# Rigidity and cohomology of Leibniz algebras

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**Abstract**

In this article, we discuss cohomology and rigidity of Leibniz algebras. We generalize Richardson's example of a rigid Lie algebra with non-trivial $H^2$ to the Leibniz setting. Namely, we consider the hemisemidirect product $\mathfrak{h}$ of a semidirect product Lie algebra $\mathfrak{g} \ltimes M_k$ of a simple Lie algebra $\mathfrak{g}$ with some irreducible $\mathfrak{g}$-module $M_k$ with a non-trivial irreducible $\mathfrak{g}$-module $I_l$. Then for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we take $M_k$ (resp. $I_l$) to be the standard irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $2k+1$ (resp. $2l+1$). Assume $k > 4l$, $k$ odd and $I \neq \mathfrak{sl}_2(\mathbb{C})$, then the Leibniz algebra $\mathfrak{h}$ is geometrically rigid and has non-trivial $HL^2$ with adjoint coefficients.

## Introduction

Let $k$ be a field. In order to study the variety of all $n$-dimensional Lie algebras over $k$, one fixes a basis $(e_i)_{i=1,\dots,n}$ in $k^n$ and represents a Lie algebra by its structure constants $(c_{ij}^k)_{i,j,k\in\{1,\dots,n\}}$ given by

$$[e_i, e_j] \;=\; \sum_{i=1}^{n} c_{ij}^k e_k.$$

These structure constants are elements of the vector space $\mathrm{Hom}(\Lambda^2(k^n), k^n)$ which must satisfy the quadratic equations

$$\sum_{p=1}^{n} (c_{jl}^p c_{ip}^k - c_{ij}^p c_{pl}^k + c_{il}^p c_{pj}^k) \;=\; 0$$

for $i, j, l, k \in \{1, \dots, n\}$ owing to the Jacobi identity. The group $\mathrm{Gl}(k^n)$ acts on the algebraic variety of structure constants by base changes. The *variety of Lie algebra laws* over $k$ is by definition the quotient of the algebraic variety defined by the above quadratic equations by the action of $\mathrm{Gl}(k^n)$. This action is usually badly behaved and the quotient has singularities, is non-reduced etc. even if $k$ is algebraically closed and of characteristic zero, which we suppose from now on. Therefore it is not a variety in the usual sense. We nevertheless continue to call it the variety of Lie algebra laws.

The analoguous picture for Leibniz algebra laws has been explored in [2]. The main change here is that the structure constants are not supposed to be anti-symmetric anymore, thus they live in $\mathrm{Hom}((k^n)^{\otimes 2}, k^n)$. Balavoine shows that a point in the variety of Leibniz algebra laws is reduced and geometrically rigid (i.e. the $\mathrm{Gl}(k^n)$-orbit of the corresponding Leibniz algebra $\mathfrak{h}$ is open in the Zariski topology) if and only if its second adjoint Leibniz cohomology space $HL^2(\mathfrak{h}, \mathfrak{h})$ is zero. The goal of the present paper is to give an example of a non-reduced point of the variety of Leibniz algebra laws, i.e. we give an example of a finite-dimensional Leibniz algebra $\mathfrak{h}$ over $k$ which is geometrically rigid, but which has $HL^2(\mathfrak{h}, \mathfrak{h}) \neq 0$.

Examples of non-reduced points in the variety of Lie algebra laws have been constructed by Richardson in [12]. Richardson shows there that for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and for $M$ the standard irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $2k+1$, the semidirect product Lie algebra $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C}) \ltimes M$ is not rigid if and only if $k = 1, 2, 3, 5$. In fact, in these dimensions, there exists a semisimple Lie algebra of dimension $6, 8, 10$ and $14$ with $\mathfrak{sl}_2(\mathbb{C})$ as a subalgebra such that the quotient module identifies with $M$. On the other hand, Richardson shows that for $k > 5$, the second adjoint Lie algebra cohomology space of $\mathfrak{g}$ is non-trivial. He concludes that for $k > 5$, $\mathfrak{g}$ represents a non-reduced point in the variety of Lie algebra laws.

Going into some more details, Richardson's rigidity result relies on his stability theorem (joint work with Stanley Page), see [10]. It says that in case some relative cohomology space $E^2(\mathfrak{g}; \mathfrak{s}, \mathfrak{g})$ is zero, the subalgebra $\mathfrak{s} \subset \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ is stable, i.e. all Lie algebra laws in some neighborhood of $\mathfrak{g}$ contain an isomorphic subalgebra which has the same brackets with the quotient module. Richardson derives from this theorem that the above Lie algebra $\mathfrak{g}$ is rigid for $k > 5$. The subalgebra $\mathfrak{g}$ in his case is the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and from here one can derive that the relative cohomology $E^2(\mathfrak{g}; \mathfrak{s}, \mathfrak{g})$ is zero.

In the Leibniz case, a non-Lie Leibniz algebra with simple quotient Lie algebra has necessarily trivial Leibniz cohomology in degrees $n \geq 2$ (see Proposition 2.7), as follows from Pirashvili's work [11]. Our idea is therefore to take a (non-Lie) Leibniz algebra whose quotient Lie algebra is of Richardson's type, i.e. a semidirect product of a simple Lie algebra with an irreducible module. We can show that this kind of Leibniz algebra still has as its adjoint Leibniz cohomology the adjoint Lie algebra cohomology of the quotient Lie algebra, see Proposition 2.13. This will then imply that in Richardson's setting, the second Leibniz cohomology of our Leibniz algebra is non-zero.

On the other hand, it is rather straightforward to generalize the proof of the Stability Theorem as its relies on three applications of the Inverse Function Theorem (see Theorem 3.3) using standard material like Massey products and the coboundary operator. Our stability theorem is Theorem 3.4. It is rather cumbersome though to satisfy the hypotheses of this theorem, namely to show that the corresponding relative cohomology space $E^2(\mathfrak{h}; \mathfrak{s}, \mathfrak{h})$ is zero. This uses some extension theory results from [8] and is done in Propositions 2.14 and 2.15. Let $M_k$ and $I_l$ be finite-dimensional non-trivial irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules of dimensions $2k+1$ resp. $2l+1$. Our main theorem reads:

**Theorem 0.1.** *Assume that $k > 4l$, $k$ odd and $I \neq \mathfrak{sl}_2(\mathbb{C})$. Let $\mathfrak{h}$ be the hemisemidirect product Leibniz algebra $\mathfrak{h} = I_l \dot{+} (\mathfrak{sl}_2(\mathbb{C}) \ltimes M_k)$ of the semidirect product Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \ltimes M_k$ with the module $I_l$ (as ideal of squares).*

*Then $\mathfrak{h}$ is a rigid Leibniz algebra with $HL^2(\mathfrak{h}, \mathfrak{h}) \neq 0$.*

The structure of the present article is as follows: In a first section, we gather preliminaries on Leibniz algebra, their modules and semidirect products. The second section is about cohomology computations. We mainly translate Pirashvili's article [11] into the cohomology language and apply it to adjoint coeffcients. The third section is then devoted to the stability theorem. The last section concludes the construction of a rigid Leibniz algebra with non-trivial $HL^2$.

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# 1 Preliminaries

## 1.1 Leibniz algebras and modules

We will always work over a field $k$ of characteristic zero. Basic material on Leibniz algebras and their modules can be found in [7].

**Definition 1.1.** *A (right) Leibniz algebra is a vector space $\mathfrak{h}$ equipped with a bilinear bracket $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ such that for all $x, y, z \in \mathfrak{h}$*

$$[[x, y], z] \;=\; [[x, z], y] + [x, [y, z]].$$

*A morphism of Leibniz algebras is as usual a linear map which respects the brackets.*

Obviously, Lie algebras are examples of Leibniz algebras, but there are non-Lie Leibniz algebras. We will try to distiguish Leibniz algebras and Lie algebras in notation by using $\mathfrak{h}$ for a representative of the former class and $\mathfrak{g}$ for a representative of the latter class.

Let us recall the notion of a semidirect product of a Lie algebra $\mathfrak{g}$ and a right $\mathfrak{g}$-module $M$.

**Definition 1.2.** *The semidirect product $\mathfrak{g} \ltimes M$ is the Lie algebra defined on the vector space $\mathfrak{g} \oplus M$ by*

$$[(x_1, m_1), (x_2, m_2)] \;=\; ([x_1, x_2], m_1 \cdot x_2 - m_2 \cdot x_1).$$

The action of Lie- or Leibniz algebras modules will be written $m \cdot x$ or sometimes with bracket notation $[m, x]$, following Loday.

**Definition 1.3.** *Let $\mathfrak{h}$ be a Leibniz algebra. A vector space $M$ is called a Leibniz $\mathfrak{h}$-module in case there exist bilinear maps $[,] : \mathfrak{h} \times M \to M$ and $[,] : M \times \mathfrak{h} \to M$ such that*

$$
\begin{aligned}
&\textit{(MLL)} \quad [m,[x,y]] \;=\; [[m,x],y] - [[m,y],x],\\
&\textit{(LML)} \quad [x,[m,y]] \;=\; [[x,m],y] - [[x,y],m],\\
&\textit{(LLM)} \quad [x,[y,m]] \;=\; [[x,y],m] - [[x,m],y].
\end{aligned}
$$

*A morphism of Leibniz $\mathfrak{h}$-modules is a linear map which respects the two brackets.*

These three conditions turn up naturally by writing what it means for an abelian extension to be a Leibniz algebra. For a Lie algebra $\mathfrak{g}$, a right $\mathfrak{g}$-module can be seen as a Leibniz $\mathfrak{g}$-module in two different ways, namely as a symmetric and as an anti-symmetric Leibniz $\mathfrak{g}$-module:

**Definition 1.4.** *(a) A Leibniz $\mathfrak{h}$-module $M$ is called symmetric in case for all $x \in \mathfrak{h}$ and all $m \in M$*

$$[m,x] \;=\; -[x,m].$$

*(b) A Leibniz $\mathfrak{h}$-module $M$ is called anti-symmetric in case for all $x \in \mathfrak{h}$ and all $m \in M$*

$$[x,m] \;=\; 0.$$

The most important example of an anti-symmetric Leibniz $\mathfrak{h}$-module is the ideal of squares $I$ or $\mathfrak{h}^{\mathrm{ann}}$, i.e. the ideal of $\mathfrak{h}$ generated by the elements of the form $[x,x]$ for $x \in \mathfrak{h}$. Indeed, $I$ becomes a Leibniz $\mathfrak{h}$-module with respect to the adjoint action and we have

$$[x,[y,y]] \;=\; [[x,y],y] - [[x,y],y] \;=\; 0.$$

The quotient of $\mathfrak{h}$ by $I$, denoted $\mathfrak{h}_{\mathrm{Lie}}$, is a Lie algebra, and we have an exact sequence of Leibniz algebras

$$0 \to \mathfrak{h}^{\mathrm{ann}} \to \mathfrak{h} \to \mathfrak{h}_{\mathrm{Lie}} \to 0,$$

which is also an abelian extension of Leibniz algebras.

Another important example of an anti-symmetric Leibniz $\mathfrak{h}$-module is the *right center* $Z_{\mathrm{right}}(\mathfrak{h})$ of the Leibniz algebra $\mathfrak{h}$. It consists by definition of the elements $z \in \mathfrak{h}$ such that for all $x \in \mathfrak{h}$ we have $[x,z] = 0$. By the above, we see that $Z_{\mathrm{right}}(\mathfrak{h})$ contains the ideal of squares, and the quotient $\mathfrak{h} \,/\, Z_{\mathrm{right}}(\mathfrak{h})$ is thus a Lie algebra.

By quotienting a Leibniz $\mathfrak{h}$-module $M$ by the submodule generated by the relations $[x,m] + [m,x]$ for all $x \in \mathfrak{h}$ and all $m \in M$, one obtains a symmetric Leibniz $\mathfrak{h}$-module $M^s$. The kernel of the projection map $M \to M^s$ is an anti-symmetric representation, which is called $M^a$. Therefore, for each Leibniz $\mathfrak{h}$-module $M$, there is a short exact sequence of Leibniz $\mathfrak{h}$-modules

$$0 \to M^a \to M \to M^s \to 0. \tag{1}$$

There is also a notion of semidirect product associated to a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $M$ which is a non-Lie Leibniz algebra, in case the action is non-trivial. This notion is due to Kinyon and Weinstein [6].

**Definition 1.5.** *The hemisemidirect product* $\mathfrak{g}\dot{+}I$ *is the Leibniz algebra defined on* $\mathfrak{g} \oplus I$ *by*

$$[(x_1, m_1), (x_2, m_2)] \;=\; ([x_1, x_2], m_1 \cdot x_2).$$

## 1.2 Extensions of Leibniz modules

Useful results on extensions of Leibniz modules of Lie algebras can be found in [8].

Let $\mathfrak{h}$ be a Leibniz algebra and let

$$0 \to M \to E \to N \to 0$$

be a short exact sequence of Leibniz $\mathfrak{h}$-modules. As a vector space, $E$ is the direct sum $E = M \oplus N$, thus we may write the Leibniz module structure in terms of product coordinates for all $m \in M$, $n \in N$ and all $x \in \mathfrak{h}$ as

$$[x, (m, n)] \;=\; ([x, m] + \phi_l(x)(n), [x, n]),$$

$$[(m, n), x] \;=\; ([m, x] + \phi_r(x)(n), [n, x]),$$

where $[x, m]$ and $[m, x]$ are determined by the Leibniz $\mathfrak{h}$-module structure on $M$, and $[x, n]$ and $[n, x]$ by the Leibniz $\mathfrak{h}$-module structure on $N$. The linear maps

$$\phi_l, \phi_r : \mathfrak{h} \to \mathrm{Hom}(N, M)$$

determine the structure of the extension $E$, and will be called *twisting functions* in the sequel. One computes easily that the three identites for a Leibniz action give the following three equations for $\phi_l$ and $\phi_r$ for $x, y \in \mathfrak{h}$ and $n \in N$:

(a) $\phi_r([x, y])(n) \;=\; [\phi_r(x)(n), y] + \phi_r(y)([n, x]) - [\phi_r(y)(n), x] - \phi_r(x)([n, y]),$

(b) $\phi_l(x)([n, y]) + [x, \phi_r(y)(n)] \;=\; [\phi_l(x)(n), y] + \phi_r(y)([x, n]) - \phi_l([x, y])(n),$

(c) $[x, \phi_l(y)(n)] + \phi_l(x)([y, n]) \;=\; \phi_l([x, y])(n) - [\phi_l(x)(n), y] - [\phi_r(y)([x, n]).$

Observe that condition (a) is the Leibniz cocycle identity for $\phi_r$ seen as a 1-cochain with values in $\mathrm{Hom}(N, M)$ in case $M$ and $N$ are symmetric $\mathfrak{h}$-modules. There is no such observation for $\phi_l$.

On the other hand, two such extensions are equivalent in case there exists a morphism of Leibniz extensions $f : E \to E'$ which makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & E \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & 0 \\
0 \arrow[r] & M \arrow[r] & E' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Writing the map $f(m, n) = (m + \bar{f}(n), n)$ in product coordinates for $E$ and $E'$, the fact that $f$ is a morphism of Leibniz $\mathfrak{h}$-modules translates into the equations:

(a) $\phi_l(x)(n) - \phi'_l(x)(n) \;=\; [x, \bar{f}(n)] - \bar{f}([x,n])$,

(b) $\phi_r(x)(n) - \phi'_r(x)(n) \;=\; [\bar{f}(n), x] - \bar{f}([n,x])$.

This description will permit us to express the non-triviality of Leibniz $\mathfrak{h}$-extensions in some special cases (i.e. for symmetric or antisymmetric modules) in terms of 1-cohomology with values in $\mathrm{Hom}(N, M)$.

# 2 Leibniz cohomology computations

In Sections 2.1, 2.2 and the first part of Section 2.3, we follow closely [11]. One objective of these sections is to translate the homology computations of Pirashvili into cohomology which we believe to be useful for further reference. Similar results can be found for example in [9] or [4]. In the second part of Section 2.3 (Proposition 2.13) and Section 2.4, we perform (presumably) new cohomology computations which we will need later on for our rigidity theorem.

## 2.1 Leibniz cohomology of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a field $k$, and $M$ a right $\mathfrak{g}$-module. Cohomology theory associates to $\mathfrak{g}$ two complexes, namely the Chevalley-Eilenberg complex

$$C^*(\mathfrak{g}, M) \;:=\; (\mathrm{Hom}(\Lambda^*\mathfrak{g}, M), d'),$$

and the Leibniz or Loday complex

$$CL^*(\mathfrak{g}, M) \;:=\; (\mathrm{Hom}(\bigotimes^* \mathfrak{g}, M), d),$$

for the Leibniz cohomology of $\mathfrak{g}$ with values in the symmetric Leibniz $\mathfrak{g}$-module $M$.

The natural epimorphism $\bigotimes^* \mathfrak{g} \to \Lambda^*\mathfrak{g}$ induces a monomorphism of complexes

$$\varphi : C^*(\mathfrak{g}, M) \hookrightarrow CL^*(\mathfrak{g}, M),$$

which is an isomorphism in degree 0 and 1. Following Pirashvili, we set:

$$C^*_{\mathrm{rel}}(\mathfrak{g}, M) \;:=\; \mathrm{coker}(\varphi), \;\text{ and }\; H^*_{\mathrm{rel}}(\mathfrak{g}, M) \;:=\; H^*(C^*_{\mathrm{rel}}(\mathfrak{g}, M), d).$$

**Proposition 2.1.** *The map $\varphi$ induces a long exact sequence in cohomology*

$$0 \to H^2(\mathfrak{g}, M) \to HL^2(\mathfrak{g}, M) \to H^0_{\mathrm{rel}}(\mathfrak{g}, M) \to H^3(\mathfrak{g}, M) \to \dots$$

The wedge product

$$m : \mathfrak{g} \otimes \Lambda^n\mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$$

gives rise to a monomorphism of complexes

$$\psi : \overline{C^*}(\mathfrak{g}, k)[-1] \hookrightarrow C^*(\mathfrak{g}, \mathfrak{g}^*).$$

**Proposition 2.2.** *The map $\psi$ induces a long exact sequence*

$$0 \to H^2(\mathfrak{g}, k) \to H^1(\mathfrak{g}, \mathfrak{g}^*) \to HR^0(\mathfrak{g}) \to H^3(\mathfrak{g}, k) \to \dots$$

Here $HR^0(\mathfrak{g})$ is the space of invariant quadratic forms on $\mathfrak{g}$, see [11] for further details. Observe that if $\mathfrak{g}$ admits an invariant inner product, one may identify the $\mathfrak{g}$-modules $\mathfrak{g}$ and $\mathfrak{g}^*$, and then the long exact sequence can be used to deduce adjoint cohomology from cohomology with values in trivial coefficents.

**Theorem 2.3.** *There exists a cohomology spectral sequence*

$$E_2^{p,q} \;=\; HR^p(\mathfrak{g}) \otimes HL^q(\mathfrak{g}, M) \;\;\Rightarrow\;\; H^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M).$$

**Corollary 2.4.** *If $H^n(\mathfrak{g}, M) = 0$ for all $n \geq 0$, then $HL^p(\mathfrak{g}, M) = 0$ for all $p \geq 0$.*

*Proof.* As in the proof of Corollary 1.3 in [11], the proof proceeds by induction, using the $E_2^{p,q}$-term of the spectral sequence and that $HL^{n+2}(\mathfrak{g}, M) \cong H^n_{\mathrm{rel}}(\mathfrak{g}, M)$ by the above long exact sequence. □

**Remark 2.5.** Observe that in general $H^n(\mathfrak{g}, M) = 0$ for one $n$ does not imply that $HL^n(\mathfrak{g}, M) = 0$. For example, the trivial Lie algebra $\mathfrak{g} = k$ has $H^2(k, k) = 0$ (and is thus Lie-rigid!), but $HL^2(k, k) = k \neq 0$, cf the remark after Proposition 1 in [5].

One may ask the general question whether for all finite-dimensional Lie algebras $\mathfrak{g} \neq k$ the hypothesis $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ implies that $HL^2(\mathfrak{g}, \mathfrak{g}) = 0$. This assertion is true for non-trivial nilpotent Lie algebras, because Théorème 2 of [3] shows that for a non-trivial nilpotent Lie algebra $\dim(H^2(\mathfrak{g}, \mathfrak{g})) \geq 2$ as its center is non-trivial. The assertion is also true for semisimple Lie algebras as we will see in the next subsection. The only remaining question is whether it is true for non-nilpotent solvable Lie algebras. This relies on the center of these, as the following theorem shows: If the center is zero, the assertion is true, if not, it should be rather easy to find a counter-example.

Let us cite a part of Theorem 2 from [5]:

**Theorem 2.6.** *Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Then $H^2(\mathfrak{g}, \mathfrak{g})$ is a direct factor of $HL^2(\mathfrak{g}, \mathfrak{g})$. Furthermore, the supplementary subspace vanishes in case the center $Z(\mathfrak{g})$ is zero.*

## 2.2 Leibniz cohomology of semisimple Lie algebras

We assume now that $\mathfrak{g}$ is a semisimple Lie algebra over a field $k$ of characteristic zero.

**Proposition 2.7.** *Let $M$ be a finite dimensional right $\mathfrak{g}$-module and $A$ be a finite dimensional Leibniz $\mathfrak{g}$-module. Then*

$$HL^n(\mathfrak{g}, M) \;=\; 0 \;\text{ for }\; n > 0, \;\text{ and }\; HL^n(\mathfrak{g}, A) \;=\; 0 \;\text{ for }\; n > 1.$$

*Proof.* It is well-known that $H^n(\mathfrak{g}, M) = 0$ for all $n \geq 0$ if $M^{\mathfrak{g}} = 0$. Therefore by Corollary 2.4, $HL^n(\mathfrak{g}, M) = 0$ for all $n \geq 0$ if $M^{\mathfrak{g}} = 0$. One can thus restrict to trivial submodules of $M$ for a general finite dimensional $\mathfrak{g}$-module $M$. We obtain

$$HL^n(\mathfrak{g}, M) \;=\; HL^n(\mathfrak{g}, k) \otimes M^{\mathfrak{g}}.$$

It is well-known that $HL^n(\mathfrak{g}, k) = 0$ for $n \geq 1$, and this shows the first statement.

For the second statement, one proceeds as in [11]. □

## 2.3 Leibniz cohomology of semisimple Leibniz algebras

In this subsection, we specialize to the adjoint representation. Let $f : \mathfrak{h} \to \mathfrak{h}_{\mathrm{Lie}}$ be the quotient morphism which sends a Leibniz algebras $\mathfrak{h}$ onto its quotient by the ideal of squares, namely the Lie algebra $\mathfrak{h}_{\mathrm{Lie}}$. We want to construct a homomorphism of cochain complexes

$$f^* : CL^*(\mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \to CL^*(\mathfrak{h}, \mathfrak{h}).$$

For this, we need to interprete the $\mathfrak{h}_{\mathrm{Lie}}$-Lie module $\mathfrak{h}$ as a Leibniz module over $\mathfrak{h}_{\mathrm{Lie}}$. It is easy to see that the restriction map $f^*$ which sends a cochain from $CL^*(\mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h})$ to the precomposition with the quotient morphism $f$ is a map of complexes in case the $\mathfrak{h}_{\mathrm{Lie}}$ is regarded as a symmetric (Leibniz) $\mathfrak{h}_{\mathrm{Lie}}$-module.

We then have a relative complex

$$CL^*(\mathfrak{h}; \mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \;:=\; \mathrm{coker}(f^* : CL^*(\mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \to CL^*(\mathfrak{h}, \mathfrak{h})).$$

**Proposition 2.8.** *The map $f$ induces a long exact sequence*

$$0 \to HL^1(\mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \to HL^1(\mathfrak{h}, \mathfrak{h}) \to HL^1(\mathfrak{h}; \mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \to HL^2(\mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) \to \dots$$

Recall the short exact sequence of Leibniz algebras induced by the above quotient morphism $f$:

$$0 \to \mathfrak{h}^{\mathrm{ann}} \to \mathfrak{h} \xrightarrow{f} \mathfrak{h}_{\mathrm{Lie}} \to 0.$$

**Theorem 2.9.** *There exists a spectral sequence*

$$E_2^{p,q} \;=\; HL^p(\mathfrak{h}, \mathfrak{h}^{\mathrm{ann}}) \otimes HL^q(\mathfrak{h}, \mathfrak{h}) \;\;\Rightarrow\;\; HL^{p+q}(\mathfrak{h}; \mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}),$$

*where $\mathfrak{h}_{\mathrm{Lie}}$ acts on $\mathfrak{h}^{\mathrm{ann}}$ by the adjoint action, lifting elements of $\mathfrak{h}_{\mathrm{Lie}}$ to $\mathfrak{h}$ using $f$.*

Observe that the filtration for the above spectral sequence is obtained as the *dual* of the filtration that Pirashvili uses.

**Proposition 2.10.** *Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra with $\mathfrak{h}_{\mathrm{Lie}} =: \mathfrak{g}$ semisimple over a field $k$ of characteristic zero. Assume that $(\mathfrak{h}^{\mathrm{ann}})^{\mathfrak{g}} = 0$. Then*

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\mathfrak{g}, \mathfrak{g}).$$

*Proof.* Observe that because $\mathfrak{g}$ is semisimple and thanks to the assumption, we have $HL^*(\mathfrak{g}, \mathfrak{h}^{\mathrm{ann}}) = 0$ for all $* \geq 0$. By Theorem 2.9 with respect to the short exact sequence of Leibniz algebras

$$0 \to \mathfrak{h}^{\mathrm{ann}} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

and the vanishing of $HL^*(\mathfrak{g}, \mathfrak{h}^{\mathrm{ann}})$, we have $HL^*(\mathfrak{h}; \mathfrak{h}_{\mathrm{Lie}}, \mathfrak{h}) = 0$. Therefore by Proposition 2.8, Leibniz cohomology of $\mathfrak{g}$ and $\mathfrak{h}$ with values in $\mathfrak{h}$ are isomorphic:

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\mathfrak{g}, \mathfrak{h}).$$

Using once again that $HL^*(\mathfrak{g}, \mathfrak{h}^{\mathrm{ann}}) = 0$, we deduce from the long exact sequence in Leibniz cohomology induced by the short exact sequence of coefficients for the Lie algebra $\mathfrak{g}$

$$0 \to \mathfrak{h}^{\mathrm{ann}} \to \mathfrak{h} \to \mathfrak{g} \to 0,$$

that

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\mathfrak{g}, \mathfrak{g}).$$

□

**Corollary 2.11.** *Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra with $\mathfrak{h}_{\mathrm{Lie}} =: \mathfrak{g}$ semisimple over a field $k$ of characteristic zero. Assume that $(\mathfrak{h}^{\mathrm{ann}})^{\mathfrak{g}} = 0$. Then*

$$HL^n(\mathfrak{h}, \mathfrak{h}) \;=\; 0 \;\; \forall n > 1.$$

*Proof.* This follows from the proposition using in addition Proposition 2.7 - note that the $\mathfrak{g}$-module $\mathfrak{g}$ in $HL^*(\mathfrak{g}, \mathfrak{g})$ is the symmetric Leibniz module $\mathfrak{g}$. □

**Remark 2.12.** (a) This corollary generalizes some of the results of [1] thoroughly. In particular, these so-called *semisimple Leibniz algebras* (i.e. finite-dimensional Leibniz algebras whose quotient Lie algebra is semisimple) $\mathfrak{h}$ are thus Leibniz-rigid.

(b) Proposition 2.10 also holds for a Leibniz algebra $\mathfrak{h}$ whose quotient Lie algebra $\mathfrak{g}$ is nilpotent in case $(\mathfrak{h}^{\mathrm{ann}})^{\mathfrak{g}} = 0$. Indeed, this is the hypothesis we need in order to apply Théorème 1 of [3].

We will need a slight extension of the results for semisimple Leibniz algebras for the cohomological assertions of the last section. Namely, let us consider a semidirect product Lie algebra $\widehat{\mathfrak{g}} = M \rtimes \mathfrak{g}$ where $\mathfrak{g}$ is a semisimple Lie algebra (over $\mathbb{C}$) and $M$ is a non-trivial finite-dimensional irreducible $\mathfrak{g}$-module. From $\widehat{\mathfrak{g}}$, we construct a Leibniz algebra $\mathfrak{h}$ which is the hemisemidirect product $\mathfrak{h} := I \dot{+} \widehat{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$ with ideal of squares $I$ which is another (not necessarily different) non-trivial finite-dimensional irreducible $\mathfrak{g}$-module.

**Proposition 2.13.** *For $\mathfrak{g}$ a semisimple Lie algebra with non-trivial finite-dimensional irreducible $\mathfrak{g}$-modules $M$ and $I$, the Leibniz algebra $\mathfrak{h} = I \dot{+} (M \rtimes \mathfrak{g})$ satisfies*

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}})$$

*where $\widehat{\mathfrak{g}} = M \rtimes \mathfrak{g}$.*

*Proof.* Let us first compute $HL^*(\widehat{\mathfrak{g}}, I)$. The Lie algebra cohomology $H^*(\widehat{\mathfrak{g}}, I)$ can be computed from the Hochschild-Serre spectral sequence associated to the short exact sequence of Lie algebras

$$0 \to M \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

This spectral sequence has the $E_1$-term

$$E_1^{p,q} \;=\; H^q(M, H^p(\mathfrak{g}, I)).$$

But $H^p(\mathfrak{g}, I) = 0$, because $\mathfrak{g}$ is semisimple and $I$ a non-trivial finite-dimensional $\mathfrak{g}$-module. We conclude that $H^n(\widehat{\mathfrak{g}}, I) = 0$ for all $n \geq 0$. By Corollary 2.4, this implies that $HL^n(\widehat{\mathfrak{g}}, I) = 0$ for all $n \geq 0$.

Now consider the exact sequence of Leibniz algebras

$$0 \to I \to \mathfrak{h} \to \widehat{\mathfrak{g}} \to 0. \tag{2}$$

It satisfies the condition of Theorem 2.9, because it is an abelian extension of Leibniz algebras. As by the above the $E_2$-term is zero, the theorem implies thus that the relative cohomology is zero:

$$HL^n(\mathfrak{h}; \widehat{\mathfrak{g}}, \mathfrak{h}) = 0$$

for all $n \geq 0$. Now the long exact sequence in Proposition 2.8 implies that

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\widehat{\mathfrak{g}}, \mathfrak{h}).$$

Therefore, using the short exact sequence (2) as a coefficient sequence, we obtain with $HL^*(\widehat{\mathfrak{g}}, I) = 0$ that

$$HL^*(\mathfrak{h}, \mathfrak{h}) \;=\; HL^*(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}})$$

as claimed. □

## 2.4 Vanishing of some relative cohomology groups

In this subsection, we consider a type of relative cohomology which will be useful in the construction of a rigid Leibniz algebra $\mathfrak{h}$ whose $HL^2(\mathfrak{h}, \mathfrak{h})$ is not zero.

Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra and $\mathfrak{s}$ be a subalgebra. Let $\mathfrak{h}^n$ be the $n$-fold Cartesian product and denote by $F_n$ the subset of $\mathfrak{h}^n$ consisting of all $(x_1, \ldots, x_n)$ satisfying the following condition: There exists at most one index $i$ such that $x_i \notin \mathfrak{s}$. Let $M$ be a Leibniz $\mathfrak{h}$-module. We define $F^n(\mathfrak{h}; \mathfrak{s}, M)$ to be the vector space of all multilinear maps $\phi : F_n \to M$. Concretely for $n = 2$, $\phi \in F^2(\mathfrak{h}; \mathfrak{s}, M)$ means that $\phi$ is defined on $(\mathfrak{h} \times \mathfrak{s}) \cup (\mathfrak{s} \times \mathfrak{h}) \cup (\mathfrak{s} \times \mathfrak{s})$. We set

$$F(\mathfrak{h}; \mathfrak{s}, M) \;=\; \bigoplus_{n \geq 0} F^n(\mathfrak{h}; \mathfrak{s}, M).$$

The coboundary operator $d$ for Leibniz cohomology is defined in the usual way. One checks that it is well-defined on $F(\mathfrak{h};\mathfrak{s},M)$. We denote by

$$E(\mathfrak{h};\mathfrak{s},M) \;=\; \bigoplus_{n\geq 0} E^n(\mathfrak{h};\mathfrak{s},M)$$

the cohomology of this complex. We will need this cohomology for adjoint coefficients, i.e. with $M=\mathfrak{h}$ and thus $E(\mathfrak{h};\mathfrak{s},\mathfrak{h})$.

Let us first analyze cocycles $\phi\in F^2(\mathfrak{h};\mathfrak{s},\mathfrak{h})$. For this, let $W$ be a supplementary subspace of $\mathfrak{s}$ in $\mathfrak{h}$. For our main application, we will have $W=M\oplus I$. The condition $d\phi=0$ means explicitly for all $x,y,z\in\mathfrak{h}$

$$[x,\phi(y,z)]+[\phi(x,z),y]-[\phi(x,y),z]-\phi([x,y],z)+\phi([x,z],y)+\phi(x,[y,z]) \;=\; 0.$$

Now $\phi$ splits into three components $f_1:\mathfrak{s}\otimes W\to\mathfrak{h}$, $f_2:W\otimes\mathfrak{s}\to\mathfrak{h}$ and $f_3:\mathfrak{s}\otimes\mathfrak{s}\to\mathfrak{h}$. In terms of these components, the cocycle condition reads

$$\begin{aligned}
\forall x,y\in\mathfrak{s},\forall z\in W:\;\; & [x,f_1(y,z)]+[f_1(x,z),y]-[f_3(x,y),z]-\\
& f_1([x,y],z)+f_2([x,z],y)+f_1(x,[y,z])=0, \qquad (3)\\
\forall x,z\in\mathfrak{s},\forall y\in W:\;\; & [x,f_2(y,z)]+[f_3(x,z),y]-[f_1(x,y),z]-\\
& f_2([x,y],z)+f_1([x,z],y)+f_1(x,[y,z])=0, \qquad (4)\\
\forall y,z\in\mathfrak{s},\forall x\in W:\;\; & [x,f_3(y,z)]+[f_2(x,z),y]-[f_2(x,y),z]-\\
& f_2([x,y],z)+f_2([x,z],y)+f_2(x,[y,z])=0.
\end{aligned}$$

Note that these conditions are very close to the conditions which we had for the twisting functions in a Leibniz $\mathfrak{h}$-extension.

Similar conditions hold for two components in $W$ and one in $\mathfrak{s}$, but these simplify, because we now impose that $[W,W]=0$ and $[\mathfrak{s},W]\subset W$ and $[W,\mathfrak{s}]\subset W$. Recall furthermore that by definition of $F(\mathfrak{h};\mathfrak{s},M)$, we have $f|_{W\otimes W}=0$. The conditions read then:

$$\begin{aligned}
\forall x,y\in W,\forall z\in\mathfrak{s}:\;\; [x,f_2(y,z)]+[f_2(x,z),y] \;&=\; 0\\
\forall x,z\in W,\forall y\in\mathfrak{s}:\;\; [x,f_1(y,z)]-[f_2(x,y),z] \;&=\; 0 \qquad (5)\\
\forall y,z\in W,\forall x\in\mathfrak{s}:\;\; [f_1(x,z),y] \;&=\; 0
\end{aligned}$$

We will use these identities in the proof of the following proposition in order to express a cocycle $\phi\in F^2(\mathfrak{h};\mathfrak{s},M)$ in terms of cocycles in ordinary Leibniz cohomology and extensions.

**Proposition 2.14.** *Let $\mathfrak{h}$ be a Leibniz algebra with subalgebra $\mathfrak{s}$. Let $M$ and $I$ be supplementary subspaces such that $\mathfrak{h}=\mathfrak{s}\oplus M\oplus I$ as a vector space. We will regard here $M$ and $I$ as quotient $\mathfrak{s}$-modules. Suppose furthermore*

*(a) $[M,M]=0$, $[\mathfrak{s},M]\subset M$, $[M,\mathfrak{s}]\subset M$, and the bracket of $\mathfrak{s}$ and $M$ is symmetric:*

$$\forall s\in\mathfrak{s},\forall w\in M:\;\; [s,w] \;=\; -[w,s],$$

*furthermore $[\mathfrak{s},I]=0$, $[I,\mathfrak{s}]\subset I$ and $I=Z_{\mathrm{right}}(\mathfrak{h})$,*

*(b)* $HL^2(\mathfrak{s},\mathfrak{h}) = 0$,

*(c)* $HL^1(\mathfrak{s}, \mathrm{Hom}(M,M)) = 0$,

*(d) all extensions of Leibniz* $\mathfrak{s}$*-modules*

$$0 \to \mathfrak{s} \oplus I \to E \to M \oplus I \to 0 \quad \text{and} \quad 0 \to M \to E' \to I \to 0$$

*split.*

*Then we have* $E^2(\mathfrak{h};\mathfrak{s},\mathfrak{h}) = 0$.

*Proof.* Let $\phi \in F^2(\mathfrak{h};\mathfrak{s},\mathfrak{h})$ with $d\phi = 0$. First of all, it follows from condition *(a)* that the three components $f_1$, $f_2$ and $f_3$ of $\phi$ satisfy the above equations. Observe that as $\phi : \mathfrak{h}^{\otimes 2} \to \mathfrak{h}$ is a cocycle, its restriction $f_3$ to $\mathfrak{s}\times\mathfrak{s}$ is a 2-cocycle on $\mathfrak{s}$ with values in $\mathfrak{h}$. It follows then from *(b)* that there exists $\psi \in F^1(\mathfrak{h};\mathfrak{s},\mathfrak{h})$ such that $\phi_1 = \phi - d\psi$ vanishes on $\mathfrak{s}\times\mathfrak{s}$. That means that for the three components of $\phi_1$ (for which we will not introduce new notations!), we can erase $f_3$ from the equations.

Let now $\pi_M : \mathfrak{h} \to M$ denote the projection of $\mathfrak{h}$ onto $M$ along $\mathfrak{s} \oplus I$. The equation for $f_2$ reads then for all $y, z \in \mathfrak{s}$ and all $x \in M$:

$$[f_2(x,z),y] - [f_2(x,y),z] - f_2([x,y],z) + f_2([x,z],y) + f_2(x,[y,z]) \;=\; 0.$$

We then use the antisymmetry of the bracket between $\mathfrak{s}$ and $M$ from condition *(a)* to write the cocycle condition for $\alpha := \pi_M \circ f_2$ for all $y, z \in \mathfrak{s}$ as

$$\alpha(-,[y,z]) \;=\; [\alpha(-,y),z] - \alpha([-,z],y) + [y,\alpha(-,z)] - \alpha([y,-],z).$$

This is the cocycle identity for 1-cocycles with values in $\mathrm{Hom}(M,M)$. By condition *(c)*, there exists $\psi_1 \in F^1(\mathfrak{h};\mathfrak{s},\mathfrak{h})$ such that $\phi_2 = \phi_1 - d\psi_1$ satisfies the following conditions: $\phi_2$ vanishes on $\mathfrak{s}\times\mathfrak{s}$ and $\pi_M \circ \phi_2$ vanishes on $M \times \mathfrak{s}$.

Now $\beta := \pi_M \circ f_1$ (where $f_1$ is the corresponding component of $\phi_2$) satisfies on the one hand by equation (5) for all $x, z \in M$ and all $y \in \mathfrak{s}$ $[x, \beta(y,z)] = 0$ and on the other hand combining equations (3) and (4) that for all $x, y \in \mathfrak{s}$ we have $[x, \beta(y,-)] = 0$. That means that $\beta$ has its values in $Z_{\mathrm{right}}(\mathfrak{h})$, but then by condition *(a)* and *(d)*, we conclude that there exists $\psi_2 \in F^1(\mathfrak{h};\mathfrak{s},\mathfrak{h})$ such that $\phi_3 = \phi_2 - d\psi_2$ satisfies the following conditions: $\phi_3$ vanishes on $\mathfrak{s}\times\mathfrak{s}$ and $\pi_M \circ \phi_3$ vanishes on $(M \oplus I) \times \mathfrak{s}$ and $\mathfrak{s} \times (M \oplus I)$.

It remains to investigate $\pi_{\mathfrak{s}\oplus I} \circ \phi_3$. Comparing the set of equations for $\pi_{\mathfrak{s}\oplus I} \circ \phi_3$ (i.e. for its two components $\pi_{\mathfrak{s}\oplus I} \circ f_1$ and $\pi_{\mathfrak{s}\oplus I} \circ f_2$) to the set of equations of an extension of Leibniz $\mathfrak{s}$-modules, we see that condition *(e)* means exactly that there exists $\psi_3 \in F^1(\mathfrak{h};\mathfrak{s},\mathfrak{h})$ such that $\phi_4 = \phi_3 - d\psi_3$ vanishes on $\mathfrak{s}\times\mathfrak{s}$, $\pi_M \circ \phi_4$ vanishes on $((M \oplus I) \times \mathfrak{s}) \cup (\mathfrak{s} \times (M \oplus I))$ and $\pi_{\mathfrak{s}} \circ \phi_4$ vanishes on $((M \oplus I) \times \mathfrak{s}) \cup (\mathfrak{s} \times (M \oplus I))$. This means that $\phi = d\psi + d\psi_1 + d\psi_2 + d\psi_3$, and we are done. $\square$

Let us now apply this proposition to a very special Leibniz algebra. Consider again a semidirect product Lie algebra $\widehat{\mathfrak{g}} = M \rtimes \mathfrak{g}$ where $\mathfrak{g}$ is a simple Lie algebra

(over $\mathbb{C}$) and $M$ is a non-trivial finite-dimensional irreducible $\mathfrak{g}$-module. From $\widehat{\mathfrak{g}}$, we construct a Leibniz algebra $\mathfrak{h}$ which is the hemisemidirect product $\mathfrak{h} := I \dot{+} \widehat{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$ with ideal of squares $I$ which is another non-trivial finite-dimensional irreducible $\mathfrak{g}$-module.

As subalgebra $\mathfrak{s}$, we take the simple Lie algebra $\mathfrak{g}$. The supplementary subspace is thus $M \oplus I$. Thus the pair $(\mathfrak{h}, \mathfrak{s})$ satisfies conditions *(a)* to *(c)* of the proposition. For condition *(d)*, we will have to restrict the irreducible $\mathfrak{g}$-modules $M$ and $I$ of the construction.

In the following, we will heavily draw on [8] in order to describe the set of Leibniz extensions $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I, M)$ and $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I \oplus M, I \oplus \mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is now supposed to be simple in order to stick to the general hypotheses of [8].

**Proposition 2.15.** *Assume that* $\dim(M) > \dim(I) \cdot \dim(\mathfrak{g})$ *and that the* $\mathfrak{g}$-*module* $I$ *is different from* $\mathfrak{g}$. *Then*

$$\mathrm{Ext}^1_{UL(\mathfrak{h})}(M \oplus I, I \oplus \mathfrak{g}) = 0 \quad \text{and} \quad \mathrm{Ext}^1_{UL(\mathfrak{h})}(I, M) = 0.$$

*Proof.* The (Leibniz) $\mathfrak{g}$-module $I \oplus \mathfrak{g}$ decomposes into the direct sum of the antisymmetric module $I$ and a symmetric module $\mathfrak{g}$. Similarly, the (Leibniz) $\mathfrak{g}$-module $I \oplus M$ decomposes into the direct sum of the antisymmetric module $I$ and a symmetric module $M$. It is thus sufficient for the first claim to compute separately $\mathrm{Ext}^1_{UL(\mathfrak{h})}(M, I)$, $\mathrm{Ext}^1_{UL(\mathfrak{h})}(M, \mathfrak{g})$, $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I, I)$ and $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I, \mathfrak{g})$.

Loday and Pirashvili show in Proposition 2.2 that there are spectral sequences

$$E_2^{p,q} = H^p(\mathfrak{g}, \mathrm{Hom}(Z, \mathrm{Ext}^q_{UL(\mathfrak{h})}(U(\mathfrak{g})^s, X))) \;\Rightarrow\; \mathrm{Ext}^{p+q}_{UL(\mathfrak{h})}(Z^s, X),$$

and

$$E_2^{p,q} = H^p(\mathfrak{g}, \mathrm{Hom}(Y, HL^q(\mathfrak{g}, X)) =) \;\Rightarrow\; \mathrm{Ext}^{p+q}_{UL(\mathfrak{h})}(Y^a, X).$$

Furthermore, they show in Proposition 2.3 that for a Leibniz $\mathfrak{g}$-module $X$, $\mathrm{Ext}^q_{UL(\mathfrak{h})}(U(\mathfrak{g})^s, X)$ for $q = 0, 1$ are the kernel and cokernel respectively of the map $f : X \to \mathrm{Hom}(\mathfrak{g}, HL^0(\mathfrak{g}, X))$ given by

$$f(x)(g) \;=\; [x, g] + [g, x], \quad \forall x \in X \;\; \forall g \in \mathfrak{g}.$$

We apply these propositions with $Z = M$ and $X = I$ or $X = \mathfrak{g}$. The space $H^1(\mathfrak{g}, \mathrm{Hom}(M, \ker(f)))$ is zero for $X = I$ and $X = \mathfrak{g}$, because $\mathfrak{g}$ is simple and $\mathrm{Hom}(M, \ker(f))$ is of finite dimension.

The space $H^0(\mathfrak{g}, \mathrm{Hom}(M, \mathrm{coker}(f)))$ is zero for $X = \mathfrak{g}$, because $HL^0(\mathfrak{g}, \mathfrak{g})$ are the left invariants and $\mathfrak{g}$ is simple (and a symmetric $\mathfrak{g}$-module).

On the other hand, for the antisymmetric $\mathfrak{g}$-module $I$, one has $HL^0(\mathfrak{g}, I) = I$. In order to have $H^0(\mathfrak{g}, \mathrm{Hom}(M, \mathrm{coker}(f))) = 0$, we need to have that the $\mathfrak{g}$-invariants of $\mathrm{Hom}(M, \mathrm{coker}(f))$, i.e. the space of $\mathfrak{g}$-morphisms from $M$ to $\mathrm{coker}(f)$, is zero. As the $\mathfrak{g}$-module $M$ is irreducible, this is the case if and only if $\mathrm{coker}(f)$ does not admit a direct factor $M$ in its decomposition into irreducibles. We can arrange this (in a rather brutal manner) by imposing that $\dim(M) > \dim(I) \cdot \dim(\mathfrak{g})$.

For the first claim, it remains to show that $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I,I) = 0$ and that $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I,\mathfrak{g}) = 0$. By the spectral sequence argument, we need to show $H^1(\mathfrak{g}, \mathrm{Hom}(I, HL^0(\mathfrak{g},X))) = 0$ and $H^0(\mathfrak{g}, \mathrm{Hom}(I, HL^1(\mathfrak{g},X))) = 0$. The first cohomology space is zero as Lie cohomology of a simple Lie algebra with values in a finite dimensional $\mathfrak{g}$-module. The second cohomology space is zero for $X = \mathfrak{g}$, because $HL^1(\mathfrak{g},\mathfrak{g}) = H^1(\mathfrak{g},\mathfrak{g})$ is zero. It remains to discuss the space $HL^1(\mathfrak{g}, I)$. The space of 1-cocycles are the maps of right $\mathfrak{g}$-modules between $\mathfrak{g}$ and $I$. By our hypotheses (i.e. because $I \neq \mathfrak{g}$ and both are simple $\mathfrak{g}$-modules), this space is zero.

For the second claim, we compute $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I,M)$ using the same methods. Here, we use that $HL^q(\mathfrak{g},M) = 0$, $q = 0,1$, for the non-trivial irreducible symetric $\mathfrak{g}$-module $M$. This shows that $\mathrm{Ext}^1_{UL(\mathfrak{h})}(I,M) = 0$. □

# 3 A stability theorem

In this section, we follow closely [10] in order to show a stability theorem for Leibniz subalgebras of a given Leibniz algebra. We work over the complex numbers $\mathbb{C}$.

Let $V$ be a finite dimensional complex vector space. Let $M$ be the algebraic variety of all Leibniz algebra structures on $V$, defined in $\mathrm{Hom}(V^{\otimes 2}, V)$ by the quadratic equations corresponding to the right Leibniz identity, i.e. for $\mu \in \mathrm{Hom}(V^{\otimes 2}, V)$, we require

$$\mu \circ \mu(x,y,z) \;:=\; \mu(\mu(x,y),z) - \mu(x,\mu(y,z)) - \mu(\mu(x,z),y) \;=\; 0$$

for all $x, y, z \in V$.

Two Leibniz algebra structures $\mu$ and $\mu'$ give rise to isomorphic Leibniz algebras $(V,\mu)$ and $(V,\mu')$ in case there exists $g \in \mathrm{Gl}(V)$ such that $g \cdot \mu = \mu'$, where the action of $\mathrm{Gl}(V)$ on $M$ is defined by

$$g \cdot \mu(x,y) \;:=\; g(\mu(g^{-1}(x), g^{-1}(y)))$$

for all $x, y \in V$.

Let $\mathfrak{h} = (V,\mu)$ be a Leibniz algebra on $V$, and consider a Leibniz subalgebra $\mathfrak{s}$ of $\mathfrak{h}$. We will denote the complex subspace of $V$ corresponding to $\mathfrak{s}$ by $S$. Let $W$ be a supplementary subspace of $S$ in $V$. For an element $\phi \in CL^2(\mathfrak{h}, V)$, we denote by $r(\phi)$ the restriction to the union of $S \otimes S$, $S \otimes W$ and $W \otimes S$. Let

$$N_1 \;:=\; \{(g,m) \in \mathrm{Gl}(V) \times M \,|\, r(g \cdot m) = r(m)\},$$

denote by $\mathrm{proj}_M : \mathrm{Gl}(V) \times M \to M$ the projection map and let $p_1 : N_1 \to M$ be the restriction of $\mathrm{proj}_M$ to $N_1$.

**Definition 3.1.** *The subalgebra $\mathfrak{s}$ is called a stable subalgebra of $\mathfrak{h}$ if $p_1$ maps every neighborhood of $(1,\mu) \in N_1$ onto a neighborhood of $\mu$ in $M$.*

**Remark 3.2.** (a) This is not the original definition of stable subalgebra in [12], but the strong version of stability which permits Page and Richardson to show the strengthened stability theorem at the end of their paper.

(b) The definition implies that if $\mathfrak{h}_1 = (V, \mu_1)$ is a Leibniz algebra sufficiently near to $\mathfrak{h}$, then $\mathfrak{h}_1$ is isomorphic to a Leibniz algebra $\mathfrak{h}_2 = (V, \mu_2)$ with the following property: for all $s \in \mathfrak{s}$ and all $x \in V$, we have $\mu_2(x, s) = \mu(x, s)$ and $\mu_2(s, x) = \mu(s, x)$.

The stability theorem below asserts that under certain cohomological conditions a given subalgebra $\mathfrak{s}$ is stable. The proof is based on the inverse and implicit function theorem, in the following algebraic geometric form. All algebraic varieties are considered to be complex and equipped with the Zariski topology. Recall that a point $x$ of an algebraic variety $X$ is called *simple* in case $\dim(X) = \dim(T_x X)$.

**Theorem 3.3.** *Let $f : X \to Y$ be a morphism of algebraic varieties, let $x \in X$ and $y = f(x) \in Y$. Suppose that $x$ (resp. $y$) is a simple point of $X$ (resp. $Y$) and that the differential $d_x f : T_x X \to T_y Y$ is surjective.*

*(a) If $U$ is a neighborhood of $x$ in $X$, then $f(U)$ is a neighborhood of $y$ in $Y$.*

*(b) There is exactly one irreducible component $X_1$ of $f^{-1}(y)$ which contains $x$.*

*Moreover, $x$ is a simple point of $X_1$ and $T_x X_1 = \ker(d_x f)$.*

*Proof.* This is Proposition 8.1 in [10]. □

**Theorem 3.4** (Stability Theorem)**.** *Let $\mathfrak{h} = (V, \mu)$ be a finite-dimensional complex Leibniz algebra, and let $\mathfrak{s}$ be a subalgebra of $\mathfrak{h}$ such that $E^2(\mathfrak{h}, \mathfrak{s}, V) = 0$. Then $\mathfrak{s}$ is a stable subalgebra of $\mathfrak{h}$.*

*Proof.* The proof procedes as in Section 11 of [10] by three distinct applications of the inverse function theorem Theorem 3.3.
**(1)** For $\phi \in \mathrm{Hom}(V^{\otimes 2}, V)$, let $\tau(\phi)$ denote the restriction of $\phi$ to $F_2$. Let

$$Z^2 := \{\theta \in F^2(\mathfrak{h}, \mathfrak{s}, V) \,|\, d\theta = 0\}$$

and let $C := \{\phi \in \mathrm{Hom}(V^{\otimes 2}, V) \,|\, \tau(\phi) \in Z^2\}$. Then $ZL^2(\mathfrak{h}, V) \subset C$. Denote by $C_0 := d(C)$ and let $C_1$ be a supplementary to $C_0$ in $B^3(L, V)$ and $C_2$ be supplementary to $B^3(L, V)$ in $\mathrm{Hom}(V^{\otimes 3}, V)$. Let $\pi_1 : \mathrm{Hom}(V^{\otimes 2}, V) \to C_1$ be the projection with kernel $C_0 \oplus C_2$. Put

$$M_1 := \{\phi \in \mathrm{Hom}(V^{\otimes 2}, V) \,|\, \pi_1(\phi \circ \phi) = 0\}.$$

Then $M \subset M_1$, because for elements $\mu \in M$, we have $\mu \circ \mu = 0$.

Let $f : \mathrm{Hom}(V^{\otimes 2}, V) \to \mathrm{Hom}(V^{\otimes 3}, V)$ be defined by $f(\phi) = \phi \circ \phi$. $f$ is a polynomial mapping. As in the proof of 6.2 in [10], we show that $d_\mu f = -d$, the Leibniz coboundary operator. This is true since the Massey product $\circ$ and

the Leibniz coboundary operator share the same relations as for Lie algebras. Let $F : \mathrm{Hom}(V^{\otimes 2}, V) \to C_1$ be the composition $\pi_1 \circ f$. Since $\pi_1$ is linear, $d_\mu F = -\pi_1 \circ d$. Hence $d_\mu F$ is surjective, because $C_1$ consists of coboundaries ($C_1$ are exactly the coboundaries which do not come from $C$!), and $\ker(d_\mu F) = C$. By Theorem 3.3, there is exactly one irreducible component $M'$ of $M_1$ which contains $\mu$. Moreover, $\mu$ is a simple point of $M'$ and $T_\mu M' = C$.
**(2)** Let $E$ be a supplementary subspace of $C$ in $\mathrm{Hom}(V^{\otimes 2}, V)$ and let $E' = \tau(E)$. We have a direct sum decomposition $F^2(\mathfrak{h}, \mathfrak{s}, V) = Z^2 \oplus E'$ in cocycles $Z^2$ and the rest $E'$. Let $\pi_Z : F^2(\mathfrak{h}, \mathfrak{s}, V) \to Z^2$ be the projection with kernel $E'$. Let $\gamma : \mathrm{Gl}(V) \to \mathrm{Hom}(V^{\otimes 2}, V)$ be the map $\gamma(g) = g \cdot \mu$. Taking $g = 1 + t\psi + \mathcal{O}(t^2)$, one obtains

$$d_1\gamma(\psi)(x, y) = \frac{d}{dt}\ g(\mu(g^{-1}(x), g^{-1}(y)))\big|_{t=0} = \psi([x, y]) - [\psi(x), y] - [x, \psi(y)],$$

for all $x, y \in V$ (with $\mu(x, y) = [x, y]$). Thus $d_1\gamma(\psi) = -d\psi$.

Let $\eta : \mathrm{Gl}(V) \times M' \to \mathrm{Hom}(V^{\otimes 2}, V)$ be defined by $\eta(g, m) = g \cdot m$ and let $\beta : \mathrm{Gl}(V) \times M' \to Z^2$ be the composition $\pi_Z \circ \tau \circ \eta$. Using the above computation for $d_1\gamma$, one obtains for $d_{(1,\mu)}\eta$

$$d_{(1,\mu)}\eta(\psi, \phi) \ = \ -d\psi + \phi.$$

Therefore $d_{(1,\mu)}\beta(\psi, \phi) \ = \ -d\tau(\psi) + \pi_Z(\tau(\phi))$. As $T_\mu M' = C$ and $\tau(C) = Z^2$, it follows that $d_{(1,\mu)}\beta$ is surjective and we can again apply Theorem 3.3: There exists exactly one irreducible component $N'$ of $\beta^{-1}(\tau(\mu))$ containing $(1, \mu)$ and $(1, \mu)$ is a simple point of $N'$ with $T_{(1,\mu)}N' = \ker(d\beta_{(1,\mu)})$.
**(3)** Let $\pi : N' \to M'$ denote the restriction of the projection

$$\mathrm{proj}_{M'} : \mathrm{Gl}(V) \times M' \to M';$$

We claim that if $E^2(\mathfrak{h}, \mathfrak{s}, V) = 0$, the differential $d_{(1,\mu)}\pi : T_{(1,\mu)}N' \to T_\mu M'$ is surjective. Indeed, let $\phi \in T_\mu M'$. By definition of $C$, $\tau(\phi) \in Z^2 = B^2(\mathfrak{h}, \mathfrak{s}, V)$ by assumption. Thus there exists $\theta \in \mathrm{Hom}(S, V)$ such that $d\theta = \tau(\phi)$. Since $\tau$ is surjective, there exists $\psi \in \mathrm{Hom}(V, V)$ (extend by zero for example!) such that $\tau(\psi) = \theta$; therefore $d\tau(\psi) = \tau(\phi) = \pi_Z(\tau(\phi))$. Consequently

$$d_{(1,\mu)}\beta(\psi, \phi) \ = \ -d\tau(\psi) + \pi_Z(\tau(\phi)) \ = \ 0.$$

It follows that $(\psi, \phi) \in T_{(1,\mu)}N' = \ker(d_{(1,\mu)}\beta)$. Since $d_{(1,\mu)}\pi(\psi, \phi) = \phi$, this shows that $d_{(1,\mu)}\pi$ is surjective. This will be used later on for the third application of the inverse function theorem.
**(4)** We claim now that there exists a neighborhood $U_3$ of $(1, \mu)$ in $\mathrm{Gl}(V) \times M$ such that if $(g, m) \in U_3$ and if $\pi_Z(\tau(g \cdot m)) = \tau(\mu)$, then $\tau(g \cdot m) = \tau(\mu)$. Indeed, let $(g, m) \in \mathrm{Gl}(V) \times M$ with $\pi_Z(\tau(g \cdot m)) = \tau(\mu)$. Then $g \cdot m = \mu + a + b$ with $a \in \ker(\tau)$ and $b \in E$ (because of the direct sum decomposition $\mathrm{Hom}(V^{\otimes 2}, V) = C \oplus E$ which maps under $\tau$ to $Z^2 \oplus E'$, and the $Z^2$-component is fixed to be $\tau(\mu)$). The restriction of $d \circ \tau$ to $E$ is a monomorphism (by definition of $C$ and

$E$). As $g \cdot m$ is a Leibniz law, we have

$$\begin{aligned} 0 &= (g \cdot m) \circ (g \cdot m) = (\mu + a + b) \circ (\mu + a + b) \\ &= -da - db + a \circ a + b \circ b + a \circ b + b \circ a. \end{aligned}$$

As $a \in \ker(\tau)$, $\tau(a \circ a) = \tau(da) = 0$. Applying $\tau$ to the preceeding equation, we obtain:

$$0 = -d\tau(b) + \tau((a+b) \circ b) + \tau(b \circ a).$$

Let $\pi_C : \mathrm{Hom}(V^{\otimes 2}, V) \to C$ be the projection with kernel $E$. We note that $\pi_C(a+b) = a$. Thus the above equation can be rewritten as:

$$0 = -d\tau(b) + \tau((a+b) \circ b) + \tau(b \circ \pi_C(a+b)).$$

For any $x \in \mathrm{Hom}(V^{\otimes 2}, V)$, let $\lambda_x : E \to \mathrm{Hom}(V^{\otimes 3}, V)$ be the linear map defined by

$$\lambda_x(\phi) := -d\tau(\phi) + \tau(x \circ \phi) + \tau(\phi \circ \pi_C(x)).$$

Then $\lambda_0 = d \circ \tau$ is a monomorphism. Thus there exists a neighborhood $J$ of 0 in $\mathrm{Hom}(V^{\otimes 2}, V)$ such that $\lambda_x$ is a monomorphism for every $x \in J$. Choose a neighborhood $U_3$ of $(1, \mu)$ in $\mathrm{Gl}(V) \times M$ such that if $(g', \mu') \in U_3$, then $(g' \cdot \mu' - \mu) \in J$. Assume now that $(g, m) \in U_3$. Then we have $g \cdot m = \mu + a + b$ and hence $x = (a+b) \in J$. We therefore obtain by the above $\lambda_x(b) = 0$. Since $\lambda_x$ is a monomorphism, it follows that $b = 0$. Thus $g \cdot m = \mu + a$ and $\tau(g \cdot m) = \tau(\mu)$ as claimed.

**(5)** Now we put together all ingredients in order to prove the theorem. By step **(3)**, $d_{(1,\mu)}\pi$ is surjective, and we may apply Theorem 3.3 again to obtain that for any neighborhood $U$ of $(1, \mu)$ in $N'$, the image $\pi(U)$ is a neighborhood of $\pi(1, \mu)$. $N' \subset \beta^{-1}(\tau(\mu))$ means that elements $(g, m)$ of $N'$ satisfy $\pi_Z(\tau(g \cdot m)) = \tau(\mu)$. By step **(4)**, we can therefore suppose that all the elements $(g, m)$ of $U$ satisfy $\tau(g \cdot m) = \tau(\mu)$. This is the claim of the stability theorem. □

# 4 A rigid Leibniz algebra with non-trivial $HL^2$

In this section, we finally obtain an analogue of Richardson's theorem [12] for Leibniz algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $M$ be an irreducible (right) $\mathfrak{g}$-module (of dimension $\geq 2$). We put $\widehat{\mathfrak{g}} = \mathfrak{g} \ltimes M$ semidirect product of $\mathfrak{g}$ and $M$ (with $[M, M] = 0$).

**Theorem 4.1** (Richardson's Theorem)**.** *Let $\widehat{\mathfrak{g}} = \mathfrak{g} \ltimes M$ as above. Then $\widehat{\mathfrak{g}}$ is not rigid if and only if there exists a semisimple Lie algebra $\widehat{\mathfrak{g}}'$ which satisfies the following conditions:*

*(a) there exists a semisimple subalgebra $\mathfrak{g}'$ of $\widehat{\mathfrak{g}}'$ which is isomorphic to $\mathfrak{g}$,*

*(b) if we identify $\mathfrak{g}'$ with $\mathfrak{g}$, then $\widehat{\mathfrak{g}}' / \mathfrak{g}'$ is isomorphic to $M$ as a $\mathfrak{g}$-module.*

*Proof.* This is Theorem 2.1 in [12]. □

Richardson shows in [12] that for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and for $M = M_k$ the standard irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $2k+1$, the Lie algebra $\widehat{\mathfrak{g}}$ is not rigid if and only if $k = 1, 2, 3, 5$. The semisimple Lie algebras $\widehat{\mathfrak{g}}'$ are in this case the standard semisimple Lie algebras of dimension $6, 8, 10$ and $14$. Observe that $\widehat{\mathfrak{g}}$ has necessarily rank 2, and these are all rank 2 semi-simple Lie algebras ($A_1 \times A_1$, $A_2$, $B_2$ and $G_2$).

We will extend Richardson's theorem to Leibniz algebras in the following sense. First of all, we will restrict to simple Lie algebras $\mathfrak{g}$. Let $I$ be another irreducible (right) $\mathfrak{g}$-module (of dimension $\geq 2$). We also set $\mathfrak{h} = \widehat{\mathfrak{g}} \dot{+} I$ the hemisemidirect product with the $\mathfrak{g}$-module $I$ (in particular $[\mathfrak{g}, I] = 0$, $[I, \mathfrak{g}] = I$ and $[M, I] = 0$). Observe that $\mathfrak{h}$ is a non-Lie Leibniz algebra with ideal of squares $I$ and with quotient Lie algebra $\widehat{\mathfrak{g}}$.

So, we have $\mathfrak{h} = \mathfrak{g} \oplus M \oplus I$ as vector spaces. Actually, we have $\mathfrak{h} = (\mathfrak{g} \dot{+} I) \ltimes M$ where the Leibniz algebra $\widehat{\mathfrak{g}}$ acts on $M$ via the quotient morphism $\widehat{\mathfrak{g}} \to \mathfrak{g}$.

In all the following, we fix the complex vector space $\mathfrak{g} \oplus M \oplus I$ and we will be interested in the different Leibniz algebra structures on $\mathfrak{g} \oplus M \oplus I$.

**Theorem 4.2.** *Assume that* $\dim(M) > \dim(I) \cdot \dim(\mathfrak{g})$ *and that the* $\mathfrak{g}$*-module* $I$ *is different from* $\mathfrak{g}$.

*Then the Leibniz algebra* $\mathfrak{h}$ *is not rigid if and only if there exists a Leibniz algebra* $\mathfrak{h}'$ *which satisfies the following conditions:*

*a) There exists a semisimple Lie subalgebra* $\mathfrak{k}$ *of* $\mathfrak{h}'$ *with a semisimple Lie subalgebra* $\mathfrak{g}' \subset \mathfrak{k}$ *which is isomorphic to* $\mathfrak{g}$*, i.e.* $\mathfrak{g} \cong \mathfrak{g}'$*;*

*b) if we identify the subalgebra* $\mathfrak{g}'$ *with* $\mathfrak{g}$ *by this isomorphism, then* $\mathfrak{k}/\mathfrak{g}$ *is isomorphic to* $M$ *as a* $\mathfrak{g}$*-module and* $\mathfrak{h}'/\mathfrak{k}$ *is isomorphic to* $I$ *as a Leibniz antisymmetric* $\mathfrak{g}$*-module.*

*Proof.* Let $V := \mathfrak{g} \oplus M \oplus I$. We will consider Leibniz algebras $\mathfrak{h}'$ on this fixed vector space $V$, i.e. $\mathfrak{h} = (V, \mu)$ and $\mathfrak{h}' = (V, \mu')$.

"$\Leftarrow$" Suppose that there exists a Leibniz algebra $\mathfrak{h}'$ which satisfies the conditions $a)$ and $b)$.

We may assume that

$$\mu(x, x') = \mu'(x, x'),\ \mu(x, m) = \mu'(x, m),\ \mu(m, x) = \mu'(m, x),\ \mu(i, x) = \mu'(i, x)$$

for all $x, x' \in \mathfrak{g}$, all $m \in M$ and all $i \in I$.

Putting $g_t(x) = x,\ g_t(m) = tm,\ g_t(i) = ti$, we have that $g_t \in \mathrm{Gl}(V)$ for all $t \neq 0$ and

$$\lim_{t \to 0} g_t \cdot \mu' = \mu.$$

Therefore, $L$ is not rigid.

"$\Rightarrow$" Let *Leib* be the set of all Leibniz algebras defined on the vector space $V$. We are considering the Leibniz subalgebra $\mathfrak{s} := \mathfrak{g}$ of $\mathfrak{h}$. It satisfies by Propositions 2.14 and 2.15 the cohomological conditions in order to apply the stability theorem. From Theorem 3.4, we therefore get the existence of a neighborhood $U$ of $\mu$ in *Leib* such that if $\mu_1 \in U$, the Leibniz algebra $L_1 = (V, \mu_1)$ is isomorphic to a Leibniz algebra $L' = (V, \mu')$ which satisfies the following conditions:

1) $\mu(x,x') = \mu'(x,x')$,

2) $\mu(x,m) = \mu'(x,m),\ \mu(m,x) = \mu'(m,x)$,

3) $\mu(i,x) = \mu'(i,x),\ \mu(x,i) = \mu'(x,i) = 0$,

for all $x, x' \in \mathfrak{g}$, all $m \in M$ and all $i \in I$.

Since $\mathfrak{h}$ is a non-Lie algebra, the Leibniz algebra $\mathfrak{h}'$ is also a non-Lie algebra. Therefore, the ideal of squares of the algebra $\mathfrak{h}'$ is also non zero. We denote it by $I'$.

From conditions 1) and 2) we conclude that $I' \cap (\mathfrak{g} + M) = \{0\}$, and thus $J := I' \cap I \neq \{0\}$.

The condition 3) implies that $J$ is right module over $\mathfrak{g}$. Since $I$ is an irreducible right $\mathfrak{g}$-module, we have $J = I$ and thus $I' = I$ as a vector spaces.

We conclude that the restriction to $\mathfrak{g} \oplus M$ places us in the Lie situation, i.e. in exactly the same situation as in Richardson's theorem. By Theorem 4.1, we thus obtain that there exists a semisimple Lie algebra $\mathfrak{k}$ with a semisimple subalgebra $\mathfrak{g}'$, isomorphic to $\mathfrak{g}$ and (when $\mathfrak{g}$ is identified with $\mathfrak{g}'$) an isomorphism of $\mathfrak{g}$-modules $\mathfrak{k}/\mathfrak{g} \cong M$. Furthermore, the conditions 1)-3) imply that $a$) and $b$) are satisfied. □

**Corollary 4.3.** *There exist rigid Leibniz algebras $\mathfrak{h}$ such that $HL^2(\mathfrak{h},\mathfrak{h}) \neq 0$.*

*Proof.* As discussed earlier, Richardson shows in [12] that for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and for $M_k$ the standard irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $2k+1$, the Lie algebra $\widehat{\mathfrak{g}}$ is not rigid if and only if $k = 1, 2, 3, 5$. He also shows that for $k > 5$ and $k$ odd, the Lie algebra cohomology of the Lie algebra $\widehat{\mathfrak{g}}$ is not zero. As candidate for our Leibniz algebra $\mathfrak{h}$ satisfying the claim of the corollary, we take as before $\mathfrak{h} = \widehat{\mathfrak{g}} \dot{+} I_l$ for some irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules $I_l \neq \mathfrak{sl}_2(\mathbb{C})$ and $M$ with dimension $\dim(M) > 3 \cdot \dim(I)$, which is assured by $k > 4l$, and $k$ odd. By Theorem 4.2, $\mathfrak{h}$ is then rigid.

On the other hand, by Proposition 2.13 and Theorem 2.6, $H^2(\widehat{\mathfrak{g}},\widehat{\mathfrak{g}})$ is a direct factor of $HL^2(\mathfrak{h},\mathfrak{h})$, thus we obtain that this Leibniz cohomology space is not zero. □

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