

where κ is the Killing form of the simple Lie algebra \mathfrak{k} , and $X, Y \in \mathfrak{g} = \mathcal{O}(\Sigma_k, \mathfrak{k})$. Elements of the central extension $\widehat{\mathfrak{g}}_\sigma$ will be denoted by $\mu k + X$, where k is the central element, $\mu \in \mathbb{C}$ and $X \in \mathcal{O}(\Sigma_k, \mathfrak{k})$.

Definition 4.1 Let $\widehat{\mathfrak{g}}_\sigma^*$ be the Fréchet space of operators $D = \lambda\partial + \xi$ where $\lambda \in \mathbb{C}$ and $\xi \in \Omega^1(\Sigma_k, \mathfrak{k})$ is a holomorphic 1-form on Σ_k . We call $\widehat{\mathfrak{g}}_\sigma^*$ the smooth dual of $\widehat{\mathfrak{g}}_\sigma$.

In order to justify the implicit duality claim in the name of $\widehat{\mathfrak{g}}_\sigma^*$, let us introduce the following bilinear form:

$$(\lambda\partial + \xi, \mu k + X) := \lambda\mu + \int_{\Sigma_k} \eta_\sigma \wedge \kappa(\xi, X) = \lambda\mu + \int_\sigma \kappa(\xi, X).$$

The Lie group $G = \mathcal{O}(\Sigma_k, K)$ acts on $\widehat{\mathfrak{g}}_\sigma^*$ by the following action, which is close to the gauge action (1) in Section 1.1:

$$f \cdot (\lambda\partial + \xi) := \lambda\partial + \delta(f) + \text{Ad}(f)^{-1}(\xi),$$

where $f \in G$. We will call this action the *coadjoint action* of G on $\widehat{\mathfrak{g}}_\sigma^*$. It identifies to the coadjoint action using the duality given by the following proposition:

Proposition 4.2 *The bilinear form*

$$(-, -) : \widehat{\mathfrak{g}}_\sigma^* \times \widehat{\mathfrak{g}}_\sigma \rightarrow \mathbb{C}$$

is a non-degenerate G -invariant pairing.

Proof. The form is non-degenerate: Indeed, consider for a given $\xi \neq 0$, $\int_\sigma \kappa(\xi, X)$ inserting an arbitrary X . Then the support $\text{supp}(\xi)$ has a non-empty intersection with $\text{im}(\sigma)$. Working locally in a suitable open set, ξ may be regarded as a \mathfrak{k} -valued holomorphic function on Σ_k . Express ξ as $\xi(z) = \sum_i f_i(z) \otimes x_i$ for holomorphic functions f_i and some elements $x_i \in \mathfrak{k}$. One may suppose that $(x_i)_{i=1}^N$ forms a basis of \mathfrak{k} and that x_{i_0} has a non-zero coefficient function in some open set. By non-degeneracy of the Killing form κ , one can find $y \in \mathfrak{k}$ with $\kappa(x_{i_0}, y) \neq 0$ and $\kappa(x_i, y) = 0$ for all $i \neq i_0$. Then take $X = 1 \otimes y$. It is clear that $(\xi, X) \neq 0$. This reasoning works for both arguments.

The form is G -invariant: Indeed, the adjoint action of G on $\widehat{\mathfrak{g}}_\sigma$ is given by

$$f \cdot (\mu k + X) := \mu k - \int_{\Sigma_k} \eta_\sigma \wedge \kappa(\delta(f), X) + \text{Ad}(f)(X).$$

With this definition, one easily computes that

$$(f \cdot (\lambda\partial + \xi), (\mu k + X)) = (\lambda\partial + \xi, f \cdot (\mu k + X)),$$

using that the Killing form is invariant under automorphisms. ■

4.2 Coadjoint orbits and flat bundles

In this subsection, we explain the link between the framework of our Section 1 and Section 3 in [EtFr94] which establishes a bijective correspondence between coadjoint orbits and flat bundles (Corollary 4.4).

The coadjoint action of G on $\widehat{\mathfrak{g}}_\sigma^*$, introduced in the previous section, leaves stable the hyperplanes

$$H_\lambda := \{ \lambda = \text{constant} \} \subset \widehat{\mathfrak{g}}_\sigma^*.$$

We will therefore restrict to H_λ in the following. Now compare the coadjoint action to the gauge action (1): In H_λ , these two actions coincide, thus the coadjoint orbits in H_λ correspond to the gauge orbits in $Z_{\text{dR}}^1(\Sigma_k, \mathfrak{k})$. The set $H_{\text{dR}}^1(\Sigma_k, \mathfrak{k})$ of gauge orbits in $Z_{\text{dR}}^1(\Sigma_k, \mathfrak{k})$ has been examined in Section 1 and identifies for Σ_k to the space of flat K -principal bundles which identifies in turn to K^ℓ / K (Corollary 1.7).

Let us show this here once again from the point of view of [EtFr94], Section 3.

Proposition 4.3 *Let $\mathcal{U} = \{U_i\}$ be a good open cover of Σ_k . Let $D = \lambda\partial + \xi \in H_\lambda$ and consider the equation*

$$\lambda(\partial\psi_i)\psi_i^{-1} + \xi = 0. \tag{3}$$

Then solutions ψ_i exist on U_i and the functions $\varphi_{ij} := \psi_i^{-1}\psi_j : U_{ij} \rightarrow K$ determine transition functions of a holomorphic K -principal bundle on Σ_k . Furthermore, the functions φ_{ij} are constant, thus the bundle is flat.

Proof. Observe that the holomorphic Maurer-Cartan equation is trivially satisfied on Σ_k as it is of complex dimension 1. Therefore solutions to equation (3) exist thanks to Theorem 1.1.

The φ_{ij} satisfy the Čech-cocycle condition on triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$

$$\varphi_{ij}\varphi_{jk}\varphi_{ki} = \psi_i^{-1}\psi_j\psi_j^{-1}\psi_k\psi_k^{-1}\psi_i = 1.$$

Therefore, they determine a holomorphic K -principal bundle. Let us show that the φ_{ij} are not only holomorphic (like in [EtFr94]), but also anti-holomorphic, and thus constant. Indeed, the transition functions φ_{ij} are also anti-holomorphic:

$$\begin{aligned} \partial(\psi_i^{-1}\psi_j) &= -\psi_i^{-1}(\partial\psi_i)\psi_i^{-1}\psi_j + \psi_i^{-1}(\partial\psi_j) \\ &= \psi_i^{-1}\xi\psi_j + \psi_i^{-1}(-\xi)\psi_j = 0 \end{aligned}$$

■

Corollary 4.4 *There is a bijective correspondence between coadjoint orbits in H_λ and gauge equivalence classes of flat connections on the trivial holomorphic K -principal bundle $K \times \Sigma_k$.*

Proof. Indeed, the previous proposition tells us that to a coadjoint orbits, we may associate an equivalence class of flat holomorphic K -principal bundles. But all these are trivial on Σ_k by Proposition 1.8. On the other hand, a flat trivial principal K -bundle has a flat connection ψ_i associated to the good cover \mathcal{U} , and $\partial(\psi_i^{-1}\psi_j) = 0$ implies that the holomorphic 1-form ξ defined by $\xi = -\lambda(\partial\psi_i)\psi_i^{-1}$ is global:

$$\begin{aligned}\partial(\psi_i^{-1}\psi_j) &= -\psi_i^{-1}(\partial\psi_i)\psi_i^{-1}\psi_j + \psi_i^{-1}(\partial\psi_j) = 0 \\ \Rightarrow (\partial\psi_i)\psi_i^{-1}\psi_j &= \partial\psi_j \\ \Rightarrow (\partial\psi_i)\psi_i^{-1} &= (\partial\psi_j)\psi_j^{-1}.\end{aligned}$$

■

Remark 4.5 For the sake of this remark, let us take $K = \mathbb{C}^*$. This group is not simple, but this will not disturb us here. We want to underline in this remark the idea that Etingof-Frenkel's construction of the holomorphic bundle above does not only construct a holomorphic bundle, but a holomorphic bundle *with a connection* ψ . We choose $K = \mathbb{C}^*$ in order to tie this construction to Deligne cohomology. Indeed, recall Theorem 2.2.12 of [Bry93]:

Theorem 4.6 *The group of isomorphism classes of pairs (L, ∇) of a line bundle L on M with connection ∇ is canonically isomorphic to the Deligne cohomology group $H^1(M, \underline{\mathbb{C}}_M^* \xrightarrow{\delta} \Omega_{\text{sm}, M, \mathbb{C}}^1)$.*

As stated, the theorem is true in the smooth setting, but it can be easily adapted to the holomorphic setting on a Stein manifold.

Our observation is that (ξ, φ_{ij}) forms a cocycle in the bicomplex of Deligne cohomology. We believe that the part of the bicomplex which is used to compute the Deligne H^1 makes also sense for a non-abelian Lie group K , while the entire bicomplex makes sense only in the abelian context, i.e. in case K is \mathbb{C}^* or more generally any abelian complex Lie group.

4.3 Description of coadjoint orbits using stabilizers

In this subsection, we adapt the idea of describing orbits by stabilizers like in [MoWe04] to our setting.

Let us denote by $\mathcal{O}_{(\xi, \lambda)}$ the orbit of the coadjoint action of $G := \mathcal{O}(\Sigma_k, K)$ on \mathfrak{g}_σ^* passing through $\lambda\partial + \xi$. Recall from the previous subsection that the set of orbits in the hyperplane H_λ may be identified to K^ℓ / K , where ℓ is the first Betti number of Σ_k and K acts on K^ℓ by conjugation in each factor.

Remark 4.7 Let us recall Frenkel's construction [Fr84] of the stabilizer of an orbit of the coadjoint action for the circle \mathbb{S}^1 . For an orbit $\mathcal{O}_{(X, \lambda)} \subset L\mathfrak{k} \oplus \mathbb{C}$ with $\lambda \neq 0$, where $L\mathfrak{k} = C^\infty(\mathbb{S}^1, \mathfrak{k})$ is the loop algebra, we can solve the differential equation

$$z' = -\frac{1}{\lambda}Xz.$$

Let $z_{(X, \lambda)}$ be its unique solution with initial condition $z_{(X, \lambda)}(0) = 1 \in K$. This is a path in K starting at $1 \in K$. Now, since $X \in L\mathfrak{k}$ is periodic, we get $z_{(X, \lambda)}(\theta + 2\pi) = z_{(X, \lambda)}(\theta)z_{(X, \lambda)}(2\pi)$. Let $\lambda\partial + Y$ be another element in the coadjoint orbit $\mathcal{O}_{(X, \lambda)}$. By the expression of the coadjoint action, X and Y are linked by $Y = gXg^{-1} - \lambda\delta(g)$ for some g in the loop group $L(K)$. Proposition 3.2.5 in [Fr84] shows that the periodicity of solutions implies the following conjugation formula

$$z_{(X, \lambda)}(\theta) = g(\theta)z_{(X, \lambda)}(\theta)g(0)^{-1}.$$

Now since g is also periodic, $z_{(X,\lambda)}(2\pi)$ and $z_{(Y,\lambda)}(2\pi)$ lie in the same conjugacy class in K . Furthermore, the stabilizer of (X, λ) in $L(K)$ is isomorphic to the stabilizer of $z_{(X,\lambda)}(2\pi)$ in K , so that we get

$$\mathcal{O}_{(X,\lambda)} = L(K) / \text{Stab}_K(z_{(X,\lambda)}(2\pi)).$$

An explicit description of the stabilizers is contained in [MoWe04].

The idea is here to use this known description of coadjoint orbits in the loop group setting by restriction our holomorphic currents to circles. For this, recall further the Lie group homomorphism from Lemma 2.9

$$\alpha_K : G \rightarrow \mathcal{C}^\infty(\mathbb{S}^1, K),$$

which is associated to $\alpha : \mathbb{S}^1 \rightarrow \Sigma_k$. In the following, we will denote $\mathcal{C}^\infty(\mathbb{S}^1, K)$ simply by $LK = K^{\mathbb{S}^1}$. Applying the construction of α_K to the ℓ embedded circles in Σ_k , denoted by α_i , which generate $\pi_1(\Sigma_k)$, we obtain a homomorphism

$$\varphi : G \rightarrow \prod_{i=1}^{\ell} K^{\mathbb{S}^1}.$$

The following proposition describes the image of the orbit $\mathcal{O}_{(\xi,\lambda)}$, seen as $G / \text{Stab}(\xi, \lambda)$, under the map induced by φ :

Proposition 4.8 *The map φ induces an injection*

$$\bar{\varphi} : G / \text{Stab}(\xi, \lambda) \rightarrow \prod_{i=1}^{\ell} K^{\mathbb{S}^1} / \prod_{i=1}^{\ell} K_{C_i},$$

where $(C_1, \dots, C_\ell) \in \text{Hom}(\pi_1(\Sigma_k), K) / K \cong K^\ell / K$ is the image of $\lambda\partial + \xi$ under the period map P (also called monodromy map in this context) and K_{C_i} is the stabilizer of the conjugacy class C_i in K , i.e. we have $K / K_{C_i} = C_i$.

Proof. (1) Definition of the map $\bar{\varphi}$.

We have to show that the map $\bar{\varphi}$ in the following diagram is well-defined.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \left(K^{\mathbb{S}^1}\right)^\ell \\ \downarrow & & \downarrow \\ G / \text{Stab}(\xi, \lambda) & \xrightarrow{\bar{\varphi}} & \left(K^{\mathbb{S}^1}\right)^\ell / \prod_{i=1}^{\ell} K_{C_i} \end{array}$$

We have to show that for all $g \in \text{Stab}(\xi, \lambda)$, $\varphi(g) \in \prod_{i=1}^{\ell} K_{C_i}$. Now, $g \in \text{Stab}(\xi, \lambda)$ means that

$$(\lambda\partial + \xi) \cdot g := \lambda\partial + (\partial g)g^{-1} + \text{Ad } g^{-1}(\xi) \stackrel{!}{=} \lambda\partial + \xi.$$

In other words,

$$\xi = (\partial g)g^{-1} + \text{Ad } g^{-1}(\xi) =: \xi \cdot g.$$

By Remark 4.7, we have a commutative diagram

$$\begin{array}{ccc}
H_{\text{dR}}^1(\Sigma_k, \mathfrak{k}) & \xrightarrow{P} & H^1(\pi_1(\Sigma_k), K) = K^\ell / K \\
\downarrow E := |_{\alpha_i} & & \downarrow \pi_i \\
H_{\text{dR,sm}}^1(\mathbb{S}^1, \mathfrak{k}) & \xrightarrow{F} & K / K
\end{array}$$

Here the map $|_{\alpha_i} : H_{\text{dR}}^1(\Sigma_k, \mathfrak{k}) \rightarrow H_{\text{dR,sm}}^1(\mathbb{S}^1, \mathfrak{k})$ restricts gauge orbits of forms on Σ_k to gauge orbits of forms on the i th circle $\alpha_i \subset \Sigma_k$. The map P is the period map from Section 1 and π_i is the projection onto the i th factor in the product. The map F is Frenkel's construction and K / K is the set of conjugacy classes of K .

By commutativity, we have on the one hand

$$\pi_i \circ P([\xi]) = C_i = F \circ E([\xi]),$$

and on the other hand

$$\pi_i \circ P([\xi \cdot g]) = C_i = F \circ E([\xi \cdot g]).$$

The equality $\xi = \xi \cdot g$ implies therefore that $\xi|_{\alpha_i}$ and $(\xi \cdot g)|_{\alpha_i}$ are sent to C_i under Frenkel's construction. We get

$$\pi_i(\varphi(g)) = g|_{\alpha_i} \in \text{Stab}(\xi|_{\alpha_i}, \lambda) = G_{C_i}.$$

(2) Injectivity of the map $\bar{\varphi}$.

Here we have to show for $f, g \in G$ with $\bar{\varphi}(f) = \bar{\varphi}(g)$ that $fg^{-1} \in \text{Stab}(\xi, \lambda)$. This means explicitly

$$\delta(f) + \text{Ad } f^{-1}\xi = \delta(g) + \text{Ad } g^{-1}\xi.$$

This equation between holomorphic functions may be shown by restriction to *one* circle α_i , because f, g are holomorphic. On one circle α_i , this is exactly the outcome of Remark 4.7. \blacksquare

Remark 4.9 In fact, it suffices to take only one factor in the product $\prod_{i=1}^{\ell} K^{\mathbb{S}^1}$, but in order to treat the circles α_i on an equal basis, we chose to take them all.

4.4 The symplectic form on the orbits

It is known since work of Kostant [Ko65], Kirillov [Ki68] and Souriau [So70] that coadjoint orbits are symplectic manifolds. Let us recall in this section the corresponding symplectic form. The main outcome of this section is that in our setting, the orbits are *holomorphic symplectic manifolds*.

Consider here one embedded circle $\alpha : \mathbb{S}^1 \rightarrow \Sigma_k$ in Σ_k . Let us distinguish α from its image $\alpha(\mathbb{S}^1) =: C \subset \Sigma_k$. Denote by

$$\varphi : \mathcal{O}(\Sigma_k, K) \rightarrow \mathcal{C}^\infty(\mathbb{S}^1, K)$$

the restriction map $f \mapsto f \circ \alpha$ to the circle C . This is a homomorphism of complex Fréchet-Lie groups. We have seen above that φ induces a map

$$\bar{\varphi} : \mathcal{O}_{(\xi, \lambda)} \rightarrow \mathcal{O}_{(\xi|_C, \lambda)},$$

the only new issue being that there is only one circle.

Let us describe tangent vectors v to the orbit $\mathcal{O}_{(\xi,\lambda)}$. The vector $v \in T_{(\lambda',\xi')} \mathcal{O}_{(\xi,\lambda)}$ is described by a smooth curve $\gamma : [-1, 1] \rightarrow \mathcal{O}_{(\xi,\lambda)}$ such that $\gamma(0) = (\lambda', \xi')$ and $v = \dot{\gamma}(0)$. As the orbit $\mathcal{O}_{(\xi,\lambda)}$ lies entirely in the hyperplane

$$H_\lambda = \{ \lambda = \text{constant} \} \subset \widehat{\mathfrak{g}}_\sigma^*,$$

we have $\lambda' = \lambda$. The curve can be taken explicitly as

$$\gamma_v(t) := (\lambda, \xi') \cdot e^{t\tilde{X}}$$

for some $\tilde{X} \in \mathcal{O}(\Sigma_k, \mathfrak{k})$.

Definition 4.10 The KKS-form ω_h on the orbit $\mathcal{O}_{(\lambda,\xi)}$ is given for all tangent vectors $v, w \in T_{(\lambda,\xi')} \mathcal{O}_{(\xi,\lambda)}$ by

$$\omega_h(v, w) := \int_C \kappa(\xi', [\tilde{X}, \tilde{Y}]),$$

where v, w are represented by $(\xi', \lambda) \cdot e^{t\tilde{X}}$ resp. $(\xi', \lambda) \cdot e^{t\tilde{Y}}$ as explained above.

As always in infinite dimensions, ω_h is a “symplectic” form in the sense that it is *weakly symplectic*, i.e. instead of demanding a *non-degenerate* skewsymmetric bilinear form, one only demands that the form induces an injection into the dual space. The form is nonetheless supposed to be closed. These properties are shown below.

In the same way, there is the standard symplectic form ω_s on the coadjoint orbits of the loop group $\mathcal{C}^\infty(\mathbb{S}^1, K)$. The relation between the two symplectic forms is simply:

Proposition 4.11 *For all tangent vectors $v, w \in T_{(\lambda,\xi')} \mathcal{O}_{(\xi,\lambda)}$, we have*

$$\omega_h(v, w) = \omega_s(T\bar{\varphi}(v), T\bar{\varphi}(w)).$$

Proof. We have

$$\begin{aligned} \omega_h(v, w) &= \int_C \kappa(\xi', [\tilde{X}, \tilde{Y}]) \\ &= \int_{\mathbb{S}^1} \kappa(\xi'|_C, T_{(\lambda,\xi')} \bar{\varphi}[\tilde{X}, \tilde{Y}]) \\ &= \int_{\mathbb{S}^1} \kappa(\xi'|_C, [T_{(\lambda,\xi')} \bar{\varphi}(\tilde{X}), T_{(\lambda,\xi')} \bar{\varphi}(\tilde{Y})]) \\ &= \omega_s(T\bar{\varphi}(v), T\bar{\varphi}(w)). \end{aligned}$$

■

Corollary 4.12 *The form ω_h is closed and weakly symplectic.*

Proof. The first of these two properties is inherited from ω_s thanks to the preceding proposition. The second assertion follows from the non-degeneracy and invariance of κ , because

$$\omega_h(v, -) = \int_C \kappa(\xi', [\tilde{X}, -]) = \int_C \kappa([\xi', \tilde{X}], -),$$

and this latter expression, seen as a map

$$T_{(\lambda,\xi')} \mathcal{O}_{(\xi,\lambda)} \rightarrow (T_{(\lambda,\xi')} \mathcal{O}_{(\xi,\lambda)})^*, \quad v \mapsto \omega_h(v, -)$$

is clearly injective. ■

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