Racks, Leibniz algebras and rack bialgebras

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0 Introduction

The triad (groups, Lie algebras, associative bialgebras) plays a prominent role in mathematics. The relations between Lie groups and Lie algebras, subsumed in *Lie theory*, are omnipresent in mathematics wherever there are symmetries. The relations often pass through associative algebras which have an additional compatible coalgebra structure, i.e. associative bialgebras. The aim of these lectures is to expose a generalized version of the triad (groups, Lie algebras, associative bialgebras), namely (racks, Leibniz algebras, rack bialgebras), and to explain the links between these structures inherited from the former, notably concerning the natural cohomologies of these structures.

The outline is the following: after recalling in Section 1 the links between groups, Lie algebras and associative bialgebras, we will introduce in Section 2 racks, Leibniz algebras and rack bialgebras and the links between these three algebraic structures. In Section 3, we will explain their cohomology theories. The main results which we discuss in Section 3 are the algebraic structures on rack cohomology related to Loday's conjectural Leibniz K-theory in Section 3.1 (Simon Covez), vanishing results for the Leibniz cohomology of nilpotent Leibniz algebras (joint work with Jörg Feldvoss) in Section 3.2, and an embedding of the Leibniz complex into the cohomology complex of rack bialgebras (Alexandre, Bordemann, Rivière, W.) in Section 3.3. Section 4 closes with final remarks and an outlook.

1 Groups, Lie algebra and associative bialgebras

Let us denote by `Grp` the category of groups, by `Lie` the category of Lie algebras over a field $\mathbb{K}$ of characteristic zero, by `Alg` the category of associative algebras over $\mathbb{K}$ and by `Bialg` the category of associative bialgebras over $\mathbb{K}$, i.e. the category of associative algebras B which carry a coassociative coproduct $\triangle : B \to B \otimes B$,

$$\triangle(b) = \sum_{(b)} b_1 \otimes b_2 = b_1 \otimes b_2,$$

(i.e. in Sweedler notation for the coproduct, we will leave out the sum-sign !) such that $\triangle$ is a morphism of associative algebras:

$$\triangle(bb') = \triangle(b)\triangle(b'),$$

where the RHS is the associative product in $B \otimes B$. This relation is sometimes called *Hopf relation*. We will often consider unital and counital bialgebras, or even Hopf algebras, for further informations about these see [12].

The categories `Grp`, `Lie` and `Bialg` are related by some functors. The functor $\mathbb{K}- : \texttt{Grp} \to \texttt{Alg}$ sends a group G to its *group algebra* $\mathbb{K}G$ over $\mathbb{K}$. $\mathbb{K}G$ is the $\mathbb{K}$-vector space with basis e_g indexed by the elements $g \in G$; it is simply the $\mathbb{K}$-linearization of the group G. We define an associative product on basis elements of $\mathbb{K}G$ by

$$e_g\, e_h := e_{gh}$$

for all elements $g, h \in G$ and extend it by $\mathbb{K}$-linearity to all elements of $\mathbb{K}G$.

In the other direction, there is the functor of units $-^{\times} : \texttt{Alg} \to \texttt{Grp}$ which sends an algebra A to the group $A^{\times}$ of its invertible elements. Note that we need a unit in the algebra A in order to have a group $A^{\times}$.

Proposition 1.1. *The functor* $\mathbb{K}- : \texttt{Grp} \to \texttt{Alg}$ *is left adjoint to the functor* $-^{\times} : \texttt{Alg} \to \texttt{Grp}$*, i.e. we have for all groups G and all (unital) algebras A a natural isomorphism*

$$\mathrm{Hom}_{\texttt{Alg}}(\mathbb{K}G, A) \cong \mathrm{Hom}_{\texttt{Grp}}(G, A^{\times}).$$

Proof. A homomorphism of algebras $f : \mathbb{K}G \to A$ is uniquely specified on the basis elements e_g for g ranging in G. The homomorphism property together with $e_g e_h = e_{gh}$ imply that we obtain a group homomorphism $\tilde{f} : G \to A^{\times}$ by defining $\tilde{f}(g) := f(e_g)$. All elements in the image of $\tilde{f}$ are invertible, thus $\tilde{f}$ has its image included in $A^{\times}$. Conversely, a group homomorphism $\tilde{f} : G \to A^{\times}$ extends to an algebra homomorphism $f : \mathbb{K}G \to A$ again because of the homomorphism property together with $e_g e_h = e_{gh}$. We leave the proof of the naturality to the interested reader. □

Now let us discuss the relation between the categories `Alg` and `Lie`. There is the functor $_{\mathrm{Lie}} : \texttt{Alg} \to \texttt{Lie}$ associating to a unital associative algebra A its underlying Lie algebra A_{Lie} given by the same vector space as A, but with the bracket

$$[a, b] := ab - ba,$$

for all elements $a, b \in A$. In the reverse direction, there is the functor of the *universal enveloping algebra* $U : \texttt{Lie} \to \texttt{Alg}$ sending a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $U\mathfrak{g}$. As before, we have

Proposition 1.2. *The functor* $U : \mathtt{Lie} \to \mathtt{Alg}$ *is left adjoint to the functor* $_{\mathrm{Lie}} : \mathtt{Alg} \to \mathtt{Lie}$*, i.e. we have for all algebras* A *and all Lie algebras* $\mathfrak{g}$ *a natural isomorphism*

$$\mathrm{Hom}_{\mathtt{Alg}}(U\mathfrak{g}, A) \cong \mathrm{Hom}_{\mathtt{Lie}}(\mathfrak{g}, A_{\mathrm{Lie}}).$$

Proof. The adjointness follows right away from the universal property of the universal enveloping algebra which reads as follows: Given any Lie algebra homomorphism $\tilde{f} : \mathfrak{g} \to A_{\mathrm{Lie}}$, there exists a unique homomorphism of associative algebras $f : U\mathfrak{g} \to A$ such that f restricts to $\tilde{f}$ on $\mathfrak{g} \subset U\mathfrak{g}$. Again, the proof of the naturality is left to the reader. □

Combining these functors, we obtain also functors between the categories $\mathtt{Lie}$ and $\mathtt{Grp}$, sending a Lie algebra $\mathfrak{g}$ to the unit group $(U\mathfrak{g})^{\times}$ of its universal enveloping algebra, and sending a group G to the Lie algebra $\mathbb{K}G_{\mathrm{Lie}}$ underlying its group algebra. Note that the group $(U\mathfrak{g})^{\times}$ is just $\mathbb{K}^{\times}$, thus not very interesting. In fact, it becomes interesting *after completion*.

Note that both left adjoints, namely $G \mapsto \mathbb{K}G$ and $\mathfrak{g} \mapsto U\mathfrak{g}$, take values in the category $\mathtt{Bialg}$ of associative bialgebras. The associative algebra $\mathbb{K}G$ becomes a bialgebra declaring that all basis elements e_g for $g \in G$ are *group-like*, i.e.

$$\triangle e_g = e_g \otimes e_g.$$

The universal enveloping algebra $U\mathfrak{g}$ carries a coproduct such that all elements x of $\mathfrak{g}$ are *primitive*, i.e. $\triangle x = x \otimes 1 + 1 \otimes x$. Thus there are two more functors, this time relating the category of bialgebra $\mathtt{Bialg}$ with $\mathtt{Grp}$ and $\mathtt{Lie}$: The functor of primitives, $P : \mathtt{Bialg} \to \mathtt{Lie}$ sends a bialgebra B to its primitives $P(B) := \{x \in B \,|\, \triangle x = x \otimes 1 + 1 \otimes x\}$. The Lie bracket underlying the associative algebra B descends to $P(B)$. There is also the functor of group-like elements $\mathtt{Bialg} \to \mathtt{Set}$, sending a bialgebra B to the set of its group-like elements $x \in B$ such that $\triangle x = x \otimes x$. For a Hopf algebra possessing an invertible antipode, this functor takes values in $\mathtt{Grp}$, see Prop. III.3.7 in [12].

A pivot in Lie theory is the relation between a Lie group and its associated Lie algebra. Here a *Lie group* is a group G which carries the structure of a differentiable manifold such that group multiplication $G \times G \to G$, $(g,h) \mapsto gh$, and inversion map $G \to G$, $g \mapsto g^{-1}$, are smooth maps with respect to the manifold structure. There are (at least) two ways of associating a Lie algebra to a Lie group. First of all, the left-invariant vector fields on G are closed under the bracket of vector fields and become thus a Lie subalgebra. But a left-invariant vector field on G is uniquely determined by its values on the tangent space T_1G at the unit $1 \in G$. Thus the tangent space T_1G becomes a Lie algebra, denoted $\mathfrak{g}$ and called the *Lie algebra* of the Lie group G. It is interesting to note that the Lie bracket can be obtained in the following way: Given two smooth curves $\gamma : [-1,1] \to G$ and $\eta : [-1,1] \to G$ with $\eta(0) = \gamma(0) = 1$ and $\dot{\gamma}(0) = x \in T_1G$, $\dot{\eta}(0) = y$, the brackets $[x,y]$ can be obtained by differentiation of the conjugation $\eta\gamma\eta^{-1}$ with respect to the two parameters of the two curves.

The second construction is a little less known, it appears e.g. in [24]. The Lie group G is in particular a *pointed manifold*, i.e. a manifold with a distinguished point $1 \in G$. As such, there is an associated bialgebra of distributions (in the sense of the continuous dual of the space of functions) on G supported in the point 1. The primitives in this bialgebra form a Lie algebra which gives back the Lie algebra $\mathfrak{g}$ of the Lie group G.

We will restrain ourselves to the integration of Lie algebras into Lie groups in the framework of Lie groups (and will not speak about the integration of Lie algebras over the complex numbers into algebraic groups). Namely, given a real finite-dimensional Lie algebra $\mathfrak{g}$, there exists a unique 1-connected Lie group G such that its Lie algebra is the Lie algebra $\mathfrak{g}$ we started with. The procedure obtaining G from $\mathfrak{g}$ is functoriel.

2 Racks, Leibniz algebras and rack bialgebras

2.1 Racks and Leibniz algebras

The notion of a rack results from axiomatizing the properties of group conjugation:

Definition 2.1. *A (left) rack consists of a set X equipped with a binary operation denoted $(x,y) \mapsto x \rhd y$ such that for all x, y, and $z \in X$, the map $y \mapsto x \rhd y$ is bijective and*

$$x \rhd (y \rhd z) \,=\, (x \rhd y) \rhd (x \rhd z).$$

The identity $x \rhd (y \rhd z) \,=\, (x \rhd y) \rhd (x \rhd z)$ is called *self-distributivity relation*, because it expresses the fact that the operation $\rhd$ is distributive with respect to itself.

The conjugation in a group G gives rise to a (left) rack operation given by $(g,h) \mapsto ghg^{-1}$. Other examples (see [10]) include the following:

Example 2.2. *Any union of conjugacy classes in a group can be viewed as a rack (although in general, it is not a group !). For example, the dihedral rack is the set of reflections in the dihedral group.*

Continuing with reflections, the plane $\mathbb{R}^2$ can be viewed as a rack with the reflection operation, i.e. for two point p and q, define $p \rhd q$ to be the point q reflected in p, meaning $p \rhd q = 2p - q$ in vector notation.

Example 2.3. *Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials. Any Λ-module M has the structure of a rack by defining $a \rhd b := ta + (1-t)b$ for all $a, b \in M$. This rack is called the Alexander rack. For $M = \mathbb{R}^2$ the plane and the action of t given by multiplication by -1, one gets back the reflection rack on the plane.*

Remark 2.4. Racks appeared in knot theory, see [10]. In fact, the fundamental rack of a codimension 2 link is a finer invariant than the corresponding group. For exemple, in the construction of the Wirtinger presentation of the knot group, one associates generators in the free group to any arc and conjugation relations to crossings. Taking instead the free rack with crossing relations expressed in terms of the rack product gives a *finer invariant*, because one did not quotient out the relations which imply that the product is a group product.

The notion of a unit leads to pointed racks.

Definition 2.5. *A pointed rack $(X, \triangleright, 1)$ consists of a set X equipped with a binary operation $\triangleright$ and an element $1 \in X$ satisfying:*

1. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$,
2. *For all $a, b \in X$, there exists a unique $x \in X$ such that $a \triangleright x = b$,*
3. $1 \triangleright x = x$ *and* $x \triangleright 1 = 1$ *for all* $x \in X$.

Once again, the conjugation rack of a group is an example of a pointed rack.

Definition 2.6. *A map $f : R \to S$ between two racks R and S is called a morphism of racks in case for all $r, r' \in R$, we have*

$$f(r \triangleright r') = f(r) \triangleright f(r').$$

In the usual way, we will speak about iso- and automorphisms of racks.

Let us denote the conjugation rack underlying the group G by $\mathrm{Conj}(G)$. Denote by `Racks` the category of racks, i.e. the category whose objects are racks and whose morphisms are rack morphisms as defined above. Then, Conj is a functor from the category of groups `Grp` to `Racks`.

Definition 2.7. *Let R be a rack. The associated group to R, denoted $\mathrm{As}(R)$, is the quotient of the free group $F(R)$ on the set R by the normal subgroup generated by the elements $xy^{-1}x^{-1}(x \triangleright y)$ for all $x, y \in R$. Denote the canonical morphism of racks by $i : R \to \mathrm{As}(R)$.*

Example 2.8. *Consider the set $R := \{x, y\}$ consisting of two elements x and y with an operation given by*

$$x \triangleright x = y, \quad x \triangleright y = x, \quad y \triangleright y = x, \quad y \triangleright x = y.$$

It is easy to verify that R is indeed a rack. The relation $x \triangleright y = xyx^{-1}$ in $\mathrm{As}(R)$ implies that $x = y$ in $\mathrm{As}(R)$. It turns out that $\mathrm{As}(R)$ is isomorphic to $\mathbb{Z}$, thus the canonical map $i : R \to \mathrm{As}(R)$ is not necessarily injective.

The importance of the associated group $\mathrm{As}(R)$ of a rack R comes from the following universal mapping property:

Lemma 2.9. *Let R be a rack and G be a group. For any morphism of racks $f : R \to \mathrm{Conj}(G)$, there exists a unique group morphism $g : \mathrm{As}(R) \to G$ such that $g \circ i = f$.*

Proof. By freeness of the free group on the set R, a morphism of racks $R \to \mathrm{Conj}(G)$ induces a group homomorphism $F(R) \to G$. This morphism sends all elements $xy^{-1}x^{-1}(x \rhd y)$ to $1 \in G$, because both the commutator in $F(R)$ and the rack product are sent to the commutator in G. The first assertion is true because $F(R) \to G$ is a group homomorphism, and the second assertion is true because $R \to \mathrm{Conj}(G)$ is a rack morphism. Thus $F(R) \to G$ factors through a morphism $\mathrm{As}(R) \to G$. This group homomorphism is unique, because it is fixed on the generators of $\mathrm{As}(R)$. ☐

From this, one can deduce that the functor $\mathrm{As} : \mathtt{Racks} \to \mathtt{Grp}$ from the category of racks to the category of groups is left adjoint to the functor $\mathrm{Conj} : \mathtt{Grp} \to \mathtt{Racks}$ which associates to a group its underlying conjugation rack.

Remark 2.10. In fact, the unit of the adjunction is just the map i. By standard arguments, the unit of the adjunction is injective, but only as a map

$$i : \mathrm{Conj}(G) \to \mathrm{Conj}(\mathrm{As}(\mathrm{Conj}(G)))$$

for a group G.

We observe that the compositions $\mathrm{Conj}(\mathrm{As}(R))$ for a racks R and $\mathrm{As}(\mathrm{Conj}(G))$ for a group G are, in general, far from being equal to R or G respectively. For example, for an abelian group A, the conjugation rack $\mathrm{Conj}(A)$ is the set A with the trivial rack product, while $\mathrm{As}(\mathrm{Conj}(A))$ is the free abelian group generated by the set A.

Definition 2.11. *Let G be a group and X be a G-set. We say that X together with a map $p : X \to G$ is an augmented rack in case it satisfies the augmentation identity, i.e.*

$$p(g \cdot x) \;=\; g\, p(x)\, g^{-1}$$

for all $g \in G$ and all $x \in X$.

We observe that for any augmented rack $p : X \to G$, one may define a rack operation on X as $x \rhd x' := p(x) \cdot x'$ for all $x, x' \in X$. Then, the map p becomes an equivariant morphism of racks (with respect to the given G-action on X and the conjugation action on the group G). Augmented racks are in fact the Yetter-Drinfel'd modules over the Hopf algebra G (in the symmetric monoidal category of sets), or in other words, the Drinfel'd center of the symmetric monoidal category of G-modules, see [11].

Remark 2.12. There are many examples of augmented racks. For example, for each rack R, the canonical morphisms $R \to \mathrm{Aut}(R)$ and $i : R \to \mathrm{As}(R)$ are augmented racks.

Definition 2.13. *Let R be a rack and X be a set. We say that R acts on X or that X is an R-set in case there are bijections $(r\cdot) : X \to X$ for all $r \in R$ such that*

$$r \cdot (r' \cdot x) \;=\; (r \rhd r') \cdot (r \cdot x)$$

for all $x \in X$ and all $r, r' \in R$.

Obviously, this definition is the prototype of a module action and adapts to different algebraic situations: In case X is an abelian group, X becomes a *rack module* over the rack R if R acts on X (in the sense of the previous definition) by $\mathbb{Z}$-linear maps $(r\cdot)$ for all $r \in R$.

Lemma 2.14. *An action of R on X is equivalent to a morphism of racks $\mu : R \to \mathrm{Bij}(X)$ with values in the conjugation rack underlying the group of bijections on X.*

Proof. Indeed, the defining equation in Definition 2.14 can be written

$$(r\cdot) \circ (r'\cdot) \circ (r\cdot)^{-1}(y) \;=\; (r \rhd r') \cdot (y)$$

for all $r, r' \in R$ and all $y \in X$. This shows that the rack product in R is sent to the rack product in $\mathrm{Conj}(\mathrm{Bij}(X))$. □

The following structure is the analogue of the semi-direct product of a group G by a G-module. As it is only "half of the structure" (the term concerning x is missing), it is termed *hemi-semidirect product*, see [13].

Definition 2.15. *Let R be a rack and X be an R-set. The hemi-semidirect product rack consists of the set $X \times R$ equipped with the rack product*

$$(x, r) \rhd (x', r') \;:=\; (r \cdot x', r \rhd r')$$

for all $x, x' \in X$ and all $r, r' \in R$.

The following definition is the Lie-group-version of a rack:

Definition 2.16. *A Lie rack is a pointed rack M with the structure of a smooth manifold such that for all $x, y \in M$, the rack operation $(x, y) \mapsto x \rhd y$ is a smooth map $M \times M \to M$ and the map $y \mapsto x \rhd y$ is a diffeomorphisms of M.*

A *morphisms of Lie racks* $\phi : M \to M'$ is a map of pointed manifolds satisfying for all $x, y \in M$ the condition $\phi(x \rhd y) = \phi(x) \rhd' \phi(y)$. The class of all Lie racks forms a category called `LieRack`.

Let us now come to Leibniz algebras: Leibniz algebras have been invented by A. M. Blokh [3] in 1965, and then rediscovered by J.-L. Loday in 1992 in the search of an explanation for the absence of periodicity in algebraic K-theory [17, p.323, Equation (10.6.1.1)']. We will come back to this point later.

Definition 2.17. *A Leibniz algebra (over $\mathbb{K}$) is a $\mathbb{K}$-vector space $\mathfrak{h}$ equipped with a linear map $[\ ,\ ] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$, written $x \otimes y \mapsto [x, y]$ such that the (left) Leibniz identity holds for all $x, y, z \in \mathfrak{h}$*

$$\big[x, [y, z]\big] = \big[[x, y], z\big] + \big[y, [x, z]\big] \tag{1}$$

A morphism of Leibniz algebras $f : \mathfrak{h} \to \mathfrak{h}'$ is a $\mathbb{K}$-linear map preserving brackets, i.e. for all $x, y \in \mathfrak{h}$ we have $f\big([x, y]\big) = \big[f(x), f(y)\big]'$. Note first that each Lie algebra is a Leibniz algebra giving rise to a functor i from the category `LieAlg` of Lie algebras to the category `Leib` of Leibniz algebras.

The following relation of Lie racks to $\mathbb{R}$-Leibniz algebras is due to M.Kinyon [14]:

Proposition 2.18. *Let $(M, 1, \rhd)$ be a Lie rack and $\mathfrak{h} = T_1 M$. Define the following bracket $[\ ,\ ]$ on $\mathfrak{h}$ by*

$$[x, y] = \frac{\partial}{\partial t} T_1 L_{a(t)}(y)\Big|_{t=0} \tag{2}$$

where $t \mapsto a(t)$ is any smooth curve defined on an open real interval containing 0 satisfying $a(0) = 1$, $(da/dt)(0) = x \in \mathfrak{h}$ and $L_{a(t)} = a(t) \rhd -$ means the left translation by $a(t)$ (with respect to the rack product $\rhd$). Then we have the following

1. *$\big(\mathfrak{h}, [\ ,\ ]\big)$ is a real Leibniz algebra.*
2. *Let $\phi : (M, 1, \rhd) \to (M', 1', \rhd')$ be a morphism of Lie racks. Then $T_1\phi : \mathfrak{h} \to \mathfrak{h}'$ is a morphism of Leibniz algebras.*

Proof. (a) Since for each $a \in M$, we have $L_a(1) = 1$, it follows that the tangent map $T_1 L_a$ maps the tangent space $T_1 M$ to $T_1 M$. Therefore the curve $t \mapsto T_1 L_{a(t)}$ is a curve of $\mathbb{R}$-linear maps $T_1 M \to T_1 M$, whence Equation (2) defines a well-defined real bilinear map $\mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$.

Let $x, y, z \in \mathfrak{h}$, and let $t \mapsto a(t)$ and $t \mapsto b(t)$ two smooth curves of an open interval (containing 0) into M such that $a(0) = 1 = b(0)$ and $(da/dt)(0) = x$, $(db/dt)(0) = y$. We compute

$$\begin{aligned}
\big[x, [y, z]\big] &= \\
&= \frac{\partial^2}{\partial s \partial t} \big(T_1 L_{a(s)} \big(T_1 L_{b(t)}(z)\big)\big)\Big|_{s,t=0} = \frac{\partial^2}{\partial s \partial t} T_1 \big(L_{a(s)} \circ L_{b(t)}\big)(z)\Big|_{s,t=0} \\
&= \frac{\partial^2}{\partial s \partial t} T_1 \big(L_{a(s) \rhd b(t)} \circ L_{a(s)}\big)(z)\Big|_{s,t=0} = \frac{\partial^2}{\partial s \partial t} \big(T_1 L_{a(s) \rhd b(t)} \big(T_1 L_{a(s)}(z)\big)\big)\Big|_{s,t=0} \\
&= \frac{\partial^2}{\partial s \partial t} T_1 L_{a(s) \rhd b(t)}\Big|_{s,t=0} \big(T_1 L_{a(0)}(z)\big) \\
&\quad + \frac{\partial}{\partial t} T_1 L_{a(0) \rhd b(t)}\Big|_{t=0} \left(\frac{\partial}{\partial s} \big(T_1 L_{a(s)}(z)\big)\right)\Big|_{s=0}.
\end{aligned}$$

Since $a(0) = 1$, we have $T_1 L_{a(0)}(z) = z$ and $a(0) \rhd b(t) = b(t)$, whence the last term equals $\big[y, [x, z]\big]$. Since for each s the curve $t \mapsto a(s) \rhd b(t)$ is equal to 1 at $t = 0$, we get

$$\frac{\partial^2}{\partial s \partial t} T_1 L_{a(s) \rhd b(t)}\Big|_{s,t=0} \big(T_1 L_{a(0)}(z)\big) \quad = \quad \left[\frac{\partial}{\partial s} T_1 L_{a(s)}(y)\Big|_{s=0}, z\right] = \big[[x, y], z\big]$$

proving the Leibniz identity.

(b) Since ϕ maps 1 to $1'$, its tangent map $T_1\phi$ maps $T_1 M$ to $T_{1'} M'$. We get for all $x, y \in \mathfrak{h} = T_1 M$, where $t \mapsto a(t)$ is a smooth curve in M with $a(0) = 1$

and $(da/dt)(0) = x$:

$$\begin{aligned} T_1\phi\big([x,y]\big) &= T_1\phi\left(\frac{\partial}{\partial t}T_1L_{a(t)}(y)\Big|_{t=0}\right) = \frac{\partial}{\partial t}\Big(T_1\big(\phi\circ L_{a(t)}\big)(y)\Big)\Big|_{t=0} \\ &= \frac{\partial}{\partial t}\Big(T_1\big(L'_{\phi(a(t))}\circ\phi\big)(y)\Big)\Big|_{t=0} = \frac{\partial}{\partial t}T_{1'}L'_{\phi(a(t))}\Big|_{t=0}\big(T_1\phi(y)\big) \\ &= \big[T_1\phi(x), T_1\phi(y)\big]. \qquad\square \end{aligned}$$

Let $\texttt{Leib}_{fd}$ denote the category of finite-dimensional real Leibniz algebras. The preceding proposition shows that there is a functor $T_*\mathcal{R} : \texttt{LieRack} \to \texttt{Leib}_{fd}$ which associates to any Lie rack M its tangent space T_1M at the distinguished point $1 \in M$ equipped with the Leibniz bracket Equation (2).

Furthermore, recall that each Leibniz algebra has two canonical subspaces

$$\begin{aligned} Q(\mathfrak{h}) &:= \Big\{x\in\mathfrak{h} \mid \exists\, N\in\mathbb{N}\setminus\{0\},\ \exists\,\lambda_1,\ldots,\lambda_N\in\mathbb{K},\ \exists\, x_1,\ldots,x_N \\ &\qquad\qquad \text{such that } x = \sum_{r=1}^{N}\lambda_r[x_r,x_r]\Big\}, && (3)\\ \mathfrak{z}(\mathfrak{h}) &:= \Big\{x\in\mathfrak{h} \mid \forall\, y\in\mathfrak{h}:\ [x,y]=0\Big\}. && (4) \end{aligned}$$

It is not hard to deduce from the Leibniz identity that both $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ are two-sided abelian ideals of $(\mathfrak{h}, [\ ,\ ])$, that $Q(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{h})$, and that the quotient Leibniz algebras

$$\overline{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h}) \quad \text{and} \quad \mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \tag{5}$$

are Lie algebras. The ideal $Q(\mathfrak{h})$ is called the *ideal of squares* and $\mathfrak{z}(\mathfrak{h})$ is called the *left center* of $\mathfrak{h}$. Since the ideal $Q(\mathfrak{h})$ is clearly mapped into the ideal $Q(\mathfrak{h}')$ by any morphism of Leibniz algebras $\mathfrak{h} \to \mathfrak{h}'$ (which is a priori not the case for $\mathfrak{z}(\mathfrak{h})$!), there is an obvious functor $\mathfrak{h} \to \overline{\mathfrak{h}}$ from the category of all Leibniz algebras to the category of all Lie algebras.

It is easy to observe that in both cases of the above Lie algebras $\overline{\mathfrak{h}}$ and $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$, the following structure is present:

Definition 2.19. *A quintuple $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot\rho\big)$ is called an augmented Leibniz algebra if the following holds:*

1. $\big(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}\big)$ *is a Lie algebra.*
2. $\mathfrak{h}$ *is a $\mathbb{K}$-vector space which is a left $\mathfrak{g}$-module via the linear map $\dot\rho : \mathfrak{g}\otimes\mathfrak{h} \to \mathfrak{h}$ written $\dot\rho_x(h) = x\cdot h$ for all $x\in\mathfrak{g}$ and $h\in\mathfrak{h}$.*
3. $p : \mathfrak{h}\to\mathfrak{g}$ *is a morphism of $\mathfrak{g}$-modules, i.e. for all $x\in\mathfrak{g}$ and $h\in\mathfrak{h}$*

$$p(x\cdot h) = \big[x, p(h)\big]_{\mathfrak{g}}. \tag{6}$$

A morphism of augmented Leibniz algebras $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot\rho\big) \to \big(\mathfrak{h}', p', \mathfrak{g}', [\ ,\ ]'_{\mathfrak{g}}, \dot\rho'\big)$ is a pair (F, f) of linear maps where $f : \mathfrak{g}\to\mathfrak{g}'$ is a morphism of Lie algebras,

$F : \mathfrak{h} \to \mathfrak{h}$ is a morphism of Lie algebra modules over f, i.e. for all $h \in \mathfrak{h}$ and $x \in \mathfrak{g}$

$$F(x \cdot h) = f(x) \cdot F(h). \tag{7}$$

Moreover the obvious diagram commutes, i.e.

$$p' \circ F = f \circ p. \tag{8}$$

The following properties are immediate from the definitions:

Proposition 2.20. *Let $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ be an augmented Leibniz algebra. Define the following bracket on $\mathfrak{h}$:*

$$[x, y]_{\mathfrak{h}} := p(x) \cdot y. \tag{9}$$

1. *$\big(\mathfrak{h}, [,]_{\mathfrak{h}}\big)$ is a Leibniz algebra on which $\mathfrak{g}$ acts as derivations. If (F, f) is a morphism of augmented Leibniz algebras, then F is a morphism of Leibniz algebras.*

2. *The kernel of p, $\ker(p)$, is a $\mathfrak{g}$-invariant two-sided abelian ideal of $\mathfrak{h}$ satisfying $Q(\mathfrak{h}) \subset \ker(p) \subset \mathfrak{z}(\mathfrak{h})$.*

3. *The image of p, $\mathrm{im}(p)$, is an ideal of the Lie algebra $\mathfrak{g}$.*

Proof. We just check the Leibniz identity: Let $x, y, z \in \mathfrak{h}$, then, writing $[\ ,\ ]_{\mathfrak{h}} = [\ ,\ ]$,

$$\begin{aligned}
\big[x, [y, z]\big] &= p(x) \cdot \big(p(y) \cdot z\big) \\
&= p(x) \cdot \big(p(y) \cdot z\big) - p(y) \cdot \big(p(x) \cdot z\big) + p(y) \cdot \big(p(x) \cdot z\big) \\
&= \big[p(x), p(y)\big]_{\mathfrak{g}} \cdot z + \big[y, [x, z]\big] \overset{(6)}{=} p\big(p(x) \cdot y\big) \cdot z + \big[y, [x, z]\big] \\
&= \big[[x, y], z\big] + \big[y, [x, z]\big].
\end{aligned}$$

□

It follows that the class of augmented Leibniz algebras forms a category `LeibA`. There is a forgetful functor from `LeibA` to `Leib` associating

$$\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big) \mapsto \big(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}}\big)$$

where the Leibniz bracket $[\ ,\ ]_{\mathfrak{h}}$ is defined in Equation (9).

On the other hand there is a functor from `Leib` to `LeibA` associating to each Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ the augmented Leibniz algebra $\big(\mathfrak{h}, p, \bar{\mathfrak{h}}, [\ ,\ ]_{\bar{\mathfrak{h}}}, \mathrm{ad}'\big)$ where $p : \mathfrak{h} \to \bar{\mathfrak{h}}$ is the canonical projection and the representation ad' of the Lie algebra $\bar{\mathfrak{h}}$ on the Leibniz algebra $\mathfrak{h}$ is defined by (for all $x, y \in \mathfrak{h}$)

$$\mathrm{ad}'_{p(x)}(y) := \mathrm{ad}_x(y) = [x, y]. \tag{10}$$

Remark 2.21. There exists an inverse to the construction in Proposition 2.18, namely an integration process which integrates finite-dimensional real (augmented) Leibniz algebras into (augmented) Lie racks. In order to make this meaningful, one also imposes that the integration process is such that finite-dimensional Lie algebras are integrated into the standard simply-connected Lie group associated to a finite-dimensional real Lie algebra. Such an integration procedure exists (see [7], [4]), but with some drawbacks: The procedure in [7] is natural and functorial and gives a nice link between Leibniz cohomology and rack cohomology. Unfortunately, it works only locally.

In the search of globalizing this local procedure, for the moment, the only known global procedure in [4] is not functorial in general. It is functorial only for morphisms of Leibniz algebras which *reduce* the spectrum of the ad-operators.

2.2 Racks, Leibniz algebras and rack bialgebras

We now come to the notion which is analogous to the notion of the group algebra in the framework of racks. First, let us recall some basic notions about coalgebras. We shall always assume that $\mathbb{Q} \subset \mathbb{K}$, i.e. that the field $\mathbb{K}$ is of characteristic zero.

Let C be a $\mathbb{K}$-vector space. Recall that a linear map $\triangle : C \to C \otimes C$ is called a *coassociative comultiplication* in case $\big(\triangle \otimes \mathrm{id}_C\big) \circ \triangle = \big(\mathrm{id}_C \otimes \triangle\big) \circ \triangle$, and the pair $(C, \triangle)$ is called a (coassociative) *coalgebra*. In Sweedler's notation, the coproduct on elements is written as

$$\triangle(a) = a_1 \otimes a_2.$$

Recall that this notation implies a sum over all tensors which form $\triangle(a)$.

Let $(C', \triangle')$ be another coalgebra. The coalgebra $(C, \triangle)$ is called *cocommutative* if $\tau \circ \triangle = \triangle$ where $\tau : C \otimes C \to C \otimes C$ denotes the canonical flip map. Recall furthermore that a linear map $\epsilon : C \to \mathbb{K}$ is called a *counit* for the coalgebra $(C, \triangle)$ in case $\big(\epsilon \otimes \mathrm{id}_C\big) \circ \triangle = \big(\mathrm{id}_C \otimes \epsilon\big) \circ \triangle = \mathrm{id}_C$. The triple $(C, \triangle, \epsilon)$ is called a *counital coalgebra*. Moreover, a counital coalgebra $(C, \triangle, \epsilon)$ equipped with an element 1 is called *coaugmented* if $\triangle(1) = 1 \otimes 1$ and $\epsilon(1) = 1 \in \mathbb{K}$. Recall that a morphism $\phi : (C, \triangle, \epsilon, 1) \to (C', \triangle', \epsilon', 1')$ of counital coaugmented coalgebras is a $\mathbb{K}$-linear map satisfying $(\phi \otimes \phi) \circ \triangle = \triangle' \circ \phi$, $\epsilon' \circ \phi = \epsilon$, and $\phi(1) = 1'$. Moreover, for any counital coaugmented coalgebra the $\mathbb{K}$-submodule of all *primitive elements* is defined by

$$P(C) := \{x \in C \mid \triangle(x) = x \otimes 1 + 1 \otimes x\}. \tag{11}$$

The analogue of the notion of a bialgebra in the framework of racks is the notion of a rack bialgebra.

Definition 2.22. *A rack bialgebra $(B, \triangle, \epsilon, 1, \mu)$ is a coassociative, counital, coaugmented coalgebra $(B, \triangle, \epsilon, 1)$ together with a product $\rhd : B \times B \to B$ which is a morphism of coalgebras (and satisfies in particular $1 \rhd 1 = 1$) such that the following identities hold for all $a, b, c \in B$*

$$\begin{aligned} 1 \rhd a &= a, & (12)\\ a \rhd 1 &= \epsilon(a)1, & (13)\\ a \rhd (b \rhd c) &= (a_1 \rhd b) \rhd (a_2 \rhd c). & (14) \end{aligned}$$

The last condition (14) is called the self-distributivity condition.

Note that we do not demand that the coalgebra B should be cocommutative. Note furthermore that the self-distributivity condition here is the *linearized* version of the self-distributivity relation of a rack.

Example 2.23. *Any coassociative, counital, coaugmented coalgebra* $(C,\triangle,\epsilon,1)$ *carries a* trivial rack bialgebra structure *defined by the left-trivial multiplicaton*

$$a \rhd b := \epsilon(a)b \tag{15}$$

which in addition is easily seen to be associative and left-unital, but in general not unital.

Example 2.24. *Let* $(H,\triangle_H,\epsilon_H,\mu_H,1_H,S)$ *be a cocommutative Hopf algebra over* $\mathbb{K}$*. Then it is easy to see (cf. also the particular case* $B=H$ *and* $\Phi=\mathrm{id}_H$ *of Proposition 2.28) that the new product* $\rhd : H\otimes H\to H$ *defined by the usual* adjoint representation

$$h \rhd h' := \mathrm{ad}_h(h') := h_1 h' S(h_2), \tag{16}$$

equips the coassociative, counital, coaugmented coalgebra $(H,\triangle_H,\epsilon_H,1_H)$ *with a rack bialgebra structure.*

In general, the adjoint representation does not preserve the coalgebra structure if no cocommutativity is assumed.

Example 2.25. *Any Leibniz algebra* $\mathfrak{h}$ *gives rise to a rack bialgebra on the* $\mathbb{K}$*-vector space* $\mathbb{K}\oplus\mathfrak{h}$ *by putting* $x\rhd y := [x,y]$ *for all* $x,y\in\mathfrak{h}$ *and that all elements of* $\mathfrak{h}$ *are primitive.*

Example 2.26. *Let* $(X,1)$ *be a pointed rack. Then there is a natural rack bialgebra structure on the vector space* $\mathbb{K}X$ *which has the elements of* X *as a basis.* $\mathbb{K}X$ *carries the usual coalgebra structure such that all* $x\in X$ *are set-like:* $\triangle(x)=x\otimes x$ *for all* $x\in X$*. The product* $\rhd$ *is then the linearization of the rack product. By functoriality,* $\rhd$ *is compatible with* $\triangle$ *and* 1*. Observe that this construction differs slightly from the construction in [5], Section 3.1.*

As in the case of Leibniz algebras and Lie racks, there is an associated *augmented* structure:

Definition 2.27. *An augmented rack bialgebra is a quadruple* (B,p,H,ℓ) *consisting of a coassociative, counital, coaugmented coalgebra* $(B,\triangle,\epsilon,1)$*, of a cocommutative Hopf algebra* $(H,\triangle_H,\epsilon_H,1_H,\mu_H,S)$*, of a morphism of coalgebras* $p:B\to H$*, and of a left action* $\ell : H\otimes B\to B$ *of* H *on* B *which is a morphism of coalgebras (i.e.* B *is a* H*-module-coalgebra) such that for all* $h\in H$ *and* $a\in B$

$$\begin{aligned} h.1 &= \epsilon_H(h)1 & (17)\\ p(h\cdot a) &= \mathrm{ad}_h\big(p(a)\big). & (18)\end{aligned}$$

where ad *denotes the usual adjoint representation for Hopf algebras, see e.g. Equation (16).*

We shall define a morphism $(B,p,H,\ell)\to(B',p',H',\ell')$ *of augmented rack bialgebras to be a pair* (ϕ,ψ) *of* $\mathbb{K}$*-linear maps where* $\phi:(B,\triangle,\epsilon,1)\to(B',\triangle',\epsilon',1')$ *is a morphism of coalgebras, and* $\psi:H\to H'$ *is a morphism of Hopf algebras such that the obvious diagrams commute:*

$$p'\circ\phi=\psi\circ p,\quad\text{and}\quad \ell'\circ(\psi\otimes\phi)=\phi\circ\ell \tag{19}$$

An immediate consequence of this definition is the following

Proposition 2.28. *Let* (B,p,H,ℓ) *be an augmented rack bialgebra. Then the coassociative, counital, coaugmented coalgebra* $(B,\epsilon,1)$ *will become a rack bialgebra by means of the product*

$$a\rhd b:=p(a)\cdot b \tag{20}$$

for all $a,b\in B$*. In particular, each cocommutative Hopf algebra* H *becomes an augmented rack bialgebra via* $(H,\mathrm{id}_H,H,\mathrm{ad})$*. In general, for each augmented rack bialgebra the map* $p:B\to H$ *is a morphism of rack bialgebras.*

Proof. We check first that $\rhd$ is a morphism of coalgebras $B\otimes B\to B$: Let $a,b\in B$, then we have, using the fact that the action ℓ and the maps p are coalgebra morphisms:

$$\begin{aligned}\triangle(a\rhd b) &= \triangle\big(p(a)\cdot b\big)=\big(p(a)_1\cdot b_1\big)\otimes\big(p(a)_2\cdot b_2\big)\\ &= \big(p(a_1)\cdot b_1\big)\otimes\big(p(a_2)\cdot b_2\big)\\ &= \big(a_1\rhd b_1\big)\otimes\big(a_2\rhd b_2\big)\end{aligned}$$

whence $\rhd$ is a morphism of coalgebras. Clearly

$$\epsilon(a\rhd b)=\epsilon\big(p(a)\cdot b\big)=\epsilon_H\big(p(a)\big)\epsilon(b)=\epsilon(a)\epsilon(b)$$

whence $\rhd$ preserves counits. Let us next compute both sides of the self-distributivity identity (14). We have for all $a,b,c\in B$

$$a\rhd(b\rhd c)=p(a).\big(p(b)\cdot c\big)=\big(p(a)p(b)\big)\cdot c,$$

and

$$\begin{aligned}(a_1\rhd b)\rhd(a_2\rhd c) &= \big(p(a_1)\cdot b\big)\rhd\big(p(a_2)\cdot c\big)\\ &= \Big(p\big(p(a_1)\cdot b\big)\Big)\cdot\big(p(a_2)\cdot c\big)\\ &= \Big(p\big(p(a_1)\cdot b\big)p(a_2)\Big)\cdot c,\end{aligned}$$

and we compute, using the fact that p is a morphism of coalgebras,

$$\begin{aligned}p\big(p(a_1)\cdot b\big)p(a_2) &= p\big(p(a)_1\cdot b\big)p(a)_2\\ &\overset{(18)}{=} \mathrm{ad}_{p(a)_1}\big(p(b)\big)p(a)_2\\ &= p(a)_1p(b)\,S\big(p(a)_2\big)p(a)_3\\ &= p(a)_1p(b)\epsilon_H\big(p(a)_2\big)\\ &= p(a)p(b),\end{aligned}$$

which proves the self-distributivity identity. Moreover we have

$$1_B \rhd a = p(1).a = 1_H.a = a,$$

and

$$a \rhd 1 = p(a).1 \stackrel{(17)}{=} \epsilon_H\big(p(a)\big)1 = \epsilon_B(a)1.$$

This shows that the coassociative, counital, coaugmented coalgebra B becomes a rack bialgebra. □

Remark 2.29. Exactly in the same way as a pointed rack gives rise to a rack bialgebra $\mathbb{K}X$, an augmented pointed rack $p : X \to G$ gives rise to an augmented rack bialgebra $p : \mathbb{K}X \to \mathbb{K}G$.

The link to Leibniz algebras is contained in the following proposition:

Proposition 2.30. *Let $(B, \triangle, \epsilon, 1, \mu)$ be a rack bialgebra. Then its subspace of all primitive elements, $P(B) =: \mathfrak{h}$, is a subalgebra with respect to to the product $\rhd$ satisfying the Leibniz identity*

$$x \rhd (y \rhd z) = (x \rhd y) \rhd z + y \rhd (x \rhd z) \tag{21}$$

for all $x, y, z \in \mathfrak{h} = P(B)$. Hence the pair $(\mathfrak{h}, [\ ,\ ])$ with $[x, y] := x \rhd y$ for all $x, y \in \mathfrak{h}$ is a Leibniz algebra over $\mathbb{K}$. Moreover, every morphism of rack bialgebras maps primitive elements to primitive elements and thus induces a morphism of Leibniz algebras.

Proof. Let $x \in \mathfrak{h}$ and $a \in B$. Since μ is a morphism of coalgebras and x is primitive, we get

$$\begin{aligned} \triangle(a \rhd x) &= (a_1 \rhd x) \otimes (a_2 \rhd 1) + (a_1 \rhd 1) \otimes (a_2 \rhd x) \\ &\stackrel{(13)}{=} \big((a_1\epsilon(a_2)) \rhd x\big) \otimes 1 + 1 \otimes \big((\epsilon(a_1)a_2) \rhd x\big) \\ &= (a \rhd x) \otimes 1 \ + \ 1 \otimes (a \rhd x), \end{aligned}$$

whence $a \rhd x$ is primitive.

It follows that $\mathfrak{h}$ is a subalgebra with respect to $\rhd$. Let $x, y, z \in \mathfrak{h}$. Then since x is primitive, it follows from $\triangle(x) = x \otimes 1 + 1 \otimes x$ and the self-distributivity identity (14) that

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (1 \rhd z) + (1 \rhd y) \rhd (x \rhd z) \stackrel{(12)}{=} (x \rhd y) \rhd z + y \rhd (x \rhd z).$$

proving the left Leibniz identity. The morphism statement is clear, since each morphism of rack bialgebras is a morphism of coalgebras and preserves primitives. □

As an immediate consequence, we get that the functor P induces a functor from the category of all rack bialgebras to the category of all Leibniz algebras over $\mathbb{K}$.

Remark 2.31. Define *set-like elements* to be elements a in a rack bialgebra B such that $\triangle(a) = a \otimes a$. Thanks to the fact that $\rhd$ is a morphism of coalgebras, the set of set-like elements $\mathrm{Slike}(B)$ is closed under $\rhd$. In fact, $\mathrm{Slike}(B)$ is a shelf (the not-necessarily-bijective version of a rack product, i.e. the map $y \mapsto x \rhd y$ is not necessarily invertible - in order to express invertibility, one needs an antipode !), and one obtains in this way a functor $\mathrm{Slike} : \mathtt{RackBialg} \to \mathtt{Shelves}$.

Proposition 2.32. *The functor of set-likes* $\mathrm{Slike} : \mathtt{RackBialg} \to \mathtt{Shelves}$ *has the functor* $\mathbb{K}- : \mathtt{Shelves} \to \mathtt{RackBialg}$ *(see Example 2.26) as its left-adjoint.*

Proof. This follows from the adjointness of the same functors, seen as functors between the categories of pointed sets and of coassociative, counital, coaugmented, cocommutative coalgebras, observing that the coalgebra morphism induced by a morphism of racks respects the rack product. □

Observe that the restriction of $\mathrm{Slike} : \mathtt{RackBialg} \to \mathtt{Shelves}$ to the subcategory of cocommutative Hopf algebras $\mathtt{Hopf}$ (where the Hopf algebra is given the rack product defined in Equation (16)) gives the usual functor of group-like elements.

We will now associate to a Leibniz algebra $\mathfrak{h}$ a rack bialgebra, i.e. construct an inverse process to Proposition 2.30. First of all, recall that each Leibniz algebra has two canonical ideals $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ (see Equations (3) and (4)) with $Q(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{h})$ such that the corresponding quotient Leibniz algebras are Lie algebras.

In order to perform the following constructions of rack bialgebras for any given Leibniz algebra $\mathfrak{h}$, choose first a two-sided ideal $\mathfrak{z} \subset \mathfrak{h}$ such that

$$Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h}), \tag{22}$$

let $\mathfrak{g}$ denote the quotient Lie algebra $\mathfrak{h}/\mathfrak{z}$, and let $p : \mathfrak{h} \to \mathfrak{g}$ be the natural projection.

The Lie algebra $\mathfrak{g}$ naturally acts as derivations on $\mathfrak{h}$ by means of (for all $x, y \in \mathfrak{h}$)

$$p(x) \cdot y := [x, y] =: \mathrm{ad}_x(y) \tag{23}$$

because $\mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$. Note that

$$\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \cong \big\{\mathrm{ad}_x \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{h}, \mathfrak{h}) \mid x \in \mathfrak{h}\big\}. \tag{24}$$

as Lie algebras. Then $(\mathfrak{h}, p, \mathfrak{g}, [,]_{\mathfrak{g}}, \cdot)$ is an augmented Leibniz algebra.

Consider now the coassociative, counital, coaugmented, cocommutative coalgebra $(B = S(\mathfrak{h}), \triangle, \epsilon, 1)$ which is actually a commutative cocommutative Hopf algebra over $\mathbb{K}$ with respect to the symmetric multiplication. The linear map $p : \mathfrak{h} \to \mathfrak{g}$ induces a unique morphism of Hopf algebras

$$\tilde{p} : S(\mathfrak{h}) \to S(\mathfrak{g}) \tag{25}$$

satisfying

$$\tilde{p}(x_1 \cdot \dots \cdot x_{\mathbb{K}}) = p(x_1) \cdot \dots \cdot p(x_{\mathbb{K}}) \tag{26}$$

for any nonnegative integer $\mathbb{K}$ and $x_1, \ldots, x_{\mathbb{K}} \in \mathfrak{h}$. In other words, the association $S : V \to S(V)$ is a functor from the category of all $\mathbb{K}$-vector spaces to the category of all commutative unital coassociative, counital, coaugmented, cocommutative coalgebras. Consider the universal enveloping algebra $U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. Since we now assume $\mathbb{Q} \subset \mathbb{K}$, the Poincaré-Birkhoff-Witt Theorem holds. More precisely, the symmetrisation map $\omega : S(\mathfrak{g}) \to U\mathfrak{g}$, defined by

$$\omega(1_{S(\mathfrak{g})}) = 1_{U\mathfrak{g}}, \quad \text{and} \quad \omega(x_1 \cdot \cdots \cdot x_{\mathbb{K}}) = \frac{1}{\mathbb{K}!} \sum_{\sigma \in S_{\mathbb{K}}} x_{\sigma(1)} \cdots x_{\sigma(\mathbb{K})}, \tag{27}$$

is an isomorphism of coassociative, counital, coaugmented, cocommutative coalgebras (in general not of associative algebras). We now need an action of the Hopf algebra $H = U\mathfrak{g}$ on B, and an intertwining map $p : B \to U\mathfrak{g}$. In order to get this, we first look at $\mathfrak{g}$-modules: The $\mathbb{K}$-module $\mathfrak{h}$ is a $\mathfrak{g}$-module by means of Equation (23), the Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module via its adjoint representation, and the linear map $p : \mathfrak{h} \to \mathfrak{g}$ is a morphism of $\mathfrak{g}$-modules since p is a morphism of Leibniz algebras. Now $S(\mathfrak{h})$ and $S(\mathfrak{g})$ are $\mathfrak{g}$-modules in the usual way, i.e. for all $\mathbb{K} \in \mathbb{N} \setminus \{0\}$, $x, x_1, \ldots, x_{\mathbb{K}} \in \mathfrak{g}$, and $h_1 \ldots, h_{\mathbb{K}} \in \mathfrak{h}$

$$x \cdot (h_1 \cdot \cdots \cdot h_{\mathbb{K}}) \quad := \quad \sum_{r=1}^{\mathbb{K}} h_1 \cdot \cdots \cdot (x \cdot h_r) \cdot \cdots \cdot h_{\mathbb{K}}, \tag{28}$$

$$x \cdot (x_1 \cdot \cdots \cdot x_{\mathbb{K}}) \quad := \quad \sum_{r=1}^{\mathbb{K}} x_1 \cdot \cdots \cdot [x.x_r] \cdot \cdots \cdot x_{\mathbb{K}}, \tag{29}$$

and of course $x \cdot 1_{S(\mathfrak{h})} = 0$ and $x \cdot 1_{S(\mathfrak{g})} = 0$. Recall that $U\mathfrak{g}$ is a $\mathfrak{g}$-module via the adjoint representation $\mathrm{ad}_x(u) = x \cdot u = xu - ux$ for all $x \in \mathfrak{g}$ and all $u \in U(\mathfrak{g})$. It is easy to see that the map $\tilde{p}$ in Equation (26) is a morphism of $\mathfrak{g}$-modules, and it is well-known that the symmetrization map ω (27) is also a morphism of $\mathfrak{g}$-modules. Define the $\mathbb{K}$-linear map $p : S(\mathfrak{h}) \to U\mathfrak{g}$ by the composition

$$p := \omega \circ \tilde{p}. \tag{30}$$

Then p is a map of coassociative, counital, coaugmented, cocommutative coalgebras and a map of $\mathfrak{g}$-modules. Thanks to the universal property of the universal enveloping algebra, it follows that $S(\mathfrak{h})$ and $U\mathfrak{g}$ are left $U\mathfrak{g}$-modules, via (for all $x_1, \ldots, x_{\mathbb{K}} \in \mathfrak{g}$, and for all $a \in S(\mathfrak{h})$)

$$(x_1 \cdots x_{\mathbb{K}}) \cdot a = x_1.(x_2.(\cdots x_{\mathbb{K}} \cdot a) \cdots) \tag{31}$$

and the usual adjoint representation (16) (for all $u \in U\mathfrak{g}$)

$$\mathrm{ad}_{x_1 \cdots x_{\mathbb{K}}}(u) = \big(\mathrm{ad}_{x_1} \circ \cdots \circ \mathrm{ad}_{x_{\mathbb{K}}}\big)(u), \tag{32}$$

and that p intertwines the $U\mathfrak{g}$-action on $C = S(\mathfrak{h})$ with the adjoint action of $U\mathfrak{g}$ on itself. Finally it is a routine check using the above identities (28) and (16) that $S(\mathfrak{h})$ becomes a module coalgebra. We can resume the preceding considerations in the following proposition:

Proposition 2.33. *Let $\mathfrak{h}$ be a Leibniz algebra over $\mathbb{K}$, let $\mathfrak{z}$ be a two-sided ideal of $\mathfrak{h}$ such that $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$, let $\mathfrak{g}$ denote the quotient Lie algebra $\mathfrak{g} := \mathfrak{h}/\mathfrak{z}$, and let $p : \mathfrak{h} \to \mathfrak{g}$ be the canonical projection.*

1. *Then there is a canonical $U(\mathfrak{g})$-action ℓ on the coassociative, counital, coaugmented, cocommutative coalgebra $B := S(\mathfrak{h})$ (making it into a module coalgebra leaving invariant 1) and a canonical lift of p to a morphism of coalgebras, $p : S(\mathfrak{h}) \to U\mathfrak{g}$ such that Equation (18) holds.*

   Hence the quadruple $(S(\mathfrak{h}), p, U\mathfrak{g}, \ell)$ is an augmented rack bialgebra whose associated Leibniz algebra is equal to $\mathfrak{h}$ and this is true independently of the choice of $\mathfrak{z}$.

   The resulting rack product $\rhd$ of $S(\mathfrak{h})$ is also independent on the choice of $\mathfrak{z}$ and is explicitly given as follows for all positive integers $\mathbb{K}, l$ and $x_1, \ldots, x_{\mathbb{K}}, y_1, \ldots, y_l \in \mathfrak{h}$:

$$\big(x_1 \cdot \dots \cdot x_{\mathbb{K}}\big) \rhd \big(y_1 \cdot \dots \cdot y_l\big) = \frac{1}{\mathbb{K}!} \sum_{\sigma \in S_{\mathbb{K}}} \big(\mathrm{ad}^s_{x_{\sigma(1)}} \circ \dots \circ \mathrm{ad}^s_{x_{\sigma(\mathbb{K})}}\big)\big(y_1 \cdot \dots \cdot y_l\big) \quad (33)$$

   where ad^s_x denotes the action of the Lie algebra $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ (see Equation (24)) on $S(\mathfrak{h})$ according to Equation (28).

2. *In case $\mathfrak{z} = Q(\mathfrak{h})$, the construction mentioned in (a) is a functor $\mathfrak{h} \to \mathsf{UAR}^\infty(\mathfrak{h})$ from the category of all Leibniz algebras to the category of all augmented rack bialgebras associating to $\mathfrak{h}$ the rack bialgebra*

$$\mathsf{UAR}^\infty(\mathfrak{h}) := (S(\mathfrak{h}), \Phi, U(\mathfrak{g}), \ell)$$

   and to each morphism f of Leibniz algebras the pair $\big(S(f), U(\overline{f})\big)$ where $\overline{f}$ is the induced Lie algebra morphism.

Proof. A great deal of the statements has already been proven in the discussion before the theorem. Note that for all $x, y \in \mathfrak{h}$ we have by definition

$$[x, y] = p(x) \cdot y = x \rhd y,$$

independently of the chosen ideal $\mathfrak{z}$. Moreover we compute

$$\begin{aligned} &\big(x_1 \cdot \dots \cdot x_{\mathbb{K}}\big) \rhd \big(y_1 \cdot \dots \cdot y_l\big) \\ &= \big((\omega \circ \tilde{p})(x_1 \cdot \dots \cdot x_{\mathbb{K}})\big) \cdot \big(y_1 \cdot \dots \cdot y_l\big) \\ &= \frac{1}{\mathbb{K}!} \sum_{\sigma \in S_{\mathbb{K}}} \big(p(x_{\sigma(1)}) \cdots p(x_{\sigma(\mathbb{K})})\big) \cdot \big(y_1 \cdot \dots \cdot y_l\big), \end{aligned}$$

which gives the desired formula since for all $x \in \mathfrak{h}$ and $a \in S(\mathfrak{h})$, we have

$$p(x) \cdot a = \mathrm{ad}^s_x(a).$$

Let us then show functoriality. For this, let $f : \mathfrak{h} \to \mathfrak{h}'$ be a morphism of Leibniz algebras, and let $\overline{f} : \overline{\mathfrak{h}} \to \overline{\mathfrak{h}'}$ be the induced morphism of Lie algebras. Hence we get

$$p' \circ f = \overline{f} \circ p \tag{34}$$

where $p' : \mathfrak{h}' \to \overline{\mathfrak{h}'}$ denotes the corresponding projection modulo $Q(\mathfrak{h}')$. Let $S(f) : S(\mathfrak{h}) \to S(\mathfrak{h}')$, $S(\overline{f}) : S(\overline{\mathfrak{h}}) \to S(\overline{\mathfrak{h}'})$, and $U(\overline{f}) : U(\overline{\mathfrak{h}}) \to U(\overline{\mathfrak{h}'})$ be the induced maps of Hopf algebras, i.e. $S(f)$ (resp. $S(\overline{f})$) satisfies Equation (26) (with p replaced by f (resp. by $\overline{f}$)), and $U(\overline{f})$ satisfies

$$U(\overline{f})\big(x_1 \cdots x_{\mathbb{K}}\big) = \overline{f}(x_1) \cdots \overline{f}(x_{\mathbb{K}})$$

for all positive integers $\mathbb{K}$ and $x_1, \ldots, x_{\mathbb{K}} \in \overline{\mathfrak{h}}$. If $\omega : S(\overline{\mathfrak{h}}) \to U(\overline{\mathfrak{h}})$ and $\omega' : S(\overline{\mathfrak{h}'}) \to U(\overline{\mathfrak{h}'})$ denote the corresponding symmetrisation maps (27), then it is easy to see from the definitions that

$$\omega' \circ S(\overline{f}) = U(\overline{f}) \circ \omega.$$

Equation (34) implies

$$\tilde{p}' \circ S(f) = S(p') \circ S(f) = S(\overline{f}) \circ S(p) = S(\overline{f}) \circ \tilde{p},$$

and composing from the left with ω' yields the equation

$$p' \circ S(f) = U(\overline{f}) \circ p. \tag{35}$$

Moreover for all $h, h' \in \mathfrak{h}$ we have, since f is a morphism of Leibniz algebras,

$$f\big(p(h) \cdot h'\big) = f\big([h, h']\big) = \big[f(h), f(h')\big]' = p'\big(f(h)\big) \cdot f(h') = \overline{f}\big(p(h)\big) \cdot f(h'),$$

hence for all $x \in \overline{h}$

$$f(x \cdot h) = \overline{f}(x) \cdot f(h),$$

and upon using Equation (28), we get for all $a \in S(\mathfrak{h})$

$$S(f)\big(x \cdot a\big) = \overline{f}(x) \cdot S(f)\big(a\big),$$

showing finally for all $u \in U\mathfrak{h}$ and all $a \in S(\mathfrak{h})$

$$S(f)\big(u \cdot a\big) = U(\overline{f})(u) \cdot S(f)\big(a\big). \tag{36}$$

Associating to every Leibniz algebra $\mathfrak{h}$ the above defined augmented rack bialgebra $(S(\mathfrak{h}), \Phi, U\overline{\mathfrak{h}}, \ell)$, and to every morphism $\psi : \mathfrak{h} \to \mathfrak{h}'$ of Leibniz algebras the pair of linear maps $\big(S(\psi), U(\overline{\psi})\big)$, we can easily check that $S(\psi)$ is a morphism of coalgebras, $U(\psi)$ is a morphism of Hopf algebras, such that the two relevant diagrams (19) commute which easily follows from (35) and (36). The rest of the functorial properties is a routine check. □

Remark 2.34. This theorem should be compared to Proposition 3.5 in [5]. In [5], the authors construct a rack bialgebra structure on $N := \mathbb{K} \oplus \mathfrak{h}$. The rack product is given by the bracket of the Leibniz algebra. We work in the preceding theorem with the whole symmetric algebra on the Leibniz algebra. Thus, in some sense, we extend their Proposition 3.5 “to all orders”.

The above rack bialgebra associated to a Leibniz algebra $\mathfrak{h}$ can be seen as one version of an *enveloping algebra* of $\mathfrak{h}$.

Let us summarize. Given a pointed rack R, there is a rack bialgebra $\mathbb{K}R$ which is the linearization of the rack R. Given a Lie rack M, its tangent space to the unit 1 is endowed with the structure of a (natural, in general non-trivial) Leibniz algebra. Leibniz algebras have "enveloping algebras" which are rack bialgebras, and every rack bialgebra has as its primitives a Leibniz algebra and as its set-likes a pointed rack. There are some other links between these structures which we do not discuss, for example the point-distributions on a Lie rack supported in 1 form a rack bialgebra. In total, the structures of rack, rack bialgebra and Leibniz algebra enjoy (largely speaking) the same links as the structures of group, bi/Hopf algebra and Lie algebra.

3 Cohomology theory of racks, Leibniz algebras and rack bialgebras

Here we come to the heart of this minicourse: The cohomology theories associated to racks, Leibniz algebras and rack bialgebras and their relations.

3.1 Cohomology of racks

As racks have been invented for the needs of knot theory, their cohomology theory or more precisely the rack cocycles are traditionally used to define knot invariants. We will not talk about this use, but rather about the use of rack cohomology in the attempt to explain the non-periodicity of algebraic K-theory. The main reference of this subsection is [6].

J.-L. Loday thought about the problem of the non-periodicity of algebraic K-theory (in contrast to the periodicity of topological K-theory), see [19]. Over the rationals, the K-theory $\mathbb{K}(A)$ of a unitary $\mathbb{K}$-algebra A can be deduced from group homology of the inductive limit group $GL(A)$ as primitives of the Hopf algebra $H_\bullet(GL(A),\mathbb{Q})$:

$$H_\bullet(GL(A),\mathbb{Q}) \simeq \Lambda(\mathbb{K}_\bullet(A)\otimes\mathbb{Q}).$$

The infinitesimal version of K-theory (invented by Loday-Quillen and Feigin-Tsygan), nowadays called cyclic homology $HC_\bullet(A)$, is related to Lie algebra homology in the same way (this is one way to define cyclic homology or the Loday-Quillen-Tsygan Theorem):

$$H_\bullet(\mathfrak{gl}(A),\mathbb{Q}) \simeq \Lambda(HC_{\bullet-1}(A)).$$

Now for the cyclic homology, one may identify the failure of periodicity. It is given by the Hochschild homology according to the Connes exact sequence:

$$\ldots \to HH_n(A) \to HC_n(A) \to HC_{n-2}(A) \to HH_{n-1}(A) \to \ldots.$$

Therefore, in order to get back to $\mathfrak{gl}(A)$, one needs to identify a homology theory associated to A which is a graded Hopf algebra whose primitives are the Hochschild groups $HH_\bullet(A)$. This is the Leibniz homology of $\mathfrak{gl}(A)$ according to the Loday-Cuvier Theorem:

$$HL_\bullet(\mathfrak{gl}(A),\mathbb{Q}) \simeq T(HH_{\bullet-1}(A)),$$

where T denotes the tensor algebra. From here, Loday drew his interest in Leibniz algebras. Thus Loday thought that in case one found an algebraic structure integrating Leibniz algebra ("coquecigrue problem"), its homology theory should then quantify the failure of periodicity of algebraic K-theory. The coquecigrue problem is at the heart of search of the relations between Leibniz algebras and racks. Loday exposed in [19] different properties such a *Leibniz K-theory* should have. My former thesis student Simon Covez showed in [24] that rack homology has all these properties but one (in fact, rack homology does not give the correct answer on abelian groups). Moreover, Simon Covez [7] showed in his thesis how the relation between rack cohomology and Leibniz cohomology permits to give a functorial answer (in terms of integration of cocycles) to the coquecigrue problem.

Let R be a rack and A be an R-module. Let us define a cochain complex $(CR^n(R,A),d_R^n)_{n\in\mathbb{N}}$ by

$$CR^n(R,A) := \mathrm{Hom}_{\mathtt{Set}}(R^n,A),$$

and

$$d_R^{n+1}f := \sum_{i=1}^{n+1}(-1)^i(d_{i,0}^{n+1}f - d_{i,1}^{n+1}f),$$

where

$$d_{i,0}^{n+1}f(x_1,\ldots,x_{n+1}) := f(x_1,\ldots,x_{i-1},x_i\rhd x_{i+1},\ldots,x_i\rhd x_{n+1}),$$

and

$$d_{i,1}^{n+1}f(x_1,\ldots,x_{n+1}) := (x_1\rhd\ldots\rhd x_i)\cdot f(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}).$$

Here $x_1\rhd\ldots\rhd x_i$ denotes by definition the composition $c_{x_1}\circ\ldots\circ c_{x_i}$ denoting the bijections $c_x : R\to R$, $c_x(y) := x\rhd y$.

The fact that $d_R^n\circ d_R^{n-1} = 0$ for all $n\geq 1$ comes from the *cubical identities*, i.e.

$$d_{i,\epsilon}^{n+1}\circ d_{j-1,\omega}^n = d_{j,\omega}\circ d_{i,\epsilon}^n$$

for all $1\leq i<j\leq n+1$ and all $\epsilon,\omega\in\{0,1\}$. The cohomology associated to this chain complex is called the *rack cohomology* of R with coefficients in A and denoted $HR^\bullet(R,A)$.

The two properties about the conjectural Leibniz K-theory which Loday conjectured and which Covez proves in [6] are:

Theorem 3.1 (Covez 2012)**.** *Let R be a rack, G a group and A an associative algebra viewed as a trivial R- and G-module. Then*

(a) *$HR^\bullet(R, A)$ is a graded dendriform algebra (and therefore becomes a graded associative algebra).*

(b) *There exists a morphism of graded algebras*

$$H^\bullet(G, A) \to HR^\bullet(\mathrm{Conj}(G), A)$$

which is injective for $\bullet = 1$.

The proofs of these two properties are very technical (in fact combinatorial). The main idea is to view the cochain complex of rack cohomology as the cubical nerve of a certain trunk associated to R. The rough explanation is the following: exactly as one may associate to a group G a simplicial object (the nerve of a certain category associated to G, leading to the classifying space BG), one may associate to a rack R a cubical object which is the nerve of a certain trunk associated to R. All the properties are shown as combinatorial properties on this cubical object.

Let us draw attention to the fact, shown and used by Simon Covez in his integration procedure of $\mathbb{R}$-Leibniz algebras into local Lie racks, that Kinyon's map from Lie racks to their tangent Leibniz algebras induces a map in cohomology, see [7].

A more recent development in this area is the collaboration Covez, Farinati, Lebed and Manchon [8]. They show (drawing also on earlier work by Serre (1951), Baues (1998), Clauwens (2011), Covez (2012) and Lebed (2017)) that:

Theorem 3.2 (CFLM 2019)**.** *(1) The rack cohomology complex with the cup product is a dg associative graded commutative algebra up to an explicit homotopy.*

(2) The rack cohomology is associative and graded commutative.

(3) In fact, the complex is dendriform and Zinbiel up to an explicit homotopy.

This new approach is based on a certain dg bialgebra

$$B(R) := \mathbb{Z}(x, e^y \,|\, x, y \in R) \,/\, \langle (x \rhd y)x - xy, e_{x \rhd y}x - xe_y \rangle,$$

with e_x of degree 1, x of degree zero for $x \in R$, $dx = 0$, $de_x = 1 - x$, x set-like, e_x primitive and $\epsilon(e_x) = 0$, $\epsilon(x) = 1$.

Furthermore, $B(R)$ is a dg $\mathrm{As}(R)$-bimodule (by multiplication on the left and on the right). The authors obtain their results by expressing rack homology with trivial coefficients and with coefficients in the monoid associated to R in terms of the homology of a quotient $\overline{B(R)}$ of $B(R)$, resp. $B(R)$ itself.

3.2 Cohomology of Leibniz algebras

The definition of the (co)homology of Leibniz algebras is due to J.-L. Loday (from his search of quantifying the failure of periodicity of algebraic K-theory) and T. Pirashvili, see [18]. We will report in this section on results from [9] (cohomology of semi-simple Leibniz algebras) and ongoing work with Jörg Feldvoss on the cohomology of nilpotent Leibniz algebras.

Recall that a (left) Leibniz algebra $\mathfrak{h}$ is a $\mathbb{K}$-vector space with a bracket which is a derivation of itself, i.e. for all $x, y, z \in \mathfrak{h}$, we have:

$$[x,[y,z]] = [[x,y],z + [y,[x,z]].$$

The ideal of squares (generated by the $[x,x]$ for $x \in \mathfrak{h}$) of $\mathfrak{h}$ is denoted $Q(\mathfrak{h})$, and the resulting quotient Lie algebra is denoted $\mathfrak{h}_{\rm Lie} := \mathfrak{h} \,/\, Q(\mathfrak{h})$.

We will briefly discuss left modules and bimodules of left Leibniz algebras. Let $\mathfrak{h}$ be a left Leibniz algebra over a field $\mathbb{K}$. A *left* $\mathfrak{h}$*-module* is a vector space M over $\mathbb{K}$ with an $\mathbb{K}$-bilinear left $\mathfrak{h}$-action $\mathfrak{h} \times M \to M$, $(x,m) \mapsto x \cdot m$ such that

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) \tag{37}$$

is satisfied for every $m \in M$ and all $x, y \in \mathfrak{h}$.

Every left $\mathfrak{h}$-module is an $\mathfrak{h}_{\rm Lie}$-module, and vice versa. Therefore left Leibniz modules are sometimes called *Lie modules*. Consequently, many properties of left Leibniz modules follow from the corresponding properties of modules for the canonical Lie algebra.

The correct concept of a module for a left Leibniz algebra $\mathfrak{h}$ is the notion of a Leibniz bimodule. An $\mathfrak{h}$*-bimodule* is a left $\mathfrak{h}$-module M with an $\mathbb{K}$-bilinear right $\mathfrak{h}$-action $M \times \mathfrak{h} \to M$, $(m,x) \mapsto m \cdot x$ such that

$$(x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy) \tag{38}$$

and

$$(m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y) \tag{39}$$

are satisfied for every $m \in M$ and all $x, y \in \mathfrak{h}$. In fact, all three identities (37), (38), and (39) are instances of the left Leibniz identity, written down for the left Leibniz algebra $\mathfrak{h} \oplus M$ which is considered as an abelian extension in the theory of non-associative algebras, where the element m coccurs on the right, in the middle, or on the left, respectively.

The usual definitions of the notions of *sub-(bi)module*, *irreducibility*, *complete reducibility*, *composition series*, *homomorphism*, *isomorphism*, etc., hold for left Leibniz modules and Leibniz bimodules.

Let $\mathfrak{h}$ be a left Leibniz algebra over a field $\mathbb{K}$, and let M be an $\mathfrak{h}$-bimodule. Then M is said to be *symmetric* if $m \cdot x = -x \cdot m$ for every $x \in \mathfrak{h}$ and every $m \in M$, and M is said to be *anti-symmetric* if $m \cdot x = 0$ for every $x \in \mathfrak{h}$ and every $m \in M$. We call

$$M_0 := \langle x \cdot m + m \cdot x \mid x \in \mathfrak{h}, m \in M \rangle_{\mathbb{F}}$$

the *anti-symmetric kernel* of M. It is known that M_0 is an anti-symmetric $\mathfrak{h}$-sub-bimodule of M such that $M_{\rm sym} := M/M_0$ is symmetric, i.e. for every Leibniz $\mathfrak{h}$-bimodule M, we have an exact sequence:

$$0 \to M_0 \to M \to M_{\rm sym} \to 0. \tag{40}$$

Recall that every left $\mathfrak{h}$-module M of a left Leibniz algebra $\mathfrak{h}$ determines a unique symmetric $\mathfrak{h}$-bimodule structure on M by defining $m \cdot x := -x \cdot m$ for every element $m \in M$ and every element $x \in \mathfrak{h}$. We will denote this symmetric $\mathfrak{h}$-bimodule by M_s. Similarly, every left $\mathfrak{h}$-module M with trivial right action is an anti-symmetric $\mathfrak{h}$-bimodule. We will denote this module by M_a. Note that for any irreducible left $\mathfrak{h}$-module M the $\mathfrak{h}$-bimodules M_s and M_a are irreducible, and every irreducible $\mathfrak{h}$-bimodule arises in this way from an irreducible left $\mathfrak{h}$-module (using the exact sequence 40).

Now let M be an $\mathfrak{h}$-bimodule and for any non-negative integer n consider the linear map $\mathrm{d}^n : \mathrm{CL}^n(\mathfrak{h}, M) \to \mathrm{CL}^{n+1}(\mathfrak{h}, M)$ defined by

$$\begin{aligned}(\mathrm{d}^n f)(x_1, \ldots, x_{n+1}) \quad := \quad & \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \\ + \quad & (-1)^{n+1} f(x_1, \ldots, x_n) \cdot x_{n+1} \\ + \quad & \sum_{1 \le i < j \le n+1} (-1)^i f(x_1, \ldots, \widehat{x}_i, \ldots, [x_i, x_j], \ldots, x_{n+1})\end{aligned}$$

for any $f \in \mathrm{CL}^n(\mathfrak{h}, M)$ and all elements $x_1, \ldots, x_{n+1} \in \mathfrak{h}$, where the term $[x_i, x_j]$ appears in the j-th position.

It is readily proved that $\mathrm{CL}^\bullet(\mathfrak{h}, M) := (\mathrm{CL}^n(\mathfrak{h}, M), \mathrm{d}^n)_{n \in \mathbb{N}_0}$ is a cochain complex, i.e., $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ for every non-negative integer n. Of course, the original idea of defining Leibniz cohomology as the cohomology of such a cochain complex for right Leibniz algebras is due to Loday and Pirashvili [18, Section 1.8]. Hence one can define the *cohomology of* $\mathfrak{h}$ *with coefficients in an* $\mathfrak{h}$*-bimodule* M by

$$\mathrm{HL}^n(\mathfrak{h}, M) := H^n(\mathrm{CL}^\bullet(\mathfrak{h}, M)) := \ker(\mathrm{d}^n) / \operatorname{im}(\mathrm{d}^{n-1})$$

for every non-negative integer n. (In this definition we use that $\mathrm{d}^{-1} := 0$.)

We now come to the cohomology results of [9] and the ongoing work with Jörg Feldvoss on the cohomology of nilpotent Leibniz algebras. All Leibniz algebras in the following will be finite dimensional.

Recall that a left Leibniz algebra $\mathfrak{h}$ is called *semi-simple* if $Q(\mathfrak{h})$ contains every solvable ideal of $\mathfrak{h}$. In particular, a finite-dimensional left Leibniz algebra $\mathfrak{h}$ is semi-simple if, and only if, $Q(\mathfrak{h}) = \mathrm{rad}(\mathfrak{h})$, where $\mathrm{rad}(\mathfrak{h})$ denotes the largest solvable ideal of $\mathfrak{h}$. Moreover, a left Leibniz algebra $\mathfrak{h}$ is semi-simple if, and only if, the canonical Lie algebra $\mathfrak{h}_{\mathrm{Lie}}$ associated to $\mathfrak{h}$ is semi-simple.

Using Pirashvili's spectral sequence [21] for the computation of Leibniz cohomology, we show:

Theorem 3.3. *Let* $\mathfrak{h}$ *be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, and let* M *be a finite-dimensional* $\mathfrak{h}$*-bimodule. Then* $\mathrm{HL}^n(\mathfrak{h}, M) = 0$ *for every integer* $n \ge 2$*, and there is a five-term exact sequence*

$$0 \to M_0 \to \mathrm{HL}^0(\mathfrak{h}, M) \to M_{\mathrm{sym}}^{\mathfrak{h}_{\mathrm{Lie}}} \to \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{h}_{\mathrm{ad}}, M_0) \to \mathrm{HL}^1(\mathfrak{h}, M) \to 0\,.$$

Moreover, if M *is symmetric, then* $\mathrm{HL}^n(\mathfrak{h}, M) = 0$ *for every integer* $n \ge 1$*.*

I will not discuss this theorem in more detail, because I had already given a talk on this theorem at Jilin in december 2020.

Our more recent results are on nilpotent Leibniz algebras, we have notably a vanishing theorem for the cohomology of nilpotent Leibniz algebras. For this, we need a *Fitting decomposition* for a Leibniz $\mathfrak{h}$-bimodule M for a Leibniz algebra $\mathfrak{h}$. For a subset $S \subset \mathfrak{h}$, denote the Fitting components of M by $M_0(\lambda_s) := \bigcup_{n\geq 1}\ker(\lambda_s^n)$ and $M_1(\lambda_s) := \bigcap_{n\geq 1}\mathrm{im}(\lambda_s^n)$ with respect to the endomorphism λ_s given by the left operation by $s \in S$. Define then the Fitting components with respect to S:

$$M_0(S) := \bigcap_{s\in S} M_0(\lambda_s)$$

and

$$M_1(S) := \sum_{s\in S} M_1(\lambda_s)\,.$$

Proposition 3.4. *Let $\mathfrak{h}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be a $\mathfrak{h}$-bimodule with associated representation (λ, μ). If S is a subset of $\mathfrak{h}$ such that the left bracket operator $L_s : \mathfrak{h} \to \mathfrak{h}$, $x \mapsto [s, x]$ is locally nilpotent for every element $s \in S$, then the following statements hold:*

(a) *$M_0(S)$ is an $\mathfrak{h}$-subbimodule of M.*

(b) *Every element of S acts locally nilpotently on $M_0(S)$ from the left and from the right.*

Moreover, if $\dim_{\mathbb{K}} M < \infty$*, then*

(c) *$M_1(S)$ is an $\mathfrak{h}$-subbimodule of M.*

(d) *$M = M_0(S) \oplus M_1(S)$.*

From this, we will now deduce a cohomology vanishing theorem.

Lemma 3.5. *Let V and W be left modules over a left Leibniz algebra $\mathfrak{h}$. If x is an element of $\mathfrak{h}$ such that*

(i) *x acts locally nilpotently on V, and*

(ii) *x acts invertibly on W,*

then x acts invertibly on $\mathrm{Hom}_{\mathbb{K}}(V, W)$*.*

The next two results are the Leibniz analogues of results that Farnsteiner obtained for Hochschild cohomology.

Proposition 3.6. *Let $\mathfrak{h}$ be a left Leibniz algebra, and let M be an $\mathfrak{h}$-bimodule with associated representation (λ, μ). If a is an element of $\mathfrak{h}$ such that*

(i) *$L_a : \mathfrak{h} \to \mathfrak{h}$, $x \mapsto [a, x]$ is locally nilpotent, and*

(ii) $\lambda_a : M \to M$ *is invertible,*

then $\mathrm{HL}^n(\mathfrak{h}, M) = 0$ *for every non-negative integer* n.

Proposition 3.7. *Let* $\mathfrak{h}$ *be a left Leibniz algebra over a field* $\mathbb{K}$*, and let* M *be an* $\mathfrak{h}$*-bimodule. If* S *is a subset of* $\mathfrak{h}$ *such that*

(i) $L_s : \mathfrak{h} \to \mathfrak{h}$, $x \mapsto [s, x]$ *is locally nilpotent for every element* $s \in S$*, and*

(ii) $\dim_{\mathbb{K}} M/M_0(S) < \infty$,

then $\mathrm{HL}^n(\mathfrak{h}, M) \cong \mathrm{HL}^n(\mathfrak{h}, M_0(S))$ *(as* $\mathbb{K}$*-vector spaces) for every non-negative integer* n.

Recall that a Leibniz algebra $\mathfrak{h}$ is called *nilpotent* if there exists a positive integer n such that any iterated bracket of n elements in $\mathfrak{h}$, no matter how associated, is zero. Let $\mathfrak{h}$ be a left Leibniz algebra. Then the *left descending central series*

$$^1\mathfrak{h} \supseteq {}^2\mathfrak{h} \supseteq {}^3\mathfrak{h} \supseteq \cdots$$

of $\mathfrak{h}$ is defined recursively by $^1\mathfrak{h} := \mathfrak{h}$ and $^{n+1}\mathfrak{h} := [\mathfrak{h}, {}^n\mathfrak{h}]$ for every positive integer n. It is clear that a Leibniz algebra is nilpotent if and only if there exists an $n \geq 1$ such that $^n\mathfrak{h} = 0$. The following immediate consequence of Proposition 3.7 is a first cohomology vanishing result for Leibniz cohomology with respect to a nilpotent subalgebra:

Theorem 3.8. *Let* $\mathfrak{h}$ *be a left Leibniz algebra over a field* $\mathbb{K}$*, and let* $\mathfrak{n}$ *be a nilpotent left ideal of* $\mathfrak{h}$*. If* M *is an* $\mathfrak{h}$*-bimodule such that* $\dim_{\mathbb{K}} M/M_0(\mathfrak{n}) < \infty$*, then* $\mathrm{HL}^n(\mathfrak{h}, M) \cong \mathrm{HL}^n(\mathfrak{h}, M_0(\mathfrak{n}))$ *(as* $\mathbb{K}$*-vector spaces) for every non-negative integer* n.

As a consequence of the previous result we obtain the following vanishing theorem for Leibniz cohomology.

Corollary 3.9. *Let* $\mathfrak{h}$ *be a left Leibniz algebra, and let* $\mathfrak{n}$ *be a nilpotent left ideal of* $\mathfrak{h}$*. If* M *is a finite-dimensional* $\mathfrak{h}$*-bimodule such that* $M^{\mathfrak{n}} = 0$*, then* $\mathrm{HL}^n(\mathfrak{h}, M) = 0$ *for every non-negative integer* n.

Theorem 3.10. *Let* $\mathfrak{h}$ *be a finite-dimensional nilpotent Leibniz algebra, and let* M *be a finite-dimensional* $\mathfrak{h}$*-bimodule. If every composition factor of* M *is non-trivial, then*

$$\mathrm{HL}^n(\mathfrak{h}, M) \cong \begin{cases} M_0 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1\,. \end{cases}$$

Moreover, if M *is symmetric, then* $\mathrm{HL}^n(\mathfrak{h}, M) = 0$ *for every non-negative integer* n.

As a sample computation, we have for example completely determined the cohomology of the trivial 1-dimensional Leibniz algebra:

Proposition 3.11. *Let $\mathfrak{h} := \mathbb{K}e$ be the one-dimensional Lie algebra, and let M be a Leibniz $\mathfrak{h}$-bimodule. Then*

$$\mathrm{HL}^n(\mathfrak{h}, M) \cong \begin{cases} M^{\mathfrak{h}} & \text{if } n = 0 \\ M^0/M\mathfrak{h} & \text{if } n \text{ is odd} \\ M^{\mathfrak{h}}/M_0 & \text{if } n \text{ is even and } n \neq 0 \end{cases}$$

(as $\mathbb{K}$-vector spaces) for every non-negative integer n, where

$$M^0 := \{m \in M \mid e \cdot m + m \cdot e = 0\}.$$

Moreover, if M is finite dimensional, then

$$M^0/M\mathfrak{h} \cong M^{\mathfrak{h}}/M_0$$

(as $\mathbb{K}$-vector spaces).

In the same spirit as Dixmier for nilpotent Lie algebras, we also have a non-vanishing theorem, but this is a different story.

3.3 Cohomology of rack bialgebras

In this last section, we will report on the deformation cohomology of rack bialgebras following [2]. Let $(R, \triangle, \epsilon, \mu, \mathbf{1})$ denote a **cocommutative** rack-bialgebra over $\mathbb{K}$, and let us also use the notation $\mu(r \otimes s)$ with the map $\mu : R \times R \to R$ for the rack product $r \triangleright s$ of two elements r and s of R. We will use the n-iterated comultiplication of r in R:

$$r^{(1)} \otimes \cdots \otimes r^{(n)} := (\triangle \otimes \mathrm{Id}^{\otimes n-1}) \circ \cdots \circ \triangle(r),$$

where we write the Sweedler index now *on top* of the elements, instead of writing it as a subscript.

Let $\mathbb{K}_\hbar = \mathbb{K}[[\hbar]]$ denote the $\mathbb{K}$-algebra of formal power series in the indeterminate $\hbar$ with coefficients in $\mathbb{K}$. If V is a vector space over $\mathbb{K}$, $V_\hbar$ stands for V$[[\hbar]]$. Recall that if W is a $\mathbb{K}$-module, a $\mathbb{K}_\hbar$-linear morphism from $V_\hbar$ to $W_\hbar$ is the same as a power series in $\hbar$ with coefficients in $\mathrm{Hom}_{\mathbb{K}}(V, W)$ via the canonical map

$$\mathrm{Hom}_{\mathbb{K}_\hbar}(V_\hbar, W_\hbar) \cong \mathrm{Hom}_{\mathbb{K}}(V, W)_\hbar.$$

This identification will be used without extra mention in the following.

Definition 3.12. *A **formal deformation of the rack product** μ is a formal power series $\mu_\hbar$*

$$\mu_\hbar := \sum_{n \geq 0} \hbar^n \mu_n$$

in $\mathrm{Hom}_{\mathbb{K}}(R \otimes R, R)_\hbar$, such that

1. $\mu_0 = \mu$,

2. $(R_\hbar, \triangle, \epsilon, \mu_\hbar, \mathbf{1})$ *is a rack bialgebra over* $\mathbb{K}_\hbar$.

As in the classical setting of deformation theory of associatice products, we will relate our deformation theory of rack products to cohomology. For this, let us first examine an introductory example:

Example 3.13. *Let* $(R, \rhd)$ *be a rack bialgebra, and suppose there exists a deformation* $\rhd_\hbar = \rhd + \hbar\omega$ *of* $\rhd$*. The new rack product* $\rhd_\hbar$ *should satisfy the self-distributivity identity, i.e. for all* $a, b, c \in R$

$$a \rhd_\hbar (b \rhd_\hbar c) \,=\, (a^{(1)} \rhd_\hbar b) \rhd_\hbar (a^{(2)} \rhd_\hbar c)$$

To the order $\hbar^0$*, this is only the self-distributivity relation for* $\rhd$*. But to order* $\hbar^1$ *(neglecting order* $\hbar^2$ *and higher), we obtain:*

$$\omega(a, b\rhd c)+a\rhd\omega(b, c) \,=\, \omega(a^{(1)}\rhd b, a^{(2)}\rhd c)+\omega(a^{(1)}, b)\rhd(a^{(2)}\rhd c)+(a^{(1)}\rhd b)\rhd\omega(a^{(2)}, c).$$

It will turn out that this is the cocycle condition for ω *in the deformation complex which we are going to define. More precisely, we will have*

1. $d_{2,0}\omega(a, b, c) \,=\, \omega(a, b \rhd c)$,
2. $d_{1,1}\omega(a, b, c) \,=\, a \rhd \omega(b, c)$,
3. $d_{1,0}\omega(a, b, c) \,=\, \omega(a^{(1)} \rhd b, a^{(2)} \rhd c)$,
4. $d^2_3\omega(a, b, c) \,=\, \omega(a^{(1)}, b) \rhd (a^{(2)} \rhd c)$,
5. $d_{2,1}\omega(a, b, c) \,=\, (a^{(1)} \rhd b) \rhd \omega((a^{(2)}, c)$.

This may perhaps help to understand the general definition of the operators $d^n_{i,\mu}$ *for* $i = 1, \ldots, n$ *and* $\mu \in \{0, 1\}$ *further down.*

On the other hand, the requirement that $\rhd_\hbar$ *should be a morphism of coalgebras (with respect to the undeformed coproduct* $\triangle$ *of* R*) means*

$$\triangle \circ \rhd_\hbar \,=\, (\rhd_\hbar \otimes \rhd_\hbar) \circ \triangle^{[2]}.$$

This reads for $a, b \in R$ *to the order* $\hbar$ *(neglecting higher powers of* $\hbar$*) as*

$$\omega(a, b)^{(1)} \otimes \omega(a, b)^{(2)} \,=\, \omega(a^{(1)}, b^{(1)}) \otimes (a^{(2)} \rhd b^{(2)}) + (a^{(1)} \rhd b^{(1)}) \otimes \omega(a^{(2)}, b^{(2)}).$$

This is exactly the requirement that ω *is a coderivation along* $\rhd = \mu$*, to be defined below.*

Recall that R being a rack bialgebra means in particular that $\mu : R^{\otimes 2} \to R$ is a morphism of coassociative coalgebras. For all positive integer n, let $\mu^n : R^{\otimes n} \to R$ be the linear map defined inductively by setting

- $\mu^1 := \mathrm{Id} : R \to R$,
- $\mu^2 := \mu : R^{\otimes 2} \to R$,

- $\mu^n := \mu \circ (\mu^1 \otimes \mu^{n-1})$, $n \geq 3$,

so that

$$\mu^n(r_1, \cdots, r_n) \,=\, r_1 \triangleright (r_2 \triangleright (\cdots \triangleright (r_{n-1} \triangleright r_n) \cdots))$$

for all $r_1, \ldots, r_n$ in R.

Proposition 3.14. *For all $n \geq 1$, the map μ^n is a morphism of coalgebras satisfying*

$$\mu^i(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i) \triangleright \mu^{n-1}(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_{i+1},\cdots,r_n) \;=\; \mu^n(r_1,\cdots,r_n), \quad (41)$$

$$\mu^n(r_1, \cdots, r_{i-1}, r_i^{(1)} \triangleright r_{i+1}, \cdots, r_i^{(n+1-i)} \triangleright r_{n+1}) \;=\; \mu^{n+1}(r_1, \cdots, r_{n+1}) \quad (42)$$

for all positive integers i and n such that $1 \leq i < n$ and for all r_1, ..., r_n in R.

Proof.

- Eqn. (41): Let us show that the assertion of Equation (41) is true for all n and i with $1 \leq i < n$ by induction over i. Suppose that the induction hypothesis is true and compute

$$\mu^i(r_1^{(1)},\cdots,r_i^{(1)},r_{i+1}) \triangleright \mu^{n-1}(r_1^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)$$
$$\left(r_1^{(1)} \triangleright \mu^i(r_2^{(1)},\cdots,r_i^{(1)},r_{i+1})\right) \triangleright \left(r_1^{(2)} \triangleright \mu^{n-2}(r_2^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)\right),$$

which gives, Thanks to the self-distributivity relation in the rack algebra R,

$$r_1 \triangleright \left(\mu^i(r_2^{(1)},\cdots,r_i^{(1)},r_{i+1}) \triangleright \mu^{n-2}(r_2^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)\right)$$
$$= r_1 \triangleright \mu^{n-1}(r_2,\cdots,r_n) \,=\, \mu^n(r_1,\cdots,r_n),$$

where we have used the induction hypothesis. This proves the assertion.

- Eqn. (42): The assertion follows here again from induction using the self-distributivity relation.

□

If $(C, \triangle_C)$ and $(D, \triangle_D)$ are two coassociative coalgebras and $\phi : C \to D$ is a morphism of coalgebras, we denote by $\mathrm{Coder}(C, V, \phi)$ the vector space of *coderivations from C to V along ϕ*, i.e. the vector space of linear maps $f : C \to D$ such that

$$\triangle_D \circ f \,=\, (f \otimes \phi + \phi \otimes f) \circ \triangle_C$$

Let us note the following permanence property of coderivations along a map under partial convolution which will be useful in the proof of the following theorem. For a coalgebra A, maps $f : A \otimes B \to V$ and $g : A \otimes C \to V$ and some product $\triangleright : V \otimes V \to V$, the *partial convolution* of f and g is the map $f \star_{\mathrm{part}} g : A \otimes B \otimes C \to V$ defined for all $a \in A$, $b \in B$ and $c \in C$ by

$$(f \star_{\mathrm{part}} g)(a \otimes b \otimes c) \,:=\, f(a^{(1)} \otimes b) \triangleright g(a^{(2)} \otimes c).$$

Lemma 3.15. *Let A, B, C and V be coalgebras, V carrying a product $\triangleright$ which is supposed to be a coalgebra morphism. Let $f : A \otimes B \to V$ be a coderivation along ϕ and $g : A \otimes C \to V$ be a coalgebra morphism. Then the partial convolution $f \star_{\mathrm{part}} g$ is a coderivation along $\phi \star_{\mathrm{part}} g$.*

Proof. We compute for all $a \in A$, $b \in B$ and $c \in C$

$$\begin{aligned}
&\triangle_V \circ (f \star_{\mathrm{part}} g)(a \otimes b \otimes c) = \triangle_V(f(a^{(1)} \otimes b) \triangleright g(a^{(2)} \otimes c)) \\
&\quad = (f(a^{(1)} \otimes b))^{(1)} \triangleright (g(a^{(2)} \otimes c))^{(1)} \otimes (f(a^{(1)} \otimes b))^{(2)} \triangleright (g(a^{(2)} \otimes c))^{(2)} \\
&\quad = (f(a^{(1)} \otimes b))^{(1)} \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes (f(a^{(1)} \otimes b))^{(2)} \triangleright g(a^{(3)} \otimes c^{(2)}) \\
&\quad = \phi(a^{(1)} \otimes b^{(1)}) \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes f(a^{(3)} \otimes b^{(2)}) \triangleright g(a^{(4)} \otimes c^{(2)}) + \\
&\quad + f(a^{(1)} \otimes b^{(1)}) \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes \phi(a^{(3)} \otimes b^{(2)}) \triangleright g(a^{(4)} \otimes c^{(2)}) \\
&\quad = (\phi \star_{\mathrm{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (f \star_{\mathrm{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) + \\
&\quad + (f \star_{\mathrm{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (\phi \star_{\mathrm{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) \\
&\quad = \big((\phi \star_{\mathrm{part}} g) \otimes (f \star_{\mathrm{part}} g) + (f \star_{\mathrm{part}} g) \otimes (\phi \star_{\mathrm{part}} g)\big) \circ \triangle_{A \otimes B \otimes C}(a \otimes b \otimes c).
\end{aligned}$$

□

Definition 3.16. *The **deformation complex of** R is the graded vector space $C^\bullet(R; R)$ defined in degree n by*

$$C^n(R; R) := \mathrm{Coder}(R^{\otimes n}, R, \mu^n)$$

endowed with the differential $d_R : C^*(R; R) \to C^{*+1}(R; R)$ defined in degree n by

$$d_R^n := \sum_{i=1}^{n} (-1)^{i+1}(d_{i,1}^n - d_{i,0}^n) \, + \, (-1)^{n+1} d_{n+1}^n$$

where the maps $d_{i,1}^n$ and $d_{i,0}^n$ are defined respectively by

$$d_{i,1}^n \omega(r_1, \cdots, r_{n+1}) := \sum_{(r_1), \cdots, (r_i)} \mu^i(r_1^{(1)}, \cdots, r_{i-1}^{(1)}, r_i) \triangleright \omega(r_1^{(2)}, \cdots, r_{i-1}^{(2)}, r_{i+1}, \cdots, r_{n+1})$$

and

$$d_{i,0}^n \omega(r_1, \cdots, r_{n+1}) := \sum_{(r_i)} \omega(r_1, \cdots, r_{i-1}, r_i^{(1)} \triangleright r_{i+1}, \cdots, r_i^{(n+1-i)} \triangleright r_{n+1})$$

and d_{n+1}^n by

$$\begin{aligned}
&d_{n+1}^n \omega(r_1, \cdots, r_{n+1}) \\
&\quad := \sum_{(r_1), \cdots, (r_{n-1})} \omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n) \triangleright \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1})
\end{aligned}$$

for all ω in $C^n(R; R)$ and $r_1, \ldots, r_{n+1}$ in R.

Theorem 3.17. *d_R is a well defined differential.*

Proof. That d_R is well defined means that it sends coderivations to coderivations. It suffices to show that this is already true for all maps $d^n_{i,1}$, $d^n_{i,0}$ and d^n_{n+1}, which is the case. For this, we use Lemma 3.15. Indeed, a cochain $\omega \in C^n(R;R)$ is a coderivation along μ^n. By Proposition 3.14, μ^n is a coalgebra morphism. On the other hand, it is clear from the formula for $d^n_{i,1}$ that $d^n_{i,1}$ is a partial convolution with respect to the first $i-1$ tensor labels of μ^i and ω. Therefore the Lemma applies to give that the result is a coderivation along the partial convolution of μ^i and μ^n, which is just μ^{n+1} again by Proposition 3.14. This shows that $d^n_{i,1}\omega$ belongs to $\mathrm{Coder}(R^{\otimes n}, R, \mu^{n+1})$ as expected. The maps $d^n_{i,0}$ and d^n_{n+1} can be treated in a similar way.

The fact that d_R squares to zero is again related to the *cubical identities* satisfied by the maps $d_{i,1}$ and the maps $d_{i,0}$, namely

$$d^{n+1}_{j,\mu} \circ d^n_{i,\nu} \;=\; d^{n+1}_{i+1,\nu} \circ d^n_{j,\mu} \quad \text{for} \quad j \leq i \quad \text{and} \quad \mu, \nu \in \{0,1\},$$

and auxiliary identities which express the compatibility of the maps $d_{i,1}$ and $d_{i,0}$ with d^n_{n+1}, and an identity involving d^n_{n+1} and d^{n+1}_{n+2}. In fact, $C^\bullet(R,R)$ becomes an *augmented cubical vector space.*

We will not show the usual cubical relations, i.e. those which do not refer to the auxiliary coboundary map d^n_{n+1}, because these are well-known to hold for rack cohomology, see [6], Corollary 3.12, and our case is easily adapted from there.

Let us show that the two following extra relations involving the extra face d^n_{n+1} hold:

$$d^{n+1}_{i,\mu} \circ d^n_{n+1} = d^{n+1}_{n+2} \circ d^n_{i,\mu} \tag{43}$$

for all $1 \leq i \leq n$ and μ in $\{0,1\}$ and

$$d^{n+1}_{n+1,0} \circ d^n_{n+1} = d^{n+1}_{n+2} \circ d^n_{n+1} + d^{n+1}_{n+1,1} \circ d^n_{n+1} \tag{44}$$

Indeed, if ω is a n-cochain and r_1, ..., r_{n+2} are elements in R, then

$$\begin{aligned}
&(d^{n+1}_{i,1} \circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2})\\
&\quad=\mu^i(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i) \rhd d^n_{n+1}\omega(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_{i+1},\cdots,r_{n+2})\\
&\quad=\mu^i(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i) \rhd \big(\omega(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_{i+1}^{(1)},\cdots,r_n^{(1)},r_{n+1})\rhd\\
&\qquad\quad \rhd \mu^n(r_1^{(3)},\cdots,r_{i-1}^{(3)},r_{i+1}^{(2)},\cdots,r_n^{(2)},r_{n+2})\big)
\end{aligned}$$

By Proposition 3.14 and the self-distributivity of the rack product, this equality can be rewritten as

$$\begin{aligned}
&(d^{n+1}_{i,1} \circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2})\\
&\quad= \big(\mu^i(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i^{(1)}) \rhd \omega(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_{i+1}^{(1)},\cdots,r_n^{(1)},r_{n+1})\big)\rhd\\
&\qquad\quad \rhd \big(\mu^i(r_1^{(3)},\cdots,r_{i-1}^{(3)},r_i^{(2)}) \rhd \mu^n(r_1^{(4)},\cdots,r_{i-1}^{(4)},r_{i+1}^{(2)},\cdots,r_n^{(2)},r_{n+2})\big)\\
&\quad= d^n_{i,1}\omega(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1}) \rhd \mu^n(r_1^{(2)},\cdots,r_n^{(2)},r_{n+2})\\
&\quad= (d^{n+1}_{n+2} \circ d^n_{i,1}\omega)(r_1,\cdots,r_{n+2}),
\end{aligned}$$

which proves that Relation (43) holds when $\mu = 1$. The case $\mu = 0$ goes as follows:

$$
\begin{aligned}
(d_{i,0}^{n+1} \circ d_{n+1}^{n}\omega)(r_1, \cdots, r_{n+2}) &= d_{n+1}^{n}\omega(r_1, \cdots, r_{i-1}, r_i^{(1)} \rhd r_{i+1}, \cdots, r_i^{(n+2-i)} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)}, \cdots, r_{i-1}^{(1)}, r_i^{(1)} \rhd r_{i+1}^{(1)}, \cdots, r_i^{(n-i)} \rhd r_n^{(1)}, r_i^{(n+1-i)} \rhd r_{n+1}) : \rhd \\
&\quad \rhd \mu^n(r_1^{(2)}, \cdots, r_{i-1}^{(2)}, r_i^{(n+2-i)} \rhd r_{i+1}^{(2)}, \cdots, r_i^{(2n-2i+1)} \rhd r_n^{(2)}, r_i^{(2n-2i+2)} \rhd r_{n+2})
\end{aligned}
$$

where we have used that the rack product is a morphism of coalgebras. Recall the following equation from Proposition 3.14:

$$
\mu^n(s_1, \cdots, s_{i-1}, s_i^{(1)} \rhd s_{i+1}, \cdots, s_i^{(n+1-i)} \rhd s_{n+1}) = \mu^{n+1}(s_1, \cdots, s_{n+1})
$$

for all s_1, ..., s_{n+1} in R and $1 \leq i \leq n$. This allows to rewrite the preceeding equality as

$$
\begin{aligned}
&(d_{i,0}^{n+1} \circ d_{n+1}^{n}\omega)(r_1, \cdots, r_{n+2}) \\
&\quad = \omega(r_1^{(1)}, \cdots, r_{i-1}^{(1)}, r_i^{(1)} \rhd r_{i+1}^{(1)}, \cdots, r_i^{(n-i)} \rhd r_n^{(1)}, r_i^{(n+1-i)} \rhd r_{n+1}) : \rhd \\
&\qquad \rhd \mu^{n+1}(r_1^{(2)}, \cdots, r_{i-1}^{(2)}, r_i^{(n+2-i)}, r_{i+1}^{(2)}, \cdots, r_n^{(2)}, r_{n+2}) \\
&\quad = d_{i,0}^{n} : \omega(r_1^{(1)}, \cdots, r_n^{(1)}, r_{n+1}) : \rhd : \mu^{n+1}(r_1^{(2)}, \cdots, r_n^{(2)}, r_{n+2}) \\
&\quad = (d_{n+2}^{n+1} \circ d_{i,0}^{n})(r_1, \cdots, r_{n+2})
\end{aligned}
$$

which proves that (43) holds when $\mu = 0$. Relation (44) relies on the fact that cochains are coderivations. Indeed,

$$
\begin{aligned}
(d_{n+1,0}^{n+1} \circ d_{n+1}^{n}\omega)(r_1, \cdots, r_{n+2}) &= d_{n+1}^{n}\omega(r_1, \cdots, r_n, r_{n+1} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n) \rhd \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n) \rhd \mu^{n+1}(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1}, r_{n+2}) \\
&= \omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n) \rhd \Big(\mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1}) \rhd \mu^n(r_1^{(3)}, \cdots, r_{n-1}^{(3)}, r_{n+2})\Big)
\end{aligned}
$$

where we have used Proposition 3.14 in the last equality. By self-distributivity of $\rhd$ and because ω is a coderivation, this gives

$$
\begin{aligned}
(d_{n+1,0}^{n+1} \circ d_{n+1}^{n}\omega)(r_1, \cdots, r_{n+2}) &= \big(\omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n)^{(1)} \rhd \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1})\big) \rhd \\
&\qquad \big(\omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n)^{(2)} \rhd \mu^n(r_1^{(3)}, \cdots, r_{n-1}^{(3)}, r_{n+2})\big) \\
&= \big(\omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n^{(1)}) \rhd \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1})\big) \rhd \\
&\qquad \big(\mu^n(r_1^{(3)}, \cdots, r_{n-1}^{(3)}, r_n^{(2)}) \rhd \mu^n(r_1^{(4)}, \cdots, r_{n-1}^{(4)}, r_{n+2})\big) \\
&\quad + \big(\mu^n(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n^{(1)}) \rhd \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1})\big) \rhd \\
&\qquad \big(\omega(r_1^{(3)}, \cdots, r_{n-1}^{(3)}, r_n^{(2)}) \rhd \mu^n(r_1^{(4)}, \cdots, r_{n-1}^{(4)}, r_{n+2})\big)
\end{aligned}
$$

Applying Proposition 3.14 again enables us to rewrite this last equality as

$$
\begin{aligned}
&(d^{n+1}_{n+1,0}\circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2})\\
&\qquad=\big(\omega(r_1^{(1)},\cdots,r_n^{(1)})\rhd\mu^n(r_1^{(2)},\cdots,r_{n-1}^{(2)},r_{n+1})\big)\rhd\mu^{n+1}(r_1^{(3)},\cdots,r_{n-1}^{(3)},r_n^{(2)},r_{n+2})\\
&\qquad\quad+\mu^{n+1}(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1})\rhd\big(\omega(r_1^{(2)},\cdots,r_n^{(2)})\rhd\mu^n(r_1^{(3)},\cdots,r_{n-1}^{(3)},r_{n+2})\big)\\
&\qquad=d^n_{n+1}\omega(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1})\rhd\mu^{n+1}(r_1^{(2)},\cdots,r_n^{(2)},r_{n+2})\\
&\qquad\quad+\mu^{n+1}(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1})\rhd d^n_{n+1}\omega(r_1^{(2)},\cdots,r_n^{(2)},r_{n+2})\\
&\qquad=\big((d^{n+1}_{n+2}\circ d^n_{n+1}+d^{n+1}_{n+1,1}\circ d^n_{n+1})\colon\omega\big)(r_1,\cdots,r_{n+2})
\end{aligned}
$$

which proves (44).

Let us show now how $d_R\circ d_R=0$ can be deduced from (43), (44) and from the cubical relations. In degree n, we have

$$
\begin{aligned}
d_R\circ d_R=&\Big(\sum_{i=1}^{n+1}(-1)^{i+1}(d^{n+1}_{i,1}-d^{n+1}_{i,0})+(-1)^{n+2}d^{n+1}_{n+2}\Big)\circ\Big(\sum_{i=1}^{n}(-1)^{i+1}(d^n_{i,1}-d^n_{i,0})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(-1)^{n+1}d^n_{n+1}\Big)\\
=&\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}(d^{n+1}_{i,1}\circ d^n_{j,1}-d^{n+1}_{i,1}\circ d^n_{j,0}-d^{n+1}_{i,0}\circ d^n_{j,1}+d^{n+1}_{i,0}\circ d^n_{j,0})\\
&+\sum_{i=1}^{n}(-1)^{n+i+1}(d^{n+1}_{n+2}\circ d^n_{i,1}-d^{n+1}_{n+2}\circ d^n_{i,0}-d^{n+1}_{i,1}\circ d^n_{n+1}+d^{n+1}_{i,0}\circ d^n_{n+1})\\
&-d^{n+1}_{n+2}\circ d^n_{n+1}-d^{n+1}_{n+1,1}\circ d^n_{n+1}+d^{n+1}_{n+1,0}\circ d^n_{n+1}
\end{aligned}
$$

The first double sum is equal to zero thanks to the cubical relations, the second sum is zero thanks to relation (43). Relation (44) implies that the last one vanishes. This shows that d_R is indeed a differential and concludes the proof of the proposition. □

Definition 3.18. *The cohomology of the deformation complex $(C^*(R;R),d_R)$ is called the adjoint cohomology of the rack bialgebra R and is denoted by $H^*(R;R)$.*

Definition 3.19. *An **infinitesimal deformation** of the rack product is a deformation of the rack product over the $\mathbb{K}$-algebra of dual numbers $\bar{\mathbb{K}}_\hbar:=\mathbb{K}_\hbar/(\hbar^2)$, i.e. a linear map $\mu_1:R^{\otimes 2}\to R$ such that $\bar{R}_\hbar:=R\otimes\bar{\mathbb{K}}_\hbar$ is a rack bialgebra over $\bar{\mathbb{K}}_\hbar$ when equipped with $\mu_0+\hbar\mu_1$.*
*Two infinitesimal deformations $\mu_0+\hbar\mu_1$ and $\mu_0+\hbar\mu_1'$ are said to be **equivalent** if there exists an automorphism $\phi:\bar{R}_\hbar\to\bar{R}_\hbar$ of the coalgebra of $(\bar{R}_\hbar,\triangle,\epsilon)$ of the form $\phi:=\mathrm{id}_R+\hbar\alpha$ such that*

$$\phi\circ(\mu_0+\hbar\mu_1)=(\mu_0+\hbar\mu_1')\circ\phi.$$

As usual, being equivalent is an equivalence relation and one has the following cohomological interpretation of the set of equivalence classes of infinitesimal deformations, denoted $Def(\mu_0,\bar{\mathbb{K}}_\hbar)$:

Proposition 3.20.
$$Def(\mu_0, \bar{\mathbb{K}}_\hbar) \,=\, H^2(R;R)$$
The identificaton is obtained by sending each equivalence class $[\mu_0 + \hbar\mu_1]$ *in* $Def(\mu_0, \bar{\mathbb{K}}_\hbar)$ *to the cohomology class* $[\mu_1]$ *in* $H^2(R;R)$.

Proof. One checks easily that the correspondence is well defined (if $\mu_0 + \hbar\mu_1$ is an infinitesimal deformation, then μ_1 is a 2-cocycle, see Example 3.13) and that it is bijective when restricted to equivalence classes. □

Remark 3.21. (a) The choice of taking coderivations in the deformation complex is explained as follows: The rack product μ is a morphism of coalgebras, and we want to deform it as a morphism of coalgebras with respect to the fixed coalgebra structure we started with. Tangent vectors to μ in $\mathrm{Hom}_{\mathrm{coalg}}(C \otimes C, C)$ are exactly coderivations along μ. This is the first step: Deformations as morphisms of coalgebras. Then as a second step, we look for 1-cocycles, meaning that we determine those morphisms of coalgebras which give rise to rack bialgebra structures. The deformation complex in [5] takes into account also the possibility of deforming the coalgebra structure, and we recover our complex by restriction.

(b) Given a Leibniz algebra $\mathfrak{h}$, there is a natural restriction map from the cohomology complex with adjoint coefficients of $\mathfrak{h}$ to the deformation complex of its augmented enveloping rack bialgebra $\mathsf{UAR}(\mathfrak{h})$. The induced map in cohomology is not necessarily an isomorphism, as the abelian case shows. Observe that the deformation complex of the rack bialgebra $\mathbb{K}R$ for a rack R does not contain the complex of rack cohomology for two reasons: First, this latter complex is ill-defined for adjoint coefficients, and second, there are not enough coderivations as all elements are set-like. A way out for this last problem would be to pass to completions.

Despite of this last remark, there is a relation between the cohomology of rack bialgebras and the cohomology of Leibniz algebras, see [16]:

Theorem 3.22. *Consider the rack bialgebra* $C = \mathbb{K} \oplus \mathfrak{h}$ *associated to a Leibniz algebra* $\mathfrak{h}$*, see Example 2.25. The Leibniz cohomology complex with values in the adjoint representation embeds into the deformation complex* $(C^*(C;C), d_C^*)$ *defined above.*

Proof. We extend Leibniz cochains $f : \mathfrak{h}^{\otimes n} \to \mathfrak{h}$ to cochains in the complex $C^*(C,C)$ with $C = \mathbb{K}\cdot 1 \oplus \mathfrak{h}$ by setting them zero on all components in $\mathbb{K}\cdot 1 \subset C$. More precisely

$$\omega((\lambda_1, x_1), \ldots, (\lambda_n, x_n)) := \mathrm{pr}_{\mathfrak{h}}(f(x_1, \ldots, x_n)),$$

for $(\lambda_i, x_i) \in k \cdot 1 \oplus \mathfrak{h}$ for all $i = 1, \ldots, n$ and with $\mathrm{pr}_{\mathfrak{h}} : k \cdot 1 \oplus \mathfrak{h} \to \mathfrak{h}$ the natural projection.

With this definition, it follows that these cochains are coderivations along μ^n, i.e.
$$\triangle_C \circ \omega = (\omega \otimes \mu^n + \mu^n \otimes \omega) \circ \triangle_{C^{\otimes n}}.$$

Indeed, when computing the iterated coproduct $\triangle_{C^{\otimes n}}(r_1, \ldots, r_n)$, the elements $r_i \in \mathfrak{h}$ are distributed among the two factors in $C^{\otimes n} \otimes C^{\otimes n}$ and all other components are filled with units. On the LHS, $\omega(r_1, \ldots, r_n)$ is primitive by construction, thus we get the two terms $\omega(r_1, \ldots, r_n) \otimes 1 + 1 \otimes \omega(r_1, \ldots, r_n)$. On the RHS, the only terms which do not vanish are those with all r_i as arguments in ω. This shows the equality.

Now we specify the different parts of the coboundary operator. We remind the reader that we write here the Sweedler indices *above* the elements as superscripts in order to distinguish them from the indices enumerating the elements.

$$\begin{aligned} d^n_{i,1}\omega(r_1, \ldots, r_{n+1}) = \\ \mu^i(r_1^{(1)}, \ldots, r_{i-1}^{(1)}, r_i) \rhd \omega(r_1^{(2)}, \ldots, r_{i-1}^{(2)}, r_{i+1}, \ldots, r_{n+1}) \\ = [r_i, \omega(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{n+1})], \end{aligned}$$

because the only contributing term is the one where all r_j are arguments of ω, i.e. all the units are in μ^i.

$$\begin{aligned} d^n_{j,0}\omega(r_1, \ldots, r_{n+1}) &= \omega(r_1, \ldots, r_{j-1}, r_j^{(1)} \rhd r_{j+1}, \ldots, r_j^{(n+1-j)} \rhd r_{n+1}) \\ &= \omega(r_1, \ldots, r_{j-1}, [r_j, r_{j+1}], r_{j+2}, \ldots, r_{n+1}) + \ldots \\ \ldots &+ \omega(r_1, \ldots, [r_j, r_{n+1}]), \end{aligned}$$

because only one of the $r_j^{(k)}$ is equal to r_j and all others are equal to 1.

$$\begin{aligned} & d^n_{n+1}\omega(r_1, \ldots, r_{n+1}) \\ &= \omega(r_1^{(1)}, \ldots, r_{n-1}^{(1)}, r_n) \rhd \mu^n(r_1^{(2)}, \ldots, r_{n-1}^{(2)}, r_{n+1}) \\ &= [\omega(r_1, \ldots, r_n), r_{n+1}], \end{aligned}$$

because this is the only term where one does not insert 1 into ω. Thus it is clear that this gives the Loday coboundary operator on Leibniz cohomology. □

In total, we therefore have a map linking the cohomology of a Lie rack to the cohomology of its tangent Leibniz algebra, and a map from Leibniz cohomology of a Leibniz algebra $\mathfrak{h}$ to the rack bialgebra cohomology of the rack bialgebra $\mathbb{K} \oplus \mathfrak{h}$. Simon Covez shows how to integrate Leibniz cocycles to rack cocycles in some very special cases [7]. Up to our knowledge, other links between these cohomologies are not known and yet to be discovered ! In order to do so, it is certainly important to consider these cohomologies *together with their links to the classical structure*, i.e. rack cohomology (of a group) together with the natural morphism from group cohomology (Simon Covez [6]), Leibniz cohomology (of a Lie algebra) with its natural links to Lie algebra cohomology (Teimuraz Pirashvili [21]) and also rack bialgebra cohomology (of a cocommutative Hopf algebra) with a morphism to/from associative bialgebra cohomology (Gerstenhaber-Schack cohomology). Some more remarks in this direction are included at the end of [16].

4 Final remarks and outlook

In order to relate these three cohomologies, one way is to take up one level of abstraction. The cohomology of Leibniz algebras is the operadic cohomology, related to the *operad* Leib, see [20]. The rack cohomology is also a Quillen cohomology, see [24]. When considering rack bialgebras, one has to consider *properads*, more general than operads (exactly as for associative bialgebras). I have a student trying to show that the properad of rack bialgebras is homotopy Koszul (which would then imply the existence of a good resolution and a conceptual way to define cohomology), but it seems this is a difficult problem. It would be interesting to have conceptual relations on the level of operads, properads or more generally monads (in order to include the case of racks) between these structures which would explain the relations observed above which make racks, Leibniz algebras and rack bialgebras so seemingly close to groups, Lie algebras and associative bialgebras.

Another related axis of research are the associated representation categories. For example, the adjunction between G-modules and $\mathbb{Z}G$-bimodules for a group G, given by the forgetful functor U towards G-modules and the tensoring by $\mathbb{Z}G$ towards $\mathbb{Z}G$-bimodules, induces isomorphisms in cohomology

$$HH^n(\mathbb{Z}G, M) \cong H^n(G, U(M))$$

for all $n \geq 0$. In order to prove this, one passes through the Ext interpretation of these cohomologies and shows that the adjunction induces a derived adjunction (on the level of derived categories). This shows that adjunctions between the representation categories can be useful in the search of relating the cohomologies. It is interesting to note that this reasoning does not work for racks and rack bialgebras. It seems that the correct way to relate racks to rack bialgebras passes through Lie racks R and distributions supported in $1 \in R$.

One issue in order to relate the representation categories is the existence of an antipode for rack bialgebras. The correct definition of such an antipode and the implications of its existence are also a promising direction of further research.

Last but not least, it would be interesting to have a theory of Malcev completions for racks, rack bialgebras and Leibniz algebras in the spirit of what is known for groups, associative bialgebras and Lie algebras as in Appendix A of [22]. In fact, one would dream of Leibniz models for rational homotopy theory which would be related to cubical simplicial sets like in Figure 1 (p. 211) of [22]. Some of the ingredients in the Leibniz version of this chain of Quillen adjoint functors are already available.

References

[1] C. Alexandre, M. Bordemann, S. Rivière, F. Wagemann, *Structure theory of rack-bialgebras.* J. Gen. Lie Theory Appl. 10 (2016), no. 1, Art. ID 1000244, 20 pp

[2] C. Alexandre, M. Bordemann, S. Rivière, F. Wagemann, *Algebraic deformation quantization of Leibniz algebras.* Comm. Algebra **46** (2018), no. 12, 5179–5201.

[3] A. M. Blokh, *On a generalization of the concept of a Lie algebra.* Dokl. Akad. Nauk. USSR 165 (1965) pp. 471–473.

[4] M. Bordemann, F. Wagemann, *Global integration of Leibniz algebras.* J. Lie Theory 27 (2017), no. 2, pp. 555–567

[5] J. Carter, A. Crans, M. Elhamdadi, M. Saito, *Cohomology of the Categorical Self-Distributivity* J. Homotopy Relat. Struct. 3 (2008), no. 1, pp. 13–63.

[6] S. Covez, *On the conjectural Leibniz cohomology for groups.* J. K-theory **10** (2012), no. 3, 519–563.

[7] S. Covez, *The local integration of Leibniz algebras.* Ann. Inst. Fourier (Grenoble) 63 (2013), no. 1, 1–35.

[8] S. Covez, M. Farinati, V. Lebed, D. Manchon, *Bialgebraic approach to rack cohomology* `arXiv:1905.02754`

[9] J. Feldvoss, F. Wagemann, *On Leibniz cohomology.* J. Algebra **569** (2021), 276–317.

[10] R. Fenn, C. Rourke, *racks and links in codimension two.* J. knot Theory Ramifications 1 (1992), no. 4, pp. 343–406

[11] P. J. Freyd, D. N. Yetter, *Braided compact closed categories with applications to low-dimensional topology.* Adv. Math. 77, no. 2 (1989), pp. 156–182

[12] C. Kassel, *Quantum groups.* Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[13] M. Kinyon, A. Weinstein, *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces.* Amer. J. Math. 123 (2001), no. 3, pp. 525–550

[14] M. K. Kinyon, *Leibniz algebras, Lie racks, and digroups.* J. Lie Theory 17 (2007), no. 1, pp. 99–114.

[15] U. Krähmer, F. Wagemann, *Racks, Leibniz algebras and Yetter-Drinfel'd modules.* Georgian Math. J. 22 (2015), no. 4, pp. 529–542.

[16] U. Krähmer, F. Wagemann, *A universal enveloping algebra for cocommutative rack bialgebras.* Forum Math. **31** (2019), no. 5, 1305–1315.

[17] J.-L. Loday, *Cyclic Homology.* Springer, Berlin, 1992.

[18] J.-L. Loday and T. Pirashvili: *Universal enveloping algebras of Leibniz algebras and (co)homology.* Math. Ann. **296** (1993), no. 1, 139–158.

[19] J.-L. Loday, *Algebraic K-theory and the conjectural Leibniz K-theory.* Special issue in honor of Hyman Bass on his seventieth birthday. Part II. K-theory **30** (2003), no. 2, 105–127.

[20] J.-L. Loday, B. Vallette, *Algebraic operads.* Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **346.** Springer, Heidelberg, 2012.

[21] T. Pirashvili, *On Leibniz homology.* Ann. Inst. Fourier (Grenoble) **44** (1994), no. 2, 401–411.

[22] D. Quillen, *Rational homotopy theory.* Ann. of Math. (2) **90** (1969), 205–295.

[23] J.-P. Serre, *Lie algebras and Lie groups.* 1964 lectures given at Harvard University. Corrected fifth printing of the second (1992) edition. Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 2006

[24] M. Szymik, *Quandle cohomology is a Quillen cohomology.* Trans. Amer. Math. Soc. **371** (2019), no. 8, 5823–5839.

[25] F. Wagemann, *Crossed modules.* De Gruyter Studies in Mathematics, 82. De Gruyter, Berlin, [2021], ©2021. xiv+393

[26] Charles Weibel, *An introduction to homological algebra* Cambridge Studies in advanced mathematics **38**, Cambridge University Press 1994