Deformation quantization of Leibniz algebras

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Abstract

In this paper, we use the local integration of a Leibniz algebra $\mathfrak{h}$ using a Baker-Campbell-Hausdorff type formula in order to deformation quantize its linear dual $\mathfrak{h}^*$. More precisely, we define a natural rack product on the set of exponential functions on $\mathfrak{h}^*$ which extends to a rack action on $\mathcal{C}^\infty(\mathfrak{h}^*)$.

Introduction

In this paper, we solve an old problem in symplectic geometry, namely we propose a way how to quantize the dual space of a Leibniz algebra $\mathfrak{h}$. This dual space $\mathfrak{h}^*$ is some kind of generalized Poisson manifold, as the bracket of $\mathfrak{h}$ is not necessarily skew-symmetric. Intimately linked to this question is the integration of Leibniz algebras.

In the search of understanding the periodicity in K-theory, J.-L. Loday introduced Leibniz algebras as non-commutative analogues of Lie algebras. More precisely, a real Leibniz algebra is a real vector space with a bracket which satisfies the (left) Leibniz identity

$$[X,[Y,Z]] \;=\; [[X,Y],Z]+[Y,[X,Z]],$$

but is not necessarily skew-symmetric. Leibniz algebras are a well-established algebraic structure generalizing Lie algebras (those Leibniz algebras where the bracket is skew-symmetric) with their own structure, deformation and homology theory. In the same way the Lie algebra homology of matrices (over a commutative ring containing the rational numbers) defines additive K-theory (i.e. cyclic homology), the Leibniz homology of matrices defines some non-commutative additive K-theory (in fact, Hochschild homology). Loday was mainly interested

in the properties of the corresponding homology theory on "group level" ("Leibniz K-Theory"), and therefore asked the question which (generalization of the structure of Lie groups) is the correct structure to integrate Leibniz algebras ?

Kinyon [15] explored Lie racks as a structure integrating Leibniz algebras. Racks are roughly speaking an axiomatization of the structure of the conjugation in a group. The rack product on a group is simply given by

$$g \triangleright h \,:=\, ghg^{-1},$$

and a general rack product on a set X is a binary operation satisfying for all $x, y, z \in X$ that $x \triangleright - : X \to X$ is bijective and the autodistributivity relation

$$x \triangleright (y \triangleright z) \,=\, (x \triangleright y) \triangleright (x \triangleright z).$$

Lie racks are the smooth analogues of racks. Kinyon showed (see Theorem 1.25) that the tangent space at the distiguished element 1 of a Lie rack carries in a natural way a Leibniz bracket. The idea is to differentiate two times the rack structure, mimicking exactly how the conjugation in a Lie group is differentiated to give first the map Ad, the adjoint action of the group on the Lie algebra, and then the Lie bracket in terms of ad, the adjoint action of the Lie algebra on itself. He did not judge racks to be the correct objects integrating Leibniz algebras. As a reason for this, he showed that all Leibniz algebras integrate into Lie racks, but in a kind of arbitrary way, as this integration does not appear to give Lie groups in case one started with a Lie algebra. It is clear (and useful as a guiding principle) that from this point of view, integrating Leibniz algebras means just an *integration of the adjoint action* of a Leibniz algebra on itself. From here stems the most important example for us of a rack product, namely

$$X \triangleright Y \,:=\, e^{\mathrm{ad}_X}(Y),$$

for all $X, Y \in \mathfrak{h}$ for a Leibniz algebra $\mathfrak{h}$.

On the other hand, Covez [10] showed in his 2010 doctoral thesis how to adapt the homological proof of Lie's Third Theorem to Leibniz algebras. Regarding a given real Leibniz algebra $\mathfrak{h}$ as an abelian extension of the Lie algebra $\mathfrak{h}_{\mathrm{Lie}}$ by its left center $Z_L(\mathfrak{h})$, he integrated Leibniz algebras into local Lie racks. The fact that this procedure works only locally stems from the fact that the Leibniz 2-cocycle governing the abelian extension is only integrated into a local rack cocycle, due to the use of open sets on the Lie group G_0 integrating $\mathfrak{h}_{\mathrm{Lie}}$ where exponential and logarithm are mutually inverse diffeomorphisms. This integration has the advantage of specializing to the conjugation racks associated to Lie groups in case the given Leibniz algebra is a Lie algebra.

Evidence that racks are the right objects integrating Leibniz algebras comes from recent work of Covez on product structures on rack homology showing that it has (some of) the expected properties of a Leibniz K-theory which were conjectured by Loday.

In the second section of our paper, we use the integration procedure of Leibniz algebras set up in Section 1 in order to develop deformation quantization of Leibniz algebras.

Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [,])$, its dual vector space $\mathfrak{g}^*$ is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and all $\xi \in \mathfrak{g}^*$ by the *Kostant-Kirillov-Souriau formula*

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.$$

Here $df(\xi)$ and $dg(\xi)$ are linear functionals on $\mathfrak{g}^*$, identified with elements of $\mathfrak{g}$.

In the same way, a general Leibniz algebra $\mathfrak{h}$ gives rise to a smooth manifold $\mathfrak{h}^*$, which carries now some kind of generalized Poisson bracket, in particular, the bracket need not be skew-symmetric. We call manifolds with such a bracket *generalized Poisson manifolds*.

It is well known that the deformation quantization of the Poisson manifold $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ is intimitely related to the integration of the bracket of $\mathfrak{g}$ into a local/formal group product via the Baker-Campbell-Hausdorff (BCH) formula. The main idea of the present paper is to use the corresponding BCH-formula for the integration of a Leibniz algebra $\mathfrak{h}$ in order to perform the corresponding deformation quantization.

The quantization technique we use relies on the quantization of special canonical relation germs, called *symplectic micromorphisms* (see [6], [7], [8], and [9]), by Fourier integral operators. In the Lie algebra case, we show that it is possible to re-interprete the Gutt star-product in terms of a symplectic micromorphism quantization, obtained by considering the cotangent lift of the local group structure on the Lie algebra. We show that this quantization method also works for Leibniz algebras, provided one takes the cotangent lift of the local rack structure for the symplectic micromorphism. This local rack structure comes from the integration procedure exposed in the first section.

The quantization of the dual of a Leibniz algebra $\mathfrak{h}^*$ that results from quantizing the symplectic micromorphism obtained from the local rack structure, is an operation

$$\triangleright : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \times \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

such that the restriction of $\triangleright$ to "unitaries" $U_\mathfrak{h} := \{E_X \mid X \in \mathfrak{h}\}$ (E_X being the exponential function on $\mathfrak{h}^*$ associated to $X \in \mathfrak{h}$) is a rack structure $\triangleright : U_\mathfrak{h} \times U_\mathfrak{h} \to U_\mathfrak{h}$. Moreover, the restriction of this operation to

$$\triangleright : U_\mathfrak{h} \times \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

should be a rack action.

Our main theorem shows exactly this:

Theorem 4.12 *The operation*

$$\triangleright_\hbar : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

defined by

$$f \triangleright_\hbar g := Q(T^*\triangleright)(f \otimes g)$$

is a quantum rack, i.e.

1. *$\triangleright_\hbar$ restricted to unitaries $U_\mathfrak{h} = \{E_X \mid X \in \mathfrak{h}\}$ is a rack structure, moreover*

$$e^{\frac{i}{\hbar}X} \triangleright_\hbar e^{\frac{i}{\hbar}Y} \;=\; e^{\frac{i}{\hbar}e^{\mathrm{ad}_X}(Y)},$$

2. *$\triangleright_\hbar$ restricted to*

$$\triangleright_\hbar : U_\mathfrak{h} \times \mathcal{C}^\infty(\mathfrak{h}^*) \to \mathcal{C}^\infty(\mathfrak{h}^*)$$

is a rack action;

$$(e^{\frac{i}{\hbar}X} \triangleright_\hbar f)(\xi) \;=\; (\mathrm{Ad}^*_{-X} f)(\xi).$$

Moreover, $\triangleright_\hbar$ coincides with the Gutt quantum rack $f \triangleright g := f *_{\mathrm{Gutt}} g *_{\mathrm{Gutt}} \overline{f}$ on unitaires in the Lie case (although it is different on the whole $\mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$).

The quantization operator Q occuring here is constructed as the Fourier integral operator associated to a generating function $S_\triangleright$ according to

$$Q(f \otimes g)(\xi) \;=\; \int_{\mathfrak{h}\times\mathfrak{h}} \widetilde{f}(X)\widetilde{g}(Y) e^{\frac{i}{\hbar}S_\triangleright(X,Y,\xi)} \frac{dXdY}{(2\pi\hbar)^n},$$

where $f, g \in \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$, $X, Y \in \mathfrak{h}$, $\xi \in \mathfrak{h}^*$ and $n = \dim(\mathfrak{h})$. The expressions $\widetilde{f}$ and $\widetilde{g}$ denote so-called asymptotic Fourier transforms, i.e. we suppressed in the notation a cutoff function with compact support around the critical points of the phase function (see equation (6)). The generating function $S_\triangleright$ relies on a Baker-Campbell-Hausdorff type formula for the Leibniz case, namely

$$S_\triangleright(X, Y, \xi) \;:=\; \langle \xi, e^{\mathrm{ad}_X}(Y)\rangle.$$

The problem of deformation quantizing a Leibniz algebra has been addressed by other authors, namely by K. Uchino in [18] in the realm of associative dialgebras.

Observe that thanks to this theorem, the general integration problem for generalized Poisson manifolds makes sense. Namely, given a generalized Poisson manifold, i.e. a manifold M together with a bracket on $\mathcal{C}^\infty(M)$ satisfying similar properties as the bracket on $\mathcal{C}^\infty(\mathfrak{h}^*)$, we may ask whether there exists a natural rack structure on the set of exponential functions which extends to a rack action on all smooth functions. Our main theorem solves this integration problem for *linear* generalized Poisson structures.

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1 Preliminaries

1.1 Derivations of Leibniz algebras

Fix a field k. Later we will specialize to $k = \mathbb{R}$ in order to speak about the exponential map (although this is not mandatory). We present here a recollection of facts from Lie algebra theory which we generalize to Leibniz algebras. The proofs are straightforward and left to the reader. Lemmas 1.3, 1.10 and 1.11 are true for general algebras, but we state them here for Leibniz algebras.

Definition 1.1. *A (left) Leibniz algebra is a k-vector space $\mathfrak{h}$ together with a k-bilinear bracket $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ such that for all $X, Y, Z \in \mathfrak{h}$*

$$[X, [Y, Z]] \;=\; [[X, Y], Z] + [Y, [X, Z]].$$

Definition 1.2. *A (left) derivation of a Leibniz algebra $\mathfrak{h}$ is a k-linear map $D : \mathfrak{h} \to \mathfrak{h}$ such that for all $X, Y \in \mathfrak{h}$*

$$D([X, Y]) \;=\; [D(X), Y] + [X, D(Y)].$$

Observe that the above left Leibniz identity means that for all $X \in \mathfrak{h}$, $\mathrm{ad}_X := [X, -]$ is a (left) derivation of the bracket. Obviously, in case the bracket is also skew-symmetric, $\mathfrak{h}$ becomes a Lie algebra and the left Leibniz identity becomes the usual Jacobi identity. As this need not be the case, the notion of Leibniz algebra generalizes the notion of Lie algebra. Observe furthermore that skew-symmetrizing the bracket of a Leibniz algebra does not necessarily give a Lie algebra, as the Jacobi identity is not necessarily satisfied.

Lemma 1.3. *For any Leibniz algebra $\mathfrak{h}$, the space of derivations $\mathrm{der}(\mathfrak{h})$ together with the bracket of derivations*

$$[D, D'] \;:=\; D \circ D' - D' \circ D,$$

forms a Lie algebra.

By the Leibniz identity, for all $X \in \mathfrak{h}$, the endomorphism ad_X is a derivation, called the *inner derivation* associated to X.

Lemma 1.4. *The subspace $\mathrm{inn}(\mathfrak{h})$ of inner derivations of a Leibniz algebra $\mathfrak{h}$ forms an ideal in the Lie algebra $\mathrm{der}(\mathfrak{h})$ of all derivations.*

Observe that the subspace $\mathrm{inn}(\mathfrak{h})$ is also the image of the map $\mathrm{ad} : \mathfrak{h} \to \mathrm{der}(\mathfrak{h})$.

Definition 1.5. *For any Leibniz algebra $\mathfrak{h}$, the quotient Lie algebra of $\mathrm{der}(\mathfrak{h})$ by the ideal of inner derivations $\mathrm{inn}(\mathfrak{h})$ is called the Lie algebra $\mathrm{out}(\mathfrak{h})$ of outer derivations of $\mathfrak{h}$.*

Definition 1.6. *The left center of a Leibniz algebra $\mathfrak{h}$ is the subspace*

$$Z_L(\mathfrak{h}) \;:=\; \{\, X \in \mathfrak{h} \,|\, [X, Y] = 0 \;\; \forall\, Y \in \mathfrak{h} \,\}.$$

Lemma 1.7. *The left center $Z_L(\mathfrak{h})$ of a Leibniz algebra $\mathfrak{h}$ is an abelian (left) ideal.*

We summarize the preceeding discussion in the following

Proposition 1.8. *For any Leibniz algebra $\mathfrak{h}$, there is a 4-term exact sequence of Leibniz algebras:*

$$0 \to Z_L(\mathfrak{h}) \to \mathfrak{h} \stackrel{\mathrm{ad}}{\to} \mathrm{der}(\mathfrak{h}) \to \mathrm{out}(\mathfrak{h}) \to 0.$$

The only Leibniz algebra in this sequence which is not necessarily a Lie algebra is $\mathfrak{h}$.

We can shorten the 4-term sequence to the following short exact sequence:

$$0 \to Z_L(\mathfrak{h}) \to \mathfrak{h} \to \mathrm{ad}(\mathfrak{h}) \to 0. \tag{1}$$

In this way, we can associate to each Leibniz algebra $\mathfrak{h}$ an abelian extension such that the quotient algebra (here $\mathrm{ad}(\mathfrak{h})$) is a Lie algebra. There are of course other choices which satisfy this requirement, but we will always use this one.

Now let us come to automorphisms and the exponential map.

Definition 1.9. *A linear map $\alpha : \mathfrak{h} \to \mathfrak{h}$ on a Leibniz algebra $\mathfrak{h}$ is called an endomorphism in case for all $X, Y \in \mathfrak{h}$:*

$$\alpha([X,Y]) \;=\; [\alpha(X), \alpha(Y)].$$

Such a map α is called an automorphism if in addition it is bijective.

We will specialize from now on to $k = \mathbb{R}$ (although this is not completely necessary).

Lemma 1.10. *Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra. For a derivation $D \in \mathrm{der}(\mathfrak{h})$, the formula*

$$\exp(D) \;:=\; \sum_{k=0}^{\infty} \frac{1}{k!} D^k$$

defines an endomorphism of $\mathfrak{h}$. Furthermore, the endomorphism $\exp(D)$ is an automorphism of $\mathfrak{h}$.

Lemma 1.11. *Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra, $\alpha \in \mathrm{Aut}(\mathfrak{h})$ be an automorphism, and $X \in \mathfrak{h}$. Then the following formula holds:*

$$\alpha \circ \exp(\mathrm{ad}_X) \circ \alpha^{-1} \;=\; \exp(\mathrm{ad}_{\alpha(X)}).$$

Recall the usual naturality properties of the exponential map of a Lie algebra $\mathfrak{g}$, see for example [19] p. 104, formula (2.13.7) and Theorem 2.13.2:

Proposition 1.12. *The exponential map $\exp : \mathfrak{g} \to G$ of a (finite-dimensional) Lie algebra $\mathfrak{g}$ into a Lie group G with Lie algebra $\mathfrak{g}$ has the following naturality properties for all $X, Y \in \mathfrak{g}$ and all $g \in G$:*

(a) $\mathrm{conj}_g(\exp(Y)) = \exp(\mathrm{Ad}_g(Y))$,

(b) $\exp(\mathrm{ad}_X)(Y) = \mathrm{Ad}_{\exp(X)}(Y)$.

Observe that as usual we do not distinguish in notation the exponential function $\exp : \mathfrak{g} \to G$ and the exponential function $\exp : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{Gl}(\mathfrak{g})$.

1.2 On the BCH-formula

Let X and Y be elements in a finite-diemnsional Lie algebra $\mathfrak{g}$ of sufficiently small norm (in order to have convergence of the following infinite series). The Baker-Campbell-Hausdorff formula (BCH-formula)

$$\mathrm{BCH}(X,Y) := X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \dots ,$$

defines a *local* Lie group structure in a neighbourhood of 0 in $\mathfrak{g}$. It is in general not a global group structure, because the bracket expression need not converge (for arbitrary elements X and Y in an arbitrary Lie algebra).

Now we pass to the conjugation with respect to the BCH-product. Note that due to associativity of the BCH-formula, we have

$$\begin{aligned} \mathrm{conj}_*(X,Y) &= \mathrm{BCH}(\mathrm{BCH}(X,Y),-X) = \\ &= \mathrm{BCH}(X,\mathrm{BCH}(Y,-X)). \end{aligned}$$

It is not so well-known that the conjugation associated to the BCH-multiplication has a much simpler formula which converges always:

Lemma 1.13. *The explicit formula BCH-conjugation* conj_* *for a Lie algebra* $\mathfrak{g}$ *is:*

$$\begin{aligned} \mathrm{conj}_*(X,Y) &= \exp(\mathrm{ad}_X)(Y) \\ &= Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \frac{1}{6}[X,[X,[X,Y]]] + \dots \end{aligned}$$

This conjugation operation is thus a perfectly *global* operation, but which is only *locally* the conjugation with respect to a group product.

Note that this operation

$$(X,Y) \mapsto \exp(\mathrm{ad}_X)(Y)$$

makes also sense for elements X, Y in any finite dimensional Leibniz algebra $\mathfrak{h}$. The exponential $\exp(\mathrm{ad}_X)$ is the (inner) automorphism (see Lemma 1.10) with respect to the (inner) derivation ad_X which is associated to each element $X \in \mathfrak{h}$.

1.3 Lie racks

Recall the notion of a rack: It comes from axiomatizing the notion of conjugation in a group and plays its role in the present context as the structure integrating Leibniz algebras.

Definition 1.14. *Let X be a set together with a binary operation denoted $(x, y) \mapsto x \triangleright y$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and for all $x, y, z \in X$,*

$$x \triangleright (y \triangleright z) \;=\; (x \triangleright y) \triangleright (x \triangleright z).$$

Then we call X (or more precisely $(X, \triangleright)$) a (left) rack.

As already mentioned, an example of a rack is the conjugation in a group G. The rack operation is in this case given by $(g, h) \mapsto ghg^{-1}$. Finite racks have served to define knot, link and tangle invariants, see for example [12]. There is also the notion of a right rack. This is by definition a set X together with a binary operation $(x, y) \mapsto x \triangleleft y$ such that all maps $x \mapsto x \triangleleft y$ are bijective and

$$(x \triangleleft y) \triangleleft z \;=\; (x \triangleleft z) \triangleleft (y \triangleleft z).$$

There are at least two ways to transform a left rack into a right rack and vice-versa. The first is to take the *opposite rack* $x \triangleleft y := y \triangleright x$, the second is to take the *inverse rack* $x \triangleleft y := (y \triangleright -)^{-1}(x)$.

Definition 1.15. *Let R be a rack and X be a set. We say that R acts on X (on the right) (or that X is a right R-set) in case for all $r \in R$, there are bijections $(\cdot r) : X \to X$ such that for all $x \in X$ and all $r, r' \in R$:*

$$(x \cdot r) \cdot r' \;=\; (x \cdot r') \cdot (r \triangleleft r').$$

There is also the notion of a left action where the corresponding identity reads

$$r \cdot (r' \cdot x) \;=\; (r \triangleright r') \cdot (r \cdot x).$$

As usual, in case X is an abelian group and the rack action is linear, we also speak of X as a right R-module.

Clearly, the adjoint action $\mathrm{Ad}_r : R \to R$ defined by $\mathrm{Ad}_r(r') := r \triangleright r'$ in a left rack R is a left action of R on itself. In the same way the adjoint action of a right rack on itself is a right action.

Lemma 1.16. *Let R be a left rack such that the underlying set is a finite dimensional vector space with linear dual R^*. Then there exists a coadjoint action $\mathrm{Ad}^* : R \times R^* \to R^*$ defined for all $r, r' \in R$ and all $f \in R^*$ by*

$$(\mathrm{Ad}^*_r(f))(r') \;:=\; f((r \triangleright -)^{-1}(r')).$$

The coadjoint action is a left action.

Proof. For the proof, write simply $r \cdot f$ for $\mathrm{Ad}^*_r(f)$. Then

$$\begin{aligned}
(r \cdot (r' \cdot f))(r \triangleright (r' \triangleright r'')) &= (r' \cdot f)((r \triangleright -)^{-1}(r \triangleright (r' \triangleright r''))) \\
&= (r' \cdot f)(r' \triangleright r'') \\
&= f((r' \triangleright -)^{-1}(r' \triangleright r'')) \\
&= f(r'').
\end{aligned}$$

We also have

$$\begin{aligned}((r \triangleright r') \cdot (r \cdot f))(r \triangleright (r' \triangleright r'')) &= ((r \triangleright r') \cdot (r \cdot f))((r \triangleright r') \triangleright (r \triangleright r'')) \\ &= (r \cdot f)(((r \triangleright r') \triangleright -)^{-1}((r \triangleright r') \triangleright (r \triangleright r''))) \\ &= (r \cdot f)(r \triangleright r'') \\ &= f((r \triangleright -)^{-1}(r \triangleright r'')) \\ &= f(r'').\end{aligned}$$

This shows that

$$r \cdot (r' \cdot f) = (r \triangleright r') \cdot (r \cdot f),$$

thus the coadjoint action is a left action. □

Remark 1.17. (a) Curiously, this does not seem to work with the opposite rack structure replacing the inverse rack structure.

(b) We did not suppose the rack structure to be linear here, but in our application of the above Lemma, this will be the case.

In the following, we will need pointed local Lie racks.

Definition 1.18. *A pointed rack $(X, \triangleright, 1)$ is a set X with a binary operation $\triangleright$ and an element $1 \in X$ such that the following axioms are satisfied:*

1. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ *for all* $x, y, z \in X$,
2. *For each* $a, b \in X$, *there exists a unique* $x \in X$ *such that* $a \triangleright x = b$,
3. $1 \triangleright x = x$ *and* $x \triangleright 1 = 1$ *for all* $x \in X$.

Once again, the conjugation rack of a group is an example of a pointed rack.

Definition 1.19. *A Lie rack X is a manifold and a pointed smooth rack, i.e. the structure maps are smooth. In particuler, the maps $x \triangleright - : X \to X$ are diffeomorphisms for all $x \in X$.*

Examples of Lie racks include obviously the conjugation racks associated to Lie groups. Another example which will play a role in the sequel is the following:

Example: Let G be a Lie group and V be a G-module (which is a k-vector space for $k = \mathbb{R}$ or $\mathbb{C}$). On $X := V \times G$, we define a binary operation $\triangleright$ by

$$(v, g) \triangleright (v', g') = (g(v'), gg'g^{-1})$$

for all $v, v' \in V$ and all $g, g' \in G$. X is a Lie rack with unit $1 := (0, 1)$ which is called a *linear Lie rack*. This is the "group-analog" of the hemi-semi-direct product of a Lie algebra with its representation (see [16], [15]), and we denote it by $V \times_{\mathrm{hs}} G$.

Let us define more generally this hemi-semi-direct product of racks:

Definition 1.20. *Let R be a rack and A be an R-set in the sense of Definition 1.15. The hemi-semi-direct product $A \times_{\mathrm{hs}} R$ of R with A is the following rack structure on the direct product set $A \times R$:*

$$(a, r) \triangleright (a', r') \;:=\; (r(a'), r \triangleright r').$$

One verifies easily that this gives indeed a rack structure.

Now let us come to digroups:

Definition 1.21. *A digroup $(H, \vdash, \dashv)$ is a set H together with two binary operations $\vdash$ and $\dashv$ satisfying the following axioms. For all $x, y, z \in H$,*

1. *$(H, \vdash)$ and $(H, \dashv)$ are semigroups,*
2. *$x \vdash (y \dashv z) \;=\; (x \vdash y) \dashv z$,*
3. *$x \dashv (y \vdash z) \;=\; x \dashv (y \dashv z)$,*
4. *$(x \dashv y) \vdash z \;=\; (x \vdash y) \vdash z$,*
5. *there exists $1 \in H$ such that $1 \vdash x \;=\; x \dashv 1 \;=\; x$ for all $x \in H$,*
6. *for all $x \in H$, there exists $x^{-1} \in H$ such that $x \vdash x^{-1} \;=\; x^{-1} \dashv x \;=\; 1$.*

An element $e \in H$ in a digroup H is called a *bar unit* in case $e \vdash x \;=\; x \dashv e \;=\; x$ for all $x \in H$. Bar units exist in a digroup, but are not necessarily unique. A digroup is a group if and only if $\vdash \;=\; \dashv$ and 1 is the unique bar unit. We call *Lie digroup* a digroup with a compatible manifold structure.

There is a digroup which resembles very much the linear Lie rack:

Remark 1.22. *Let G be a Lie group and M be a G-module (which is a k-vector space for $k = \mathbb{R}$ or $\mathbb{C}$). Define on $H := M \times G$ the structure of a digroup by*

$$(u, g) \vdash (v, h) \;:=\; (g(v), gh)$$

and

$$(u, g) \dashv (v, h) \;:=\; (u, gh)$$

for all $u, v \in M$ and all $g, h \in G$. Then $M \times G$ is a Lie digroup with distinguished bar unit $(0, 1)$. The inverse of an element (u, g) is $(0, g^{-1})$. This Lie digroup is called the linear Lie digroup associated to G and M.

Digroups give rise to racks in the following way:

Proposition 1.23. *Let $(H, \vdash, \dashv)$ be a digroup and put*

$$x \triangleright y \;:=\; x \vdash y \dashv x^{-1} \tag{2}$$

for all $x, y \in H$. Then $(H, \triangleright)$ is a rack, pointed in 1. Moreover, in case $(H, \vdash, \dashv)$ is a Lie digroup (i.e. all structures are smooth), $(H, \triangleright)$ is a Lie rack.

In the case of the example in Remark 1.22, the obtained Lie rack is the above described linear Lie rack $M \times_{\mathrm{hs}} G$. In this sense every linear Lie rack "comes from" a linear Lie digroup.

Remark 1.24. *There are several ways to construct a group out of a rack X. The associated group $As(X)$ is the quotient of the free group on X by the normal subgroup generated by the set $\{(xy^{-1}x^{-1})(x \triangleright y) \,:\, x, y \in X\}$. For pointed racks, one modifies this definition such that 1 becomes the unit of $As(X)$.*

1.4 From split Leibniz algebras to Lie racks

In this subsection, we summarize Kinyon's approach [15] to the integration of (split) Leibniz algebras by Lie racks.

Kinyon shows in [15] the following theorem which is at the heart of all our attempts to integrate Leibniz algebras.

Theorem 1.25. *Let $(X, \triangleright, 1)$ be a Lie rack, and let $\mathfrak{h} := T_1X$. Then there exists a bilinear map $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ such that*

1. *$(\mathfrak{h}, [,])$ is a (left) Leibniz algebra,*
2. *for each $x \in X$, the tangent map $\Phi(x) := T_1\phi(x)$ of the left translation map $\phi(x) : X \to X$, $y \mapsto x \triangleright y$, is an automorphism of $(\mathfrak{h}, [,])$,*
3. *if $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$ is defined by $Y \mapsto \mathrm{ad}_X(Y) := [X, Y]$, then $\mathrm{ad} = T_1\Phi$.*

Let us recall its proof for the sake of self-containedness:

Proof. We have for all $x \in X$, $\phi(x)(1) = x \triangleright 1 = 1$, thus $\Phi(x) := T_1\phi$ is an endomorphism of $\mathfrak{h} := T_1X$. As each $\phi(x)$ is invertible, we have $\Phi(x) \in \mathrm{Gl}(\mathfrak{h})$. Now the map $\Phi : X \to \mathrm{Gl}(\mathfrak{h})$ satisfies $\Phi(1) = \mathrm{id}$, thus we may differentiate again in order to obtain $\mathrm{ad} : T_1X \to \mathfrak{gl}(\mathfrak{h})$. Now we set

$$[X, Y] \;:=\; \mathrm{ad}_X(Y)$$

for all $X, Y \in \mathfrak{h} = T_1X$. In terms of the left translations $\phi(x)$, the rack identity can be expressed by the equation

$$\phi(x)(\phi(y)(z)) \;=\; \phi(\phi(x)(y))(\phi(x)(z)).$$

We differentiate this equation at $1 \in X$ first with respect z, then with respect to y to obtain

$$\Phi(x)\,([Y, Z]) \;=\; [\Phi(x)(Y), \Phi(x)(Z)]$$

for all $x \in X$ and all $Y, Z \in \mathfrak{h}$. This expresses the fact that for each $x \in X$, $\Phi(x) \in \mathrm{Aut}(T_1X, [,])$. Finally, we differentiate this last equation at 1 with respect to x to obtain

$$[X, [Y, Z]] \;=\; [[X, Y], Z] + [Y, [X, Z]]$$

for all $X, Y, Z \in \mathfrak{h}$, This shows that $\mathfrak{h}$ is a left Leibniz algebra. $\square$

Example: In the special case of a linear Lie rack, we obtain the hemi-semi-direct product Leibniz algebra $\mathfrak{h} = V \times_{\mathrm{hs}} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G, endowed with the bracket:

$$[(v, X), (v', X')] \,=\, (X(v'), [X, X']).$$

The G-module V is here seen as a $\mathfrak{g}$-module in the usual way.

Remark 1.26. *Note that the passage form a Lie rack to its tangent Leibniz algebra is a functor, as in the case for Lie algebras.*

Kinyon's main result in [15] is the integration of split Leibniz algebras (i.e. those isomorphic to a hemi-semi-direct product Leibniz algebra) into linear Lie racks and thus into Lie digroups.

Theorem 1.27 (Kinyon)**.** *Let $\mathfrak{h}$ be a split Leibniz algebra. Then there exists a linear Lie digroup with tangent Leibniz algebra isomorphic to $\mathfrak{h}$.*

Remark 1.28. *In fact, Simon Covez showed in his (unpublished) Master thesis that conversely, in case a Leibniz algebra integrates into a Lie digroup, it must be split over some ideal containing the ideal of squares (more precisely, it is split over the ideal* $\ker(T_1 i)$ *where i is the inversion map of the digroup).*

1.5 Integration of Leibniz algebras

Here we present an approach to the integration of Leibniz algebras. It is closely related to work by H. Bass (unpublished), referred to in [12], and M. Kinyon [15].

This approach builds on a remark by H. Bass in the Lie algebra case, referred to in [12], and is already contained in [15] (end of Section 3), but Kinyon believed this integration to be too arbitrary, as it does not necessarily yield Lie groups in the case of Lie algebras.

Let $\mathfrak{h}$ be a finite-dimensional real Leibniz algebra.

Theorem 1.29. *On the vector space $\mathfrak{h}$, there exists a Lie rack structure which is given by*

$$(X, Y) \mapsto \exp(\mathrm{ad}_X)(Y) \,=:\, X \rhd Y$$

for all $X, Y \in \mathfrak{h}$. Observe that it is analytic in the first and linear in the second variable. This Lie rack structure has the following properties:

1. *In case $\mathfrak{h}$ is a Lie algebra, the corresponding Lie rack structure is locally the conjugation rack structure with respect to to a Lie group structure.*

2. *The Lie rack structure is globally described by a BCH-type formula.*

Proof. Note that by Lemma 1.10, $Y \mapsto \exp(\mathrm{ad}_X)(Y)$ is an automorphism of $\mathfrak{h}$. The fact that the binary operation

$$(X, Y) \mapsto X \rhd Y = \exp(\mathrm{ad}_X)(Y)$$

is a rack product is easily computed directly (using Lemma 1.11):

$$\begin{aligned}(X \triangleright Y) \triangleright (X \triangleright Z) &= \exp(\mathrm{ad}_{\exp(\mathrm{ad}_X)(Y)})(\exp(\mathrm{ad}_X)(Z)) \\ &= \Big(\exp(\mathrm{ad}_X) \circ \exp(\mathrm{ad}_Y) \circ \exp(-\mathrm{ad}_X) \circ \exp(\mathrm{ad}_X)\Big)(Z) \\ &= \Big(\exp(\mathrm{ad}_X) \circ \exp(\mathrm{ad}_Y)\Big)(Z) \\ &= X \triangleright (Y \triangleright (Z)).\end{aligned}$$

The BCH-formula which is referrred to in the statement is contained in Lemma 1.13, while the local Lie group structure in the case of a Lie algebra is given by the BCH-product. The BCH-type formula boils down in our case to an exponential formula. □

A drawback of this approach is that in the case of a Lie algebra, the space is only *locally* a Lie group, but not necessarily globally.

2 Deformation quantization of Leibniz algebras

2.1 Motivation

Recall that given a finite-dimensional real Lie algebra $(\mathfrak{g}, [,])$, its dual vector space $\mathfrak{g}^*$ is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and all $\xi \in \mathfrak{g}^*$ by the *Kostant-Kirillov-Souriau formula*

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.$$

Here $df(\xi)$ and $dg(\xi)$ are linear functionals on $\mathfrak{g}^*$, identified with elements of $\mathfrak{g}$. The goal of the second part of this article is to define deformation quantization for an analoguous bracket on the dual of a Leibniz algebra.

Let $(\mathfrak{h}, [,])$ be a (left, real, finite-dimensional) Leibniz algebra. Its linear dual $\mathfrak{h}^*$ is still a smooth manifold. The smooth functions $\mathcal{C}^\infty(\mathfrak{h}^*)$ on $\mathfrak{h}^*$ have a natural bracket

$$\{\ ,\ \} : \mathcal{C}^\infty(\mathfrak{h}^*) \times \mathcal{C}^\infty(\mathfrak{h}^*) \to \mathcal{C}^\infty(\mathfrak{h}^*).$$

Namely for all $f, g \in \mathcal{C}^\infty(\mathfrak{h}^*)$ and all $\xi \in \mathfrak{h}^*$

$$\{f, g\}(\xi) := -\langle \xi, [df(0), dg(\xi)] \rangle \tag{3}$$

At this stage, it may seem arbitrary that in comparison to the above bracket on the dual of a Lie algebra, we evaluated the first variable in 0 and that there is a minus sign. It is an outcome (see Theorem 2.12) of our deformation quantization procedure that this is the bracket which we are deforming. We will not introduce a different notation for this generalized bracket. We hope it will be clear from the context which bracket we will consider. Observe however

that this generalized bracket does not reproduce the usual Poisson bracket in the special case of a Lie algebra.

One readily verifies that this bilinear bracket satisfies the (right) Leibniz rule for all $f, g, h \in \mathcal{C}^\infty(\mathfrak{h}^*)$:

$$\{f, gh\} \;=\; \{f, g\}h + g\{f, h\}. \tag{4}$$

Remark 2.1. Observe that there is a remainder of the left Leibniz rule, too. Vector fields are derivations on the algebra of functions. Tangent vectors are *pointwise derivations*, i.e. the derivation property holds when interpreted as immediately followed by evaluation in a point. In this sense, the Leibniz rule in the first variable of $\{-,-\}$ holds when interpreted as immediately followed by evaluation in 0.

On the other hand, the bracket does not satisfy anymore the (left) Leibniz identity $\{f, \{g, h\}\} \;=\; \{\{f, g\}, h\} + \{g, \{f, h\}\}$ and it is certainly not necessarily skew-symmetric.

Remark 2.2. It is natural that the generalized bracket should satisfy much weaker conditions than a Poisson bracket on a smooth manifold. Indeed, it is shown in [13] that a bracket on a commutative associative algebra (in characteristic zero, without zero divisors) which satisfies the Leibniz rule in both variables and the Leibniz identity is necessarily skew-symmetric. We thank K. Uchino for bringing this fact to our attention.

We call the bracket in (3) *Leibniz-Poisson* bracket.

Definition 2.3. *A generalized Poisson manifold is a smooth manifold M whose space of smooth functions $\mathcal{C}^\infty(M)$ is endowed with a bilinear bracket*

$$\{\,,\,\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

satisfying the above property (4).

The notion of star-product [1] is closely related to the notion of Poisson manifold.

Definition 2.4. *A star-product $*$ on a Poisson manifold $(M, \{,\})$ is a formal deformation $*_\epsilon$ of the commutative associative product on $\mathcal{C}^\infty(M)$, i.e. an associative product*

$$f *_\epsilon g \;=\; fg + \epsilon B_1(f, g) + \ldots + \epsilon^n B_n(f, g) + \ldots$$

such that the $B_n(-,-)$'s are bidifferential operators for all $n \geq 1$ and that the constant function 1 is a unit.

Namely, given a star-product $*_\epsilon$ on M,

$$f *_\epsilon g \;=\; fg + \epsilon B_1(f, g) + \ldots,$$

the antisymmetrization of the first terms yields a Poisson bracket on M:

$$\{f,g\} \,=\, B_1(f,g) - B_1(g,f).$$

One says that the star-product $*_\epsilon$ *quantizes* the Poisson bracket in this case.

Conversely, M. Kontsevich [17] showed that any Poisson bracket can be quantized (non uniquely) into a star-product.

On the other hand, the quantization of a Lie algebra $\mathfrak{g}$ is known to be (roughly) the data of a $*$-algebra $\mathcal{A}_\mathfrak{g}$ for which the *self-adjoint elements*

$$U_\mathfrak{g} \,=\, \{a \in \mathcal{A}_\mathfrak{g} \,|\, a^* = a\,\}$$

form a group isomorphic to the Lie group integrating $\mathfrak{g}$.

A model for this quantizing $*$-algebra is the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$; another one is the convolution algebra $C(\mathfrak{g})$ of compactly supported continuous functions on the integrating group. Deformation quantization, by considering $*$-algebras quantizing the Poisson structure on the dual space $\mathfrak{g}^*$, gives yet a third model.

Namely the *Gutt $*$-algebra* $(\mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]], *_{\mathrm{Gutt}})$ (see [14]) where $\epsilon = \frac{\hbar}{2i}$ quantizes $\mathfrak{g}^*$ in the sense of deformation quantization and also has the following properties:

1. The complex conjugation is the involution of the $*$-algebra

$$\overline{f *_{\mathrm{Gutt}} g} \,=\, \overline{g} *_{\mathrm{Gutt}} \overline{f}.$$

2. $U_\mathfrak{g} \,=\, \{E_X \,|\, X \in \mathfrak{g}\}$ where $E_X(\xi) = e^{\frac{i}{\hbar}\langle X,\xi\rangle}$ and

$$E_X *_{\mathrm{Gutt}} E_Y \,=\, E_{BCH(X,Y)} \quad \text{and} \quad \overline{E_X} = E_{-X}.$$

Thus $U_\mathfrak{g}$ is isomorphic to the formal/local group $(\mathfrak{g}, BCH)$ integrating $(\mathfrak{g}, [\,,\,])$.

In everything that follows, one can always exchange the expansion parameters ϵ and $\hbar$ using the formula $\epsilon = \frac{\hbar}{2i}$.

In this section, we aim at quantizing a Leibniz algebra $\mathfrak{h}$ using techniques similar to deformation quantization. As we will see, what we obtain is an operation

$$\rhd : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \times \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

such that the restriction of $\rhd$ to $U_\mathfrak{h} = \{E_X \,|\, X \in \mathfrak{h}\}$ is a rack structure $\rhd : U_\mathfrak{h} \times U_\mathfrak{h} \to U_\mathfrak{h}$.

Moreover, the restriction of this operation to

$$\rhd : U_\mathfrak{h} \times \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

is a rack action.

Remark 2.5. *In the case of a Lie algebra $(\mathfrak{g}, [\,,\,])$, one can obtain such a quantum rack $\rhd_{\mathrm{Gutt}}$ from the Gutt star-product; namely*

$$f \rhd_{\mathrm{Gutt}} g \,:=\, f *_{\mathrm{Gutt}} g *_{\mathrm{Gutt}} \overline{f},$$

whose restriction to the exponentials is

$$e^{\frac{i}{\hbar}X} \triangleright_{\mathrm{Gutt}} e^{\frac{i}{\hbar}X} \;=\; e^{\frac{i}{\hbar}X} *_{\mathrm{Gutt}} e^{\frac{i}{\hbar}Y} *_{\mathrm{Gutt}} e^{-\frac{i}{\hbar}X} \;=\; e^{\frac{i}{\hbar}\mathrm{conj}_*(X,Y)},$$

where we have used Lemma 1.13. In the same vein, we obtain

$$e^{\frac{i}{\hbar}X} \triangleright_{\mathrm{Gutt}} g \;=\; g + \epsilon B^1_X(g) + \epsilon^2 B^2_X(g) + \cdots,$$

where the B^n_X are certain differential operators depending on $X \in \mathfrak{g}$. The quantum rack we will obtain in the case of a general Leibniz algebra will not coincide with the one in the Lie algebra case, but their restrictions on exponentials will.

We start by reinterpreting the Gutt star-product quantizing a Lie algebra $(\mathfrak{g}, [\,,\,])$ as the quantization of the symplectic micromorphism obtained by the cotangent lift of the group operation $m : G \times G \to G$ on the integrating Lie group. We will then follow a similar strategy for Leibniz algebras $\mathfrak{h}$ by quantizing the corresponding micromorphism obtained by the cotangent lift of the integrating rack structure $\triangleright : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$.

2.2 Gutt star-product as the quantization of a symplectic micromorphism

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra with integrating Lie group G. The *cotangent lift* T^*m of the group operation $m : G \times G \to G$ is the Lagrangian submanifold

$$T^*m \;:=\; \big\{\big((g, T^*_g R_h \xi), (h, T^*_h L_g \xi), (gh, \xi)\big) :\; g, h \in G,\, \xi \in T^*_{gh}G\big\}$$

of $\overline{T^*G} \times \overline{T^*G} \times T^*G$, where $R_h : G \to G$ and $L_g : G \to G$ are the usual right and left translations on G, respectively. As usual, $\overline{M}$ for a symplectic manifold M denotes the manifold M endowed with the opposite symplectic structure. The cotangent lift T^*m is actually the graph of the global symplectic groupoid

$$T^*G \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{g}^*$$

integrating the Poisson manifold $\mathfrak{g}^*$. We refer the reader to [4] and [20] for more details on the relationships between integrated Poisson data and Lagrangian submanifolds.

Seeing T^*m as a canonical relation from $T^*G \times T^*G$ to T^*G in the symplectic category, one wishes to associate to it a Fourier integral operator (depending on a parameter $\hbar$) from some $L^2(\mathfrak{g}^*) \otimes L^2(\mathfrak{g}^*) \to L^2(\mathfrak{g}^*)$ whose asymptotic expansion in the limit $\hbar \to 0$ would yield a star-product, in the spirit of [21] and [22] (see also [20] for a more recent exposition).

This is in general a very hard problem analytically, and it turns out that one is more lucky by only looking at the germ of

$$T^*m \subset \overline{T^*G} \times \overline{T^*G} \times T^*G$$

around the graph of the diagonal map $\triangle_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*$, where we consider $\mathfrak{g}^*$ as an embedded lagrangian submanifold in T^*G, namely the fiber over the identity element. Namely, as shown in [8], this germ is a symplectic micromorphism, which is readily quantizable by Fourier integral operators (see [9]).

Symplectic micromorphisms

Let us recall the definition of a symplectic micromorphism (see [6], [7], [8], and [9] for more details) as well as some aspect of their quantization.

Definition 2.6. *A symplectic micromorphism* $([L], \phi)$ *from a symplectic microfold* $[M, A]$ *(i.e. a germ of a symplectic manifold around a Lagrangian submanifold* $A \subset M$*, called the core of the microfold) to a symplectic microfold* $[N, B]$ *is the data of a Lagrangian submanifold germ* $[L]$ *in* $\overline{M} \times N$ *around the graph* $\mathrm{gr}(\phi)$ *of a smooth map* $\phi : A \to B$ *such that the intersection* $L \cap (A \times B) = \mathrm{gr}(\phi)$ *is clean for a representative* $L \in [L]$*.*

The symplectic micromorphisms are the morphisms of a category, *the microsymplectic category*. We denote them by $([L], \phi) : [M, A] \to [N, B]$, and, when the symplectic microfold is $[T^*A, A]$, we simply write T^*A.

An important example of symplectic micromorphisms comes from cotangent lifts of smooth maps between manifolds. Namely, if $\phi : B \to A$ is a smooth map, then the conormal bundle $N^*\mathrm{gr}\phi$ of the graph of ϕ is a lagrangian submanifold of $T^*(A \times B)$. Using the identification (Schwartz transform) between this last cotangent bundle and $\overline{T^*A} \times T^*B$, the conormal bundle to the graph yields a symplectic micromorphism, which we denote by $T^*\phi : T^*A \to T^*B$, by taking the germ of the resulting lagrangian submanifold

$$\Big\{\Big(\big(p_A,\, \phi(x_B)\big),\, \big((T^*_{x_B}\phi)p_A, x_B\big)\Big) : (p_A, x_B) \in \phi^*(T^*A)\Big\}$$

around the graph of ϕ, and where (p_A, x_A) and (p_B, x_B) are the canonical coordinates on T^*A and T^*B respectively.

When the target and source symplectic microfold cores are *euclidean* (i.e. when $A = \mathbb{R}^k$ and $B = \mathbb{R}^l$ for some $k \geq 1$ and $l \geq 1$), a symplectic micromorphism from T^*A to T^*B can be associated with a family of formal Fourier integral operators from $C^\infty(A)[[\hbar]]$ to $C^\infty(B)[[\hbar]]$ using the symplectic micromorphism generating function (see [9] for a general theory of symplectic micromorphism quantization).

Namely, as shown in [7], when the target and source symplectic microfold cores are euclidean any symplectic micromorphism $([L], \phi)$ from T^*A to T^*B can be described by a generating function germ $[S_L] : \phi^*(T^*A) \to \mathbb{R}$ around the zero section of the pullback bundle $\phi^*(T^*A)$ as follows: There is a representative $L \in [L]$ such that

$$\Big\{\Big(\big(p_A,\, \frac{\partial S_L}{\partial p_A}(p_A, x_B)\big),\, \big(\frac{\partial S_L}{\partial x_B}(p_A, x_B),\, x_B\big)\Big) : (p_A, x_B) \in W\Big\},$$

where W is an appropriate neighborhood of the zero section in $\phi^*(T^*A)$. This generating function S_L is unique if one requires that it satisfies the property $S_L(0, x) = 0$. The geometric condition on the cleanness of the intersection in the definition above can be expressed in terms of the generating function as follows:

$$\frac{\partial S_L}{\partial p_A}(p_A, 0) = \phi(x_B) \quad \text{and} \quad \frac{\partial S_L}{\partial x_B}(0, x_B) = 0. \tag{5}$$

In this light, one can see S_L as a deformation of the cotangent lift generating function, which is the first term of S_L in a Taylor expansion:

$$S_L(p_A, x_B) = \langle p_A, \phi(x_B)\rangle + \mathcal{O}(p_A^2).$$

Remark 2.7. *Conversely, any generating function germ* $[S] : \phi^*(T^*A) \to \mathbb{R}$ *satisfying conditions* (5) *defines uniquely a symplectic micromorphism* $([L_S], \phi) : T^*A \to T^*B$.

Now, using the generating function S_L of the symplectic micromorphism $([L], \phi)$, one can construct a formal operator

$$\begin{array}{rcl} C^\infty(A)[[\hbar]] & \longrightarrow & C^\infty(B)[[\hbar]] \\ \psi & \mapsto & Q([L], \phi)\psi \end{array}$$

by taking the stationary phase expansion of the following oscillatory integral ($n = \dim(A)$):

$$\int_{T^*A} \chi(p_A, x_A)\psi(x_A) e^{\frac{i}{\hbar}(S_L(p_A, x_B) - p_A x_A)} \frac{dx_A dp_A}{(2\pi\hbar)^n}, \tag{6}$$

where χ is a cutoff function with compact support around the critical points of the phase $S_L(p_A, x_B) - p_A x_A$ (with respect to the integration variables) and with value 1 on this critical locus, which is nothing but the points in $\{(0, \phi(x_B)) : x_B \in B\}$. Since the critical locus is contained in the zero section, the asymptotic expansion does not depend on the cutoff functions and, hence, is well-defined. To simplify the notation, we will abuse it slightly, and write from now on (with $T^*A = \mathbb{R}^{2n}$):

$$(Q([L], \phi)\psi)(x_B) = \int_{\mathbb{R}^n} \widetilde{\psi}(p_A) e^{\frac{i}{\hbar} S_L(p_A, x_B)} \frac{dp_A}{(2\pi\hbar)^{n/2}},$$

to mean the asymptotic expansion above, and where $\widetilde{\psi}(p_B)$ is the asymptotic Fourier transform of ψ; namely,

$$\widetilde{\psi}(p_A) = \int_{\mathbb{R}^n} \chi(p_A, x_A)\psi(x_A) e^{-\frac{i}{\hbar} p_A x_A} \frac{dx_A}{(2\pi\hbar)^{n/2}}.$$

Back to the Gutt star-product

Let us now apply the discussion of the previous subsection to the quantization of the linear Poisson structure on the dual of a Lie algebra $\mathfrak{g}$. Consider first the integrating Lie group G. Taking the cotangent lift of the group operation $m : G \times G \to G$ yields a symplectic micromorphism

$$([T^*m], \triangle_{\mathfrak{g}^*}) : [T^*G, \mathfrak{g}^*] \otimes [T^*G, \mathfrak{g}^*] \to [T^*G, \mathfrak{g}^*],$$

where we take the core in the source and target symplectic microfolds to be not the cotangent bundle zero section G, but rather the fiber above the identity, i.e.

the dual of the Lie algebra. Identifying $[T^*G, \mathfrak{g}^*]$ with $[T^*\mathfrak{g}^*, \mathfrak{g}^*]$ (which we will denote simply by $T^*\mathfrak{g}^*$) using the Lagrangian embedding germ

$$[T^*\mathfrak{g}^*, \mathfrak{g}^*] \to [T^*G, \mathfrak{g}^*], \quad (X, \xi) \mapsto (\exp(X), (T_1^* L_{\exp(X)})^{-1}\xi),$$

the Lagrangian germ $[T^*m]$ becomes the cotangent lift of the local group operation $BCH : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and $([T^*m], \Delta_{\mathfrak{g}^*})$ becomes a symplectic micromorphism from $T^*\mathfrak{g}^* \otimes T^*\mathfrak{g}^*$ to $T^*\mathfrak{g}^*$, whose underlying Lagrangian submanifold germ coincides with the multiplication of the local symplectic groupoid integrating the linear Poisson structure on $\mathfrak{g}^*$.

This local/formal symplectic groupoid is described in [5], where it is shown that T^*m can be described in term of the following generating function germ

$$S(X, Y, \xi) \,=\, \Big\langle \xi, BCH(X, Y) \Big\rangle$$

as follows:

$$T^*m \,=\, \left\{ \left((X, \frac{\partial S}{\partial X}), (Y, \frac{\partial S}{\partial Y}), (\frac{\partial S}{\partial \xi}, \xi) \right) : (X, Y, \xi) \in W \right\}$$

where W is an appropriate neighborhood of the zero section in $T^*\mathfrak{g}^* \oplus T^*\mathfrak{g}^*$.

Once the generating function of a symplectic micromorphism is computed, it is easy to obtain a family of (formal) Fourier integral operators quantizing it as explained in the previous subsection. In the case at hand, we obtain the following family of formal operators

$$Q(T^*m) : \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]]$$

of the form (in the previously established notation):

$$Q(T^*m)(f \otimes g)(\xi) \,=\, \int_{\mathfrak{g}\times\mathfrak{g}} \widetilde{f}(X)\widetilde{g}(Y) e^{\frac{i}{\hbar}S(X,Y,\xi)} \frac{dX dY}{(2\pi\hbar)^n}, \tag{7}$$

where n is the dimension of $\mathfrak{g}$.

When S is the generating function of $([T^*m], \triangle_{\mathfrak{g}^*})$, we have that

$$f * g \,=\, Q(T^*m)(f \otimes g)$$

coincides with the Gutt star-product [2, 3, 14]. For other star-products in integral form on duals of Lie algebras as in (7), we refer the reader to the work of Ben Amar [2, 3].

2.3 Quantizing a Leibniz algebra

Let $(\mathfrak{h}, [,])$ be a Leibniz algebra and $(\mathfrak{h}, \triangleright)$ its integrating Lie rack from Section 1. The idea is to quantize the Lagrangian relation

$$T^*\triangleright : T^*\mathfrak{h} \times T^*\mathfrak{h} \to T^*\mathfrak{h}$$

as we did for the group operation in the case of a Lie algebra.

In the case at hand, the integrating Lie rack from Section 1 reads

$$\triangleright : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}, \quad (X, Y) \mapsto e^{\mathrm{ad}_X}(Y) =: \mathrm{Ad}_X(Y).$$

The first step is to take the cotangent lift of the rack operation and compute its generating function:

Proposition 2.8. *The cotangent lift of $\triangleright$ yields a symplectic micromorphism*

$$T^*\triangleright : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \to T^*\mathfrak{h}^*$$

with generating function

$$S_\triangleright(X, Y, \xi) \;:=\; \langle \xi, \mathrm{Ad}_X(Y)\rangle.$$

Proof. Consider the generating function

$$\begin{aligned} S_\triangleright(X, Y, \xi) &:= \langle \xi, \mathrm{Ad}_X(Y)\rangle \\ &= \langle \xi, Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots\rangle \end{aligned}$$

We will denote the variables by $(X, Y) =: P$ and ξ, and write accordingly $S_\triangleright(X, Y, \xi) = S_\triangleright(P, \xi)$.

As shown in [7] Sections 3.1 and 3.2 (see also [5] Section 1.2), a generating function of the type

$$\begin{aligned} S_\triangleright(P, \xi) &= \langle \xi, Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots\rangle \\ &= \langle \Phi(\xi), P\rangle + \mathcal{O}(P^2) \end{aligned}$$

where $\Phi : \mathfrak{h}^* \to \mathfrak{h}^* \times \mathfrak{h}^*$, $\Phi(\xi) = (0, \xi)$, yields a symplectic micromorphism

$$([L_S], \Phi) : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \to T^*\mathfrak{h}^*$$

where

$$\begin{aligned} L_S &= \left\{ \left((X, \frac{\partial S_\triangleright}{\partial X}), (Y, \frac{\partial S_\triangleright}{\partial Y}), (\frac{\partial S_\triangleright}{\partial \xi}, \xi) \right) \mid \xi \in \mathfrak{g}^*, \;\; X, Y \in \mathfrak{g} \right\} \\ &= \{((X, \langle [X, Y], \xi\rangle), (Y, \mathrm{Ad}^*_X(\xi)), (\mathrm{Ad}_X(Y), \xi)) \mid \xi \in \mathfrak{g}^*, \;\; X, Y \in \mathfrak{g}\} \end{aligned}$$

which one recognizes to be the cotangent lift of the map $(X, Y) \mapsto \mathrm{Ad}_X(Y)$. □

Remark 2.9. *If $\mathfrak{g}$ is a Lie algebra, then $\mathrm{Ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of the local/formal group $(\mathfrak{g}, BCH)$ on $\mathfrak{g}$ by Lemma 1.13. The cotangent lift of this action is a Hamiltonian action of $(\mathfrak{g}, BCH)$ on $T^*\mathfrak{g}$, given by $T^*\mathrm{Ad}_X : T^*\mathfrak{g} \to T^*\mathfrak{g}$ for all $X \in \mathfrak{g}$. This Hamiltonian action has an equivariant momentum map $J : T^*\mathfrak{g} \to \mathfrak{g}^*$ given by $J(Y, \xi) = \langle \xi, \mathrm{ad}_Y\rangle$, i.e.*

$$\langle X, J(Y, \xi)\rangle \;=\; \langle \xi, [Y, X]\rangle.$$

Under the identification $T^*\mathfrak{g} \cong T^*\mathfrak{g}^*$ ($\cong \mathfrak{g} \times \mathfrak{g}^*$), the cotangent lift $T^*\mathrm{Ad}_X$ gives a Hamiltonian action of $(\mathfrak{g}, BCH)$ on $T^*\mathfrak{g}^*$. This yields an action of the (local) symplectic groupoid $T^*\mathfrak{g}^* \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{g}^*$ on $J : T^*\mathfrak{g}^* \to \mathfrak{g}^*$ whose graph

$$\rho_{\mathrm{Ad}} : T^*\mathfrak{g}^* \, {}_s\!\times_J T^*\mathfrak{g}^* \to T^*\mathfrak{g}^*$$

is a (germ of a) Lagrangian submanifold yielding the symplectic micromorphism (as explained in [7])

$$T^*\mathrm{Ad} \,=\, \Big\{\Big(\big(X,\, J(\mathrm{Ad}_{-X}(Y), \mathrm{Ad}^*_X(\xi)\big),\, \big(Y,\xi\big),\, \big(\mathrm{Ad}_{-X}(Y), \mathrm{Ad}^*_X(\xi)\big)\Big) \,:\, X, Y, \xi\Big\},$$

which we can simplify using the equivariance of the moment map J:

$$\begin{aligned}
\langle X, J(\mathrm{Ad}_{-X}(Y), \mathrm{Ad}^*_X(\xi))\rangle &= \langle X, J(T^*\mathrm{Ad}_X(Y,\xi))\rangle \\
&= \langle X, \mathrm{Ad}^*_X J(Y,\xi)\rangle \\
&= \langle \mathrm{Ad}_X X, J(Y,\xi)\rangle \\
&= \langle X, J(Y,\xi)\rangle,
\end{aligned}$$

where we have used that $[X, X] = 0$ in the Lie algebra $\mathfrak{g}$. Therefore, we obtain

$$T^*\mathrm{Ad} \,=\, \Big\{\Big((X,\, J(Y,\xi)),\, (Y,\xi),\, T^*\mathrm{Ad}_X(Y,\xi)\Big) \,:\, (X,Y,\xi) \in T^*\mathfrak{g}^* \oplus T^*\mathfrak{g}^*\}.$$

Under the identification $T^*\mathfrak{g} \cong T^*\mathfrak{g}^* \cong \mathfrak{g} \times \mathfrak{g}^*$, we have that

$$T^*\mathrm{Ad}_X = T^*\mathrm{Ad}^*_X,$$

i.e. the cotangent lift of the adjoint action and that of the coadjoint action coincide. Thus quantizing $\triangleright : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ should be the same as quantizing the coadjoint action $\mathrm{Ad}^*_X : \mathfrak{h}^* \to \mathfrak{h}^*$. Observe that switching to Leibniz algebras, the adjoint action Ad_X becomes a left rack action in the sense of Definition 1.15. Therefore the coadjoint action Ad^*_X becomes naturally a left rack action on $\mathfrak{h}^*$ via the formula

$$(\mathrm{Ad}^*_X(f))(Y) \,:=\, f((X \triangleright -)^{-1}Y),$$

see Lemma 1.16. Hence the stage is set to study the object which should replace the symplectic groupoid $T^*\mathfrak{g}^* \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{g}^*$ in the context of deformation quantization of Leibniz algebras. We will do this in subsequent work.

We are now ready to quantize $T^*\triangleright : T^*\mathfrak{h} \otimes T^*\mathfrak{h} \to T^*\mathfrak{h}$. As before, the family of semi-classical Fourier integral operators quantizing the symplectic micromorphism is given by

$$Q(T^*\triangleright)(f \otimes g)(\xi) \,=\, \int_{\mathfrak{g}\times\mathfrak{g}} \widetilde{f}(X)\widetilde{g}(Y) e^{\frac{i}{\hbar} S_{\triangleright}(X,Y,\xi)} \frac{dX dY}{(2\pi\hbar)^n},$$

where $\widetilde{f}$ and $\widetilde{g}$ are the asymptotic Fourier transforms (with a cutoff function involved, see Equation (6)).

Theorem 2.10. *The operation*

$$\triangleright_\hbar : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

defined by

$$f \triangleright_\hbar g \;:=\; Q(T^*\triangleright)(f \otimes g)$$

is a quantum rack, i.e.

(1) $\triangleright_\hbar$ *restricted to* $U_\mathfrak{h} = \{E_X \mid X \in \mathfrak{h}\}$ *is a rack structure and*

$$e^{\frac{i}{\hbar}X} \triangleright_\hbar e^{\frac{i}{\hbar}Y} \;=\; e^{\frac{i}{\hbar}\mathrm{conj}_*(X,Y)},$$

(2) $\triangleright_\hbar$ *restricted to* $\triangleright_\hbar : U_\mathfrak{h} \times \mathcal{C}^\infty(\mathfrak{h}^*) \to \mathcal{C}^\infty(\mathfrak{h}^*)$ *is a rack action and*

$$(e^{\frac{i}{\hbar}X} \triangleright_\hbar f)(\xi) \;=\; (\mathrm{Ad}^*_{-X} f)(\xi).$$

Moreover, $\triangleright_\hbar$ coincides with the Gutt quantum rack $f \triangleright g := f *_{\mathrm{Gutt}} g *_{\mathrm{Gutt}} \overline{f}$ on the restrictions in the Lie case (although it is different on the whole $\mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$).

Remark 2.11. *Actually, Property (2) in the theorem above holds also for square integrable functions, and we even obtain a unitary rack action:*

$$\triangleright_\hbar : U_\mathfrak{h} \times L^2(\mathfrak{h}^*) \to L^2(\mathfrak{h}^*).$$

Proof. The first property follows from the fact that exponentials Fourier transform to delta functions:

$$\begin{aligned}
\left(e^{\frac{i}{\hbar}\bar{X}} \triangleright_\hbar e^{\frac{i}{\hbar}\bar{Y}}\right)(\xi) &= \int \widetilde{e^{\frac{i}{\hbar}\bar{X}}}\,\widetilde{e^{\frac{i}{\hbar}\bar{Y}}}\, e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= (2\pi\hbar)^{\dim(\mathfrak{h})} \int \delta_{\bar{X}}(X)\delta_{\bar{Y}}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= e^{\frac{i}{\hbar}\langle \mathrm{Ad}_{\bar{X}}(\bar{Y}), \xi\rangle} \;=\; e^{\frac{i}{\hbar}\langle \mathrm{conj}_*(\bar{X},\bar{Y}), \xi\rangle}.
\end{aligned}$$

Now $\triangleright_\hbar$ satisfies the rack identity on $U_\mathfrak{h}$, because conj_* does. Furthermore,

$$E_Y \mapsto E_X \triangleright_\hbar E_Y = E_{\mathrm{conj}_*(X,Y)}$$

is bijective for all $X \in \mathfrak{h}$, because $Y \mapsto \mathrm{conj}_*(X,Y)$ is. It is also clear from the formula above that this rack structure coincides with the Gutt rack structure in the case of a Lie algebra.

The second property also follows from the fact that exponentials Fourier-transform to delta functions:

$$\begin{aligned}
\left(e^{\frac{i}{\hbar}\bar{X}} \triangleright_\hbar f\right)(\xi) &= \int \widetilde{e^{\frac{i}{\hbar}\bar{X}}}\, \widetilde{f}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= (2\pi\hbar)^{(\dim(\mathfrak{h}))/2} \int \delta_{\bar{X}}(X) \widetilde{f}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= \frac{1}{(2\pi\hbar)^{(\dim(\mathfrak{h}))/2}} \int \widetilde{f}(Y) e^{\frac{i}{\hbar}\langle \mathrm{Ad}^*_{-X}\xi, Y\rangle} dY \\
&= f(\mathrm{Ad}^*_{-X}\xi).
\end{aligned}$$

One sees that this defines a rack action from the fact that the coadjoint action Ad^*_{-X} is a rack action. $\square$

Let us now show that the first term of the quantized bracket is indeed the bracket (3). For an oscillatory integral as the above expression for $f \triangleright_\hbar g$, there is a well defined procedure of expansion in terms of Feynman graphs, in case the integral has a unique, non-degenerate critical point. This procedure is for example explained in [11].

Theorem 2.12. *(a) The above oscillatory integral $f \triangleright_\hbar g$ has a unique, non-degenerate critical point and admits thus a Feynman expansion in terms of graphs.*

(b) The first term of the formal expansion of

$$(f \triangleright_\hbar g)(\xi) \;=\; \int \chi\, f(\bar{\xi}) g(\bar{\eta}) e^{\frac{i}{\hbar}(-\bar{X}\bar{\zeta} - \bar{Y}\bar{\eta} + \langle \xi, \exp(\mathrm{ad}_{\bar{X}})(\bar{Y})\rangle)} \frac{d\bar{X}\, d\bar{Y}\, d\bar{\zeta}\, d\bar{\eta}}{(2\pi\hbar)^n} \qquad (8)$$

in powers of $\hbar$ is the Leibniz-Poisson bracket (3), i.e.

$$\{f, g\}(\xi) \;=\; -\langle \xi, [df(0), dg(\xi)]\rangle.$$

Proof. Observe that in equation (8), we wrote out explicitely the asymptotic Fourier transforms of f and g (using the cutoff function χ, see Equation (6)). The total phase of the above oscillatory integral is thus

$$S_\xi(\bar{X}, \bar{Y}, \bar{\zeta}, \bar{\eta}) \;=\; -\bar{X}\bar{\zeta} - \bar{Y}\bar{\eta} + \langle \xi, \exp(\mathrm{ad}_{\bar{X}})(\bar{Y})\rangle.$$

The phase $S_\xi(\bar{X}, \bar{Y}, \bar{\zeta}, \bar{\eta})$ has

$$c_\xi \;=\; (\bar{X} = 0, \bar{Y} = 0, \bar{\zeta} = 0, \bar{\eta} = \xi)$$

as its unique critical point. This means that for any given ξ, c_ξ is unique within the points $c := (\bar{X}, \bar{Y}, \bar{\zeta}, \bar{\eta})$ such that

$$\frac{\partial S_\xi}{\partial \bar{X}}(c) \;=\; 0, \quad \frac{\partial S_\xi}{\partial \bar{Y}}(c) \;=\; 0, \quad \frac{\partial S_\xi}{\partial \bar{\zeta}}(c) \;=\; 0, \quad \frac{\partial S_\xi}{\partial \bar{\eta}}(c) \;=\; 0.$$

The critical point c_ξ is easily computed from the partial derivatives. It turns out that

$$\frac{\partial S_\xi}{\partial \bar{X}}(c) \;=\; -\bar{\zeta} + T_1, \quad \frac{\partial S_\xi}{\partial \bar{Y}}(c) \;=\; -\bar{\eta} + T_2, \quad \frac{\partial S_\xi}{\partial \bar{\zeta}}(c) \;=\; -\bar{Y}, \quad \frac{\partial S_\xi}{\partial \bar{\eta}}(c) \;=\; -\bar{X},$$

where the term T_1 is the derivative of $\bar{X} \mapsto \langle \xi, \exp(\mathrm{ad}_{\bar{X}})(\bar{Y})\rangle$ and the term T_2 is the derivative of $\bar{Y} \mapsto \langle \xi, \exp(\mathrm{ad}_{\bar{X}})(\bar{Y})\rangle$. One concludes from setting the third and fourth equation equal to zero that $\bar{X} = \bar{Y} = 0$. The first term of $Y \mapsto \langle \xi, \exp(\mathrm{ad}_{\bar{X}})(\bar{Y})\rangle$ is $\xi\bar{Y}$, thus the constant term in T_2 is ξ. All other terms in T_1 and T_2 are zero at the critical point due to $\bar{X} = \bar{Y} = 0$. In conclusion $c_\xi \;=\; (\bar{X} = 0, \bar{Y} = 0, \bar{\zeta} = 0, \bar{\eta} = \xi)$.

The Hessian of S_ξ at the critical point c_ξ reads in block notation

$$D^2 S_\xi(c_\xi) \;=\; \begin{pmatrix} 0 & c_{ij}^k \xi_k & -1 & 0 \\ c_{ij}^k \xi_k & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

where c_{ij}^k are the structure constants of the Leibniz algebra $\mathfrak{h}$ and in Einstein convention, the sum over repeated indices is understood.

Denoting the matrix $D^2 S_\xi(c_\xi)$ simply by B, it is evident that $\det(B) = 1$, thus the critical point c_ξ is non-degenerate. Moreover, the signature of B is 0. The Feynman expansion (cf [11]) therefore reads

$$\begin{aligned} I(\hbar) &= (f \rhd_\hbar g)(\xi) = \\ &= \frac{e^{\frac{i\pi}{4}\mathrm{sign}(B)}}{\sqrt{|\det(B)|}} \Big(\sum_{\Gamma \in G_{3\geq}(2)} \frac{(i\hbar)^{|\mathrm{E}_\Gamma| - |\mathrm{V}_\Gamma^{\mathrm{int}}|}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(S_\xi; f, g) \Big) \\ &= \sum_{\Gamma \in G_{3\geq}(2)} \frac{(i\hbar)^{|\mathrm{E}_\Gamma| - |\mathrm{V}_\Gamma^{\mathrm{int}}|}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(S_\xi; f, g). \end{aligned}$$

These sums are sums over the set $G_{3\geq}(2)$ of Feynman graphs Γ with 2 external vertices and internal vertices of valence greater or equal to 3. For the definition of a Feynman graph, we refer the reader to [11]. $|\mathrm{E}_\Gamma|$ is the cardinality of the set of edges of Γ, $\mathrm{V}_\Gamma^{\mathrm{int}}$ is the set of internal vertices of Γ. $\mathrm{Aut}(\Gamma)$ is the number of symmetries of Γ. To each Γ, one associates an amplitude $F_\Gamma(S_\xi; f, g)$ in a way which is specified in *loc. cit.*. Namely, $F_\Gamma(S_\xi; f, g)$ is a product of two partial derivatives of S_ξ (represented by the internal vertices) and partial derivatives of f and g (represented by the external vertices) all of which are evaluated at the critical point c_ξ and contracted using the matrix B^{-1} which reads

$$B^{-1} \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -c_{ij}^k \xi_k \\ 0 & -1 & -c_{ij}^k \xi_k & 0 \end{pmatrix}.$$

Observe that it is the contraction with B^{-1} (where the terms $-c_{ij}^k \xi_k$ carry a minus sign !) which renders the Leibniz-Poisson bracket negative.

The first terms of the expansion of (8) in powers of $\hbar$ read therefore

$$(f \rhd_\hbar g)(\xi) \;=\; f(0) g(\xi) + \frac{i}{\hbar} \{f, g\}(\xi) + \mathcal{O}(\hbar),$$

where

$$\{f, g\}(\xi) \;=\; -\sum_{i,j,k} c_{i,j}^k \frac{\partial f}{\partial \xi_i}(0) \frac{\partial g}{\partial \xi_j}(\xi) \xi_k,$$

as in formula (3). □

Remark 2.13. (a) It is rather straight forward to compute the terms in this starproduct, the graphs which we have to consider are rather easy. For example, there are no inner loops.

(b) The zeroth term of the expansion, i.e. the product $f \otimes g \mapsto f(0)g(\xi)$, is actually associative.

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