Tensor structure on $k\mathcal{C}$-mod and cohomology rings

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1. Basic concepts

- $\mathcal{C}$ a small category (we will assume it is **finite** throughout this talk);

- k a field and $Vect_k$ the category of k-vector spaces.

There are three well defined mathematical subjects:

- the category algebra $k\mathcal{C}$;

- the functor category $Vect_k^{\mathcal{C}}$;

- the classifying space $B\mathcal{C}$.

The k-**category algebra** is defined as a k-vector space $k\,\mathsf{Mor}\,\mathcal{C}$, in which the multiplication is given on base elements by

$$\alpha * \beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ are composable in } \mathcal{C}; \\ 0\ , & \text{otherwise.} \end{cases}$$

- an identity $1_{\mathcal{C}} = \sum_{x \in \mathsf{Ob}\,\mathcal{C}} 1_x$;
- an equivalence $k\mathcal{C}$-mod $\simeq Vect_k^{\mathcal{C}}$ (B. Mitchell 1972) given by : $M \mapsto F_M, F_M(x) = 1_x \cdot M$ and $F \mapsto \oplus_{x \in \mathsf{Ob}\,\mathcal{C}} F(x)$. This equivalence gives $k\mathcal{C}$-mod a monoidal category structure: there exists a tensor product $\hat{\otimes}_k$ and a tensor identity $\underline{k}$;
- a co-multiplication $\Delta : k\mathcal{C} \to k\mathcal{C} \otimes_k k\mathcal{C}$,

  $$\Delta(\textstyle\sum_i \lambda_i \alpha_i) = \textstyle\sum_i \lambda_i \alpha_i \otimes \alpha_i,$$

  induced by the canonical diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

2. Tensor structure on $k\mathcal{C}$-mod

Let $M, N \in k\mathcal{C}$-mod. The (internal) tensor product $M\hat{\otimes}_k N \in k\mathcal{C}$-mod is defined by

$$(M\hat{\otimes}_k N)(x) = M(x) \otimes_k N(x),$$

where $x \in \mathrm{Ob}\,\mathcal{C}$. Note that $M\hat{\otimes}_k N \subset M \otimes_k N$ as vector spaces, and the category algebra acts on it via $\Delta : k\mathcal{C} \to k\mathcal{C}\hat{\otimes}_k k\mathcal{C} \subset k\mathcal{C} \otimes_k k\mathcal{C}$.

There is a **tensor identity** $\underline{k} \in k\mathcal{C}$-mod, which as an element in $Vect_k^{\mathcal{C}}$ is the constant functor taking k as its values. The tensor identity $\underline{k}$ is also called the **trivial module** of $k\mathcal{C}$.

Lemma The trivial module can be realized by $k\,\mathrm{Ob}\,\mathcal{C}$. In fact, there exists a surjection

$$\epsilon : k\mathcal{C} \to k\,\mathrm{Ob}\,\mathcal{C},$$

defined on base elements by $\epsilon(\alpha) = t(\alpha)$, the target of the morphism α. This gives $k\,\mathrm{Ob}\,\mathcal{C}$ a $k\mathcal{C}$-module structure and it is isomorphic to $\underline{k}$.

The map $\epsilon : k\mathcal{C} \to \underline{k}$ plays the role of a co-unit.

The algebra $k\mathcal{C}$ is *almost a cocommutative bialgebra* in the sense that we can produce the following structure maps: 1. co-associativity

$$\begin{tikzcd}
k\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & k\mathcal{C} \otimes k\mathcal{C} \arrow[d, "\Delta\otimes 1"] \\
k\mathcal{C} \otimes k\mathcal{C} \arrow[r, "1\otimes\Delta"'] & k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C};
\end{tikzcd}$$

2. **co-unitary property**

$$\begin{tikzcd}
 & k\mathcal{C} \arrow[dl, "\cong"'] \arrow[d, "\Delta"'] \arrow[dr, "\cong"] & \\
\underline{k}\hat{\otimes} k\mathcal{C} & k\mathcal{C}\hat{\otimes} k\mathcal{C} \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] & k\mathcal{C}\hat{\otimes}\underline{k};
\end{tikzcd}$$

3. **co-commutativity**

$$\begin{tikzcd}
 & k\mathcal{C} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
k\mathcal{C} \otimes k\mathcal{C} \arrow[rr, "T"'] & & k\mathcal{C} \otimes k\mathcal{C};
\end{tikzcd}$$

4. multiplication and co-multiplication

$$\begin{tikzcd}
k\mathcal{C} \otimes k\mathcal{C} \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & k\mathcal{C} \arrow[dd, "\Delta"] \\
k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \arrow[d, "1\otimes T\otimes 1"'] & \\
k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \otimes k\mathcal{C} \arrow[r, "\mu\otimes\mu"'] & k\mathcal{C} \otimes k\mathcal{C};
\end{tikzcd}$$

5. unit and co-multiplication:

$$\begin{tikzcd}
& k \otimes k \arrow[dl, "\mu"'] \arrow[dr, "\pi\circ(\iota\otimes\iota)"] & \\
k\mathcal{C} \arrow[rr, "\Delta"'] & & k\mathcal{C}\widehat{\otimes}k\mathcal{C};
\end{tikzcd}$$

where ι is the inclusion $k = k1_{\mathcal{C}} \hookrightarrow k\mathcal{C}$ and π is the truncation map $k\mathcal{C} \otimes k\mathcal{C} \to k\mathcal{C}\widehat{\otimes}k\mathcal{C}$.

6. unit and co-unit

$$\begin{tikzcd}
& k \arrow[dl, "\iota"'] \arrow[dr, "\eta"] & \\
k\mathcal{C} \arrow[rr, "\epsilon"'] & & \underline{k},
\end{tikzcd}$$

where the k-linear map $\eta : k = k1_{\mathcal{C}} \to \underline{k}$ is defined by $1_{\mathcal{C}} \mapsto \sum_{x\in \mathrm{Ob}\,\mathcal{C}} x$.

Remark a) We do not have a bialgebra in the usual sense; b) There is **no antipode** map in general. This is a significant difference between a category algebra and a cocommutative Hopf algebra.

3. Cohomology rings

We call $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k},\underline{k})$ the **ordinary cohomology ring** of $k\mathcal{C}$ (with Yoneda splice).

There is also a **Hochschild cohomology ring** $\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C},k\mathcal{C})$. Here $k\mathcal{C}^e$ is the category algebra of $\mathcal{C}^e := \mathcal{C}\times\mathcal{C}^{op}$, which is isomorphic to the enveloping algebra $(k\mathcal{C})^e := k\mathcal{C}\otimes_k (k\mathcal{C})^{op}$.

Note that $k\mathcal{C}\in k\mathcal{C}^e$-mod $\simeq Vect_k^{\mathcal{C}^e}$ as a functor is described by $k\mathcal{C}(x,y) = k\,\mathrm{Hom}_{\mathcal{C}}(y,x)$.

Main Theorem There is a natural split surjective algebra homomorphism

$$\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C},k\mathcal{C}) \to \mathrm{Ext}^*_{k\mathcal{C}}(\underline{k},\underline{k}).$$

Remark Note that $k\mathcal{C}^e$-mod is a monoidal category w.r.t. the tensor product $\otimes_{k\mathcal{C}}$ and tensor identity $k\mathcal{C}$. It has another monoidal category structure w.r.t. $\hat{\otimes}_k$ and tensor identity $\underline{k}$.

Let $M, M', N, N' \in k\mathcal{C}$-mod. We will define the cup product to be

$$\cup{:}\mathsf{Ext}^i_{k\mathcal{C}}(M,N)\otimes\mathsf{Ext}^j_{k\mathcal{C}}(M',N')\to\mathsf{Ext}^{i+j}_{k\mathcal{C}}(M\hat{\otimes}M',N\hat{\otimes}N').$$

This gives $\mathsf{Ext}^*_{k\mathcal{C}}(\underline{k},\underline{k})$ a ring structure and its action on $\mathsf{Ext}^*_{k\mathcal{C}}(M,N)$ for arbitrary $M, N \in k\mathcal{C}$-mod. Note that $\mathsf{Ext}^*_{k\mathcal{C}}(\underline{k},F) \cong \mathsf{H}^*(\mathcal{C};F) \cong \varprojlim^*_{\mathcal{C}} F$ for any $F \in k\mathcal{C}$-mod.

Let $\mathbb{C}$ and $\mathbb{D}$ be two complexes of $k\mathcal{C}$-modules.

- The product $(\mathbb{C}\hat{\otimes}\mathbb{D})_n = \oplus_{i+j=n}\mathbb{C}_i\hat{\otimes}\mathbb{D}_j$, with the differential (a natural transformation)

  $$\partial_x(a\otimes b) = \partial^{\mathbb{C}}_x a\otimes b + (-1)^{\mathsf{deg}(a)} a\otimes\partial^{\mathbb{D}}_x b,$$

  where $a \in \mathbb{C}_i(x)$ and $b \in \mathbb{D}_j(x)$, for each $x \in \mathsf{Ob}\,\mathcal{C}$.

- A Künneth formula: for each integer n

  $$\mathsf{H}_n(\mathbb{C}\hat{\otimes}\mathbb{D}) \cong \oplus_{i+j=n}\mathsf{H}_i(\mathbb{C})\hat{\otimes}\mathsf{H}_j(\mathbb{D}).$$

Suppose $\zeta \in \mathsf{Ext}^m_{k\mathcal{C}}(M, N)$ is represented by

$$0 \to N \to L_{m-1} \to \cdots \to L_0 \to M \to 0,$$

and $\zeta' \in \mathsf{Ext}^n_{k\mathcal{C}}(M', N')$ is represented by

$$0 \to N' \to L'_{n-1} \to \cdots \to L'_0 \to M' \to 0.$$

We construct an exact sequence

$$0 \to N \widehat{\otimes} N' \to (L_{m-1} \widehat{\otimes} N) \oplus (N \widehat{\otimes} L'_{n-1}) \to \cdots$$
$$\to L_0 \widehat{\otimes} L'_0 \to M \widehat{\otimes} M' \to 0,$$

and it is defined to be the cup product of ζ and ζ', $\zeta \cup \zeta' \in \mathsf{Ext}^{m+n}_{k\mathcal{C}}(M \widehat{\otimes} M', N \widehat{\otimes} N')$.

Lemma Let ζ, ζ' be as above. The cup product $\zeta \cup \zeta'$ is the Yoneda splice of

$$\zeta \widehat{\otimes} \mathsf{Id}_{N'} \in \mathsf{Ext}^i_{k\mathcal{C}}(M \widehat{\otimes} N', N \widehat{\otimes} N')$$

with

$$\mathsf{Id}_M \widehat{\otimes} \zeta' \in \mathsf{Ext}^j_{k\mathcal{C}}(M \widehat{\otimes} M', M \widehat{\otimes} N').$$

It says that "cup product=Yoneda splice" on $\mathsf{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$. We demonstrate they are the same as the "simplicial cup product".

Proposition With the above cup product,

$$\mathsf{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}) \cong \mathsf{H}^*(B\mathcal{C}, k)$$

as algebras.

The ring $\mathsf{H}^*(B\mathcal{C}, k)$ is computed from the nerve $N_*\mathcal{C}$ of $\mathcal{C}$.

- $N_0\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$;
- $N_n\mathcal{C} = \{x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n | \alpha_i \in \mathrm{Mor}\,\mathcal{C}\}$, if $n > 0$.

On the simplicial complex $kN_*\mathcal{C} \to 0$, for each base element $x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n \in kN_n\mathcal{C}$,

$$\begin{aligned} & \delta(x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n) \\ = \ & \textstyle\sum_{i=0}^{n}(-1)^i x_0 \overset{\alpha_1}{\to} \cdots \to \widehat{x_i} \to \cdots \overset{\alpha_n}{\to} x_n. \end{aligned}$$

The cohomology ring $\mathsf{H}^*(B\mathcal{C}, k)$ is the homology of the cochain complex $0 \to \mathrm{Hom}_k(kN_*\mathcal{C}, k)$, and the cup product is given by the Alexander-Whitney map on $kN_*\mathcal{C} \to 0$.

In order to compare $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$ with $\mathrm{H}^*(B\mathcal{C}, k)$, We note that each $kN_n\mathcal{C}$ has a $k\mathcal{C}$-module structure, and especially $kN_0\mathcal{C} = k\,\mathrm{Ob}\,\mathcal{C} \cong \underline{k}$.

The bar resolution $\mathcal{B}^{\mathcal{C}}_* = \mathcal{B}_*$ of $\underline{k}$. For each $x \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{B}_n(x)$ is the k-vector space with base elements of the form

$$x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n \overset{\alpha}{\to} x$$

with $x_i, x \in \mathrm{Ob}\,\mathcal{C}$, $\alpha_i, \alpha \in \mathrm{Mor}(\mathcal{C})$, and a non-negative $n \in \mathbb{Z}$. The differential, as a natural transformation, is defined subsequently as

$$\begin{aligned} & \delta_x(x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n \overset{\alpha}{\to} x) \\ = \ & \textstyle\sum_{i=0}^{n} (-1)^i x_0 \overset{\alpha_1}{\to} \cdots \to \widehat{x_i} \to \cdots \overset{\alpha_n}{\to} x_n \overset{\alpha}{\to} x. \end{aligned}$$

The complex of $k\mathcal{C}$-modules $\mathcal{B}_* \to \underline{k} \to 0$ is a projective resolution, and for any $M \in k\mathcal{C}$-mod

$$\mathrm{Hom}_{k\mathcal{C}}(\mathcal{B}_n, M) \cong \prod_{x_0 \to x_1 \to \cdots \to x_n \in N_n\mathcal{C}} M(x_n).$$

Note that $\mathcal{B}_0 = k\mathcal{C}$.

Using the bar resolution $\mathcal{B}_* \to \underline{k} \to 0$, we can describe the cup product on $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$.

A diagonal approximation map $D : \mathcal{B}_* \to \mathcal{B}_* \widehat{\otimes} \mathcal{B}_*$, as a natural transformation, is given by

$$D_x(x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_n}{\to} x_n \overset{\alpha}{\to} x)$$

$$= \sum_{i=0}^{n} (x_0 \overset{\alpha_1}{\to} \cdots \to x_i \overset{\alpha \cdots \alpha_{i+1}}{\to} x) \otimes (x_i \overset{\alpha_{i+1}}{\to} \cdots \overset{\alpha_n}{\to} x_n \overset{\alpha}{\to} x),$$

for any $x \in \mathrm{Ob}\,\mathcal{C}$ and integer n.

- $\mathrm{Hom}_{k\mathcal{C}}(\mathcal{B}_*, \underline{k}) \cong \mathrm{Hom}_k(kN_*\mathcal{C}, k)$ as complexes;

- D corresponds to the Alexander-Whitney map.

These imply that $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}) \cong \mathrm{H}^*(B\mathcal{C}, k)$ as algebras.

4. Main theorem

There exists a natural split surjective algebra homomorphism

$$\phi_{\mathcal{C}} : \mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \to \mathsf{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}).$$

- This algebra homomorphism is given by $- \otimes_{k\mathcal{C}} \underline{k}$;
- known to be true for groups (Cartan-Eilenberg 1956 etc) and posets (Gerstenhaber-Shack 1986), with distinct proofs;
- regarding groups and posets as small categories, our result generalizes both well known theorems.

Key features of the proof:

- Use $F(\mathcal{C})$, the category of factorizations in $\mathcal{C}$, introduced by D. Quillen. It generalizes and replaces the diagonal subgroup $\Delta G \subset G \times G$ when $\mathcal{C} = G$ is a group. Indeed there exists a commutative diagram

$$\begin{tikzcd}
F(\mathcal{C}) \arrow[r, "\tau=(t,s)"] \arrow[dr, "t"'] & \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op} \arrow[d, "pr"] \\
 & \mathcal{C}
\end{tikzcd} .$$

  It was shown by Quillen that $t : F(\mathcal{C}) \to \mathcal{C}$ induces a homotopy equivalence.

- Use the left Kan extensions LK_τ, LK_{pr} and LK_t, which generalize various inductions (e.g. $\uparrow_{\Delta G}^{G\times G}$).

The preceding diagram gives rise to new commutative diagrams of module categories and functors among them:

$$\begin{tikzcd}
kF(\mathcal{C})\text{-mod} & & k\mathcal{C}^e\text{-mod} \arrow[ll, "\mathrm{Res}_\tau"'] \\
& k\mathcal{C}\text{-mod} \arrow[ul, "\mathrm{Res}_t"] \arrow[ur, "\mathrm{Res}_{pr}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
kF(\mathcal{C})\text{-mod} \arrow[rr, "LK_\tau"] \arrow[dr, "LK_t"'] & & k\mathcal{C}^e\text{-mod} \arrow[dl, "LK_{pr}"] \\
& k\mathcal{C}\text{-mod} &
\end{tikzcd}$$

Here Res_τ is the functor induced by τ by precomposition, and is called the **restriction** along τ. The functor LK_τ is the well known left adjoint of it, called the **left Kan extension** of τ. The other two pairs of functors are constructed in the same way over t and pr, respectively.

- Res is always exact and preserves $\underline{k}$;

- LK always preserves projectives.

Moving to cohomology:

1. LK_t maps the bar resolution
$$\mathcal{B}_*^{F(\mathcal{C})} \to \underline{k} \to 0$$
to a projective resolution of $k\mathcal{C}$-modules
$$LK_t\mathcal{B}_*^{F(\mathcal{C})} \to LK_t\underline{k} \cong \underline{k} \to 0;$$

2. LK_τ takes the above bar resolution to a projective resolution of $k\mathcal{C}^e$-modules
$$LK_\tau\mathcal{B}_*^{F(\mathcal{C})} \to LK_\tau\underline{k} \cong k\mathcal{C} \to 0;$$

3. LK_{pr} furthermore maps the projective resolution of $k\mathcal{C}$ in 2) to the projective resolution of the $k\mathcal{C}$-module $\underline{k}$ in 1).

4. $\mathrm{Res}_\tau \mathrm{k}\mathcal{C} \cong \underline{k} \oplus \mathrm{N}_{\mathcal{C}}$, as a $kF(\mathcal{C})$-module.

Along with the adjunctions with corresponding restrictions, the previous observations on the three left Kan extensions lead to a commutative diagram of cohomology rings

$$\begin{tikzcd}
\mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k},\underline{k}) \arrow[rr, "\tau^*"] \arrow[dr, "\cong", "t^*"'] & & \mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C},k\mathcal{C}) \arrow[dl, "pr^*=\phi_{\mathcal{C}}"] \\
& \mathsf{Ext}^*_{k\mathcal{C}}(\underline{k},\underline{k}) &
\end{tikzcd} .$$

The map pr^* is the same as the one induced by $-\otimes_{k\mathcal{C}}\underline{k}$ and is often written as $\phi_{\underline{k}}$ or $\phi_{\mathcal{C}}$. The map t^* is an isomorphism. Then $\tau^*(t^*)^{-1}$ becomes a right inverse to $pr^*=\phi_{\mathcal{C}}$.

Remark The map t^* can be identified with the homomorphism $\mathsf{H}^*(BF(\mathcal{C}),k)\to\mathsf{H}^*(B\mathcal{C},k)$ induced by the topological map $Bt : BF(\mathcal{C})\to B\mathcal{C}$ which is a homotopy equivalence, Let f, g be two cocycles representing two cohomology classes. Then we construct the following diagram

$$\begin{tikzcd}
f \cup g : & \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "D^{F(\mathcal{C})}"] & \mathcal{B}_*^{F(\mathcal{C})} \widehat{\otimes} \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "f \otimes g"] & \underline{k} \widehat{\otimes} \underline{k} \cong \underline{k} \\
 & & \arrow[r, phantom, "\Downarrow LK_\tau"] & \\
LK_\tau(f \cup g) : & LK_\tau \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "LK_\tau(D^{F(\mathcal{C})})"] \arrow[d, equal] & LK_\tau(\mathcal{B}_*^{F(\mathcal{C})} \widehat{\otimes} \mathcal{B}_*^{F(\mathcal{C})}) \arrow[r, "LK_\tau(f \otimes g)"] \arrow[d, dotted, "\Theta_\tau"'] & LK_\tau(\underline{k} \widehat{\otimes} \underline{k}) \cong k\mathcal{C} \arrow[d, dotted, "\Theta_0"', "\cong"] \\
LK_\tau(f) \cup LK_\tau(g) : & LK_\tau \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, dotted, "D^{\mathcal{C}^e}"'] & LK_\tau \mathcal{B}_*^{F(\mathcal{C})} \otimes_{k\mathcal{C}} LK_\tau \mathcal{B}_*^{F(\mathcal{C})} \arrow[r, "LK_\tau(f) \otimes LK_\tau(g)"'] & LK_\tau(\underline{k}) \otimes_{k\mathcal{C}} LK_\tau(\underline{k}) \cong k\mathcal{C}.
\end{tikzcd}$$

The commutativity of the lower two rows shows that τ^* preserves cup products.

More generally, for any $M \in k\mathcal{C}^e$-mod,

$$\mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M) \cong \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau M).$$

The splitting of $\mathsf{Res}_\tau k\mathcal{C} \cong \underline{k} \oplus N_{\mathcal{C}}$ induces a surjective homomorphism $\rho : \mathsf{Res}_\tau k\mathcal{C} \widehat{\otimes} \mathsf{Res}_\tau M \to \mathsf{Res}_\tau M$, and

$$\rho^* : \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau k\mathcal{C} \widehat{\otimes} \mathsf{Res}_\tau M) \to \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau M).$$

The latter fits into the following commutative diagram

$$\begin{tikzcd}
\mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau k\mathcal{C}) \otimes \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau M) \arrow[r, "\cup"] \arrow[d, equal] & \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau k\mathcal{C} \widehat{\otimes} \mathsf{Res}_\tau M) \arrow[r, "\rho^*"] & \mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau M) \arrow[d, equal] \\
\mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \otimes \mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M) \arrow[r, "\cup"'] & \mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C} \otimes_{k\mathcal{C}} M) \arrow[r, "\cong"'] & \mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M).
\end{tikzcd}$$

Since $\mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \underline{k})$ is a direct summand of $\mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \mathsf{Res}_\tau k\mathcal{C})$, it exhibits the action of $\mathsf{Ext}^*_{kF(\mathcal{C})}(\underline{k}, \underline{k})$ (and $\mathsf{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$) on $\mathsf{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, M)$.

5. Examples

Example 1 Let k be a field of characteristic 2 and $\mathcal{C}$ the following category

$$\begin{tikzcd}
x \arrow[loop left, "1_x"] \arrow[r, "\alpha"] & y \arrow[loop above, "1_y"] \arrow[loop below, "g"']
\end{tikzcd}$$

with $g^2 = 1_y$ and $\alpha = g\alpha$.

1. Two indecomposable projective: $P_{x,k} = k1_x + k\alpha$ and $P_{y,k} = k1_y + kg$;

2. Two simples of dimension one: $S_{x,k}, S_{y,k}$.

The trivial module is projective, $\underline{k} \cong P_{x,k}$. Hence $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, M) \cong M(x)$ for any $M \in k\mathcal{C}$-mod. Especially $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}) \cong k$. However, both modules $\mathrm{Ext}^*_{k\mathcal{C}}(S_{x,k}, S_{y,k})$ and $\mathrm{Ext}^*_{k\mathcal{C}}(S_{y,k}, S_{y,k})$ are infinite dimensional.

Example 2 A category by Aurélien Djament: $\mathcal{E}$ as follows

$$\begin{tikzcd} x \arrow[loop, out=170, in=120, "h"'] \arrow[loop, out=60, in=10, "1_x"'] \arrow[loop, out=190, in=240, "g"] \arrow[loop, out=350, in=300, "gh"] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & y \arrow[loop right, "\{1_y\}"] \end{tikzcd},$$

where $g^2 = h^2 = 1_x, gh = hg, \alpha h = \beta g = \alpha$, and $\alpha g = \beta h = \beta$. We can compute the mod-2 ordinary and Hochschild cohomology rings.

The ordinary cohomology ring is computed as $\mathsf{H}^*(B\mathcal{E}, k)$ using the fact that $B\mathcal{E} \simeq B(\mathbb{Z}_2 \times \mathbb{Z}_2)/B\mathbb{Z}_2$. Using a long exact sequence we find that $\mathsf{Ext}^*_{k\mathcal{E}}(\underline{k}, \underline{k})$ is a subring of the polynomial ring $\mathsf{H}^*(\mathbb{Z}_2 \times \mathbb{Z}_2, k) \cong k[u, v]$ with $\deg u = \deg v = 1$:

$$\mathsf{Ext}^*_{k\mathcal{E}}(\underline{k}, \underline{k}) \cong \begin{cases} k, & \text{at degree zero;} \\ uk[u, v]\ , & \text{at positive degree.} \end{cases}$$

The ordinary cohomology ring has no nilpotents and is not finitely generated.

By our Main Theorem, there exists a split surjection

$$\phi_{\mathcal{E}} : \mathsf{Ext}^*_{k\mathcal{E}^e}(k\mathcal{E}, k\mathcal{E}) \to \mathsf{Ext}^*_{k\mathcal{E}}(\underline{k}, \underline{k}).$$

Thus **the Hochschild cohomology ring of $k\mathcal{E}$ modulo nilpotents is not finitely generated** either. In fact, in this case, the kernel of $\phi_{\mathcal{E}}$ is explicitly calculated and consists of nilpotents only. As a reminder, $k\mathcal{E}$ is not Hopf. It is not even self-injective.

This above example has been modified by Nicole Snashall:

There exists a 7-dimensional Koszul algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated, independent of the choice of the base field. This algebra is isomorphic to $k\mathcal{E}$ when $\mathrm{ch}k = 2$.

The category of factorizations in $\mathcal{E}$, $F(\mathcal{E})$, has the following shape

$$\begin{tikzcd}
 & & {[\alpha]} \arrow[rr, shift left] & & {[\beta]} \arrow[ll, shift left] & & \\
{[1_x]} \arrow[urr] \arrow[urrrr] \arrow[dr, shift left] & & & & & & {[1_y]} \arrow[ull] \arrow[ullll] \\
 & {[h]} \arrow[ul, shift left] \arrow[uur] \arrow[uurrr] \arrow[drr, shift left] & & & & {[gh]} \arrow[uulll] \arrow[uul] \arrow[dll, shift left] & \\
 & & & {[g]} \arrow[uuul] \arrow[uuur] \arrow[ull, shift left] \arrow[urr, shift left] & & &
\end{tikzcd},$$

in which $[1_x] \cong [h] \cong [g] \cong [gh]$ and $[\alpha] \cong [\beta]$. For the purpose of computation, we use the skeleton $F'(\mathcal{E})$ of $F(\mathcal{E})$. Note that $F(\mathcal{E}) \simeq F'(\mathcal{E})$. Hence $BF(\mathcal{E}) \simeq BF'(\mathcal{E})$ and $kF(\mathcal{E})$-mod $\simeq kF'(\mathcal{E})$-mod.

$$\begin{tikzcd}
 & {[\alpha]} \arrow[loop above, "\{(1_y,1_x^{op})\}"] & \\
{[1_x]} \arrow[ur, "\{(\alpha,1_x^{op}),(\alpha,h^{op}),(\beta,g^{op}),(\beta,(gh)^{op})\}"] \arrow[loop below, "\{(1_x,1_x^{op}),(h,h^{op}),(g,g^{op}),(gh,(gh)^{op})\}"] & & {[1_y].} \arrow[ul, "\{(1_y,\alpha^{op})\}"'] \arrow[loop below, "\{(1_y,1_y^{op})\}"]
\end{tikzcd}$$

In the above category, next to each arrow is the set of homomorphisms in $F'(\mathcal{E})$ from one object to another.

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