BARRIER RESONANCES IN STRONG MAGNETIC FIELDS

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Abstract

In this paper, we study the resonances of Schrödinger operators with a homogeneous magnetic field when the field strength tends to the infinity. We prove the existence of resonances and give complete asymptotic expansions for the energy and the width of resonances in the case that the electric potential has a high barrier.

1 Introduction

The Schrödinger operator with a homogeneous magnetic field pointing at z-direction in $\mathbf{R}^3$ can be written as:

$$P(B) = (D_x - By)^2 + D_y^2 + D_z^2 + ZV(x, y, z)$$

where $D_v = -i\frac{\partial}{\partial_v}$, $Z > 0$ is the nuclear charge and $B > 0$ is proportional to the magnetic field strength. In the case of a hydrogen atom, $V = -1/r$. The study of the Schrödinger operator of hydrogen atom in strong magnetic field, called quantum diamagnetic Kepler problem, is particularly interesting, because it is a model for illustrating several phenomena of "quantum chaos". See [11] and the references therein. In trying to understand the properties of Schrödinger operators in strong magnetic field, we studied in [20] the resonances in strong magnetic field for Schrödinger operators with Stark effect in two dimensional case. The present work aims at studying the resonances of Schrödinger operators in strong magnetic fields in three dimensional case with a smooth, even analytical potential, avoiding thereby further complications caused by Coulombic singularity. Note that for quantum diamagnetic Kepler problem, the existence of resonances is strongly suggested by numerical results in physical literature. See for example [9] and [11].

A direct mathematical study of resonances for $P(B)$ might be subtle. In this paper, we consider the case that Z is a large parameter and denote hereafter $P(B)$ by $P(B,Z)$. It is a physical folklore that most interesting phenomena of this problem appear in the regime where B and Z are of the same magnitude. See the work of Lieb, Solovej and Yngvason [15] on asymptotics of the sum of negative eigenvalues of semiclassical Schrödinger operators with a strong magnetic field. Note that dividing $P(B,Z)$ by Z, one sees that the semiclassical parameter h in their work corresponds to $1/\sqrt{Z}$ in our case. In this work, we are able to treat the case $Z \sim \mu B$ for μ appropriately small. See the condition (1.4) below.

Assume that V is a smooth potential tending to zero at the infinity and satisfies:

$$|\partial_x^\alpha V(x)| \leq C_\alpha < x >^{-|\alpha|}, x \in \mathbf{R}^3, \tag{1.1}$$

with $< x >= (1 + |x|^2)^{1/2}$. Suppose that V has a holomorphic extension in the magnetic field direction $z = x_3$ into the domain $\mathbf{K}_1 \equiv \{z \in \mathbf{C}; |\Im z| \leq \delta|\Re z| + R\}$ for some $\delta, R > 0$ and

$$\text{(1.1) is satisfied for } z \in \mathbf{K}_1. \tag{1.2}$$

In this paper, we define the resonances of $P(B,Z)$ through complex dilation in z-variable. Note that we can equally use the theory of Helffer-Sjöstrand [13] or method of analytical distortion to define resonances of $P(B,Z)$. All these definitions are equivalent in our case. We choose to use analytical dilation because it is the simplest one and is sufficient to obtain detailed informations on energy and width of resonances in the situations studied in this paper. For $\theta \in \mathbf{C}$ with $|\theta|$ small,

$$P(B, Z; \theta) \equiv (D_x - By)^2 + D_y^2 + e^{-2\theta} D_z^2 + ZV(x, y, e^\theta z) \qquad (1.3)$$

is well defined. For $\Im\theta > 0$, the essential spectrum of $P(B, Z; \theta)$ is given by

$$\Sigma_e(\theta) = \cup_{k \in \mathbf{N}} \{(2k+1)B + \lambda e^\theta; \lambda \geq 0\}.$$

It can be readily verified that the discrete eigenvalues of $P(B, Z; \theta)$ outside $\Sigma_e(\theta)$, $\Im\theta > 0$, are independent of θ and coincide with poles of the meromorphic extension of $P(B, Z)$ from the upper-half complex plane. These eigenvalues of $P(B, Z; \theta)$ are called resonances of $P(B, Z)$. Note that our assumptions on V do not exclude the existence of embedded eigenvalues of $P(B, Z)$ in $[B, +\infty[$. See [4]. But our results on width of resonances will show that the eigenvalues of $P(B, Z)$ are really absent near the energy under consideration, at least for B large enough.

Let us explain the ideas for studying the discrete spectrum of $P(B, Z; \theta)$ near a barrier top of ZV. See Section 3 for details on the geometrical conditions on V. In the following, θ is fixed with $\Im\theta > 0$ as big as possible in accordance with the analyticity assumption (1.2). Assume that the nuclear charge is not too strong in comparing with the magnetic field:

$$1 \ll Z < \mu B, \text{ where } \mu > 0 \text{ satisfies: } \mu \|V_\theta\| < 2\Im\theta, \qquad (1.4)$$

where $V_\theta = V(x, y, e^\theta z)$. Making use of the strong field reduction (see [7],[14], [20]), we reduce the study of the spectrum of $P(B, Z; \theta)$ near $(2n+1)B + ZE$ to the analysis of the spectrum near ZE of a pseudo-differential operator, whose leading term is

$$Q = e^{-2\theta} D_z^2 + ZV(x, hD_x, e^\theta z)$$

where $h = 1/B$. In this paper, we denote by $a(x, D_x)$ the pseudodifferential operator with Weyl symbol $a(x, \xi)$ defined by:

$$a(x, D_x) f(x) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{i(x-y)\cdot\xi} a((x+y)/2, \xi) f(y) \, dy d\xi,$$

for $f \in \mathcal{S}(\mathbf{R}^n)$. Q is a one dimensional Schrödinger operator with operator-valued potential. Note that for each fixed z, the operator $V(x, hD_x, z)$ is compact and has only discrete eigenvalues near $E > 0$. It is believed that it is the interaction of these curves of eigenvalues which produces the resonances in strong magnetic field. However a mathematical realization of this idea appears difficult, because the spacing between the eigenvalues of $V(x, hD_x, z)$ is small, typically, of the order $O(h)$, while the spectrum of resulting effective Hamiltonian can be only distinguished at the level $1/\sqrt{Z} \sim O(h^{1/2})$. Our approach is to first study the resonances of

$$Q(x, \xi) = D_z^2 + ZV(x, \xi, z)$$

where x and ξ are regarded as parameters. The top resonances of this kind of operators were studied by several authors ([3], [8],[10],[19]) and the regime $Z \to \infty$ corresponds to the semiclassical limit. So we can fairly well understand the dependence of the resonances of $Q(x, \xi)$ on x, ξ. The resulting effective resonant Hamiltonian is a non self-adjoint h-pseudodifferential operator whose eigenvalues are resonances of original problem. Note that it is well known that resonances of Schrödinger operators can be identified with eigenvalues of a family of non-self adjoint operators (see [2], [6]). But in our case, the effective resonant Hamiltonian is θ-independent. On computing its eignvalues in the limit $h \to 0$, we can obtain complete asymptotic expansions for the energy and the width of resonances.

The organization of the paper is as follows: In Section 2, we reduce the resonance problem in strong magnetic field to a problem in semiclassical limit. Here the condition (1.4) is needed in order to separate contributions coming from different Landau levels. In Section 3, we study the spectral properties of the dilated Hamiltonian near the top of the potential. We establish results on the exponential decay of eigenfunctions and stability estimates on the resolvent. The main results of the paper are proven in Section 4, where we give complete expansions in $Z^{-1/2}$ and B^{-1} for resonances of $P(B, Z; \theta)$. In particular, we see that the width of resonances grows with $\sqrt{Z}$ and the eigenvalues of $P(B, Z)$ are absent near $(2n + 1)B + ZE$ for any $n \in \mathbf{N}$. Finally in Appendix, we sketch some results on the symbolic calculus and the continuity of h-pseudo-differential operators with operator-valued symbol depending on another small parameter.

2 The Strong Field Reduction

To make the strong field reduction to $P(B, Z; \theta)$, we use symplectic changes of coordinates and write it in the form

$$P(B, Z; \theta) = B(D_y^2 + y^2) + e^{-2\theta} D_z^2 + ZV(x - h^{1/2} D_y, h^{1/2} y + hD_x, e^\theta z). \quad (2.1)$$

Here $h = 1/B$. Let ϕ_n be the n-th normalized Hermite function:

$$(D_y^2 + y^2)\phi_n = (2n+1)\phi_n, \|\phi_n\| = 1.$$

Consider the Grushin problem:

$$\mathcal{P}(\lambda) = \begin{pmatrix} P(B, Z; \theta) - \lambda & .\phi_n \\ < ., \phi_n >_Y & 0 \end{pmatrix} : D \oplus L^2(\mathbf{R}^2_{xz}) \to L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^2_{xz}), \quad (2.2)$$

for $\lambda \in \Omega \equiv \{\mu \in \mathbf{C}; \mu = \nu_1 + e^{-2\theta}\nu_2, \nu_j \in \mathbf{R}$ such that $|\nu_1 - (2n+1)B| < aB, |\nu_2| < CB\}$, where $0 < a < 2$, $C > 0$, D is the domain of the Laplacian and $< ., . >_Y$ denotes the scalar product in y-variable. Let Π denote the spectral projector associated to ϕ_n in $L^2(\mathbf{R}^3)$:

$$\Pi f =< f(x, ., z), \phi_n >_Y \phi_n(y).$$

Put $\hat{\Pi} = 1 - \Pi$. Denote by $\hat{R}(B, Z; \lambda) = (\hat{\Pi} P(B, Z; \theta)\hat{\Pi} - \lambda)^{-1}\hat{\Pi}$ the reduced resolvent.

Lemma 2.1 *Under the assumption (1.4), the reduced resolvent $\hat{R}(B, Z; \lambda)$ exists for $\lambda \in \Omega$ with Ω defined as above for some $a > 1$. We have the estimate:*

$$\|\hat{R}(B, Z; \lambda)\| \le C/B, \quad \textit{uniformly in } B, Z \textit{ and } \lambda \in \Omega. \quad (2.3)$$

Proof. Let $\hat{R}_0(B; \lambda)$ denote the reduced resolvent with $V \equiv 0$. Then we have:

$$\|\hat{R}_0(B; \lambda)\| \le 1/(\sin(2\Im\theta)(2 - a)B), \text{ for } \lambda \in \Omega.$$

Making use of (1.4), one derives:

$$\|Z\tilde{V}_\theta \hat{R}_0(B; \lambda)\| \le \frac{\mu\|\tilde{V}_\theta\|}{\sin(2\Im\theta)(2 - a)} = \frac{\mu\|V_\theta\|}{\sin(2\Im\theta)(2 - a)} < 1, \quad (2.4)$$

if we choose $a > 1$ close to 1. Here $\tilde{V}_\theta = V(x - h^{1/2}D_y, hD_x + h^{1/2}y, e^\theta z)$ with $h = 1/B$. The estimate (2.3) follows readily. □

Theorem 2.2 *Under the above assumptions, the Grushin problem (2.2) has a unique solution for* $\lambda \in \Omega$ *and is given by:*

$$\mathcal{E} = \begin{pmatrix} E(B,Z;\lambda) & E_+(B,Z;\lambda) \\ E_-(B,Z;\lambda) & E_{+-}(B,Z;\lambda) \end{pmatrix}, \tag{2.5}$$

where $E_{+-}(B,Z;\lambda)$ *is an h-pseudodifferential operator of the form:*

$$\begin{aligned} E_{+-}(B,Z;\lambda) &= \lambda - \{(2n+1)B + e^{-2\theta}D_z^2 + ZV(x,hD_x,e^\theta z) \\ &\quad + Z\sum_{j=1}^{\infty} h^j a_j(x,hD_x,e^\theta z,h^{1/2}e^{-\theta}D_z;hZ,h\lambda)\}, \end{aligned} \tag{2.6}$$

each a_j *being a bounded symbol, holomorphic in* hZ *and* $h\lambda$*.*

Proof. We first construct an approximate inverse to (2.2). Let

$$\mathcal{E}_0 = \begin{pmatrix} \hat{R}(B,Z;\lambda) & .\phi_n \\ <.,\phi_n>_Y & \lambda - \{(2n+1)B + e^{-2\theta}D_z^2 + Z<\tilde{V}_\theta(.\phi_n),\phi_n>_Y\} \end{pmatrix}.$$

Then we can check that $\mathcal{P}(\lambda)\mathcal{E}_0 = 1 + K_1$, where

$$K_1 = \begin{pmatrix} Z[\Pi,\tilde{V}_\theta]\hat{R}(B,Z;\lambda) & Z\hat{\Pi}(\tilde{V}_\theta.\phi_n) \\ 0 & 0 \end{pmatrix}.$$

Since Π is an operator with kernel in $\mathcal{S}(\mathcal{R}^2)$, one can estimate $[\Pi,\tilde{V}_\theta] = O(B^{-1/2})$. From Lemma 2.1, it follows that

$$\|Z[\Pi,\tilde{V}_\theta]\hat{R}(B,Z;\lambda)\| \leq CB^{-1/2}, \tag{2.7}$$

uniformly in $Z < \mu B$ and $\lambda \in \Omega$. This proves that $1 + K_1$ is invertible, if $B >> 1$ is large enough. Similarly, we can construct a left inverse to $\mathcal{P}(\lambda)$, which shows the inversibility of $\mathcal{P}(\lambda)$ for $\lambda \in \Omega$.

The inverse of $1 + K_1$ can be explicitly written down:

$$(1+K_1)^{-1} = \begin{pmatrix} a(\lambda) & -Za(\lambda)\hat{\Pi}(\tilde{V}_\theta.\phi_n) \\ 0 & 1 \end{pmatrix},$$

where

$$a(\lambda) = (1 - Z[\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda))^{-1}.$$

So we can calculate the inverse $\mathcal{P}(\lambda)^{-1} = \mathcal{E}_0(1 + K_1)^{-1}$ and give explicit formulas for $E(B, Z; \lambda)$, $E_\pm(B, Z; \lambda)$ and $E_{+-}(B, Z; \lambda)$ in (2.5). In the following, the expicilt formulas for $E(B, Z; \lambda)$ and $E_\pm(B, Z; \lambda)$ are not needed. We just indicate that they are holomorphic in $\lambda \in \Omega$ and refer the reader to [20] for detailed expressions in a similar situation. Let us write down the formula for $E_{+-}(B, Z; \lambda)$:

$$\begin{aligned} E_{+-}(B, Z; \lambda) &= \lambda - \{(2n+1)B + e^{-2\theta}D_z^2 + Z < \tilde{V}_\theta(.\phi_n), \phi_n >_Y \\ &\quad -Z < a(\lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n), \phi_n >_Y\}. \end{aligned} \tag{2.8}$$

Making use of the fact that ϕ_n is an eigenfunction of the harmonic oscillator with eigenvalue $2n + 1$, we can show as in [14] that $< \tilde{V}_\theta(.\phi_n), \phi_n >_Y$ is an h-pseudo-differential operator whose symbol is given by:

$$V(x, \xi, e^\theta z) + h\frac{2n+1}{4}\Delta_{x\xi}V(x, \xi, e^\theta z) + O(h^2). \tag{2.9}$$

To show that $-Z < a(\lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n), \phi_n >_Y$ is an h-pseudo-differential operator of the form:

$$Z\sum_{j=1}^{\infty} h^j b_j(x, hD_x, e^\theta z, h^{1/2}e^{-\theta}D_z; hZ, h\lambda), \tag{2.10}$$

we expand $a(\lambda)$ in powers of $Z[\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda)$:

$$a(\lambda) = 1 + \sum_{j\geq 1}(Z[\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda))^j.$$

By (2.7), this series is convergent in the norm of bounded operators. Clearly, $< \hat{\Pi}\tilde{V}_\theta(.\phi_n), \phi_n >_Y = 0$. Let us look at the term

$$Z < [\tilde{V}_\theta, \hat{\Pi}]\hat{R}(B, Z; \lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n), \phi_n >_Y$$

in details. Making use of the symbolic calculus of pseudo-differential operators with operator-valued symbols (see [5]),we can construct a parametrix for $\hat{R}(B, Z; \lambda)$ and show that $\hat{R}(B, Z; \lambda)$ is an h-pseudo-differential operator in x and z variables with $\mathcal{L}(L^2(\mathbf{R}_y))$-values symbol. In particular, its leading symbol is given by:

$$B^{-1}(\hat{\Pi}(D_y^2 + y^2 + he^{-2\theta}\eta^2 + hZV(x - h^{1/2}D_y, h\xi + h^{1/2}y, e^\theta z))\hat{\Pi} - h\lambda)^{-1}\hat{\Pi}.$$

A direct computation shows that $Z^2 < [\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n), \phi_n >_Y$ is an operator of the form (2.10).

For the later use, let us give more informations on the leading symbol of $Z^2 < [\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n)$. We write:

$$\begin{aligned} \tilde{V}_\theta &= V(x, hD_x, e^\theta z) \\ &\quad + h^{1/2}y(\partial_\xi V)(x, hD_x, e^\theta z) - h^{1/2}D_y(\partial_x V)(x, hD_x, e^\theta z) + O(h). \end{aligned}$$

The pieces involving $V(x, hD_x, e^\theta z)$ are equal to 0. So we obtain:

$$< [\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda)\hat{\Pi}\tilde{V}_\theta(.\phi_n, \phi_n >_Y = h^2 R_1 + h^3 R_2,$$

where R_2 is an operator of the form (2.10) and R_1 is an h-pseudo-differential operator with symbol r_1 of the form:

$$r_1(x, \xi, e^\theta z, e^{-\theta}\eta; hZ, h\lambda) = (\partial_x V)^2 r_{11} + \partial_x V \partial_\xi V r_{12} + (\partial_\xi V)^2 r_{22}. \quad (2.11)$$

Here $\partial_x V$, $\partial_\xi V$ are functions of $(x, \xi, e^\theta z)$, the r_{ij}'s are bounded symbols. Recall that the quantization rule here is: $(x, \xi) \to (x, hD_x)$ and $(z, \eta) \to (z, h^{1/2}D_z)$.

By the same method, we can study the terms related to

$$(Z[\tilde{V}_\theta, \Pi]\hat{R}(B, Z; \lambda))^j, j \geq 2.$$

By (2.7), these terms do not contribute to a_1 in (2.6). Note that by the arguments of [14], the half-integer powers of h disappear, due to the special properties of ϕ_n. This proves (2.6) and therefore finishes the proof of Theorem 2.2. $\square$

Note the $\mathcal{E}_0$ depends holomorphically on $\lambda \in \Omega$. Since $P(B, Z)$ has only discrete spectrum in Ω, we derive from Theorem 2.2 and the resolvent formula given in [13] the following

Corollary 2.3 *Under the assumption of Theorem 2.2, $\lambda \in \Omega$ is an eigenvalue of $P(B,Z,\theta)$ if and only if 0 is an eigenvalue of $E_{+-}(B,Z;\lambda)$ and the algebraic multiplicities are the same.*

By Corollary 2.3 , we are led to studying a non linear eigenvalue problem. From now on, we slightly change the notations in variables and put:

$$Q(h,Z;\theta) = e^{-2\theta}D_y^2 + ZV(x,hD_x,e^{\theta}y) + hZV_1(x,hD_x,e^{\theta}y), \qquad (2.12)$$

where $V_1 = (2n+1)/4\Delta_{x\xi}V(x,\xi,z)$. ¿From the proof of Theorem 2.2, one has:

$$a_1 = \frac{2n+1}{4}\Delta_{x\xi}V + hZr_1,$$

where r_1 is given by (2.11). This gives:

$$E_{+-}(B,Z;\lambda) = \lambda - \{(2n+1)B + Q(h,Z;\theta) + Z^2h^2R_1 + O(Zh^2)\}$$

where R_1 is the operator with symbol r_1 and $\|O(Zh^2)\| \leq CZh^2$ uniformly in $\lambda \in \Omega$. We shall show that $(hZ)^2R_1$ does not contribute to the leading terms of the resonances of $P(B,Z)$. Note also that the eigenvalues of $Q(h,Z;\theta)$ are resonances of the self-adjoint operator

$$Q(h,Z) = D_y^2 + ZV(x,hD_x,y) + hZV_1(x,hD_x,y). \qquad (2.13)$$

In the next section, we shall first study a one-dimensional resonance problem for a Schrödinger operator :

$$Q(x,\xi;h,Z) = D_y^2 + ZV(x,\xi,y) + hZV_1(x,\xi,y) \qquad (2.14)$$

depending on parameters x,ξ. By quantizing the curves of resonances, we obtain effective resonant Hamiltonians, which are non self-adjoint operators independent of θ and whose eigenvalues determine the leading terms of resonances of $P(B,Z)$.

3 Effective Resonant Hamiltonians

To obtain precise informations on resonances of $P(B,Z)$, we need to introduce additional geometrical assumptions on the potential V. To simplify notations, we denote $w=(x,\xi,y)\in\mathbf{R}^3$. Suppose that V has a maximum $E>0$ at some point w_0. By a translation, we can assume that $w_0=0$. Suppose that

$$\text{the Hessian of } V \text{ at } 0 \text{ is negative definite,} \tag{3.1}$$

and

$$2\eta^2 - y\partial_y V(w) > 0, \text{ for } (\eta,w)\in S_E \equiv \{\eta^2+V(w)=E\} \text{ with } w\neq 0. \tag{3.2}$$

(3.2) is a kind of non-trapping assumption and it implies in particular that for $\Im\theta>0$ small enough, $e^{-2\theta}\eta^2+V(x,\xi,e^\theta y)-E$ is elliptic except for $w=0$.

Let $H(w)$ denote the Hessian of V at w. By (3.1), we see that $H(w)<0$, for $|w|<\delta$ for some $\delta>0$. In particular,

$$H_y \equiv \partial_y^2 V(w) < 0, |w|<\delta.$$

Since $\partial_y V(w)=0$, when $w=0$, by the Theorem of implicit functions, for x,ξ small, the equation $\partial_y V(x,\xi,y)=0$ has a unique solution in y given by $y=y(x,\xi)\in\mathbf{R}$ with $y(0,0)=0$. Define:

$$G(x,\xi)=V(x,\xi,y(x,\xi)), H(x,\xi)=H_y(x,\xi,y(x,\xi)).$$

To simplify notations, we shall assume in the following that $y(x,\xi)\equiv 0$ for (x,ξ) near 0, which is equivalent with:

$$\partial_y V(x,\xi,0)=0, \qquad \text{for } (x,\xi) \text{ near } 0. \tag{3.3}$$

With a slight modification, our methods developped below work also in general situation. In fact in most cases, it is sufficient to use a translation in y when $|(x,\xi)|$ is small.

Let q_θ be the leading symbol of $Q(h,Z;\theta)/Z$ (cf. (2.12)):

$$q_\theta = e^{-2\theta}\eta^2 + V(x,\xi,e^\theta y).$$

For (η, w) with $|\Re q_\theta - E| < \epsilon$ and $|w| > \delta > 0$, one has:

$$|\Im q_\theta| \geq c\Im\theta, \text{ for some } c > 0, \tag{3.4}$$

due to the assumption (3.2). This shows that for $|w| > \delta$, we always have:

$$|q_\theta - E| > c\Im\theta \tag{3.5}$$

For $|w| \leq \delta$, we use the development of V in y near $y = 0$:

$$V(w) = G(x, \xi) + H(x, \xi)y^2/2 + O(y^3).$$

Then it is clear that for $\delta > 0$ small enough, one has:

$$-\Im(q_\theta - E) \geq c\Im\theta(\eta^2 + y^2), \text{ for } |w| < \delta. \tag{3.6}$$

Now by an argument of perturbation, we derive that there exists a non-negative function, $f(y)$, usually depending on $\Im\theta$ such that $f(y) > c > 0$ for $|y| > \delta$ and $f(y) \sim cy^2$ for y near 0 and (3.5), (3.6) are still satisfied with η replaced by $\eta + if'(y)$. Remark that we only require the derivative f' of f to be small, so we can take $f(y) \sim c\Im\theta|y|$, $c > 0$, for $|y|$ large. As in [20], $f(.)$ can be used to measure the exponential decay of the eigenfunctions of $Q_\theta(x, \xi)$ defined by:

$$Q_\theta(x, \xi) = \hbar^2 e^{-2\theta} D_y^2 + V(x, \xi, e^\theta y) + \hbar V_1(x, \xi, e^\theta y), \text{ where } \hbar = \frac{1}{\sqrt{Z}}. \tag{3.7}$$

Here $V_1 = a_1$. See Proposition 2.2. Note that $Q_\theta(x, \xi)$ is a one-dimensional semiclassical Schrödinger operator with parameters x, ξ. So We can easily analyze the dependence of its eigenvalues on x, ξ. The following result can be proven as in [20].

Theorem 3.1 *Let $\Omega_1 = \{z \in \mathbf{C}; |\Re z - E| \leq 1/C\Im\theta, |\Im z| \leq C\hbar\}$. Here $C > 1$ is large enough. Let $\delta, \hbar_0 > 0$ be small enough. Assume that $u(\hbar)$ is a normalized eigenfunction of $Q_\theta(x, \xi)$:*

$$Q_\theta(x, \xi)u(\hbar) = \lambda u(\hbar), \|u(\hbar)\| = 1, \text{ and } \lambda \in \Omega_1.$$

Then there exists $C > 0$ such that

$$\hbar\|\nabla_y(e^{f/\hbar}u(\hbar))\| + \|e^{f/\hbar}u(\hbar)\| \leq C, \text{ uniformly in } 0 < \hbar < \hbar_0, |x| + |\xi| < \delta.$$

Here $f > 0$ is the function constructed above and $\|.\|$ denotes the norm in $L^2(\mathbf{R}_y)$.

Proof. We can compute that the $\hbar$-symbol of $e^{f/\hbar}Q_\theta e^{-f/\hbar}$ is given by:

$$(\eta + if'(y))^2 + V(w) + O(\hbar).$$

Let χ_1, χ_2 be a partition of unity on $\mathbf{R}_y$ with supp$\chi_1 \subset \{|y| < \delta_1\}$, $\chi_1(y) = 1$ for $|y| < \delta_1/2$ and $\chi_1^2 + \chi_2^2 \equiv 1$. On $supp\chi_2$, $e^{f/\hbar}Q_\theta e^{-f/\hbar} - z, z \in \Omega_1$, is elliptic. On constructing a parametrix, we can show that

$$(\chi_2(y) + O(\hbar))(e^{f/\hbar}u(\hbar)) = 0 \tag{3.8}$$

Making use of (3.6) and splitting the region $\{|y| < \delta_1\}$ according to $\{|y| > R\hbar^{1/2}\}$ or $\{|y| \le R\hbar^{1/2}\}$, we can prove as in [20] that

$$\|\chi_1 e^{f/\hbar}u(\hbar)\|^2 \le C(R) + \frac{C'}{R}\|e^{f/\hbar}u(\hbar)\|^2. \tag{3.9}$$

Here C' is independent $R, \hbar$. Taking R large enough, we derive from (3.8) and (3.9) that

$$\|e^{f/\hbar}u(\hbar)\| \le C,$$

uniformly in $\hbar$. The desired estimate follows from the ellipticity of the equation. $\square$

By theorem 3.1, we see that the energy of the eigenfunctions of $Q_\theta(x,\xi)$ is exponentially small outside a neighbourhood of zero. So to compute the eigenvalues of $Q_\theta(x,\xi)$, we need just to study this operator near 0. Note that we are now studying a one-dimensional problem of barrier top resonances, which was already discussed in more general situations in [8] and [19]. Here the main attention will be paid to the dependence of these eigenvalues on x, ξ. Let $T_0(x,\xi)$ be defined by:

$$T_0(x,\xi) = \hbar^2 e^{-2\theta} D_y^2 + G(x,\xi) + \frac{H(x,\xi)}{2} e^{2\theta} y^2. \tag{3.10}$$

The spectrum of $T_0(x,\xi)$ is purely discrete and is independent of θ so long as $|\theta|$ is small and $\Im\theta > 0$:

$$\sigma(T_0(x,\xi)) = \{\lambda_j^0 = G(x,\xi) - i(2j+1)\sqrt{|H(x,\xi)|/2}\;\hbar, j \in \mathbf{N}\} \tag{3.11}$$

Assume that $\lambda_j^0 \notin \partial\Omega_1$ for all j. Put $\Sigma_0 = \{\lambda_j^0; \lambda_j^0 \in \Omega_1\}$.

Theorem 3.2 *There exists a bijection* $b : \Sigma_0 \to \sigma(Q_\theta(x,\xi)) \cap \Omega_1$ *such that*

$$b(\lambda) - \lambda = O(\hbar^{3/2}), \forall \lambda \in \Sigma_0,$$

uniformly in x, ξ *with* $(x,\xi) \in B_\delta \equiv \{|x| + |\xi| < \delta\}$. *In addition, all eigenvalues of* $Q_\theta(x,\xi)$ *are algebraically simple and have complete asymptotic expansions of the form:*

$$\lambda_j(x,\xi;h,\hbar) \sim \mu_j(x,\xi;\hbar) + \sum_{k=1}^{\infty} h^k \mu_{jk}(x,\xi;\hbar) \tag{3.12}$$

for $(x,\xi) \in B_\delta$, *where each* $\mu_{jk}(x,\xi;\hbar)$ *has a complete expansion in powers of* $\hbar$ *and in particular*

$$\mu_j(x,\xi;\hbar) = G(x,\xi) - i(2j+1)\sqrt{\frac{|H(x,\xi)|}{2}}\hbar + O(\hbar^2)$$

for $(x,\xi) \in B_\delta$.

Proof. The first part of the theorem was proven in [8, 19]. The asymptotic expansions in powers of h follow from an argument of perturbation and Theorem 3.2 and the asymptotic expansions in powers $\hbar$ were proven in [19]. Note that the half-integral powers of $\hbar$ disappear, because by the first part of the theorem, the eigenvalues of $Q_\theta(x,\xi)$ are simple. □

Remark 1. In the case that $y(x,\xi) \not\equiv 0$, we should replace $T_0(x,\xi)$ by $\hat{T}_0(x,\xi)$ defined by:

$$\hat{T}_0(x,\xi) = \hbar^2 e^{-2\theta} D_y^2 + G(x,\xi) + \frac{H(x,\xi)}{2}(e^\theta y - y(x,\xi))^2.$$

By the translation analyticity of harmonic oscillator, we see that $\sigma(\hat{T}_0(x,\xi)) = \sigma(T_0(x,\xi))$. So one can prove that Theorem 3.2 also holds in general case.

From Theorem 3.2, it results readily that $\lambda_j(x,\xi;h,\hbar)$ is smooth in $(x,\xi) \in B_\delta$. To obtain detailed results on resonances of $P(B,Z)$, we need to know more informations on the geometrical form of the curves of eigenvalues.

Lemma 3.3 *Assume that*

$$\nabla_{x,\xi} V(0,0,y) = 0, \forall y \in \mathbf{R}. \tag{3.13}$$

Then for all j, $\mu_j(x,\xi;\hbar)$ has a non-degenerate critical point at $(0,0)$:

$$\nabla_{x,\xi}\mu_j(0,0;\hbar) = 0, \text{ and } det H_j(0;\hbar) \neq 0, \text{ for } \hbar \text{ small enough}, \tag{3.14}$$

where $H_j(,.;\hbar)$ is the Hessian of $\mu_j(.,.;\hbar)$.

Proof. We remark first that $\mu_j(.,.)$ are eigenvalues of $\tilde{Q}(x,\xi) \equiv e^{-2\theta}\hbar^2 D_y^2 + V(x,\xi,e^\theta y)$. Let $u_j(x,\xi)$ be an eigenfunction of $\tilde{Q}(x,\xi)$:

$$\tilde{Q}(x,\xi)u_j(x,\xi) = \mu_j(x,\xi;\hbar)u_j(x,\xi), \text{ with } < u_j(x,\xi), u_j^*(x,\xi) >= 1,$$

where u_j^* is an eigenfunction of $\tilde{Q}(x,\xi)^*$ with eigenvalue $\bar{\mu}_j(x,\xi)$. We can choose $u_j(x,\xi), u_j^*(x,\xi)$ locally smoothly in x,ξ. Then we can compute

$$\begin{aligned} &\nabla\mu_j(x,\xi) \\ &= \nabla < \tilde{Q}(x,\xi)u_j(x,\xi), u_j^*(x,\xi) > \\ &= < (\nabla\tilde{Q}(x,\xi))u_j, u_j^* > + \mu_j(x,\xi;\hbar)(< \nabla u_j, u_j^* > + < u_j, \nabla u_j^* >) \\ &= < (\nabla V(x,\xi,e^\theta .))u_j, u_j^* > . \end{aligned}$$

Since $V(w)$ is analytic in y, (3.13) implies that $\nabla V(0,0,e^\theta y) \equiv 0$. This shows that 0 is a critical point of $\mu_j(x,\xi;\hbar)$ for all j. The non-degeneracy follows from the asymptotic expression for $\mu_j(x,\xi;\hbar)$, because the Hessian of $G(x,\xi)$ at 0 is non-degenerate. $\square$

Remark that $G(x,\xi)$ has a non-degenerate maximum at 0. Making use of the asymptotic expression for $\mu_j(x,\xi;\hbar)$, we derive readily that $\Re\mu_j(x,\xi;\hbar)$ has a non-degenerate maximum at 0. Put

$$E_j(\hbar) = \mu_j(0,0;\hbar). \tag{3.15}$$

Then $E_j(\hbar) = E - i(2j+1)\sqrt{\frac{|H(0,0)|}{2}}\hbar + O(\hbar^2)$. We define:

$$\Omega^j = \{z \in \mathbf{C}; |z - E_j(\hbar)| < Ch\}, \tag{3.16}$$

where $C > 0$ can be arbitrarily large. For $k \neq j$,

$$\begin{aligned} & |\lambda_k(x,\xi;h,\hbar) - E_j(\hbar)| \\ \geq \; & 2|k-j|\sqrt{\frac{|H(0,0)|}{2}}\hbar - C'\hbar(x^2+\xi^2) - C''h \\ \geq \; & C\hbar, \end{aligned}$$

for $(x,\xi) \in B_\delta$ with $\delta > 0$ small enough. Here we used the fact that $h = O(\hbar^2)$. This shows $dist(\lambda_k(x,\xi;h,\hbar),\Omega^j) > c\hbar$ for $k \neq j$. Now let $u_j(x,\xi), u_j^*(x,\xi)$ be eigenfunctions of $Q_\theta(x,\xi), Q_\theta(x,\xi)^*$ associated with eigenvalues $\lambda_j(x,\xi;h,\hbar)$, $\bar{\lambda}_j(x,\xi;h,\hbar)$, respectively. Schrinking B_δ if necessary, we can assume that they are both in $C^\infty(B_\delta; L^2(\mathbf{R}_y))$. We normalize them so that $< u_j(x,\xi), u_j^*(x,\xi) >= 1$. Define the spectral projector $\pi(x,\xi)$ of $Q_\theta(x,\xi)$ by:

$$\pi(x,\xi) =< ., u_j^*(x,\xi) > u_j(x,\xi), \quad (x,\xi) \in B_\delta.$$

Let $\hat{\pi}(x,\xi) = 1 - \pi(x,\xi)$. We define the reduced resolvent:

$$\hat{r}(x,\xi;z) = (\hat{\pi}(x,\xi)Q_\theta(x,\xi)\hat{\pi}(x,\xi) - z)^{-1}\hat{\pi}(x,\xi),$$

for $z \in \Omega^j$.

Proposition 3.4 *With the above notations, we have:*

$$\|\hat{r}(x,\xi;z)\|_{\mathcal{L}(L^2(\mathbf{R}_y))} \leq C\hbar^{-1}, \tag{3.17}$$

uniformly in $h, \hbar$ and $(x,\xi) \in B_\delta$.

Proof. Let χ_1, χ_2 be a partition of unity on $\mathbf{R}_y$ with $supp\chi_2 \subset \{|y| > \epsilon\}$ and $\chi_2 = 1$ on $\{|y| > 2\epsilon\}$. Since $|w| > \epsilon' > 0$, for $y \in supp\chi_2, (x,\xi) \in B_\delta$, as shown before, we have in this case: $|q_\theta - z| > C(\theta) > 0$. Let A be the pseudo-differential operator with Weyl symbol $\chi_2(q_\theta - z)^{-1}$. Then making use of results on usual symbolic calculus (see [18]), we obtain:

$$\hat{r}(x,\xi)\chi_2 = \hat{\pi}(x,\xi)A + \hat{r}(x,\xi)O(\hbar). \tag{3.18}$$

On $supp\chi_1$, we approximate Q_θ by T_0 defined in (3.10). Let $\hat{r}_0(x,\xi;z)$ be the reduced resolvent of $T_0(x,\xi)$ defined as above with the spectral projector

$\pi_0(x,\xi)$ of $T_0(x,\xi)$ associated with λ_j^0. Then by the rescaling $y \to \hbar^{1/2}y$, we have:

$$\|y^k(\hbar D_y)^l \hat{r}_0(x,\xi;z)\| \le \hbar^{-1+(k+l)/2}, z \in \Omega^j, \tag{3.19}$$

for $k+m \le 2$. One can verify:

$$\begin{aligned} \hat{r}(x,\xi;z)\chi_1 &= \chi_1\hat{r}_0(x,\xi;z) + \pi(x,\xi)\chi_1\hat{r}_0(x,\xi;z) \\ &\quad -\hat{r}(x,\xi;z)\chi_1\pi_0(x,\xi) + \hat{r}(x,\xi;z)(Q_\theta\chi_1 - \chi_1 T_0)\hat{r}_0(x,\xi;z). \end{aligned} \tag{3.20}$$

Making use of (3.19) and the fact that $|y| < 2\epsilon$ for $y \in supp\chi_1$, we derive:

$$\begin{aligned} &\|(Q_\theta\chi_1 - \chi_1 T_0)\hat{r}_0(x,\xi;z)\| \\ &\le C\hbar^{1/2} + C'\|y^3\chi_1\hat{r}_0(x,\xi;z)\| \\ &\le C\hbar^{1/2} + C''\epsilon \end{aligned}$$

where C'' is indpendent of ϵ and $\hbar$. From (3.18) and (3.20), we obtain:

$$\hat{r}(x,\xi;z) = \hat{\pi}A + \chi_1\hat{r}_0(x,\xi;z) + \hat{r}(x,\xi;z)B,$$

where B is bounded by $O(\hbar^{1/2}) + C''\epsilon$. So $1-B$ is invertible, if $\epsilon, \hbar$ are small enough. This proves the proposition. $\Box$

¿From the proof of Proposition 3.4, we see that

$$\|\partial_{x\xi}^\alpha \hat{r}(x,\xi)\| \le C_\alpha(1+|x|+|\xi|)^{-|\alpha|},$$

for $|x|+|\xi| > \delta/2$.

Corollary 3.5 *With the notations of Proposition 3.4, we have, for $R > 1$ large enough,*

$$\|\partial_{x\xi}^\alpha \hat{r}(x,\xi;z)\| \le \begin{cases} C_\alpha \hbar^{-(1+|\alpha|)/2}, & R\hbar^{1/4} \le |x|+|\xi| \le \delta/2, \\ C_\alpha \hbar^{-1-3|\alpha|/4}, & |x|+|\xi| < R\hbar^{1/4}, \end{cases}$$

for any $\alpha \in \mathbf{N}^2$.

Proof. Note that for w near 0 and $|x|+|\xi| \ge R\hbar^{1/4}$, we have $V(x,\xi,y) \le E - cR\hbar^{1/2}$, for some $c > 0$ and

$$|q_\theta - z| \ge cR\hbar^{1/2} > 0, \text{ for } z \in \Omega^j.$$

As in Appendix, we see that the norm of the $\hbar$ pseudo-differential operator with symbol $(q_\theta - z)^{-1}$ is bounded by $O(1/(R\hbar^{1/2}))$. Therefore for R large enough, we can construct an approximation for $\hat{r}(x,\xi)$ and derive that

$$\|\hat{r}(x,\xi)\| \le C\hbar^{-1/2}, \qquad |(x,\xi)| > R\hbar^{1/4}.$$

The first part of the corollary follows. To prove the second part, we note that by the proof of Proposition 3.4, we have the resolvent equation

$$\hat{r} = (\hat{\pi}A + \chi_1\hat{r}_0)(1-B)^{-1}. \tag{3.21}$$

So it is sufficient to prove the corollary for $\hat{r}_0$. Since $\hat{\pi}_0 T_0 = T_0 - \lambda_j^0\hat{\pi}_0$, we can compute:

$$\partial_{x\xi}\hat{r}_0 = -\hat{r}_0(\partial_{x\xi}T_0 - \partial_{x\xi}(\lambda_j^0))\hat{r}_0.$$

Since $\partial_{x\xi}T_0 - \partial_{x\xi}(\lambda_j^0\hat{\pi}_0) = O(|x|+|\xi|)$, we obtain:

$$\|\partial_{x\xi}\hat{r}_0(x,\xi)\| \le C\hbar^{-7/4}$$

for $|x|+|\xi| < R\hbar^{1/4}$. Now the second part of the corollary follows from an induction on $|\alpha|$. $\square$

Now let $E_{+-}(B,Z;\lambda)$ be constructed in Theorem 2.2. Assume that $(2n+1)B + ZE \in \Omega$. Then for B large, one has: $(2n+1)B + Z\Omega^j \subset \Omega$. We denote:

$$F_{+-}(h,\hbar;z) = \frac{1}{Z}E_{+-}(B,Z;\lambda), \text{ with } h = \frac{1}{B}, \hbar = \frac{1}{\sqrt{Z}}, \lambda = (2n+1)B + Zz.$$

Let $u_j(x,\xi)$ and $u_j^*(x,\xi), (x,\xi) \in B_\delta$, be the same as in the definition of $\hat{\pi}(x,\xi)$ in Proposition 3.4. We can take $u_\infty \in \mathcal{S}(\mathbf{R}_y)$ independent of (x,ξ) such that $\|u_\infty\| = 1$ and for some $0 < \epsilon < 1/2$,

$$|<u_\infty, u_j(x,\xi)>| < \epsilon \tag{3.22}$$

$$|<u_\infty, u_j^*(x,\xi)>| < \epsilon \tag{3.23}$$

for all $(x,\xi) \in B_\delta$. In fact, we just need to take

$$u_\infty = R^{-1/2}v(./R), \text{ where } v \in \mathcal{S}, v(y) = 0, \text{ for } |y| < 1, \text{ and } \|v\| = 1.$$

Then by the exponential decay of eigenfunctions (Theorem 3.1), we obtain:

$$|<u_\infty, u_j(x,\xi)>| = O(e^{-c/\hbar}), \ |<u_\infty, u_j^*(x,\xi)>| = O(e^{-c/\hbar}),$$

for $(x,\xi) \in B_\delta$. The desired result follows.

Now let χ be a cut-off function in (x,ξ) such that $0 \leq \chi \leq 1$ and $supp\chi \subset B_\delta, \chi = 1$ on $B_{\delta/2}$. We define:

$$u(x,\xi) = \chi(x,\xi)u_j(x,\xi) + (1-\chi)(x,\xi)u_\infty,$$
$$u^*(x,\xi) = \chi(x,\xi)u_j^*(x,\xi) + (1-\chi)(x,\xi)u_\infty. \tag{3.24}$$

By (3.22) and (3.23), u and u^* are normalizable. So we can assume without loss that

$$< u(x,\xi), u^*(x,\xi) >= 1, \qquad \forall (x,\xi) \in \mathbf{R}^2.$$

Let $R_+ : L^2(\mathbf{R}_x) \to D_0, R_- : L^2(\mathbf{R}^2_{xy}) \to L^2(\mathbf{R}_x)$ be defined by:

$$(R_+f)(x,y) = \frac{1}{2\pi}\iint_{\mathbf{R}^2} e^{i(x-x').\xi}u(\frac{x+x'}{2}, h\xi; y)f(x')dx'd\xi$$

for $f \in \mathcal{S}(\mathbf{R}_x)$ and

$$(R_-g)(x) = \frac{1}{2\pi}\iint e^{i(x-x').\xi}(\int_{\mathbf{R}_y} \bar{u}^*(\frac{x+x'}{2}, h\xi)g(x,y)dy)dx'd\xi$$

for $g \in \mathcal{S}(\mathbf{R}_x; L^2(\mathbf{R}_y))$. Here D_0 is the domain of D_y^2 in $L^2(\mathbf{R}^2)$. It is clear that R_+, R_- extend to bounded operators in associated spaces. For $z \in \Omega^j$, consider the operator $\mathcal{Q}(z) : D_0 \oplus L^2(\mathbf{R}_x) \to L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}_x)$ defined by:

$$\mathcal{Q}(z) = \begin{pmatrix} F_{+-}(h,\hbar;z) & R_+ \\ R_- & 0 \end{pmatrix}.$$

To describe the inverse of $\mathcal{Q}(z)$, introduce classes of operator-valued symbols depending on the parameter $\hbar$.

Definition 1. Let H_1, H_2 be two Hilbert spaces. For $m \in \mathbf{R}, \rho \in [0,1]$, we denote by $\mathbf{S}^{m,\rho}(H_1,H_2)$ the set of functions $a(\hbar) \in C^\infty(\mathbf{R}^{2n}_{x\xi}; \mathcal{L}(H_1,H_2))$ satisfying the estimates:

$$\|\partial^\alpha a(x,\xi;\hbar)\| \leq C_\alpha \hbar^{m-\rho|\alpha|}, (x,\xi) \in \mathbf{R}^{2n},$$

for $0 < \hbar < \hbar_0$. In the case $\rho = 1$, we denote $\mathbf{S}^{m,\rho}(H_1,H_2)$ by $\mathbf{S}^m(H_1,H_2)$ and in the case $H_1 = H_2 = \mathbf{C}$, we denote $\mathbf{S}^{m,\rho}(H_1,H_2)$ by $S^{m,\rho}$.

By Corollary 3.5, we see that $\hat{r} \in \mathbf{S}^{-1,3/4}(H_1, H_2)$ with $H_1 = L^2(\mathbf{R}_y)$, $H_2 = D(D_y^2)$. In Appendix, we shall give some results on the symbolic calculus of h-pseudo-differential operators with symbols in $\mathbf{S}^{m,\rho}(H_1, H_2)$. Let κ be the ratio of nuclear charge to magnetic force: $\kappa = Z/B = h/\hbar^2$. In the following all estimates are uniform with respect to κ.

Theorem 3.6 *There exists $h_0 > 0$ such that $\mathcal{Q}(z)$ is invertible for all $0 < h < h_0, z \in \Omega^j$ and $0 < \hbar < \hbar_0$ with $\hbar_0$ small. Let $\mathcal{G}(z)$ denote its inverse. Then $\mathcal{G}(z)$ is holomorphic in $z \in \Omega^j$ and one has:*

$$\mathcal{G}(z) = \left(\begin{array}{cc} G_0(z) & G_+(z) \\ G_-(z) & G_{+-}(z) \end{array} \right), \tag{3.25}$$

where $G_{+-}(z)$ is an h-pseudo-differential operator with symbol $g(.,.;h,\hbar,\kappa,z) \in S^0$ given by:

$$g(x,\xi;h,\hbar,\kappa,z) \sim \sum_{k=0}^{\infty} h^k g_k(x,\xi;\hbar,\kappa,z),$$

*where $g_j(.,.;\hbar,\kappa,z) \in S^{3/2-j,3/4}, j \geq 2$ and $g_0(.,.;\hbar,\kappa,z), g_1(.,.;\hbar,\kappa,z)$ are bounded symbols in usual sense (*i.e. *all derivatives are uniformly bounded in $\hbar$). In particular, g_0 is independent of κ and*

$$g_0(x,\xi;\hbar,z) = \mu_j(x,\xi;\hbar) - z,$$

for $(x,\xi) \in B_{\delta/2}$.

Proof. Remark first that according to (2.6),

$$F_{+-}(h,\hbar;z) = z - \{Q_\theta + Zh^2 R_1 + O(h^2)\} = z - \{Q_\theta + O(h)\},$$

where $Q_\theta = \frac{1}{Z}Q(h,Z;\theta)$ with $Q(h,Z;\theta)$ given by (2.12) and R_1 is an operator whose symbol r_1 is given by (2.11). The operator-valued symbol of $\mathcal{Q}(z)$ is given by:

$$\mathcal{Q}(x,\xi;z) = \left(\begin{array}{cc} z - F(x,\xi;z) & r_+(x,\xi) \\ r_-(x,\xi) & 0 \end{array} \right) : D_0 \oplus \mathbf{C} \to L^2(\mathbf{R}_y) \oplus \mathbf{C}. \tag{3.26}$$

Here

$$\begin{aligned} F(x,\xi;z) &= Q_\theta(x,\xi)+O(h) \\ r_+(x,\xi):\mathbf{C}\ni c &\to r_+(x,\xi)c\equiv cu(x,\xi)>\in L^2, \\ r_-(x,\xi):L^2\ni f &\to r_-(x,\xi)f\equiv<f,u^*(x,\xi)>\in \mathbf{C}. \end{aligned}$$

We propose to prove that $\mathcal{Q}(x,\xi;z)$ is invertible for all (x,ξ) and $z\in\Omega^j$. Let π' be defined by:

$$(\pi'(x,\xi)f)(y)=<f,u^*(x,\xi)>u(x,\xi).$$

Then $\pi'(x,\xi)=\pi(x,\xi)$, for $(x,\xi)\in B_{\delta/2}$. Put $\hat{\pi}'=1-\pi'$. Applying Proposition 3.4, we see that the reduced resolvent:

$$\hat{R}(x,\xi)\equiv(z-\hat{\pi}'F(x,\xi;z)\hat{\pi}')^{-1}\hat{\pi}'$$

exists for all $(x,\xi)\in B_{\delta/2}$ and $z\in\Omega^j$ and satisfies the estimates of Corollary 3.5. Let $\mathcal{G}_0(.)$ be defined by:

$$\mathcal{G}_0(x,\xi;z)=\begin{pmatrix} \hat{R}(x,\xi) & r_+(x,\xi) \\ r_-(x,\xi) & \lambda_j(x,\xi;h,\hbar)-z \end{pmatrix},\ \text{for } (x,\xi)\in B_{\delta/2}.$$

Then it is easy to check that

$$\mathcal{Q}(x,\xi;z)\mathcal{G}_0(x,\xi;z)=1+O(h),\quad \mathcal{G}_0(x,\xi;z)\mathcal{Q}(x,\xi;z)=1+O(h),$$

for $(x,\xi)\in B_{\delta/2}$. Therefore we conlude that the inverse of $\mathcal{Q}(x,\xi;z)$ exists in this region and is equal to $\mathcal{G}_0(x,\xi;z)(1+O(h))$. For $|x|+|\xi|\geq\delta/2$, one has seen that $Q_\theta(x,\xi)-z$ is invertible for $z\in\Omega^j$, if $\hbar$ is small enough and the inverse is uniformly bounded in $(x,\xi)\notin B_{\delta/2}$. By an argument of perturbation, we derive that $(z-F(x,\xi;z))^{-1}$ exists and satisfies:

$$\|\partial^\alpha_{x\xi}(z-F(x,\xi;z))^{-1}\|\leq C_\alpha$$

for $(x,\xi)\notin B_{\delta/2}$. Consider the problem

$$\left\{\begin{array}{lll} (z-F(x,\xi;z))u & + \ r_+(x,\xi)v & =u', \\ \quad r_-(x,\xi)u & & =v', \end{array}\right. \tag{3.27}$$

where $u' \in L^2$ and $v' \in \mathbf{C}$. For $(x,\xi) \notin B_{\delta/2}$, (3.27) is equivalent with

$$\left\{ \begin{array}{rcl} u & = & (z - F(x,\xi;z))^{-1}(u' - r_+(x,\xi)v) \\ r_-(z - F(x,\xi;z))^{-1} r_+ v & = & r_-(z - F(x,\xi;z))^{-1}u' - v'. \end{array} \right. \tag{3.28}$$

Note that $r_-(x,\xi)(z - F(x,\xi;z))^{-1}r_+(x,\xi)$ is a complex number and we can estimate:

$$\begin{aligned} & r_-(x,\xi)(z - F(x,\xi;z))^{-1}r_+(x,\xi) \\ & = \; < (z - F(x,\xi;z))^{-1}u(x,\xi), u^*(x,\xi) > \\ & = \; \frac{1}{z - \tilde{\chi}\lambda_j}\{< u, u^* > - < (z-F)^{-1}u_\infty, u^* > \tilde{\chi}\lambda_j(1-\chi)\} + O(\hbar^2) + o_R(1) \\ & = \; \frac{1}{z - \tilde{\chi}\lambda_j} + O(\hbar^2) + o_R(1) \neq 0, \end{aligned}$$

for $\hbar$ and $1/R$ small enough. Here $\tilde{\chi}$ is supported in $B_\delta, 0 \leq \tilde{\chi} \leq 1$ and $\tilde{\chi} = 1$ on $supp\chi$. In the above equalities, we have used the estimates:

$$\begin{aligned} F(x,\xi;z)u & = \chi\lambda_j u_j + O(\hbar^2) + o_R(1) \\ & = \tilde{\chi}\lambda_j u - \tilde{\chi}\lambda_j(1-\chi)u_\infty + O(\hbar^2) + o_R(1), \qquad \text{and} \\ F(x,\xi;z)u_\infty & = O(\hbar^2) + o_R(1), \end{aligned}$$

with $\lim_{R\to\infty} o_R(1) = 0$, due to the construction of u_∞. Remark that $\Re\lambda_j < E - c_0$, $c_0 > 0$, for $(x,\xi) \notin B_{\delta/2}$, so $z - \lambda_j \neq 0$, for $z \in \Omega^j$.

Put $c = 1/((r_-(z-F)^{-1}r_+)(x,\xi;z))$. Then equations (3.28) have a unique solution for each couple of $(u',v') \in L^2(\mathbf{R}_y) \oplus \mathbf{C}$:

$$\left\{ \begin{array}{rcl} u & = & (z-F)^{-1}((1 - cr_-(z-F)^{-1})u' + cr_+(x,\xi)v') \\ v & = & cr_-(z-F)^{-1}u' - cv'. \end{array} \right. \tag{3.29}$$

This proves that $\mathcal{Q}(x,\xi;z)$ is also invertible for $(x,\xi) \notin B_{\delta/2}$ and the inverse is given by:

$$\begin{aligned} & \mathcal{Q}^{-1}(x,\xi;z) \\ & = \begin{pmatrix} (z-F)^{-1} - c(z-F)^{-1}\pi'(z-F)^{-1} & c(z-F)^{-1}r_+ \\ cr_-(z-F)^{-1} & -c \end{pmatrix} \end{aligned} \tag{3.30}$$

for $(x,\xi) \notin B_{\delta/2}$.

Now let $\mathcal{G}_1(z)$ denote the h-pseudo-differential operator with operator-valued symbol $\mathcal{Q}^{-1}(x,\xi;z)$. Since $\|\partial^\alpha_{x\xi}\mathcal{Q}^{-1}(x,\xi;z)\|$ is uniformly bounded for $(x,\xi) \notin B_{\delta/2}$ and bounded by $O(\hbar^{-1-3|\alpha|/4})$ for $(x,\xi) \in B_{\delta/2}$, we can apply the results on symbolic calculus given in the Appendix (see the remark after the proof of Proposition A.2) to obtain that

$$\mathcal{Q}(z)\mathcal{G}_1(z) = 1 + hR',$$

where R' is a pseudo-differential operator satisfying the estimate:

$$\|hR'\| \leq Ch^{1/4}.$$

This proves the existence of a right inverse, if $h > 0$ is small enough. By the same method we can construct a left inverse, hence prove the inversibility of $\mathcal{Q}(z), z \in \Omega^j$.

To compute the symbols of the inverse $\mathcal{G}(z)$, we use the formula:

$$\begin{aligned} \mathcal{G}(z) &= \mathcal{G}_1(z)(1+hR')^{-1} \\ &= \mathcal{G}_1(z)(1+\sum_{j=1}^{N}(-1)^j(hR')^j + (-1)^{N+1}(hR')^{N+1}(1+hR')^{-1}. \end{aligned}$$

The leading symbol of R' is given by:

$$r'(x,\xi) \equiv i/2\{\mathcal{Q}(x,\xi;z), \mathcal{Q}^{-1}(x,\xi;z)\}.$$

Here $\{.,.\}$ denotes the Poisson bracket. Note that r' is uniformly bounded on $\mathbf{R}^2 \setminus B_\epsilon$, for any $\epsilon > 0$. For $(x,\xi) \in B_\epsilon$, $\mathcal{Q}^{-1}(x,\xi) = \mathcal{G}_0(x,\xi) + O(h/\hbar)$. Therefore we have:

$$r'(x,\xi) = i/2\{\mathcal{Q}(x,\xi), \mathcal{G}_0\} + O(1),$$

for $(x,\xi) \in B_\epsilon$. By a direct computation, we obtain:

$$r'(x,\xi) = \frac{i}{2}\begin{pmatrix} -\partial_\xi V_\theta \partial_x \hat{r} + \partial_x V_\theta \partial_\xi \hat{r} & 0 \\ \partial_\xi r_- \partial_x \hat{r} - \partial_x r_- \partial_\xi \hat{r} & 0 \end{pmatrix} + O(1),$$

where $V_\theta = V(x,\xi,e^\theta y)$, $\hat{r}$ is the same as in (3.17) and $O(1)$ is a term bounded in usual sense. ¿From here we conclude that g_1 is a bounded symbol.

To estimate g_2, we compute

$$\mathcal{G}_0 r' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} + O(1),$$

where

$$\begin{aligned} a_{11} &= \{\hat{r}(-\partial_\xi V_\theta \partial_x \hat{r} + \partial_\xi r_+ \partial_x r_-) + r_+ \partial_\xi r_- \partial_x \hat{r}\} - O(x \leftrightarrow \xi), \\ a_{12} &= \hat{r}(-\partial_\xi V_\theta \partial_x r_+ + \partial_x \lambda_j \partial_\xi r_+) - O(x \leftrightarrow \xi), \\ a_{21} &= -r_- \partial_\xi V_\theta \partial_x \hat{r} - O(x \leftrightarrow \xi). \end{aligned}$$

In the above equations, $O(x \leftrightarrow \xi)$ denotes a term having the same expression as that just preceding it with ∂_x, ∂_ξ interchanged. Making use of the identity $\hat{F} = F - \pi F \pi + O(h^2)$ and the estimate $\partial_{x\xi} V_\theta = \partial_{x\xi} V(x, \xi, 0) + O(y^2|(x,\xi)|)$ for (x, ξ) near 0 and y near 0, we deduce from Corollary 3.5 that

$$a_{11} = O(\hbar^{-3/2}), \quad a_{12} = O(\hbar^{-1/2}), \quad a_{21} = O(\hbar^{-1/2}).$$

Consequently the element in the right lower corner of $\mathcal{G}_0 r' r'$ is in $S^{-1/2,3/4}$, by applying the result of Corollary 3.5. This proves: $g_2 \in S^{-1/2,3/4}$. The results on $g_k, k > 2$ follow by an induction. $\square$

¿From Theorem 3.6, we derive that 0 is an eigenvalue of $E_{+-}(\lambda)$ if and only if 0 is an eigenvalue of $G_{+-}(z)$ with $\lambda = (2n+1)B + Zz, z \in \Omega^j$ and the multiplicities are the same (if we count them appropriately). Note that by the construction of the inverse to $\mathcal{Q}(z)$, one has:

$$g_0(x, \xi; \hbar, z) = \begin{cases} \mu_j(x, \xi; \hbar) - z, & (x,\xi) \in B_{\delta/2}, \\ \tilde{\chi} \mu_j(x, \xi; \hbar) - z + o_R(1) & (x, \xi) \in \mathbf{R}^2 \setminus B_{\delta/2}. \end{cases} \tag{3.31}$$

This shows that $g_0(.,.; \hbar, z), z \in \Omega^j$, is elliptic outside any small neighbourhood of 0, provided $\hbar > 0$ is small enough.

4 Barrier Resonaces

In this Section we want to study a non-linear eigenvalue problem related to $G_{+-}(z)$ for $z \in \Omega^j$. Due to the preparations made in the preceding Section, we can now easily give complete asymptotic expansions for the eigenvalues of G_{+-} near E. See [20] and also [12] for self-adjoint problems.

To symplify notations, we denote:

$$F(h,z) = g_0(x, hD_x; \hbar, z) + hg_1(x, hD_x; \hbar, \kappa, z).$$

To study the eigenvalues of F for $z \in \Omega^j$, we look at its microlocalization near 0. Recall that $g_0 = \mu_j - z$ for (x,ξ) near 0 and a_1 in (2.6) is given by: $a_1 = (2n+1)/4V_1 + \kappa r_1$. By (2.11) and (3.13), $r_1 \equiv 0$, if $(x,\xi) = (0,0)$. ¿From the computations made in the proof of Theorem 3.6, one can check that

$$\nu_1(\hbar) \equiv g_1(0,0;\hbar,\kappa,z) \tag{4.1}$$

is independent of κ and z. We write:

$$g_0(x,\xi;\hbar,z) = E_j(\hbar) - z + \frac{1}{2} < A(x,\xi),(x,\xi) > + O((x,\xi)^3)$$

for (x,ξ) near 0. Here A is the Hessian of μ_j at 0. So $\Re A$ is negatively definite and $\Im A = O(\hbar)$. It is clear that

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \text{ with } a, b, c, \in \mathbf{C}.$$

Lemma 4.1 *Let F_0 denote the maximal closed extension of the h-(pseudo-)differential operator with symbol $f = 1/2 < A(x,\xi),(x,\xi) >$. Then F_0 has purely disrete spectrum and*

$$\sigma(F_0) = \{h(2k+1)\nu_0; k \in \mathbf{N}\}.$$

Here $\nu_0 = -1/2(det\, A)^{1/2}$ and the branch of $(.)^{1/2}$ is chosen so that $\Re(det\, A)^{1/2} > 0$.

Proof. Since $\Re f \leq -c_0(x^2+\xi^2), c_0 > 0$, it is clear that F_0 has only discrete spectrum. To compute its eigenvalues, we decompose the symbol f as:

$$f(x,\xi) = ax^2 + 2bx\xi + c\xi^2 = (c\xi + \lambda x)(\xi + \mu x),$$

where we take:

$$\lambda = \frac{a}{\mu}, \qquad \mu = \frac{b - i\sqrt{det\,A}}{c}.$$

with $\Re\sqrt{det\,A} > 0$. Since $\Im b, \Im c$ are of order $O(\hbar)$ and $\Re c < 0$, we see that $\Im\mu < 0$, so the equation: $(hD_x + \mu x)u = 0$ has a solution in L^2 given by:

$$\psi_0 = e^{-i\mu x^2/(2h)}. \tag{4.2}$$

Let $a = hD + \mu x, a' = chD + \lambda x$. Then we can compute:

$$F_0 = a'a + h\nu_0, \qquad [a, a'] = 2h\nu_0. \tag{4.3}$$

So we derive easily that $\psi_n \equiv a'^n\psi_0, n \in \mathbf{N}$, is an eigenfunction of F_0 with eigenvalue $h(2n+1)\nu_0$. We leave the reader the care to check that these are the only eigenvalues of F_0. See also [17] for harmonic oscillator. $\Box$

It can be seen from the construction of $\nu_0,\ \nu_1$ that they depend on j and $\hbar$, but are independent of h and κ. Now let Ω_k be the domain:

$$\Omega_k = \{z \in \mathbf{C}; |z - E_j(\hbar) - h((2k+1)\nu_0 + \nu_1)| < \epsilon h\},$$

for $\epsilon > 0$ small enough. As in [20], we can show that outside the union of all Ω_k, there is no resonance of $P(B, Z)$.

Proposition 4.2 *For $h > 0$ small, there exists exactly one point $z = z(h, \hbar, \kappa)$ $\in \Omega_k$ such that 0 is a simple eigenvalue of $G_{+-}(z)$ and one has:*

$$z = E_j(\hbar) + h((2k+1)\nu_0 + \nu_1) + O(h^{3/2}).$$

Proof. We denote: $P_0 \equiv F_0 + h\nu_1 + E_j(\hbar), E_{jk} = E_j(\hbar) + h((2k+1)\nu_0 + \nu_1)$. Let u, u^* be eigenfunctions of P_0 and P_0^* associated with eigenvalues E_{jk} and $\bar{E}_{jk}$, respectively. Assume that u and u^* are appropriately normalized. Then as in [20], we can show that the operator:

$$\mathcal{A}(z) \equiv \begin{pmatrix} G_{+-}(z) & .u \\ < ., u^* > & 0 \end{pmatrix} : L^2(\mathbf{R}) \oplus \mathbf{C} \to L^2(\mathbf{R}) \oplus \mathbf{C} \tag{4.4}$$

is invertible for $z \in \Omega_k$. Since u and u^* behave like $O(e^{-cx^2/h})$, $c > 0$, we can derive that the inverse of $\mathcal{A}(z)$ is given by

$$\mathcal{B}(z) = \begin{pmatrix} B_0(z) & B_+(z) \\ B_-(z) & B(z) \end{pmatrix},$$

where $B(z)$ is holomorphic function in $z \in \Omega_k$ and $B(z) = z - E_{jk} + O(h^{3/2})$. By Riemann-Roche Theorem, $B(z)$ has a simple zero inside Ω_k. The desired results now follow from the arguments of [13]. $\Box$

Making use of the eigenvalues and eigenfunctions of P_0 as first approximations, we can construct an asymptotic solution to the non-linear eigenvalue problem:

$$G_{+-}(\tilde{E})\tilde{u} = O(h^\infty), \qquad \tilde{E} \in \Omega_k$$

in the form

$$\begin{aligned} \tilde{E} &\sim E_{jk} + \sum_{l=2}^{\infty} h^l e_l, \\ \tilde{u} &\sim \psi_0 \sum_{l=0}^{\infty} h^j u_j, \end{aligned}$$

where ψ_0 is given in Lemma 4.1. To accomplish this, we use the smoothness of $G_{+-}(z)$ in $z \in \Omega_k$ and the Taylor expansion of the symbol of $G_{+-}(z)$ at $(x, \xi) = (0, 0)$ to write $G_{+-}(z)$ as a formal differential operator. Note that

$$\|x^k (hD_x)^j \psi_0 u_l\| = O(h^{(k+j)/2+1/4}),$$

for any bounded smooth function u_l non-oscillatory in h. This allows us to construct by an induction in j the u_j's in a small neighbourhood of $x = 0$. To make a sense out of these formal constructions, we need to introduce a microlocal cut-off of the form $\chi(x)\chi(hD_x)$, where χ is supported in a small neighbourhood of 0 and $\chi = 1$ near 0. Taking notice that ψ_0 and its Fourier transform are exponentially decreasing (in h and $|x|$) outside a small neighbourhood of 0, the remainder related to the cut-off is negligible. Note in particular that we do not need the analyticity of the symbol of $G_{+-}(z)$ in (x, ξ), since we are contented with a remainder estimate of the form $O(h^\infty)$ and the results on symbolic calculus of usual pseudo-differential operators are sufficient for us. The details of the construction of asymptotic solutions are similar as in [12], [16],[20] and will not be given here. Now studying a Grushin problem similar to (4.4) with u, u^* replaced by $\tilde{u}, \tilde{u}^*$, we can show that the unique resonance of $P(B, Z)$ inside Ω_k is given by $\tilde{E} + O(h^\infty)$. Summing up, we have proven the main result of this paper:

Theorem 4.3 *Under the assumptions (1.1), (1.2), (3.1), (3.2) and (3.13), the resonaces of $P(B, Z)$ in $(2n+1)B + Z\Omega^j$ exist for all n and j. They are all simple and are given by complete asymptotic expansion:*

$$\begin{aligned} z_{njk}(h, \hbar, \kappa) \quad &\sim \quad (2n+1)B + Z\{E_j(\hbar) + h((2k+1)\nu_0(\hbar) + \nu_1(\hbar)) \\ &\qquad + \sum_{l=2}^{\infty} h^l \nu_{jkln}(\hbar, \kappa)\}, \end{aligned}$$

where the asymptotic expansion in h is uniform in $\hbar, \kappa$ and each $\nu_{jkln}(\hbar, \kappa)$ has a complete expansion in powers of $\hbar$. In particular, we have a complete asymptotic expansion for the width of resonances and the leading term is given by:

$$\begin{aligned} &\Im z_{njk}(h, \hbar, \kappa) \\ &= \quad Z\Im\{E_j(\hbar) + h((2k+1)\nu_0(\hbar) + \nu_1(\hbar))\} + O(Zh^2) \\ &= \quad -Z^{1/2}(2j+1)\sqrt{\frac{|H(0,0)|}{2}} + O(1). \end{aligned}$$

Here $h = 1/B$, $\hbar = 1/Z^{1/2}$ and $\kappa = Z/B$.

Since embedded eigenvalues of $P(B, Z)$ in $(2n+1)B + Z\Omega^j$ appear as discrete real eigenvalues of $P(B, Z; \theta), \Im\theta > 0$, Theorem 4.3 implies that the embedded eigenvalues of $P(B, Z)$ in $((2n+1)B + Z\Omega^j) \cap \mathbf{R}$ are absent. Note that under the assumptions of this work, there may be embedded eigenvalues below each Landau level. See [4] for an example. Finally, let us indicate that if $\kappa = Z/B$ is also a small parameter, we can then expand the reduced resolvent $\hat{R}(B, Z; \lambda)$ in terms of the free resolvent $\hat{R}_0(B, \lambda)$. In this way, one can show that $\nu_{jkln}(\hbar, \kappa)$ also admits complete asymptotic expansion in powers of κ.

A Some Remarks on Symbolic Calculus

The classes of symbols used in Section 3 are not standard, in the sense that on one hand, the symbols take values of operators and on the other hand, they depend on a small parameter $\hbar$, over which we loss the control with each derivation. For these reasons, we shall sketch two results on h-pseudo-differential operators with symbols in these particular classes. In the following we always assume that $h = O(\hbar^2)$, as is the case in the preceding sections.

Proposition A.1 *Let $a \in \mathbf{S}^m(H_1, H_2)$, $m \in \mathbf{R}$. Let $a(x, hD_x)$ denote the h-pseudo-differential operator with Weyl symbol a. Then we have:*

$$\|a(x, hD_x)\|_{\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)} \leq C\hbar^m,$$

uniformly in h. Here $\mathbf{H}_j = L^2(\mathbf{R}^n; H_j)$, $j = 1, 2$.

Proof. Remark that $a(x, hD)$ is unitarily equivalent to $a(h^{1/2}x, h^{1/2}D)$ and the derivatives of $a(h^{1/2}x, h^{1/2}\xi)$ are all of the order $O(\hbar^m)$. Then it is sufficient to apply the known results in [5], [18]. □

Proposition A.2 *Let $a_1 \in \mathbf{S}^{m_1,\rho}(H_2, H_3)$, $a_2 \in \mathbf{S}^{m_2,\rho}(H_1, H_2)$, $m_1, m_2 \in \mathbf{R}, \rho \in]0, 1[$. Then $b(x, hD) \equiv a_1(x, hD)a_2(x, hD)$ is an h-pseudo-differential operator with symbol $b \in \mathbf{S}^{m_1+m_2,\rho}(H_1, H_3)$ and*

$$b(x, hD) = \sum_{j=0}^{N} h^j b_j(x, hD) + h^{N+1} R_N(h), \tag{A.1}$$

where

$$b_j(x, \xi) = \sum_{|\alpha|+|\beta|=j} \frac{(-1)^{|\beta|}}{2^j \alpha!\beta!} \partial_\xi^\alpha D_x^\beta a_1 \partial_\xi^\beta D_x^\alpha a_2$$

and $R_N(h)$ satisfies the estimate:

$$\|R_N(h)\| \leq C_N \hbar^{m_1+m_2-2\rho(N+1)}, \tag{A.2}$$

uniformly in h.

Proof. We need only give the proof for (A.2). See [5] for the formula on symbolic calculus. Since each $h^{j-N-1}b_j(x, hD)$ satisfies (A.2) for $j > N+1$, it suffices to show:

$$\|R_N(h)\| \le C_N \hbar^{m_1+m_2-2\rho(N+1)} h^{-N_0}, \tag{A.3}$$

for some N_0 independent of N. (A.3) can be proven by making use of explicit expression for the integral kernel of $R_N(h)$ and methods of oscillatory integrals. The details are omitted here. See [18]. $\square$

Note that if one of the a_j's, say a_1, is in $\mathbf{S}^{m_1,0}$, then we can prove by the same methods that the remainder $R_N(h)$ in (A.1) satisfies:

$$\|R_N(h)\| \le C\hbar^{m_1+m_2-\rho(N+1)},$$

uniformly in h.

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