On Hopf 2-algebras

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Abstract

Our main goal in this paper is to translate the diagram below relating groups, Lie algebras and Hopf algebras to the corresponding 2-objects, i.e. to categorify it. This is done interpreting 2-objects as crossed modules and showing the compatibility of the standard functors linking groups, Lie algebras and Hopf algebras with the concept of a crossed module. One outcome is the construction of an enveloping algebra of the string Lie algebra of Baez-Crans [BaeCra04], another is the clarification of the passage from crossed modules of Hopf algebras to Hopf 2-algebras.

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Introduction

One of the most impressive theorems in the theory of Lie groups is Lie's third theorem: the possibility to integrate a real or complex (finite-dimensional) Lie algebra in a unique way into a connected, simply connected Lie group. Algebraically, some aspects of this integration process are captured in the following diagram:

$$\begin{tikzcd}
\mathtt{Lie} \arrow[r, leftrightarrow, shift right, "U"', "P"] \arrow[d, leftrightarrow] & \mathtt{ccHopf} \arrow[d, leftrightarrow] \\
\mathtt{Grp} \arrow[r, leftrightarrow, shift right, "k[-]"', "\chi"] & \mathtt{cHopf}
\end{tikzcd}$$

Here `Lie` is the category of Lie algebras over the field k with $k = \mathbb{R}$ or $k = \mathbb{C}$, `Grp` is the category of groups, supposed to be finite or connected algebraic (in which case we assume $k = \mathbb{C}$), when we apply the functor $k[-]$ of (regular) functions, `Hopf` is the category of k-Hopf algebras, and `ccHopf` and `cHopf` are its subcategories of cocommutative resp. commutative Hopf algebras. The functors U and P are those of the enveloping algebra and of primitives, and χ is

the functor of characters. The functors on the RHS of the diagram stipulate duality (linear duality in the case of finite dimensional Hopf algebras, restricted duality in the case of graded Hopf algebras with finite dimensional graded components, or continuous duality in the case of complete topological Hopf algebras, depending on the context) whereas those on the LHS stipulate integration and derivation. Recall that the integration process consists in associating to a finite dimensional Lie algebra $\mathfrak{g}$ first its universal enveloping algebra $U\mathfrak{g}$. The latter is seen as the continuous dual of the completion of the space of functions on some formal group law (the space of "point distributions"), see [Ser64]. Therefore in order to integrate $\mathfrak{g}$, one dualizes $U\mathfrak{g}$, which gives a completion of the Hopf algebra of formal functions, and finally recovers the group as group-like elements or characters in this Hopf algebra of functions. According to the properties of the initial Hopf algebra and the geometric requirements of the reader, this gives then a formal group, an algebraic group or a finite group. For reference, the formal case may be found in [Ser64].

On the other hand, in recent times, *categorification*, i.e. the passage to categorified algebraic structures, plays a growing rôle in algebra and geometry. Here categorification means the replacement of the underlying sets in some algebraic structure by categories, and maps between these sets by functors. For example, instead of regarding a group in `Sets`, the category of sets (which is the ordinary concept of a group), one considers a group object in the category `Cat` of (small) categories. In this way one arrives at the notion of a 2-group.

There is recent intense mathematical research striving to understand how to integrate Lie 2-algebras into Lie 2-groups, see for example [BCSS07], [Get09], [Hen08], [Woc08]. The purpose of this article is to establish the above diagram in the context of 2-groups and Lie 2-algebras, which can be seen as some answer to the integration problem (see theorem 6 in section 6 in order to have a precise statement). The diagram is obtained as the union of propositions 1 to 4. Thus our main result reads:

Theorem 1. *The functors U, P, $k[-]$ and χ, which we introduced above, extend to the following diagram between categories of strict 2-objects:*

$$\begin{tikzcd}
2-\mathtt{Lie} \arrow[r, leftrightarrow, shift right, "U"', "P"] \arrow[d, leftrightarrow] & 2-\mathtt{ccHopf} \arrow[d, leftrightarrow] \\
2-\mathtt{Grp} \arrow[r, leftrightarrow, shift right, "k[-]"', "\chi"] & 2-\mathtt{cHopf}
\end{tikzcd}$$

One aspect of this theorem is that it supplies the article [FLN07] with a huge amount of examples; namely, all strict Lie 2-algebras give rise to their kind of cat^1-Hopf algebras (and, by the way, all semi-strict Lie 2-algebras of [BaeCra04] can be strictified).

The main method that we use in the proof of theorem 1 is the translation of 2-groups and Lie 2-algebras into a different algebraic structure, namely the

structure of a *crossed module*. Let us explain this structure in the context of groups. A *crossed module of groups* is a homomorphism of groups $\mu : M \to N$ together with an action α of N on M by automorphisms, denoted by $\alpha : m \mapsto {}^n m$ for $n \in N$ and $m \in M$, such that

(a) $\mu({}^n m) = n\mu(m)n^{-1}$ and

(b) ${}^{\mu(m)}m' = mm'm^{-1}$.

It is well known, see [Lod82] [Mac97] [Por09], that the category of strict 2-groups and the category of crossed modules of groups are equivalent. This is discussed in more detail in section 2. Usually, this equivalence is formulated as an equivalence between 2-categories, but in order to keep the abstract categorical formalism to a minimum, we will stick to ordinary categories here. We believe that the extension to 2-categories is straight forward.

Similarly, the category of strict Lie 2-algebras and the category of crossed modules of Lie algebras are equivalent (see [BaeCra04] and section 3), and a similar statement also holds for cocommutative Hopf algebras [FLN07].

Thus in order to achieve our task to transpose the above diagram into 2-structures, we focus on the compatibility of the concept of a crossed module with the above standard functors U, P, $k[-]$ and χ. This is done in sections 4 and 5. In section 5, due to the duality coming into play when passing to the right bottom corner in the above diagram, we explore the dual notion of crossed modules of Hopf algebra, namely, *crossed comodules of Hopf algebras* (see definition 13). We believe that this new algebraic structure gives an equivalent way of formulating Hopf 2-algebras.

One outcome of the discussion of compatibility of the concept of a crossed module with standard functors is that the definition of a crossed module in [FLN07] (taken over slightly generalized in our article as definition 1) does not seem to be too far from the "right" categorification. Let us denote such a crossed module of Hopf algebras by $\gamma : B \to H$. Definition 1 imposes compatibility relations between the module structure (of H on B) and the Hopf algebra structure on B, namely, B has to be an H-module algebra, an H-module coalgebra and the antipode of B has to be a morphism of H-modules. We show that these conditions are the natural reflection of the fact that in the case of a crossed module of Lie algebras (see definition 11) $\mu : \mathfrak{m} \to \mathfrak{n}$, the action has to be an action by derivations, i.e. for all $m, m' \in \mathfrak{m}$ and all $n \in \mathfrak{n}$:

$$n \cdot [m, m'] = [n \cdot m, m'] + [m, n \cdot m'],$$

and in the case of a crossed module of groups $\mu : M \to N$, the action has to be an action by automorphisms of groups, i.e. for all $m, m' \in M$ and all $n \in N$:

$${}^n(mm') = ({}^n m)({}^n m').$$

Our main point is that this compatibility between the module structure (of H on B) and the Hopf algebra structure on B is necessary in case one demands

the crossed modules of Lie algebras and of groups to have similar compatibility conditions.

This is important to note. Indeed, the obstacle to define a crossed module of associative algebras corresponding to a crossed module of Lie algebras by taking the functor U term by term, is that the action is by derivations of the associative product (see section 4). Thus the term by term U-image of a crossed module of Lie algebras does not satisfy the condition of compatibility for a crossed module of associative algebras $\rho : R \to A$, which reads for all $a \in A$ and all $r, r' \in R$:

$$a(rr') \,=\, (ar)r', \quad (rr')a \,=\, r(r'a),$$

see [DIK08]: the naive belief that a crossed module of Hopf algebras is in particular a crossed module of associative algebras which is simultaneously a crossed module of coassociative coalgebras is wrong.

On the other hand, the association of a crossed module of associative algebras corresponding to a crossed module of Lie algebras may be seen as a map on the level of cocycles whose induced map in cohomology

$$\phi : H^3(\mathfrak{g}, M^{\mathrm{ad}}) \to HH^3(U\mathfrak{g}, M)$$

is known to be an isomorphism. To our knowledge, no natural map on cocycles inducing ϕ is known.

In future work, we plan to drop the commutativity/cocommutativity condition in the definitions of a crossed (co)module of Hopf algebras and to show in this more general framework the same equivalence of categories as in [FLN07].

Let us note some by-products of our study: let G be a connected, simply connected, complex simple Lie group and $\mathfrak{g}$ its Lie algebra. It is well known that the de Rham cohomology group $H^3(G)$ is one dimensional and isomorphic to $H^3(\mathfrak{g})$, and that this space is generated by the Cartan cocycle $< [,], >$, which is manufactured from the Killing form $<,>$ and the bracket $[,]$ on $\mathfrak{g}$. In [Wag06], the second named author gives an explicit crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$ which represents the cohomology class of $< [,], >$ (via the bijection between equivalence classes of crossed modules $\mu : \mathfrak{m} \to \mathfrak{n}$ with fixed kernel $\ker(\mu) = V$ and fixed cokernel $\operatorname{coker}(\mu) = \mathfrak{g}$ and $H^3(\mathfrak{g}, V)$). On the other hand, the *string group* associated to G is its 3-connected cover, see [BCSS07], [Hen05]. This string group is only defined up to homotopy and cannot be realized as a strict, finite dimensional Lie 2-group, but only as an infinite dimensional Lie 2-group [BCSS07] or non-strict [Woc09]. Applying the functor U to our crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$ gives by proposition 1 a crossed module of Hopf algebras which is a natural candidate for an enveloping algebra of the string Lie algebra (see remark 8), explicitly:

Corollary 1. *There is a crossed module of Hopf algebras*

$$\mu : S(N(0)^\sharp) \to S(N(0)^\sharp) \otimes_\alpha U\mathfrak{g}$$

which is a natural algebraic candidate for the enveloping algebra of the string Lie 2*-algebra in the sense that its underlying vector spaces are at most countably infinite dimensional.*

This restriction on the dimension of the underlying vector spaces is important when one wants to perform algebraic constructions with the enveloping algebra, like for example pass to the dual crossed comodule. On the other hand, it is well-known that a finite dimensional construction is not possible, cf [Hoch54]. A similar statement holds for any finite dimensional semi-strict Lie 2-algebra, see theorem 5.

We leave open the question of defining an enveloping algebra of a Lie 2-algebra as a left adjoint 2-functor (and refer to remark 8 for the notations used in the preceding corollary).

A second by-product is a sort of Kostant's theorem for irreducible Hopf 2-algebras in characteristic 0 (see remark 4). Indeed, the ordinary Kostant theorem (asserting that an irreducible Hopf algebra in characteristic 0 is isomorphic to the universal enveloping algebra of its primitives) and our techniques show how to find a crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ for a given crossed module of irreducible Hopf algebras $\gamma : B \to H$ such that $\gamma : B \to H$ is isomorphic to $U(\mu : \mathfrak{m} \to \mathfrak{n})$.

Corollary 2. *An irreducible cocommutative Hopf* 2*-algebra is equivalent to an enveloping Hopf* 2*-algebra, i.e. to a Hopf* 2*-algebra of the form* $U(\mu : \mathfrak{m} \to \mathfrak{n})$*, where* $\mu : \mathfrak{m} \to \mathfrak{n}$ *is a crossed module of Lie algebras.*

A third by-product is a new approach to the equivalence between crossed modules of Hopf algebras and Hopf 2-algebras (called cat^1-Hopf algebras in) [FLN07]. Namely, we use the functors U and P to reduce the problem to Lie algebras where the equivalence is well known. We get in this way:

Corollary 3. *When restricting to the subcategory of irreducible cocommutative Hopf algebras, the notions of crossed module and of* cat^1*-Hopf algebra are equivalent.*

Open questions abound - let us state only two of them which will guide our future research in this field: how to construct a cohomological interpretation of crossed modules of Hopf algebras, inspired by the fact that crossed modules of Lie algebras, groups or associative algebras are classified (up to equivalence) by 3-cohomology classes ? How to quantize Hopf 2-algebras, or, in other words, how to deform our crossed modules of enveloping algebras to get some quantum 2-groups ? Some very partial answers to these questions are contained in remarks 6 and 7 in section 4.

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1 Preliminaries

Here we collect standard notations and definitions for studying Hopf algebras coming from groups and Lie algebras, and their crossed modules. We introduce in definition 1 the notion of a *crossed module of Hopf algebras* which is slightly more general than definition 12 given in [FLN07] (see remark 1).

Let us denote by τ the twist in the symmetric monoidal category of vector spaces over a commutative field k of characteristic 0. We refrain from formulating our entire paper relative to an arbitrary underlying symmetric monoidal category, but we believe that this generalization is straight forward. In geometric situations, we will always suppose $k = \mathbb{C}$. All Lie and Hopf algebras are supposed to be algebras over k. A Hopf algebra H over k is given by $(H, \mu_H, \eta_H, \triangle_H, \epsilon_H, S_H)$, where μ_H is the associative product, η_H its unit, $\triangle_H$ the coassociative coproduct, ϵ_H its counit, and S_H is the antipode. In case the Hopf algebra we work with is clear from the context, we feel free to drop the corresponding index (and write for example μ instead of μ_H).

Let H be a Hopf algebra. Then H becomes a left H-module by the *adjoint action* which is defined for $h, k \in H$ as follows:

$$\mathrm{ad}_H(h)(k) \;=\; \mu \circ (\mu \otimes S) \circ (\mathrm{id}_H \otimes \tau_{H,H}) \circ (\triangle \otimes \mathrm{id}_H)(h \otimes k).$$

Recall the notions of an H-module algebra and an H-module coalgebra: given a Hopf algebra H and an associative algebra (A, η, μ), A is called a *left H-module algebra* in case the vector space underlying A is a left H-module and the action is compatible with the algebra structure in the sense that μ and η are morphisms of H-modules (where H acts on $A \otimes A$ and k using $\triangle_H$ and ϵ_H respectively). Similarly, a coalgebra $(C, \epsilon, \triangle)$ is a *left H-comodule coalgebra* in case the vector space C carries a left coaction of H and ϵ and $\triangle$ are morphisms of H-comodules. In an analogous way, one defines a *left H-comodule algebra* and a *left H-module coalgebra*. Some of the corresponding diagrams are spelt out in detail in the book of Kassel [Kas95]. Everything extends in an obvious way to right modules.

Definition 1. *Let $\gamma : B \to H$ be a morphism of Hopf algebras. The morphism $\gamma : B \to H$ is a crossed module of Hopf algebras in case*

(i) (B, ϕ_B) is a left H-module and a left H-module algebra and H-module coalgebra,

(ii) $\gamma \circ \phi_B \;=\; \mu_H \circ (\mu_H \otimes S_H) \circ (\mathrm{id}_H \otimes \tau_{H,H}) \circ (\triangle_H \otimes \gamma)$, or $\gamma \circ \phi_B \;=\; \mathrm{ad}_H \circ (\mathrm{id}_H \otimes \gamma)$ in shorthand notation, i.e. γ is a morphism of H-modules where H carries the H-module structure given by the adjoint action ad_H,

(iii) $\phi_B \circ (\gamma \otimes \mathrm{id}_B) \;=\; \mu_B \circ (\mu_B \otimes S_B) \circ (\mathrm{id}_B \otimes \tau_{B,B}) \circ (\triangle_B \otimes \mathrm{id}_B)$, or $\phi_B \circ (\gamma \otimes \mathrm{id}_B) \;=\; \mathrm{ad}_B$ in shorthand notation (This is sometimes called Peiffer identity.)

Definition 2. *A morphism $(\rho, \sigma) : (\gamma : B \to H) \to (\gamma' : B' \to H')$ between crossed modules of Hopf algebras $\gamma : B \to H$ and $\gamma' : B' \to H'$ is a pair of*

morphisms of Hopf algebras $\rho : B \to B'$ and $\sigma : H \to H'$ such that the following diagrams commute:

$$\begin{tikzcd} B \arrow[r, "\rho"] \arrow[d, "\gamma"] & B' \arrow[d, "\gamma'"] \\ H \arrow[r, "\sigma"] & H' \end{tikzcd} \qquad \begin{tikzcd} B \otimes H \arrow[r, "\rho\otimes\sigma"] \arrow[d, "\phi_B"] & B' \otimes H' \arrow[d, "\phi'_B"] \\ B \arrow[r, "\rho"] & B' \end{tikzcd}$$

It is easy to show that crossed modules of Hopf algebras form a category with respect to this notion of morphisms.

Remark 1.

(a) Observe that the definition is not autodual: B carries only the structure of an H-module - the definition does not demand a comodule structure.

(b) Observe that one cannot associate naively a four term exact sequence of Hopf algebras to a crossed module of Hopf algebras: condition (ii) does not imply that the image of γ is an associative ideal (while it is always a coideal).

Remark 2.
This definition differs from definition 12 given in [FLN07]: we do not make any assumption on cocommutativity nor impose constraints coming from compatibility with cocommutativity (condition (i) of definition 12 in [FLN07]). We do not ask the antipode S_B to be a morphism of H-modules either (condition (iii) of definition 12 in [FLN07]).

Let us denote `Lie`, `Hopf`, `Grp`, `algGrp` the categories of k-Lie algebras, resp. k-Hopf algebras, resp. groups, resp. (connected) algebraic groups defined over $\mathbb{C}$. Speaking of algebraic groups will always imply that the ground field k is $\mathbb{C}$, the field of complex numbers. Let us furthermore denote by `ccHopf` and `cHopf` the full subcategories of `Hopf` consisting of cocommutative resp. commutative Hopf algebras. Recall the enveloping functor

$$U : \texttt{Lie} \to \texttt{Hopf},$$

which associates to a Lie algebra $\mathfrak{g}$ its universal enveloping algebra $U\mathfrak{g}$; the functor of primitives

$$P : \texttt{Hopf} \to \texttt{Lie},$$

which associates to a Hopf algebra its Lie algebra of primitive elements (i.e. the $h \in H$ such that $\triangle h = 1 \otimes h + h \otimes 1$); the functor of regular functions

$$k[-] : \texttt{algGrp} \to \texttt{cHopf},$$

which associates to a connected algebraic group G its Hopf algebra of regular functions $k[G]$; the functor of characters

$$\chi : \texttt{Hopf} \to \texttt{Grp},$$

which associates to a Hopf algebra H the group of its characters $\chi(H)$ (i.e. of algebra morphisms from H to k). Recall that a commutative Hopf algebra H which is finitely generated as an algebra gives rise to an affine algebraic group, see [Swe69], section 6.3, p. 123. The associated algebraic group is $\chi(H)$. Recall furthermore that the functors U and P are equivalences of categories in characteristic 0 when one restricts to the full subcategory of *irreducible* cocommutative Hopf algebras (see [Swe69] theorem 13.0.1, p. 274).

2 On 2-groups

The subject of this section is the categorification of the notion of a group, a categorified group being a 2-group. This matter is well known and we refer to [Mac97], chapter XII, [Lod82] and [Por09].

To categorify an algebraic notion, one looks at this kind of object not in the category of sets, but in the category of (small) categories. For example, a categorified group, or 2-group, is a group object in the category of categories. Amazingly, this is the same as a category object in the category of groups, see [Mac97] p. 269.

Definition 3. *A 2-group is a category object in the category* `Grp`*, i.e. it is the data of two groups G_0, the group of objects, and G_1, the group of arrows, together with group homomorphisms $s, t : G_1 \to G_0$, source and target, $i : G_0 \to G_1$, inclusion of identities, and $m : G_1 \times_{G_0} G_1 \to G_1$, the categorical composition (of arrows) which satisfy the usual axioms of a category.*

We should emphasize, however, that the 2-groups introduced here are *strict* 2-groups, i.e. in the categorified version all laws are verified as equations. In other categorifications, one relaxes some or all laws to hold only up to natural transformation, transformations which should then satisfy coherence conditions. This leads to much more general notions (like coherent 2-groups, weak 2-groups, see [BaeLau04]), but does not occupy us here.

Recall the notion of a morphism between strict 2-groups:

Definition 4. *A morphism $F : (G_0, G_1) \to (H_0, H_1)$ between 2-groups (G_0, G_1) and (H_0, H_1) is a functor internal to the category* `Grp`*, i.e. the data of morphisms of groups $F_0 : G_0 \to H_0$ and $F_1 : G_1 \to H_1$ such that the following diagrams commute:*

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "F_1 \times F_1"] \arrow[d, "m_G"] & H_1 \times_{H_0} H_1 \arrow[d, "m_H"] \\ G_1 \arrow[r, "F_1"] & H_1 \end{tikzcd} \qquad \begin{tikzcd} G_1 \arrow[r, "s,t"] \arrow[d, "F_1"] & G_0 \arrow[r, "i"] \arrow[d, "F_0"] & G_1 \arrow[d, "F_1"] \\ H_1 \arrow[r, "s,t"] & H_0 \arrow[r, "i"] & H_1 \end{tikzcd}$$

With this notion of morphisms, strict 2-groups form a category. 2-groups also form a 2-category (cf [Por09], [Mac97]), but we will stick to the easiest categorical framework in which our article works.

The notion of a crossed module of groups has already been defined in the introduction. We recall it here for the convenience of the reader:

Definition 5. *A crossed module of groups is a homomorphism of groups $\mu : M \to N$ together with an action α of N on M by automorphisms, denoted by $\alpha : m \mapsto^n m$ for $n \in N$ and $m \in M$, such that*

(a) $\mu({}^n m) = n\mu(m)n^{-1}$ *and*

(b) ${}^{\mu(m)}m' = mm'm^{-1}$.

Recall the notion of a morphism of crossed modules of groups:

Definition 6. *A morphism $(\rho, \sigma) : (\mu : M \to N) \to (\mu' : M' \to N')$ between crossed modules of groups $(\mu : M \to N)$ and $(\mu' : M' \to N')$ is a pair of group homomorphisms $\rho : M \to M'$ and $\sigma : N \to N'$ such that the following diagrams commute:*

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\mu"] & M' \arrow[d, "\mu'"] \\ N \arrow[r, "\sigma"] & N' \end{tikzcd} \qquad \begin{tikzcd} M \times N \arrow[r, "\rho\times\sigma"] \arrow[d, "\alpha"] & M' \times N' \arrow[d, "\alpha'"] \\ M \arrow[r, "\rho"] & M' \end{tikzcd}$$

Crossed modules form a category with this notion of morphisms. They also form a 2-category, as explained for example in [Por09]. The following theorem can be found in [Mac97], chapter XII, [Lod82], and in much more detail in [Por09].

Theorem 2. *The categories of 2-groups and of crossed modules of groups are equivalent.*

Remark 3.
One interesting point about this theorem is that the categorical composition $m : G_1 \times_{G_0} G_1 \to G_1$ does not represent additional structure, but is already encoded in the group law of G_1, namely, one has

$$g \circ f := m(f, g) = f(i(b))^{-1} f,$$

where $t(f) = b$; this formula (which involves only the group multiplication in G_1 on the RHS) is shown, for example, in chapter XII of [Mac97]. Thus the data of two groups G_0, G_1 and morphisms $s, t : G_1 \to G_0$ and $i : G_0 \to G_1$ satisfying the usual axioms of source, target and object inclusion in a category is already equivalent to the data of a crossed module.

3 On Lie 2-algebras

Here we recall in the same way as in the previous section the correspondence between crossed modules of Lie algebras and strict Lie 2-algebras. Note that this correspondence does not occur as stated in the literature, but is treated for semi-strict Lie 2-algebras in [BaeCra04].

In order to define a Lie 2-algebra, we first define a 2-vector space (over the ground field k).

Definition 7. *A 2-vector space is a category object in the category* `Vect` *of vector spaces, i.e. it is the data of a vector space V_1 of arrows, a vector space V_0 of objects, and of linear maps $s, t : V_1 \to V_0$, $i : V_0 \to V_1$ and $m : V_1 \times_{V_0} V_1 \to V_1$ satisfying the usual axioms of a category.*

2-vector spaces come together with their natural notion of morphisms:

Definition 8. *A morphism $F : V \to V'$ between 2-vector spaces V and V' is a linear functor, i.e. the data of two linear maps $F_0 : V_0 \to V_0'$ and $F_1 : V_1 \to V_1'$ such that the following diagrams commute:*

$$\begin{tikzcd} V_1 \times_{V_0} V_1 \arrow[r, "F_1 \times F_1"] \arrow[d, "m_V"] & V_1' \times_{V_0'} V_1' \arrow[d, "m_{V'}"] \\ V_1 \arrow[r, "F_1"] & V_1' \end{tikzcd} \qquad \begin{tikzcd} V_1 \arrow[r, "s,t"] \arrow[d, "F_1"] & V_0 \arrow[r, "i"] \arrow[d, "F_0"] & V_1 \arrow[d, "F_1"] \\ V_1' \arrow[r, "s,t"] & V_0' \arrow[r, "i"] & V_1' \end{tikzcd}$$

Once again, the fact that we are dealing with strict 2-vector spaces implies that we have a category of 2-vector spaces (with the above notion of morphisms).

It is shown in lemma 6 of [BaeCra04] that once again, the categorical composition m is redundant data, i.e. can be recovered by the vector space structure. This can also be seen as a special case of the corresponding result of the previous section. More explicitly, writing elements $f, g \in V_1$ as $f = \vec{f} + i(s(f))$ and $g = \vec{g} + i(s(g))$, one has

$$g \circ f \;:=\; m(f, g) \;=\; i(s(f)) + \vec{f} + \vec{g}$$

(Note that we have the usual convention for the composition of maps, while Baez and Crans have the categorical convention.)

Definition 9. *A Lie 2-algebra is a category object in the category* `Lie` *of Lie algebras, i.e. it is the data of a Lie algebra $\mathfrak{g}_1$ of arrows, a Lie algebra $\mathfrak{g}_0$ of objects, and of Lie algebra morphisms $s, t : \mathfrak{g}_1 \to \mathfrak{g}_0$, $i : \mathfrak{g}_0 \to \mathfrak{g}_1$ and $m : \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$ satisfying the usual axioms of a category.*

Lie 2-algebras come together with the appropriate notion of morphisms:

Definition 10. *A morphism $F : \mathfrak{g} \to \mathfrak{h}$ between (strict) Lie 2-algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a morphism of the underlying 2-vector spaces such that the linear maps F_0 and F_1 are morphisms of Lie algebras.*

Comparing this definition to the corresponding one in [BaeCra04] (def. 23), one sees that F_2 is always the identity in our setting.

In particular, a Lie 2-algebra is a 2-vector space and the categorical composition $m : \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$ can be recovered from the 2-vector space structure (the fact that the so-defined m is a morphism of Lie algebras is easily verified).

Let us also recall the definition of a crossed module of Lie algebras, for more information on these see [Wag06].

Definition 11. *A crossed module of Lie algebras is a homomorphism of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action, denoted $\alpha(m, n) = n \cdot m$, of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations such that for all $m, m' \in \mathfrak{m}$ and all $n \in \mathfrak{n}$*

(a) $\mu(n \cdot m) = [n, \mu(m)]$ *and*

(b) $\mu(m) \cdot m' = [m, m']$.

Definition 12. *A morphism $(\rho, \sigma) : (\mu : \mathfrak{m} \to \mathfrak{n}) \to (\mu' : \mathfrak{m}' \to \mathfrak{n}')$ of crossed morphisms of Lie algebras is a pair of morphisms of Lie algebras $\rho : \mathfrak{m} \to \mathfrak{m}'$ and $\sigma : \mathfrak{n} \to \mathfrak{n}'$ such that the following diagrams commute:*

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\rho"] \arrow[d, "\mu"] & \mathfrak{m}' \arrow[d, "\mu'"] \\ \mathfrak{n} \arrow[r, "\sigma"] & \mathfrak{n}' \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{m} \times \mathfrak{n} \arrow[r, "\rho\times\sigma"] \arrow[d, "\alpha"] & \mathfrak{m}' \times \mathfrak{n}' \arrow[d, "\alpha'"] \\ \mathfrak{m} \arrow[r, "\rho"] & \mathfrak{m}' \end{tikzcd}$$

Theorem 3. *The categories of strict Lie 2-algebras and of crossed modules of Lie algebras are equivalent.*

Proof. Given a Lie 2-algebra $s, t : \mathfrak{g}_1 \to \mathfrak{g}_0$, the corresponding crossed module is defined by

$$\mu := t|_{\ker(s)} \,:\, \mathfrak{m} := \ker(s) \to \mathfrak{n} := \mathfrak{n}.$$

The action of $\mathfrak{n}$ on $\mathfrak{m}$ is given by

$$n \cdot m := [i(n), m],$$

for $n \in \mathfrak{n}$ and $m \in \mathfrak{m}$ (where the bracket is taken in $\mathfrak{g}_1$). This is well defined and an action by derivations. Axiom (a) follows from

$$\mu(n \cdot m) = \mu([i(n), m]) = [\mu \circ i(n), \mu(m)] = [n, \mu(m)].$$

Axiom (b) follows from

$$\mu(m) \cdot m' = [i \circ \mu(m), m'] = [i \circ t(m), m']$$

by writing $i \circ t(m) = m + r$ for $r \in \ker(t)$ and by using that $\ker(t)$ and $\ker(s)$ in a Lie 2-algebra commute (shown in Lemma 1 after the proof).

On the other hand, given a crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, associate to it

$$s, t \,:\, \mathfrak{n} \ltimes \mathfrak{m} \to \mathfrak{n}$$

by $s(n,m) = n$, $t(n,m) = \mu(m)+n$, $i(n) = (n,0)$, where the semi-direct product Lie algebra $\mathfrak{n}\ltimes\mathfrak{m}$ is built from the given action of $\mathfrak{n}$ on $\mathfrak{m}$ by $[(n_1,m_1),(n_2,m_2)] := ([n_1,n_2]_{\mathfrak{n}}, n_1\cdot m_2 - n_2\cdot m_1)$. Let us emphasize that this semi-direct product concerns *only* the Lie algebra structure on $\mathfrak{n}$ and the $\mathfrak{n}$-module structure on $\mathfrak{m}$; the bracket on $\mathfrak{m}$ is lost. (Nevertheless, the bracket on $\mathfrak{m}$ in a crossed module $\mu : \mathfrak{m} \to \mathfrak{n}$ is encoded in the action and the morphism μ by axiom (b).) The composition of arrows is already encoded in the underlying structure of 2-vector space, as remarked before the statement of the theorem. □

Lemma 1. $[\ker(s), \ker(t)] = 0$ *in a strict Lie* 2*-algebra.*

Proof. The fact that the composition of arrows is a homomorphism of Lie algebras gives the following "middle four exchange" property

$$[g_1, g_2] \circ [f_1, f_2] = [g_1 \circ f_1, g_2 \circ f_2]$$

for composable arrows $f_1, f_2, g_1, g_2 \in \mathfrak{g}_1$. Now suppose that $g_1 \in \ker(s)$ and $f_2 \in \ker(t)$. Then denote by f_1 and by g_2 the identity (w.r.t. the composition) in $0 \in \mathfrak{g}_0$. As these are identities, we have $g_1 = g_1 \circ f_1$ and $f_2 = g_2 \circ f_2$. On the other hand, i is a morphism of Lie algebras and sends $0 \in \mathfrak{g}_0$ to the $0 \in \mathfrak{g}_1$. Therefore we may conclude

$$[g_1, f_2] = [g_1 \circ f_1, g_2 \circ f_2] = [g_1, g_2] \circ [f_1, f_2] = 0.$$

□

4 Crossed modules of Lie and Hopf algebras

The goal of this section is to study the compatibility of the notion of crossed module with the standard functors U and P between the categories of Lie algebras and cocommutative Hopf algebras. The main proposition shows how to associate a Hopf 2-algebra to a Lie 2-algebra.

Lemma 2. *An action of a Lie algebra* $\mathfrak{g}$ *by derivations (of the Lie bracket) on a Lie algebra* $\mathfrak{h}$ *extends to an action by derivations (of the associative product) of* $U\mathfrak{g}$ *on* $U\mathfrak{h}$*.*

Proof. This is prop. 2.4.9 (i), p. 81, in [Dix74]. □

Lemma 3. *Let* $\mathfrak{g}$ *be a Lie algebra. An associative algebra* A *is an* $U\mathfrak{g}$*-module algebra if and only if the vector space* A *is a* $\mathfrak{g}$*-module such that elements of* $\mathfrak{g}$ *act by derivations (of the associative product of* A*).*

Proof. This is lemma V.6.3, p. 108, in [Kas95]. □

Proposition 1. *The functor* U *(see section* 1*) extends to a functor (still denoted* U*) from the category of crossed modules of Lie algebras to the catoegory of crossed modules of (cocommutative) Hopf algebras.*

Proof. Let $\mu : \mathfrak{m} \to \mathfrak{n}$ be a crossed module of Lie algebras. By functoriality, we get a homomorphism of associative algebras $\gamma := U\mu : U\mathfrak{m} \to U\mathfrak{n}$. By the previous two lemmas, $U\mathfrak{n}$ acts (by derivations of the associative product) on $U\mathfrak{m}$ and with this action, $U\mathfrak{m}$ becomes an $U\mathfrak{n}$-module algebra. Let us show that $U\mathfrak{m}$ is also an $U\mathfrak{n}$-module coalgebra, i.e. that the coproduct $\triangle_{U\mathfrak{m}}$ and the counit $\epsilon_{U\mathfrak{m}}$ are $U\mathfrak{n}$-module homomorphisms. Recall for this the action of $n \in U\mathfrak{n}$ on the tensor product

$$n \cdot m \otimes m' := \triangle_{U\mathfrak{n}} n \cdot m \otimes m',$$

where $m, m' \in U\mathfrak{m}$. Now for primitive elements $m \in \mathfrak{m} \subset U\mathfrak{m}$ and $n \in \mathfrak{n} \subset U\mathfrak{n}$, one has

$$\triangle_{U\mathfrak{n}} n \cdot \triangle_{U\mathfrak{m}} m = \triangle_{U\mathfrak{m}}(n \cdot m),$$

because $\mathfrak{n}$ acts trivially on k. The general case is obtained using induction and the fact that the coproducts are algebra morphisms. The counit $\epsilon : U\mathfrak{m} \to k$ is clearly a morphism.

It remains to show properties (ii) and (iii). These two follow from the properties (a) and (b) of a crossed module (see definition 11). Indeed, on primitives, identity (ii) *is* identity (a) and identity (iii) *is* identity (b). The general case follows from induction.

One word about the morphism-side of the statement: a morphism of crossed modules $(\rho, \sigma) : (\mu : \mathfrak{m} \to \mathfrak{n}) \to (\mu' : \mathfrak{m}' \to \mathfrak{n}')$ is sent to a morphism of crossed modules of Hopf algebras $(U\rho, U\sigma) : (U\mu : U\mathfrak{m} \to U\mathfrak{n}) \to (U\mu' : U\mathfrak{m}' \to U\mathfrak{n}')$. This renders the first diagram in Definition 2 commutative. In order to render the second one commutative, one should use the natural isomorphism $\zeta_{\mathfrak{m},\mathfrak{n}} : U(\mathfrak{m} \times \mathfrak{n}) \to U\mathfrak{m} \otimes U\mathfrak{n}$ and the actions $\phi_{U\mathfrak{m}} : U\mathfrak{n} \otimes U\mathfrak{m} \to U\mathfrak{m}$ given by $\phi_{U\mathfrak{m}} = \alpha \circ \zeta^{-1}_{\mathfrak{m},\mathfrak{n}}$, where α is the action of $\mathfrak{n}$ on $\mathfrak{m}$ from the crossed module. □

Proposition 2. *The functor P (see section 1) extends to a functor (still denoted P) from the category of crossed modules of Hopf algebras to the category of crossed modules of Lie algebras.*

Proof. Let $\gamma : B \to H$ be a crossed module of Hopf algebras (definition 1). The set of primitives $P(B)$ and $P(H)$ of B and H are Lie algebras, and the restriction of γ to $P(B)$ is a Lie algebra morphism $\gamma : P(B) \to P(H)$. By hypothesis, we have a map $\zeta : H \otimes B \to B$ which is an action of H on B. Restrict it to $\bar{\zeta} : H \otimes P(B) \to B$. $\bar{\zeta}$ takes its values in $P(B)$, because $(\star)$ the coproduct of B is a morphism of H-modules:

$$\begin{aligned}
\Delta(h \cdot b) &\overset{(\star)}{=} \Delta(h) \cdot \Delta b \\
&= (\sum h' \otimes h'') \cdot (\sum b' \otimes b'') \\
&\overset{b \in P(B)}{=} \sum (h' \cdot 1 \otimes h'' \cdot b + h' \cdot b \otimes h'' \cdot 1) \\
&= \sum (\epsilon(h') \cdot 1 \otimes h'' \cdot b + h' \cdot b \otimes \epsilon(h'') \cdot 1) \\
&= 1 \otimes (\sum (\epsilon(h')h'') \cdot b) + ((\sum \epsilon(h'')h') \cdot b) \otimes 1) \\
&= 1 \otimes (h \cdot b) + (h \cdot b) \otimes 1)
\end{aligned}$$

Then we may restrict $\bar{\zeta}$ further to $\tilde{\zeta} : P(H) \otimes P(B) \to P(B)$. The "associativity" of $\tilde{\zeta}$ is clear (and implies the property of a Lie algebra action), but we have to show that the action $\tilde{\zeta}$ is by derivations. This follows from the hypothesis that the multiplication of B is a morphism of H-modules.

As we have already mentioned, properties (ii) and (iii) of a crossed module of Hopf algebras imply, when restricted to the primitives, properties (a) and (b) of a crossed module of Lie algebras.

The morphism-side of the statement is straight forward. □

Remark 4.
Let us deduce from the above propositions some equivalence between irreducible cocommutative Hopf 2-algebras over a field of characteristic 0 and enveloping 2-algebras.

Indeed, it is well known that the functors U and P are equivalences of categories in characteristic 0 when one restricts the category of Hopf algebras to the full subcategory of irreducible cocommutative Hopf algebras (see [Swe69] theorem 13.0.1, p. 274). Call a Hopf 2-algebra *irreducible* in case its corresponding crossed module of Hopf algebras is composed of irreducible Hopf algebras. One deduces then from proposition 1:

Theorem 4. *An irreducible cocommutative Hopf 2-algebra is isomorphic to an enveloping Hopf 2-algebra, i.e. to a Hopf 2-algebra of the form $U(\mu : \mathfrak{m} \to \mathfrak{n})$, where $\mu : \mathfrak{m} \to \mathfrak{n}$ is a crossed module of Lie algebras.*

Remark 5.
In a very similar manner as the proofs of propositions 1 and 2, it is easy to show that in case $\mu : M \to N$ is a crossed module of groups, the induced morphism between group algebras $\mu : kM \to kN$ is a crossed module of Hopf algebras. The fact that kM is a kN-module algebra comes from the fact that the N-action on M is by automorphisms, and that kM is a kN-module coalgebra is always the case when one linearizes an action of a group on a set, see [Kas95] p.203. Taking group-like elements is the way back to the crossed module of groups.

For infinite groups, the corresponding group algebras do not (in general) reflect all features of the given group. Therefore one often passes to topological versions, like the C^*-algebra associated to a group. The correspondence between crossed modules of groups and crossed modules of their group algebras respects this kind of topological versions. In case one sees the C^*-algebra associated to a group not as some completion of the group algebra, but as a space of characters on the group, it is closer to the constructions in section 5, but the same remark applies.

As an example, consider the group S^1 and the dense subgroup $\mathbb{Z}$ which is the image of the embedding $\mathbb{Z} \to S^1$, $k \mapsto e^{i\lambda k}$ where $\lambda / 2\pi$ is an irrational number. The quotient $S^1 / \mathbb{Z}$ is an abelian group, but carries the discrete topology, thus the associated C^*-algebra is trivial. It is explained in [BloWei08] (see also references therein) that a way to associate meaningful C^*-algebras to this example goes under the name *Hopfish algebras*. We will not go into the definition of a Hopfish algebra and refer to *loc. cit.*.

The point we want to make here (we owe this remark to Chenchang Zhu) is that another way around the problem of associating a C^*-algebra to the quotient $S^1 / \mathbb{Z}$ is to take the crossed module of C^*-algebras associated to the crossed module of groups $\mathbb{Z} \to S^1$. Its corresponding 2-group (see section 1) is then $S^1 \ltimes \mathbb{Z} \to S^1$ (which may be regarded as a groupoid). In other words, this crossed module of C^*-algebras replaces the C^*-algebra of the groupoid $S^1 \ltimes \mathbb{Z} \to S^1$. A more precise link between the C^*-algebra of the groupoid and the crossed module of C^*-algebras needs to be investigated. Note that this framework applies for any group G and for any normal subgroup N to the crossed module $N \to G$ or the corresponding 2-group/groupoid $G \ltimes N \to G$.

Remark 6.
Let us comment on the possibility of a "standard cohomological framework" for crossed modules of Hopf algebras. In the case of Lie algebras, (discrete) groups or associative algebras, taking sections one may associate to a crossed module a 3-cohomology class which expresses the equivalence class of the given crossed module. As we stated before, this scheme needs modification in the context of Hopf algebras. One way around this problem is to transfer using the functor U and proposition 1 the equivalence relation for crossed modules of Lie algebras and their cohomology classes directly to Hopf algebras. This may not be the most canonical way to do so, but it sets at least compatibility conditions which one might want to impose on some cohomological description of crossed modules of Hopf algebras.

Remark 7.
Let us emphasize that Baez and Crans [BaeCra04], section 6, thought of $\{\mathfrak{g}_\hbar\}_\hbar$ as a one-parameter deformation of Lie 2-algebras of the "trivial" Lie 2-algebra $\mathfrak{g}$. From this point of view, the family of crossed modules of Hopf algebras $U\mathfrak{g}_\hbar$ may be regarded as a quantum 2-group, i.e. a non-trivial deformation of Hopf 2-algebras.

In order to explain in which sense this deformation is non-trivial, we need the 2-categorical structure. Each semi-strict Lie 2-algebra $\mathfrak{g}_\hbar$ is non-strict and only (non-trivially) equivalent to a strict one ("strictification"), while $\mathfrak{g}_0$ is strict.

5 Crossed (co)modules of groups and Hopf algebras

The goal of this section is to study the compatibility of the notion of crossed module with the standard functors $k[-]$ and χ between the categories of algebraic groups and Hopf algebras. This needs the introduction of the new notion of *crossed comodule* of Hopf algebras (definition 13) dual to the concept of crossed module of Hopf algebras. The main results of this section (propositions 3 and 4) together with classical equivalence between algebraic goups and their rings of functions give a correspondence between commutative Hopf 2-algebras (in their incarnation as crossed comodules) and connected algebraic 2-groups.

Definition 13 (Definition of a crossed comodule of Hopf algebras.)**.** *Let $\zeta : K \to L$ be a morphism of Hopf algebras. The morphism ζ defines a crossed comodule of Hopf algebras in case*

(i) (L, ρ_L) *is at the same time a left K-comodule algebra and a left K-comodule coalgebra,*

(ii) $\rho_L \circ \zeta \;=\; (\mathrm{id}_K \otimes \zeta) \circ \mathrm{coad}_K$,

(iii) $(\zeta \otimes \mathrm{id}_L) \circ \rho_L \;=\; \mathrm{coad}_L$.

Definition 14. *A morphism $(\rho, \sigma) : (\zeta : K \to L) \to (\zeta' : K' \to L')$ of crossed comodules of Hopf algebras $\zeta : K \to L$ and $\zeta' : K' \to L'$ is a pair of morphisms of Hopf algebras $\rho : K \to K'$ and $\sigma : L \to L'$ such that the following diagrams are commutative:*

$$\begin{tikzcd} K \arrow[r, "\rho"] \arrow[d, "\zeta"] & K' \arrow[d, "\zeta'"] \\ H \arrow[r, "\sigma"] & H' \end{tikzcd} \qquad\qquad \begin{tikzcd} L \arrow[r, "\sigma"] \arrow[d, "\rho_L"] & L' \arrow[d, "\rho_{L'}"] \\ K \otimes L \arrow[r, "\rho\otimes\sigma"] & K' \otimes L' \end{tikzcd}$$

It is easy to show that crossed comodules form a category with this notion of morphisms.

We recall for (i) that, by definition, (L, ρ_L) is a left K-comodule coalgebra if Δ_L is a comodule morphism, i.e. satisfies $(\mathrm{id}_K \otimes \Delta_L) \circ \rho_L \overset{\diamondsuit}{=} \rho_{L\otimes L} \circ \Delta_L$ with $\rho_{L\otimes L} := (\mu_K \otimes \mathrm{id}_L \otimes \mathrm{id}_L) \circ (\mathrm{id}_K \otimes \tau \otimes \mathrm{id}_L) \circ (\rho_L \otimes \rho_L)$.

In (ii) and (iii) coad_L and coad_K denote the adjoint coaction (and not the coadjoint action !) defined as $\mathrm{coad}_L \;:=\; (\mu_L \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \tau) \circ (\triangle_L \otimes S_L) \circ \triangle_L$.

The need for definition 13 relies on the fact that the notion of a Hopf algebra is auto-dual, while that of a crossed module is not. The existence of both notions of crossed modules and comodules shows that there are (at least) two natural ways to categorify the notion of a Hopf algebra.

The next proposition generates natural examples of crossed comodules of Hopf algebras, namely when one considers the space of functions on an algebraic 2-group.

Proposition 3. *The functor $k[-]$ extends to a functor (still denoted $k[-]$) from the category of crossed modules of connected algebraic groups to the category of crossed comodules of Hopf algebras.*

Proof. Let $\mu : M \to N$ be a crossed module of connected algebraic groups. We get a morphism of commutative Hopf algebras $\zeta := \mu^* : k[N] \to k[M]$. The action of N on M is transformed into a coaction of $k[N]$ on $k[M]$: $\rho_{k[M]} : k[M] \to k[N] \otimes k[M]$ is just the dual of the action map $N \times M \to M$. Now, $k[M]$ is a $k[N]$-comodule coalgebra, because of the fact that N acts on M by automorphisms.

On the other hand, $k[M]$ is a $k[N]$-comodule algebra, because the multiplication on $k[M]$ is the dual of the diagonal $M \to M \times M$, $m \mapsto (m, m)$.

Finally, the identities (ii) and (iii) of crossed comodule of Hopf algebras are consequences (dualization) of the properties (a) and (b) of the crossed module $\mu : M \to N$.

The morphism-part of the statement is straight forward. □

We didn't say much about the equivalence of identities (ii) and (iii) and properties (a) and (b) here: one point of view is that (a) and (b) can be formulated in terms of morphisms (suppressing objects), and then this equivalence is a formal duality, i.e. one simply applies the functor $k[-]$. Another point of view is to write everything in terms of objects – this has the advantage of being more explicit. We will follow this line in the proof of proposition 4 below.

Now, conversely, one can obtain obtain 2-groups from crossed comodules of Hopf algebras. This is the content of the following proposition.

Proposition 4. *The functor χ extends to a functor (still denoted χ) from the category of crossed comodules of commutative Hopf algebras to the category of crossed modules of groups.*

Its proof will be given after having introduced some notations and the preliminary lemma 4.

For the rest of this section, K and L will be two Hopf algebras, $\rho_L : L \mapsto K \otimes L$ a linear map, $M := \chi[L]$, $N := \chi[K]$, $\eta_K, \gamma_K \in N$, ϕ_L, $\psi_L \in M$.

One defines the product $\star_{\rho_L}$ of N on M by

$$\eta_K \star_{\rho_L} \phi_L := \mu_k \circ (\eta_K \otimes \phi_L) \circ \rho_L.$$

In case $K = L$ and $\rho_L = \Delta_L$ (resp. $L = K$ and $\rho_L = \Delta_K$, resp. $L = K$ and $\rho_L = \mathrm{coad}_K$), we will simplify the notation $\star_{\rho_L}$ to $\star_L$ (resp. $\star_K$, resp. $\star_{\widetilde{\rho_K}}$). From now on, (L, ρ_L) will be a left K-comodule algebra and coalgebra.

Lemma 4. *(α) $(M, \star_L)$ is a group with inverse $S_L^\star$,*

(β) $\star_{\rho_L}$ defines a group action of N on M,

(γ) $\star_{\rho_L}$ is an action by automorphisms,

(δ) $\eta_K \star_{\widetilde{\rho_K}} \gamma_K = \eta_K \star \gamma_K \star \eta_K^{-1}$.

Proof. (α) and (β) are proven using definitions and coassociativity of Δ_L and ρ_L. The proof of (γ), i.e. N acts on M by automorphisms, is a little more

involved and is based on the fact ($\Diamond$) that Δ_L is a comodule map:

$$
\begin{aligned}
\eta_K \star_{\rho_L} (\phi_L \star_L \psi_L) &\overset{\star_{\rho_L}}{=} \mu_k(\eta_K \otimes (\phi_L \star_L \psi_L)) \circ \rho_L \\
&\overset{\star_L}{=} \mu_k(\mathrm{id}_k \otimes \mu_k) \circ (\eta_K \otimes \phi_L \otimes \psi_L) \circ (\mathrm{id}_K \otimes \Delta_L) \circ \rho_L \\
&\overset{(\Diamond)}{=} \mu_k(\mathrm{id}_k \otimes \mu_k) \circ (\eta_K \otimes \phi_L \otimes \psi_L) \circ \rho_{L\otimes L} \circ \Delta_L \\
&\overset{\rho_{L\otimes L}}{=} \mu_k(\mathrm{id}_k \otimes \mu_k) \circ (\eta_K \otimes \phi_L \otimes \psi_L) \\
&\qquad \circ(\mu_K \otimes \mathrm{id}_L \otimes \mathrm{id}_L) \circ (\mathrm{id}_K \otimes \tau \otimes \mathrm{id}_L) \circ (\rho_L \otimes \rho_L) \circ \Delta_L \\
&\overset{\eta_K}{=} \mu_k(\mathrm{id}_k \otimes \mu_k) \circ (\mu_k(\eta_K \otimes \eta_K) \otimes \phi_L \otimes \psi_L) \\
&\qquad \circ(\mathrm{id}_K \otimes \tau \otimes \mathrm{id}_L) \circ (\rho_L \otimes \rho_L) \circ \Delta_L \\
&\overset{\tau}{=} \mu_k(\mu_k \otimes \mu_k) \circ (\eta_K \otimes \phi_L \otimes \eta_K \otimes \psi_L) \circ (\rho_L \otimes \rho_L) \circ \Delta_L \\
&\overset{\star_L}{=} (\mu_k \circ (\eta_K \otimes \phi_L) \circ \rho_L) \star_L (\mu_k \circ (\eta_K \otimes \psi_L) \circ \rho_L) \\
&\overset{\star_{\rho_L}}{=} (\eta_K \star_{\rho_L} \phi_L) \star_L (\eta_K \star_{\rho_L} \psi_L).
\end{aligned}
$$

We finally prove (δ):

$$
\begin{aligned}
\eta_K \star_{\rho_K} \gamma_K &\overset{\star_{\rho_K}}{=} \mu_k(\eta_K \otimes \gamma_K) \circ \mathrm{coad}_K \\
&\overset{coad_K}{=} \mu_k(\eta_K \otimes \gamma_K) \circ (\mu_K \otimes \mathrm{id}_K) \circ (\mathrm{id}_K \otimes \tau) \circ (\triangle_K \otimes S_K) \circ \triangle_K \\
&\overset{\mu_K}{=} \mu_k(\mu_k \otimes \mathrm{id}_k) \circ (\eta_K \otimes \eta_K \otimes \gamma_K) \circ (\mathrm{id}_K \otimes \tau) \circ (\triangle_K \otimes S_K) \circ \triangle_K \\
&\overset{\tau}{=} \mu_k(\mu_k \otimes \mathrm{id}_k) \circ (\eta_K \otimes \gamma_K \otimes \eta_K) \circ (\triangle_K \otimes S_K) \circ \triangle_K \\
&\overset{\star_K}{=} \eta_K \star \gamma_K \star \eta_K^{-1}.
\end{aligned}
$$

□

We can now turn to the proof of proposition 4.

Proof. Let $\zeta : K \to L$ be a crossed comodule of commutative Hopf algebras K and L. By functoriality, we get a group homomorphism $\mu := \zeta^* : M \to N$. We already know from lemma 4 that $\star_{\rho_L}$ is an action by automorphisms, so it only remains to show that identities (a) and (b) hold.

Identity (a) is obtained from axiom (ii):

$$\begin{aligned}
\mu(\eta_K \star_{\rho_L} \psi_L) &\overset{\mu}{=} (\eta_K \star_{\rho_L} \psi_L) \circ \zeta \\
&\overset{\star_{\rho_L}}{=} \mu_k(\eta_K \otimes \psi_L) \circ \rho_L \circ \zeta \\
&\overset{(ii)}{=} \mu_k(\eta_K \otimes \psi_L) \circ (\mathrm{id}_K \otimes \zeta) \circ \mathrm{coad}_K \\
&\overset{\mu}{=} \mu_k(\eta_K \otimes \mu(\psi_L)) \circ \mathrm{coad}_K \\
&\overset{\star_{\widetilde{\rho_K}}}{=} (\eta_K \star_{\rho_K} \mu(\psi_L))
\end{aligned}$$

Identity (b) is obtained from axiom (iii) and lemma 4 (ρ):

$$\begin{aligned}
\mu(\phi_L) \star_{\rho_L} \psi_L &\overset{\mu}{=} (\phi_L \circ \zeta) \star_{\rho_L} \psi_L \\
&\overset{\star_{\rho_L}}{=} \mu_k(\phi_L \otimes \psi_L) \circ (\zeta \otimes id_L) \circ \rho_L \\
&\overset{(iii)}{=} \mu_k(\phi_L \otimes \psi_L) \circ \mathrm{coad}_L \\
&\overset{(\delta)}{=} \phi_L \star \psi_L \star \phi_L^{-1}.
\end{aligned}$$

□

6 Integration of Lie 2-algebras

In this section, we explain a general scheme of how to integrate (some class of) semi-strict Lie 2-algebras. This is motivated by the following example of a crossed module of Hopf algebras:

Remark 8.
Let us treat in some detail the example of the crossed module of Hopf algebras corresponding to the crossed module of Lie algebras which represents the generator of $H^3(\mathfrak{g}, \mathbb{C})$, where $\mathfrak{g}$ is any simple complex finite dimensional Lie algebra. This leads to the definition of the enveloping 2-algebra of the string Lie algebra of Baez and Crans [BaeCra04].

Denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and choose a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\mathfrak{n}$ be the nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{n} \oplus \mathfrak{h} = \mathfrak{b}$ as vector spaces. Denote by $M(\lambda)$ the Verma module of $\mathfrak{g}$ of highest weight λ. Namely $M(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module given by the trivial action of $\mathfrak{n}$ and the $\mathfrak{h}$-action via $\lambda \in \mathfrak{h}^*$. $M(\lambda)$ possesses a unique maximal proper submodule $N(\lambda)$ and the quotient $L(\lambda)$ is therefore irreducible. Denote by $M(\lambda)^\sharp$, $N(\lambda)^\sharp$ and $L(\lambda)^\sharp$ the restricted duals of these graded $\mathfrak{g}$-modules. We have by definition a short exact sequence of $\mathfrak{g}$-modules:

$$0 \to L(\lambda)^\sharp \to M(\lambda)^\sharp \to N(\lambda)^\sharp \to 0,$$

which specializes to

$$0 \to \mathbb{C} \to M(0)^\sharp \to N(0)^\sharp \to 0$$

for $\lambda = 0$. On the other hand, there is a 2-cocycle $\alpha' \in C^2(\mathfrak{g}, N(0)^\sharp)$ which gives an abelian extension

$$0 \to N(0)^\sharp \to N(0)^\sharp \times_{\alpha'} \mathfrak{g} \to \mathfrak{g} \to 0.$$

It is shown in [Wag06] that the splicing together of these two sequences gives a crossed module

$$\mu : N(0)^\sharp \to N(0)^\sharp \times_{\alpha'} \mathfrak{g},$$

which represents a generator of $H^3(\mathfrak{g}; \mathbb{C})$.

Another crossed module representing a generator of $H^3(\mathfrak{g}; \mathbb{C})$ is known [Nee06]: denoting by $P\mathfrak{g}$ and $\Omega\mathfrak{g}$ the path- and loop Lie algebra corresponding to $\mathfrak{g}$, there is a (general) extensions of Lie algebras

$$0 \to \Omega\mathfrak{g} \to P\mathfrak{g} \to \mathfrak{g} \to 0,$$

adapting the path-loop fibration to the Lie algebra setting. But a crossed module is nothing but a central extension of an ideal of some Lie algebra (cf [Nee06]), and thus the standard central extension of $\Omega\mathfrak{g}$ gives rise to a crossed module

$$0 \to \mathbb{C} \to \Omega\mathfrak{g} \to P\mathfrak{g} \to \mathfrak{g} \to 0.$$

It is shown in [Nee06] that the corresponding class generates $H^3(\mathfrak{g}; \mathbb{C})$.

Recall further from [BaeCra04] that semi-strict Lie 2-algebras are categorified Lie algebras with a functorial bracket which is strictly antisymmetric, but satisfies the Jacobi identity only up to a Jacobiator. This Jacobiator gives a 3-cocycle θ of one of the underlying Lie algebras $\mathfrak{g}$ with values in some $\mathfrak{g}$-module V. The data $(\mathfrak{g}, V, [\theta])$ then completely specifies the equivalence class of the given semi-strict Lie 2-algebra. In the end of their paper [BaeCra04], Baez and Crans construct a family $\mathfrak{g}_\hbar$ of semi-strict Lie 2-algebras whose equivalence classes are given in this sense by the triplets $(\mathfrak{g}, \mathbb{C}, [\hbar\theta])$, where here θ is the Cartan cocycle, i.e. $\theta = \langle [,], \rangle$ for $\langle , \rangle$ the Killing form of the simple complex finite dimensional Lie algebra $\mathfrak{g}$, and $\hbar$ is a scalar.

Recall now from [BCSS07] that the Lie 2-algebras $\mathfrak{g}_\hbar$ are linked to the *string group* (for $\hbar = \pm 1$). For a given simply connected, simple Lie group G, there is a topological group $\hat{G}$ obtained by killing the third homotopy group of G. This group $\hat{G}$ is, by analogy with the case of $G = \mathrm{Spin}(n)$, called the string group of G. Now the authors of [BCSS07] construct an infinite-dimensional Lie 2-group, whose Lie 2-algebra is equivalent to $\mathfrak{g}_\hbar$ and whose geometric realization is (homotopy equivalent to) $\hat{G}$ for $\hbar = \pm 1$.

Direct application of proposition 1 gives us a crossed module of Hopf algebras

$$\gamma : S(N(0)^\sharp) \to S(N(0)^\sharp) \otimes_{\hbar\alpha'} U\mathfrak{g} \tag{1}$$

Here S denotes the symmetric algebra on a k-vector space. The above mentioned crossed module corresponding to the path-loop fibration also gives rise

to an enveloping algebra, but observe that it is not as algebraic as (1) and thus less meaningful (what is the enveloping algebra of a topological Lie algebra ?) from the point of view of representation theory (while it has the advantage of being more geometric). More precisely, in our crossed module we take enveloping algebras of *at most countably infinite dimensional* vector spaces, while the underlying vector spaces in the crossed module stemming from the path-loop fibration are not countably infinite dimensional.

The construction which we cited here as an exemple works for any 3-cohomology class of a finite dimensional Lie algebra. Therefore we deduce the following theorem (cf [BaeCra04] for missing notations) directly from proposition 1.

Theorem 5. *For any skeletal semistrict Lie 2-algebra corresponding to a triple* $(\mathfrak{g}, V, [\theta] \in H^3(\mathfrak{g}, V))$*, there is a crossed module of Hopf algebras*

$$\gamma : S(I) \to S(Q) \otimes_{\alpha''} U\mathfrak{g} \tag{2}$$

which is a natural algebraic candidate for the enveloping algebra of the given Lie 2-algebra in the sense that all its underlying vector spaces are at most countably infinite dimensional.

Here Q and I are $\mathfrak{g}$-modules such that I is injective and such that there is a short exact sequence of $\mathfrak{g}$-modules:

$$0 \to V \to I \to Q \to 0.$$

The class of the 2-cocycle $\alpha'' \in Z^2(\mathfrak{g}, Q)$ corresponds to the class of the given 3-cocycle θ under the corresponding connecting homomorphism. The above construction (1) is a special case. It is well known that in the context of remark 8, there cannot be a purely finite dimensional construction in general (see [Hoch54]).

The properties of the preceeding theorem are important, because they permit to establish a duality theory for the mentioned kind of enveloping algebras for skeletal semi-strict Lie 2-algebras (up to equivalence of Lie 2-algebras, cf [BaeCra04]).

Recall how finite dimensional Lie algebras are linked to formal groups (in characteristic zero), see [Ser64]. Take a (finite dimensional real) Lie group G with unit 1, and complete the algebra of germs of functions around 1 by the adic topology. Its continuous dual may be identified to the Hopf algebra U of point distributions. U is isomorphic to $U\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Synthetically, this can be understood as the conjunction of linear duality and the Milnor-Moore theorem. In this way, Serre establishes an equivalence of categories between the category of formal groups and the category of finite dimensional Lie algebras.

Now this also works backwards: starting with a finite dimensional Lie algebra $\mathfrak{g}$, one forms the universal enveloping algebra $U\mathfrak{g}$. Its dual is the completion of a commutative Hopf algebra, from which one may extract the group by taking characters.

This scheme of reasoning obviously also works for crossed modules of Lie algebras; this is our first procedure to integrate Lie 2-algebras. One has to apply the functor U in order to obtain a crossed modules of Hopf algebras by proposition 1. Then one passes to the continuous dual, which is naturally a crossed comodule of commutative Hopf algebras (see section 5). Each constituant of this crossed comodule is the completion of the commutative Hopf algebra of functions on some formal group, or of germs of functions in an identity-neighborhood of some Lie group G. The results of section 5 now imply that this crossed comodule gives rise to a crossed module of groups, which may then be interpreted as some 2-group. In this way, the above theorem permits to integrate strict Lie 2-algebras, i.e. crossed modules of Lie algebras, where all constituants are finite dimensional Lie algebras. A general semi-strict Lie 2-algebra is always equivalent to a strict one ("strictification"), thus this scheme permits to integrate semi-strict Lie 2-algebras which admit a strictification consisting of finite dimensional Lie algebras. We summarize the above discussion in the following theorem:

Theorem 6. *Every semi-strict Lie 2-algebra which admits a strictification consisting of finite dimensional Lie algebras can be integrated into a finite dimensional strict Lie 2-group whose associated Lie 2-algebra is equivalent to the given one.*

Let us remark that Behrang Noohi has a different point of view which also leads to an integration procedure for this kind of Lie 2-algebras, see [Noo09].

Let us finally discuss the integrability of the crossed module of Lie algebras

$$\mu : N(0)^\sharp \to N(0)^\sharp \times_{\alpha'} \mathfrak{g},$$

to which the theorem does not apply. An integration of the more geometric crossed module $\Omega\mathfrak{g} \to P\mathfrak{g}$ is the object of [BCSS07]. In fact, the crossed module μ is not integrable (at least, not in a naïve way).

Indeed, as explained in [Wag06], the crossed module μ which also represents the Cartan class in $H^3\mathfrak{g}, \mathbb{C})$, is spliced together from an abelian extension

$$0 \to (U\mathfrak{g}^+)^* \to (U\mathfrak{g}^+)^* \times_\alpha \mathfrak{g} \to \mathfrak{g} \to 0,$$

and the short exact sequence of (adjoint) $\mathfrak{g}$-modules

$$0 \to \mathbb{C} \to (U\mathfrak{g})^* \to (U\mathfrak{g}^+)^* \to 0.$$

Let G be the connected, simply connected compact Lie group corresponding to $\mathfrak{g}$. The non-integrability of μ is due to the fact that the $\mathfrak{g}$-module $U\mathfrak{g}$ (with the action by left multiplication) does not integrate into a G-module.

Even if it did, the integration would not have been this simple: in fact, the resulting crossed module of groups must contain a topologically non-trivial fiber bundle. If not, there would be a globally smooth section of the crossed module, and its 3-class would be presented by a globally smooth cocycle. This

is impossible, because the smooth cohomology of G is trivial (see for example the references to Hu's, van Est's or Mostow's work in [BaeLau04]).

Notwithstanding this example, in principle crossed modules of the above kind which are spliced together from an abelian extension and a short exact sequence of modules may be integrated as follows – this is our (sketched) second integration procedure, which applies (in principle) to infinite dimensional Lie algebras: first integrate the modules to locally convex G-modules, and suppose that the 2-cocycle from the abelian extension is continuous. Integrate it into a locally smooth group 2-cocycle using [Nee04]. Obstructions to this step reside in $\pi_1(G)$ and $\pi_2(G)$ (see *loc. cit.*) and vanish therefore. Finally, splice together the exact sequences to obtain a crossed module of groups.

7 Relation to previous work on crossed modules of Hopf algebras

The only work on crossed modules of Hopf algebras we know about is [FLN07]. Its main result, theorem (14), states the equivalence of categories between the category of cat^1–Hopf algebras and the category of crossed modules of Hopf algebras. The definition of pre-cat^1–Hopf algebras that it offers can be generalised as follows:

Definition 15. *Let A and H be two Hopf algebras, $s, t : A \to H$ and $e : H \to A$ be morphisms of Hopf algebras. $\overline{A} := (A, H, s, t, e)$ is a pre-*cat^1*-Hopf algebra if*

$$s \circ e = t \circ e = id_H.$$

Remark 9.
The definition of a pre-cat^1-Hopf algebra given in [FLN07] requires in addition cocommutativity of H together with the conditions ii) $(s \otimes A) \circ \tau_{A,A} \circ \Delta_A = (s \otimes A) \circ \Delta_A$ and iii)$(t \otimes A) \circ \tau_{A,A} \circ \Delta_A = (t \otimes A) \circ \Delta_A$.

Then, in definition (9) of [FLN07], cat^1-Hopf algebras are defined as monoids in a certain monoidal category $\mathcal{PC}^1_H$ whose objects are precisely pre-cat^1-Hopf algebras in the sense of [FLN07] (we will not recall the monoidal structure here).

The proof of the equivalence between the category of cat^1-Hopf algebras and the category of crossed modules of Hopf algebras (theorem (14) in [FLN07]) is rather technical and involves abstract tools taken from other works. We propose here a different approach to this equivalence.

Theorem 7. *When restricting to the subcategory of irreducible cocommutative Hopf algebras, the categories of crossed modules of Hopf algebras and the category of pre-*cat^1*-Hopf algebra are equivalent.*

Proof. Given a pre-cat^1-Hopf algebra $s, t : A \to H$, $e : H \to A$ where H and A are irreducible cocommutative Hopf algebras (over the characteristic zero field k), we have $A = U(P(A))$ and $H = U(P(H))$ (see [Swe69] theorem 13.0.1, p. 274), and the morphisms are also induced from morphisms of Lie algebras.

Denote by $\mathfrak{g}_1 := P(A)$ and $\mathfrak{g}_0 := P(H)$, then we have a Lie 2-algebra $\mathfrak{g}_1 \to \mathfrak{g}_0$, because the categorical composition m is already encoded in the 2-vector space structure (see section 3). This Lie 2-algebra corresponds then to a crossed module of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, and applying the functor U, we get the crossed module of Hopf algebras associated to the given pre-cat^1-Hopf algebra $s, t : A \to H$ by proposition 1.

Conversely, given a crossed module of irreducible cocommutative Hopf algebras $\gamma : B \to H$, we can associate to it in the same way a pre-cat^1-Hopf algebra $s, t : A \to H$, $e : H \to A$ using proposition 2. □

Remark 10.
In the same way, one can show that the category of crossed comodules of commutative Hopf algebras and the category of commutative Hopf 2-algebras (meaning explicitly pre-cat^1-Hopf algebras with commutativity replacing the cocommutativity requirement) are equivalent when restricting to the subcategory of those Hopf algebras which are finitely generated as an algebra. It suffices to apply the functors $k[-]$ and χ to translate the problem into groups, where we then apply theorem 2.

Remark 11.
In conjunction with theorem (14) of *loc. cit.*, our proposition shows that when one restricts the monoidal category $\mathcal{P}C^1_H$ to irreducible cocommutative Hopf algebras, every object becomes a monoid. In particular, this implies that the characterization of monoids given in proposition ([FLN07], 14 ii)) should be automatically satisfied.

Remark 12.
The previous remark can be seen as the incarnation in the Hopf level of remark 3. A natural question at this stage is whether this property holds true if one drops the irreducibility property. In the terms of [FLN07], is every object of $\mathcal{P}C^1_H$ a monoid ?

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