Some limiting situations with semilinear elliptic equations

In this minicourse, we will consider the following equation :

$$(P_\lambda) \qquad \begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \text{ regular bounded domain } \subset \mathbb{R}^N \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where f is a positive, nondecreasing convex function in $[0, \infty)$ and λ is a positive constant.

It is easy to prove that there is a critical value $\lambda^* \in (0, \infty)$ such that no classical solution exists for $\lambda > \lambda^*$, while minimal positive solution u_λ exists for $\lambda \in (0, \lambda^*)$. So a natural question is to ask what happens when λ tends to λ^*, and for the limiting problem (P_{λ^*}). More precisely, we are interested in the asymptotic behavior of u_λ; the existence and regularity problem of solution for (P_{λ^*}); and some information of infinite branches for the solution curve.

We will show the results for two situations of very different natures, the quasilinear case and the superlinear case, i.e. when we have respectively

$$\lim_{t\to\infty} \frac{f(t)}{t} < \infty \quad \text{or} \quad \lim_{t\to\infty} \frac{f(t)}{t} = \infty.$$

In particular, we will show the use of many basic tools in the study of semilnear elliptic partial differential equations.