
Volume 317 Number 1, November 1, 2007 ISSN 0021-8693
ELSEVIER
JOURNAL OF
ALGEBRA
Editor-in-Chief
Michel Broué
Editor of the section on Computational Algebra
Gerhard Hiss
Founding Editor
G. Higman, 1964–1984
Editor-in-Chief
W. Feit, 1985–2000
Editorial Board
Nicolás Andruskiewitsch
Luchezar L. Avramov
Eva Bayer-Fluckiger
Corrado de Concini
William Crawley-Boevey
Steven Dale Cutkosky
Kent R. Fuller
Dihua Jiang
Masaki Kashiwara
E.I. Khukhro
Kazuhiko Kurano
Gus I. Lehrer
Martin Liebeck
Peter Littelmann
Susan Montgomery
Laurent Moret-Bailly
Jan Saxl
Leonard L. Scott, Jr.
Vera Serganova
Aner Shalev
Ronald Solomon
J.T. Stafford
Gernot Stroth
Michel Van den Bergh
Efim Zelmanov
Section on Computational Algebra
Jon Carlson
John Cremona
Patrick Dehornoy
Harm Derksen
Meinolf Geck
Derek Holt
William M. Kantor
Reinhard Laubenbacher
Gunter Malle
Eamonn O'Brien
Bruno Salvy
Jean-Yves Thibon

Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 317 (2007) 153–183
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Representations of categories and their applications

Fei Xu *

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Received 7 July 2006
Available online 31 August 2007
Communicated by Michel Broué

Abstract

We define for each small category $\mathcal{C}$ a category algebra $R\mathcal{C}$ over a base ring R and study its representations. When $\mathcal{C}$ is an EI-category, we develop a theory of vertices and sources for $R\mathcal{C}$-mod, which parameterizes the indecomposable $R\mathcal{C}$-modules. As a main application, we use our theory to find formulas for computing higher (inverse) limits over $\mathcal{C}$.

Keywords: Representations of categories; Category algebras; Relative projectivity; Vertices and sources; Higher limits; Cohomology ring

1. Introduction

Let $\mathcal{C}$ be a small category and R a commutative ring with an identity. The *category algebra* $R\mathcal{C}$ is a free R-module whose basis is the set of all morphisms of $\mathcal{C}$. The product on the basis elements of $R\mathcal{C}$ is defined by composition and then it is linearly extended to a product on all elements of $R\mathcal{C}$. The category algebra is introduced in the first place as a tool to study representations and cohomology of the category on which it is defined. A *representation* of $\mathcal{C}$ over R is a covariant functor from the small category to the category of R-modules, i.e. R-mod. The starting point of our research is the following result of Mitchell.

Proposition 1.1. *(See Mitchell [25].) For any small category $\mathcal{C}$ with finitely many objects, the category of covariant* (*respectively contra-variant*) *functors from $\mathcal{C}$ to R-mod can be identified with the unital left* (*respectively right*) *$R\mathcal{C}$-modules.*

* Current address: Department of Mathematical Sciences, University of Aberdeen, Meston Building, Aberdeen AB24 3UE, Scotland, UK.
E-mail address: fxu@maths.abdn.ac.uk.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.07.021

When the category in question is a group or a partially ordered set (poset), the category algebra is either a group algebra or an incidence algebra and our work does not provide many new results for such a category algebra. It is the recent active investigations of categories associated with groups that motivates our research. We have in mind the constructions of group modules using representations of the Tits building of a Chevalley group by Ronan and Smith [32], the homology decompositions of classifying spaces of groups using representations of orbit categories and other categories by Dwyer [11], the theory of p-local finite groups of Broto–Levi–Oliver [8], as well as the work of Grodal [17], Jackowski–McClure–Oliver [21], Linckelmann [24] and Symonds [34]. The purpose of this paper is to provide a general framework for studying representations and cohomology of abstract small categories and to prove some general results which can be used to understand the representation and cohomology theory of the particular categories we mentioned above.

To achieve the goal, we need to extract some fundamental properties from the prototypes we have got and use them as axioms to construct a class of abstract small categories, which should be general enough to contain all existing examples while on the other hand is specific enough for us to develop the representation and cohomology theory. This class of categories will be the class of finite EI-categories which we now define. A small category $\mathcal{C}$ is said to be *finite* if its morphisms form a finite set $\mathrm{Mor}\,\mathcal{C}$. It implies that the set of objects $\mathrm{Ob}\,\mathcal{C}$ is finite and that $R\mathcal{C}$ is of finite R-rank with an identity $1 = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} 1_x$. A small category $\mathcal{C}$ is *EI* if every endomorphism is an isomorphism. Any EI-category $\mathcal{C}$ has a prominent property that there exists a natural preorder on its set of objects: if $x, y \in \mathrm{Ob}\,\mathcal{C}$ then $x \leqslant y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$. The preorder allows us to obtain a filtration for any $R\mathcal{C}$-module M: $\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ so that every factor concentrates on a single isomorphism class of objects of $\mathcal{C}$ (that is, as a functor each factor takes non-zero values only at a single isomorphism class of objects). The preorder also allows us to construct very useful auxiliary full subcategories such as $\mathcal{C}_{\leqslant x}$ for every x, where $\mathrm{Ob}\,\mathcal{C}_{\leqslant x} = \{y \in \mathrm{Ob}\,\mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(y, x) \neq \emptyset\}$. Similar constructions include $\mathcal{C}_{<x}$, $\mathcal{C}_{>x}$ and $\mathcal{C}_{\geqslant x}$ for each and every $x \in \mathrm{Ob}\,\mathcal{C}$.

Lück has classified the projective and simple $R\mathcal{C}$-modules for EI-categories, and has studied the restriction of projective modules to certain subalgebras in [23]. The reader can find a short description of his key results in Section 3.1. In this paper we exploit further properties and other aspects of the representations of finite EI-categories, with the applications to computing higher limits in mind.

Let $\mathcal{D}$ be a subcategory of a small category $\mathcal{C}$. Then $R\mathcal{D}$ is a subalgebra of $R\mathcal{C}$. There are two naturally defined functors: the restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}}: R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ and the induction $\uparrow^{\mathcal{C}}_{\mathcal{D}}: R\mathcal{D}\text{-mod} \to R\mathcal{C}\text{-mod}$. The induction is the usual tensor product $R\mathcal{C} \otimes_{R\mathcal{D}} -$, while the restriction is more sophisticatedly defined since $R\mathcal{C}$ is not a unital $R\mathcal{D}$-module. The reader is advised to read from Definition 2.2.2 through Proposition 2.2.3 for more information. We call an $R\mathcal{C}$-module M relatively *$\mathcal{D}$-projective* (or projective relative to $\mathcal{D}$) if M is isomorphic to a direct summand of $M{\downarrow^{\mathcal{C}}_{\mathcal{D}}}{\uparrow^{\mathcal{C}}_{\mathcal{D}}}$. When $\mathcal{D} \subset \mathcal{C}$ is a *full* subcategory, we have the following results.

Theorem 1.2. *Let $\mathcal{D}$ be a full subcategory of a finite EI-category $\mathcal{C}$. Then*

(1) *$M{\downarrow^{\mathcal{C}}_{\mathcal{D}}} \in R\mathcal{D}$-mod is indecomposable if $M \in R\mathcal{C}$-mod is indecomposable and relatively $\mathcal{D}$-projective*;
(2) *$N{\uparrow^{\mathcal{C}}_{\mathcal{D}}} \in R\mathcal{C}$-mod is indecomposable and relatively $\mathcal{D}$-projective if $N \in R\mathcal{D}$-mod is indecomposable*;
(3) *if $M \in R\mathcal{C}$-mod is relatively $\mathcal{D}$-projective then $M{\downarrow^{\mathcal{C}}_{\mathcal{D}}}{\uparrow^{\mathcal{C}}_{\mathcal{D}}} \cong M$.*

The above theorem gives us a parametrization of $R\mathcal{C}$-modules through the full subcategories of $\mathcal{C}$. Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. We define $R\mathcal{C}_{\mathcal{D}}$-mod to be the full subcategory of $R\mathcal{C}$-mod consisting of all modules which are relatively $\mathcal{D}$-projective.

Theorem 1.3. *Let $\mathcal{D}$ be a full subcategory of a finite EI-category $\mathcal{C}$. Then $R\mathcal{D}$-mod is equivalent to $R\mathcal{C}_{\mathcal{D}}$-mod.*

A subcategory $\mathcal{D}$ is called *convex* if the composite of any two morphisms $\alpha, \beta \in \operatorname{Mor}\mathcal{C}$ being a morphism in $\operatorname{Mor}\mathcal{D}$ implies both α and β belong to $\operatorname{Mor}\mathcal{D}$.

Theorem 1.4. *Let $\mathcal{C}$ be a finite EI-category and M an indecomposable $R\mathcal{C}$-module. Then there exists the smallest full convex subcategory of $\mathcal{C}$, relative to which M is projective.*

The subcategory in Theorem 1.4 is called the vertex of M, denoted by $\mathcal{V}_M$. The restriction of M, $M\downarrow^{\mathcal{C}}_{\mathcal{V}_M}$, is an indecomposable $R\mathcal{V}_M$-module by Theorem 1.2 and is called the source for M. Just as in group representation theory, there is a trivial module (or constant functor) $\underline{R}$, which sends every object to R and every morphism to the identity and plays an important role in this paper. For any functor $\iota : \mathcal{D} \to \mathcal{C}$ and any $x \in \operatorname{Ob}\mathcal{C}$ there exists an *overcategory* denoted by $\iota\downarrow_x$ (Mac Lane [26]), which can be used to define the left Kan extension $\varinjlim_{\iota\downarrow_?} -$ of ι. When $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion, the left Kan extension is isomorphic to the induction $\uparrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{C} \otimes_{R\mathcal{D}} - : R\mathcal{D}\text{-mod} \to R\mathcal{C}\text{-mod}$. Using (3) of Theorem 1.2, we can give the following characterization of the full subcategories $\mathcal{D}$, relative to which $\underline{R}$ is projective.

Proposition 1.5. *Suppose $\mathcal{C}$ is a finite EI-category and $\mathcal{D}$ is a full subcategory. Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be the inclusion. Then the $R\mathcal{C}$-module $\underline{R}$ is relatively $\mathcal{D}$-projective if and only if every $\iota\downarrow_x$, $x \in \operatorname{Ob}\mathcal{C}$, is connected.*

The above characterization, however, is not easy to use in practice to narrow down the possible choices of the vertex of $\underline{R}$. Inspired by the work of Puig [29], Thévenaz [35] and Symonds [34], we define an object $x \in \operatorname{Ob}\mathcal{C}$ to be weakly essential if $\mathcal{C}_{<x}$ is empty or has more than one component. Following Symonds [34], the full subcategory consisting of all such objects in $\operatorname{Ob}\mathcal{C}$ is named $\operatorname{Wess}_0(\mathcal{C})$. There is a larger subcategory, denoted by $\operatorname{Wess}(\mathcal{C})$, containing $x \in \operatorname{Ob}\mathcal{C}$ such that $\mathcal{C}_{<x}$ is not contractible, see Quillen [30,31], Bouc [6] and Symonds [34].

Proposition 1.6. *Let $\mathcal{C}$ be a finite EI-category. Then* $\operatorname{Wess}_0(\mathcal{C}) \subset \mathcal{V}_{\underline{R}}$.

In Section 3.6 we will elaborate on this point and prove that if $(\mathcal{C}, \mathcal{I})$ is a finite category with subobjects then $\mathcal{C}$ is EI, $\mathcal{I}$ is a poset and $\operatorname{Wess}_0(\mathcal{I})$ completely determines $\mathcal{V}_{\underline{R}}$. The other category $\operatorname{Wess}(\mathcal{I})$ is as equally important as $\operatorname{Wess}_0(\mathcal{I})$ in this situation, see Propositions 3.5.6 and 3.5.7.

Since $R\mathcal{C}$-mod is an abelian category with enough projectives and injectives, we can consider the groups $\operatorname{Ext}^*_{R\mathcal{C}}(M, N)$ for two arbitrary $R\mathcal{C}$-modules. When $M = N$, $\operatorname{Ext}^*_{R\mathcal{C}}(M, M) := \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_{R\mathcal{C}}(M, M)$ possesses a ring structure with the multiplication given by the Yoneda splice. The following theorem is proved by using Theorem 1.2 and the Eckmann–Shapiro Lemma (see Benson [4]).

Theorem 1.7. *Let $\mathcal{D}$ be a full subcategory of a finite EI-category $\mathcal{C}$ and M an $R\mathcal{C}$-module which is relatively $\mathcal{D}$-projective. If $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module then we have*

$$\operatorname{Ext}^*_{R\mathcal{C}}(M, N) \cong \operatorname{Ext}^*_{R\mathcal{D}}\big(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, N\downarrow^{\mathcal{C}}_{\mathcal{D}}\big),$$

for any $R\mathcal{C}$-module N. When $M = N$, this is a ring isomorphism.

It is well known that $\varprojlim^*_{\mathcal{C}} N \cong \operatorname{Ext}^*_{R\mathcal{C}}(\underline{R}, N)$ for any $R\mathcal{C}$-module N. In this case, Theorem 1.7 becomes a result of Jackowski and Słomińska [22]. The representation theory of an EI-category $\mathcal{C}$ enables us to describe projective resolutions of $R\mathcal{C}$-modules, and hence leads us to some other interesting results. In the following proposition, $S_{x,V}$ and $S_{y,U}$ are two arbitrary simple $R\mathcal{C}$-modules. The structure of simple $R\mathcal{C}$-modules and their projective covers was studied by Lück [23] and is described in Theorem 3.1.2 and the paragraph after it.

Proposition 1.8. *Let $\mathcal{C}$ be an EI-category and R a commutative ring. Then*

$$\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,U}) \cong \operatorname{Ext}^*_{R\mathcal{C}^y_x}(S_{x,V}, S_{y,U}),$$

where $\mathcal{C}^y_x = \mathcal{C}_{\geqslant x} \cap \mathcal{C}_{\leqslant y}$. Especially we have $\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{x,U}) \cong \operatorname{Ext}^*_{R\operatorname{Aut}_{\mathcal{C}}(x)}(V, U)$.

Using our knowledge about the minimal resolutions, we can also show when $R\mathcal{C}$ has finite global dimension.

Theorem 1.9. *Let $\mathcal{C}$ be a finite EI-category. Then $R\mathcal{C}$ has finite global dimension if and only if for all $x \in \operatorname{Ob}\mathcal{C}$, $|\operatorname{Aut}_{\mathcal{C}}(x)|^{-1} \in R$.*

This paper is organized as follows. In Section 2 we define category algebras and their representations and go over some basic homological properties of category algebras. The central piece of the paper is Section 3 on the representation theory of EI-categories. We will give a description of the simple and projective $R\mathcal{C}$-modules, as well as the restriction of projective modules to subalgebras. Our theory of vertices and sources will be developed, and we will apply it to establish reduction formulas for computing Ext groups. In Section 4 we describe the minimal projective resolutions for $R\mathcal{C}$-modules and consider their applications. For general background in homological algebra, the reader is referred to Hilton and Stammbach [18] and Mac Lane [26,27]. For representation theory used in this paper, one may consult Benson [4,5] and Webb [38]. For other works related to our subject, besides the ones cited in the paper, one can try Broto–Levi–Oliver [7], Jackowski–McClure [20] and Villarroel-Webb [37].

2. Category algebras and basic properties

Throughout this paper, the base ring R is always a commutative ring with an identity. A module will be a finitely generated left module, if it is not otherwise specified.

2.1. Definition

Definition 2.1.1. Let $\mathcal{C}$ be a category and R a commutative ring. The category algebra $R\mathcal{C}$ is the free R-module whose basis is the set of morphisms of $\mathcal{C}$. We define a product on the basis elements of $R\mathcal{C}$ by

$$f * g = \begin{cases} f \circ g, & \text{if } f \text{ and } g \text{ can be composed in } \mathcal{C}, \\ 0, & \text{otherwise} \end{cases}$$

and then extend this product linearly to all elements of $R\mathcal{C}$. With this product, $R\mathcal{C}$ becomes an associative R-algebra.

A category $\mathcal{C}$ is said to be *finite* if all its morphisms form a finite set $\mathrm{Mor}(\mathcal{C})$. Note that this implies $\mathcal{C}$ has finitely many objects (i.e. $\mathrm{Ob}\,\mathcal{C}$ is finite), and that $R\mathcal{C}$ is of finite R-rank. If $\mathrm{Ob}\,\mathcal{C}$ is finite, it is easy to see that $\sum_{x\in\mathrm{Ob}\,\mathcal{C}} 1_x$ is the identity of $R\mathcal{C}$ where 1_x is the identity of $\mathrm{Aut}_{\mathcal{C}}(x)$.

We say $\mathcal{C}$ is connected if $\mathcal{C}$ as a (directed) graph is connected. Every category $\mathcal{C}$ is a disjoint union of connected components $\mathcal{C} = \bigcup_{i\in J} \mathcal{C}_i$, where each $\mathcal{C}_i$ is a connected full subcategory and J is an index set. As a consequence the category algebra $R\mathcal{C}$ becomes a direct sum of ideals $R\mathcal{C}_i$, $i \in J$. Thus in order to study the properties of $R\mathcal{C}$ it suffices to study the properties of each $R\mathcal{C}_i$. For simplicity and some technical reasons we make the following assumption.

Convention. In this paper, we assume $\mathcal{C}$ is connected.

2.2. Representations of categories

We shall show that a fundamental property of a category algebra $R\mathcal{C}$ is that it provides a mechanism for discussing representations of $\mathcal{C}$, in a sense which we now define.

Definition 2.2.1. A representation of a category $\mathcal{C}$ over a commutative ring R is a covariant functor $M : \mathcal{C} \to R\text{-mod}$.

The functor category $(R\text{-mod})^{\mathcal{C}}$ is an abelian category with enough projectives and injectives so we can talk about subfunctors and quotient functors and do homological algebra on it. As we have mentioned in the introduction to this paper, Mitchell [25] proved the category of covariant functors from $\mathcal{C}$ to R-mod is equivalent to the category of left $R\mathcal{C}$-modules, i.e. $(R\text{-mod})^{\mathcal{C}} \simeq R\mathcal{C}\text{-mod}$, if $\mathrm{Ob}\,\mathcal{C}$ is finite. The functors which establish the equivalences are described as follows. If $F \in (R\text{-mod})^{\mathcal{C}}$ is a covariant functor, then we define an $R\mathcal{C}$-module M_F to be $M_F = \bigoplus_{x\in\mathrm{Ob}\,\mathcal{C}} M(x)$. Conversely if $M \in R\mathcal{C}\text{-mod}$, we can define a functor F_M such that $F_M(x) = 1_x \cdot M$. Since our real intention is to study finite categories, for convenience we usually will not distinguish $R\mathcal{C}$-mod and $(R\text{-mod})^{\mathcal{C}}$ if it does not cause any serious trouble.

Throughout this paper, we are going to use $\underline{R}$ to denote the constant functor or trivial module. For any category $\mathcal{C}$, $\underline{R} : \mathcal{C} \to R\text{-mod}$, is defined by $\underline{R}(x) = R$ for all $x \in \mathrm{Ob}\,\mathcal{C}$ and $\underline{R}(f) = \mathrm{Id}$ for all $f \in \mathrm{Mor}\,\mathcal{C}$.

Any group G can be regarded as a category $\hat{G}$ with only one object $*$, whose morphisms are the elements of G. The group algebra RG is the same as the category algebra $R\hat{G}$, and a left RG-module M is a representation of $\hat{G}$ in an obvious way. The trivial $R\hat{G}$-module $\underline{R}$ is exactly the trivial module of RG. As further examples of category algebras we observe that when $\mathfrak{q}$ is a quiver, the category algebra of the free category $\mathcal{C}_{\mathfrak{q}}$ generated by $\mathfrak{q}$ (see Mac Lane [26]) is the same as the path algebra of the quiver $\mathfrak{q}$, and that when Γ is a poset the incidence algebra of Γ (see [9]) is the same as the category algebra $R\Gamma$.

Definition 2.2.2. Suppose $\mu : \mathcal{D} \to \mathcal{C}$ is a (covariant) functor. We define $\mathrm{Res}_\mu : (R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{D}}$ to be the restriction along μ. Given a functor $M \in (R\text{-mod})^{\mathcal{C}}$, we have $\mathrm{Res}_\mu M = M \circ \mu \in (R\text{-mod})^{\mathcal{D}}$.

Given a functor $\mu : \mathcal{D} \to \mathcal{C}$, the restriction $\mathrm{Res}_\mu : (R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{D}}$ has a counterpart, also denoted by Res_μ, between the corresponding module categories: $\mathrm{Res}_\mu : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$. In fact, if $M = \bigoplus_{x\in\mathrm{Ob}\,\mathcal{C}} M(x)$ is an $R\mathcal{C}$-module, then $\mathrm{Res}_\mu M = \bigoplus_{y\in\mathrm{Ob}\,\mathcal{D}} M(\mu(y)) \in R\mathcal{D}\text{-mod}$. On the other hand $\mu : \mathcal{D} \to \mathcal{C}$ extends linearly to a natural map of R-modules $\bar{\mu} : R\mathcal{D} \to R\mathcal{C}$, which is not necessarily an algebra homomorphism, and here is a simple example. Let $\mathcal{D}$ be a category with two objects and only identity maps, and let $\mathcal{C}$ be a category with one object and the identity map along with the unique functor $\mu:\mathcal{D}\to\mathcal{C}$. Then the map $\bar{\mu}:R\mathcal{D}\to R\mathcal{C}$ is not an algebra homomorphism for the product of the two morphisms in $\mathcal{D}$ is zero while the product of their images is not. When $\bar{\mu}$ is an algebra homomorphism, it induces the representation-theoretic restriction $\downarrow^{R\mathcal{C}}_{R\mathcal{D}}: R\mathcal{C}\text{-mod}\to R\mathcal{D}\text{-mod}$.

Proposition 2.2.3. *A functor $\mu:\mathcal{D}\to\mathcal{C}$ extends linearly to an algebra homomorphism $\bar{\mu}:R\mathcal{D}\to R\mathcal{C}$ if and only if μ is injective on* $\operatorname{Ob}\mathcal{D}$. *When this happens, the induced functor followed by $1_{R\mathcal{D}}$, $1_{R\mathcal{D}}\cdot\downarrow^{R\mathcal{C}}_{R\mathcal{D}}: R\mathcal{C}\text{-mod}\to R\mathcal{D}\text{-mod}$, is exactly* Res_μ.

Proof. We know $\mu(\beta\alpha)=\mu(\beta)\mu(\alpha)$ for any pair of composable morphisms α,β in $\mathcal{D}$. The injectivity of μ implies two morphisms $\alpha,\beta\in\operatorname{Mor}\mathcal{D}$ are composable if and only if $\mu(\alpha),\mu(\beta)\in\operatorname{Mor}\mathcal{C}$ are composable.

If μ is injective on $\operatorname{Ob}\mathcal{D}$, then we define a map $\bar{\mu}:R\mathcal{D}\to R\mathcal{C}$ as the linear extension of functor μ, i.e., $\bar{\mu}(\sum_i r_i\alpha_i)=\sum_i r_i\bar{\mu}(\alpha_i)$ for any $r_i\in R,\alpha_i\in\operatorname{Mor}(\mathcal{D})$. This $\bar{\mu}$ is indeed an algebra homomorphism because our previous observation of μ implies $\bar{\mu}((\sum_j r_j\beta_j)(\sum_i r_i\alpha_i))=\bar{\mu}(\sum_j r_j\beta_j)\bar{\mu}(\sum_i r_i\alpha_i)$ is always true.

On the other hand if the linear extension $\bar{\mu}:R\mathcal{D}\to R\mathcal{C}$ is an algebra homomorphism then we must have $\bar{\mu}(0)=0$ and then $\bar{\mu}(1_x)\bar{\mu}(1_y)=\bar{\mu}(1_x\cdot 1_y)=0$ unless $x=y$. This implies that μ is injective on $\operatorname{Ob}\mathcal{D}$. $\square$

In Section 3, we will take $\mathcal{D}$ to be a full subcategory of $\mathcal{C}$ and $\mu=\iota$, the inclusion. Then the restriction $\operatorname{Res}_\iota: R\mathcal{C}\text{-mod}\to R\mathcal{D}\text{-mod}$ is determined by the algebra homomorphism $\bar{\iota}:R\mathcal{D}\to R\mathcal{C}$, hence by $\iota:\mathcal{D}\to\mathcal{C}$. For this reason we do not distinguish Res_ι and $\downarrow^{R\mathcal{C}}_{R\mathcal{D}}$, and will write $\downarrow^{R\mathcal{C}}_{R\mathcal{D}}$ and Res_ι as $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ which is common in representation theory.

Proposition 2.2.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be equivalent small categories. Then*

(1) *$(R\text{-mod})^{\mathcal{C}}\simeq(R\text{-mod})^{\mathcal{D}}$, an equivalence which sends the constant functor to the constant functor. If both* $\operatorname{Ob}\mathcal{C}$ *and* $\operatorname{Ob}\mathcal{D}$ *are finite then $R\mathcal{C}$ and $R\mathcal{D}$ are Morita equivalent*; *and*
(2) *the nerves $N\mathcal{C}$ and $N\mathcal{D}$ are homotopy equivalent.*

Proof. We prove the first assertion. The second is well-known and a proof of it can be found in Baues and Wirsching [3].

We show the two functor categories $(R\text{-mod})^{\mathcal{C}}$ and $(R\text{-mod})^{\mathcal{D}}$ are equivalent. Then it implies the module categories $R\mathcal{C}$-mod and $R\mathcal{D}$-mod are equivalent, hence $R\mathcal{C}$ and $R\mathcal{D}$ are Morita equivalent. In fact if $\mu:\mathcal{D}\to\mathcal{C}$ and $\nu:\mathcal{C}\to\mathcal{D}$ are equivalences, we have $\operatorname{Res}_\mu\operatorname{Res}_\nu\cong\operatorname{Id}_{R\mathcal{D}}:(R\text{-mod})^{\mathcal{D}}\to(R\text{-mod})^{\mathcal{D}}$ because of the following diagram

$$\begin{array}{ccc}
M(\nu\mu(x))=(\operatorname{Res}_\mu\operatorname{Res}_\nu M)(x) & \xrightarrow{\cong} & (\operatorname{Id}_{\mathcal{C}'} M)(x)=M(x)\\
\Big\downarrow {\scriptstyle M(\nu\mu(\alpha))=(\operatorname{Res}_\mu\operatorname{Res}_\nu M)(\alpha)} & & \Big\downarrow {\scriptstyle (\operatorname{Id}_{\mathcal{C}'} M)(\alpha)=M(\alpha)}\\
M(\nu\mu(y))=(\operatorname{Res}_\mu\operatorname{Res}_\nu M)(y) & \xrightarrow[\cong]{} & (\operatorname{Id}_{\mathcal{C}'} M)(y)=M(y)
\end{array}$$

where $M \in (R\text{-mod})^{\mathcal{D}}$, $\alpha x \to y \in \operatorname{Mor}\mathcal{D}$ and $\mathrm{Id}_{R\mathcal{D}}$ is the identity functor. Similarly we can show $\mathrm{Res}_\nu \mathrm{Res}_\mu \cong \mathrm{Id}_{R\mathcal{C}}(R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{C}}$. Clearly the constant functor restricts to the constant functor always. □

2.3. Basic homological properties

The category $R\mathcal{C}$-mod is abelian and has enough projective and injectives so we can consider the Ext groups $\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$ for $M, N \in R\mathcal{C}$-mod. For any $M \in R\mathcal{C}$-mod $\mathrm{Ext}^*_{R\mathcal{C}}(M,M)$ has a ring structure with product given by the Yoneda splice, but it is the case where $M = \underline{R}$ that is of great interest to us.

Definition 2.3.1. We call the ring $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},\underline{R}) = \bigoplus_{i\geqslant 0}\mathrm{Ext}^i_{R\mathcal{C}}(\underline{R},\underline{R})$ the cohomology ring of the category algebra $R\mathcal{C}$. The product in this ring is defined by the Yoneda splice.

There exists a ring isomorphism

$$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},\underline{R}) \cong \mathrm{H}^*\big(|\mathcal{C}|, R\big),$$

where $|\mathcal{C}|$ stands for the topological realization of the nerve $N\mathcal{C}$. Thus we will also call the ring defined above as the cohomology ring of $\mathcal{C}$ with coefficients in R. When $\mathcal{C}$ is a finite group and R is Noetherian, its cohomology ring is finitely generated by a theorem of Evens [14] and Venkov [36]. However, the finite generation is not true in general for finite categories, see Xu [40].

Direct computation of cohomology groups is in general very difficult, and so people have been searching for reduction formulas. It is well known that if a functor $\mu:\mathcal{D}\to\mathcal{C}$ has a left adjoint $\nu:\mathcal{C}\to\mathcal{D}$, then Res_ν is also the left adjoint of Res_μ, and hence $\mathrm{Ext}^*_{R\mathcal{C}}(\mathrm{Res}_\nu M, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M, \mathrm{Res}_\mu N)$ for any $M \in R\mathcal{D}$-mod and $N \in R\mathcal{C}$-mod, because both Res_μ and Res_ν are exact (see for example Jackowski–McClure–Oliver [21, II, Proposition 5.1]). If μ is indeed an equivalence, we have the following result.

Lemma 2.3.2. *Let $\mu:\mathcal{D}\to\mathcal{C}$ be an equivalence of two small categories. Then we have*

$$\mathrm{Ext}^*_{R\mathcal{C}}(M,N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\mathrm{Res}_\mu M, \mathrm{Res}_\mu N),$$

for $M, N \in R\mathcal{C}$-mod. In particular there is a ring isomorphism

$$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},\underline{R}) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R},\underline{R}).$$

Proof. The thing is, $R\mathcal{C}$ and $R\mathcal{D}$ are Morita equivalent by the functor Res_μ, which takes $\underline{R}$ to $\underline{R}$. □

The groups $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M)$ are very useful to us, since they can be used to recover the cohomology theory of small categories that has been discussed in various places in the literature, see Baues–Wirsching [3], Generalov [16] and Oliver [28].

Definition 2.3.3. Let $\mathcal{C}$ be a small category. The nth cohomology group, $\mathrm{H}^n(\mathcal{C}, M)$, of $\mathcal{C}$ with coefficients in module $M \in R\mathcal{C}$-mod is defined by

$$\mathrm{H}^n(\mathcal{C}, M) := \mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, M).$$

It was shown by Roos [33] and Gabriel–Zisman [15] that

$$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M) \cong \varprojlim\nolimits^*_{\mathcal{C}} M,$$

the higher inverse limits of M over $\mathcal{C}$. The computation of higher limits occupies an important place in group cohomology theory so we record some relevant results below.

Let $\mathcal{C}$ and $\mathcal{D}$ be small categories equipped with a functor $\mu : \mathcal{D} \to \mathcal{C}$. For each $y \in \mathrm{Ob}\,\mathcal{C}$, the overcategory $\mu{\downarrow_y}$ (comma category in Mac Lane [26]) consists of objects (x, α), where $x \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mu(x), y)$. A morphism from (x, α) to (x', α') in the overcategory is given by a morphism $\beta \in \mathrm{Hom}_{\mathcal{D}}(x, x')$, which satisfies $\alpha\mu(\beta) = \alpha'$. We can define a functor $\mu{\downarrow_?} : \mathcal{C} \to$ sCat (the category of small categories), and thus a functor $C_*(\mu{\downarrow_?}) : \mathcal{C} \to R\mathcal{C}$-Cplx (the category of complexes of $R\mathcal{C}$-modules) through the simplicial complexes given by the overcategories. When $\mathcal{D} = \mathcal{C}$ and $\mu = \mathrm{Id}$, we normally write $\mathrm{Id}(\mathcal{C}){\downarrow_y}$ as $\mathcal{C}{\downarrow_y}$ for any $y \in \mathrm{Ob}\,\mathcal{C}$. It is well known that $\{C_*(\mathcal{C}{\downarrow_?})\}$ is a projective resolution (the bar resolution) of $\underline{R} \in R\mathcal{C}$-mod, see for instance Grodal [17].

The restriction $\mathrm{Res}_\mu : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$, induced by any functor $\mu : \mathcal{D} \to \mathcal{C}$, always has a left adjoint, called the left Kan extension $K : R\mathcal{D}\text{-mod} \to R\mathcal{C}\text{-mod}$ and defined by

$$K(M)(y) = \varinjlim\nolimits_{\mu\downarrow_y} M \circ \pi,$$

where $M \in R\mathcal{D}$-mod, $y \in \mathrm{Ob}\,\mathcal{C}$ and $\pi : \mu{\downarrow_y} \to \mathcal{D}$ is the projection $(x, \alpha) \mapsto x$. Let $\mathcal{P} \to \underline{R} \to 0$ be a projective resolution of the $R\mathcal{D}$-module $\underline{R}$. Then

$$\varprojlim\nolimits^n_{\mathcal{D}} \mathrm{Res}_\mu M \cong \mathrm{Ext}^n_{R\mathcal{D}}(\underline{R}, \mathrm{Res}_\mu M) \cong \mathrm{H}_n\big(\mathrm{Hom}_{R\mathcal{C}}\big(K(\mathcal{P}), M\big)\big).$$

When $\mathcal{P} = C_*(\mathcal{D}{\downarrow_?})$, it was known to Dwyer–Kan [12] and Hollender–Vogt [19] that $K(C_*(\mathcal{D}{\downarrow_?})) \cong C_*(\mu{\downarrow_?})$, while the latter will become a projective resolution of the $R\mathcal{C}$-module $\underline{R}$ if it is exact. A category $\mathcal{E}$ is called *R-acyclic* if its reduced homology groups $\tilde{\mathrm{H}}_*(|\mathcal{E}|, R)$ vanish.

Proposition 2.3.4. *(See Jackowski–Słomińska [22, 5.4].) Let $\mu : \mathcal{D} \to \mathcal{C}$ satisfy the condition that every $\mu{\downarrow_y}$, $y \in \mathrm{Ob}\,\mathcal{C}$, is R-acyclic. Then $\varprojlim^*_{\mathcal{C}} M \cong \varprojlim^*_{\mathcal{D}} \mathrm{Res}_\mu M$ for any $R\mathcal{C}$-module M.*

Note that if all the overcategories $\mu{\downarrow_y}$, $y \in \mathrm{Ob}\,\mathcal{C}$, are contractible, then $|\mathcal{D}| \simeq |\mathcal{C}|$ by Quillen's Theorem A [30]. When we turn to our representation-theoretic settings and assume $R\mathcal{C}$ is a right (non-unital) $R\mathcal{D}$-module, Res_μ becomes the usual restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ whose left adjoint is the induction $\uparrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{C} \otimes_{R\mathcal{D}} -$. Under the circumstances, both Lemma 2.3.2 and Proposition 2.3.4 can be rewritten using $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ and $\downarrow^{\mathcal{C}}_{\mathcal{D}}$.

3. EI-categories, relative projectivity, vertices and sources

In this section, we investigate the representation theory of EI-categories and its applications to cohomology theory, especially to the computation of higher limits. We always assume the base ring R is a field or a complete discrete valuation ring, in order to have the unique decomposition property for every $R\mathcal{C}$-module. When R is a field of characteristic $p > 0$, we denote it by $\mathbb{F}_p$ (instead of $\mathbb{F}_q$ for $q = p^n$, etc.), and require it to be large enough (e.g. algebraically closed, etc.) if necessary.

Definition. An EI-category is a small category $\mathcal{C}$ in which all endomorphisms are isomorphisms.

Some of the general theory of EI-categories can be found in the book of tom Dieck [10, (I.11)], much of which was due to Lück [23]. One of the important features of EI-categories is described as follows. Given an EI-category $\mathcal{C}$, there is a preorder defined on $\mathrm{Ob}\,\mathcal{C}$, that is, $y \leqslant x$ if and only if $\mathrm{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$. Let $[y]$ be the isomorphism class of an object $y \in \mathrm{Ob}\,\mathcal{C}$. This preorder induces a partial order on the set $\mathrm{Is}\,\mathcal{C}$ of isomorphism classes of $\mathrm{Ob}\,\mathcal{C}$ (specified by $[y] \leqslant [x]$ if and only if $\mathrm{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$), which plays an important role in studying representations and cohomology of EI-categories. Because of the existence of an order for the isomorphism classes of objects in any EI-category, EI-categories are sometimes referred to as "ordered categories" by some authors, see Oliver [28] and Jackowski–Słomińska [22]. For any EI-category $\mathcal{C}$, any subcategory $\mathcal{D}$ and any object $x \in \mathrm{Ob}\,\mathcal{C}$, we can define a full subcategory $\mathcal{D}_{\leqslant x} \subset \mathcal{D}$ consisting of all $y \in \mathrm{Ob}\,\mathcal{D}$ such that $y \leqslant x$, or equivalently $\mathrm{Hom}_{\mathcal{D}}(y, x) \neq \emptyset$. Similarly we can define other full subcategories of $\mathcal{D}$: $\mathcal{D}_{<x}$, $\mathcal{D}_{\geqslant x}$ and $\mathcal{D}_{>x}$.

Convention. In the rest of this article, we are going to assume that $\mathcal{C}$ is finite (i.e. $\mathrm{Mor}(\mathcal{C})$ is finite). Then $R\mathcal{C}$ becomes an R-algebra of finite rank and $1_{R\mathcal{C}}$ is a sum of primitive orthogonal idempotents.

When we consider a full subcategory $\mathcal{D}$ of an EI-category $\mathcal{C}$, we suppose $\mathcal{D}$ has the following property: if $x \in \mathrm{Ob}\,\mathcal{D}$, then $[x] \subset \mathrm{Ob}\,\mathcal{D}$, where $[x]$ is the isomorphism class of x in $\mathcal{C}$.

Let $\mathcal{C}$ be an EI-category and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. The second condition in the convention (on $\mathcal{D}$) is a natural requirement, which will not change the nature of any questions to be considered here and does protect us from some unnecessary non-essential technical troubles. The first reason is that, if we are to investigate an $R\mathcal{C}$-module M, then (as a functor) M has to take an isomorphic value on every object of an isomorphism class of objects in $\mathcal{C}$, and the second is that for any full subcategory $\mathcal{D} \subset \mathcal{C}$, there always exists a natural full subcategory $\mathcal{E}$ such that $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}$, $\mathcal{E} \simeq \mathcal{D}$ and $\mathcal{E}$ meets our convention.

3.1. Projective modules and simple modules

Now we start describing the projective and simple modules for an EI-category. The base ring R is assumed to be a field or a complete discrete valuation ring.

Proposition 3.1.1. *(See Lück [23].) Any projective $R\mathcal{C}$-module is isomorphic to a direct sum of indecomposable projective modules of the form $R\mathcal{C} \cdot e$, where $e \in R\,\mathrm{Aut}_{\mathcal{C}}(x)$ is a primitive idempotent, for some $x \in \mathrm{Ob}\,\mathcal{C}$.*

Since each indecomposable projective module is a direct summand of some $R\mathcal{C} \cdot 1_x = R\,\mathrm{Hom}_{\mathcal{C}}(x, -)$, $x \in \mathrm{Ob}\,\mathcal{C}$, and all the non-isomorphisms in $\mathrm{Hom}_{\mathcal{C}}(x, -)$ span a submodule that is contained in the radical of $R\mathcal{C} \cdot 1_x$, $R\mathcal{C} \cdot 1_x$ is the projective cover of a semi-simple module, which is non-zero only on the isomorphism class $[x]$.

Theorem 3.1.2. *(See Lück [23].) Let $\mathcal{C}$ be an EI-category. For each object $x \in \mathrm{Ob}\,\mathcal{C}$ and simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module V there is a simple $R\mathcal{C}$-module M such that $[x] \in \mathrm{Is}\,\mathcal{C}$ is exactly the set of objects on which M is non-zero, and $M(x) = V$. On the other hand, if M is a simple $R\mathcal{C}$-module, then there exists a unique isomorphism class of objects $[x] \in \mathrm{Is}\,\mathcal{C}$ on which M is non-zero, and furthermore each $M(x)$ is a simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. These two processes are inverse to each other. Thus the isomorphism classes of the simple $R\mathcal{C}$-modules biject with the pairs $([x], V)$, where $x \in \mathrm{Ob}\,\mathcal{C}$ and V is a simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$ module, taken up to isomorphism.*

We denote a simple $R\mathcal{C}$-module by $S_{x,V}$, if it corresponds to a pair $([x], V)$ where V is a simple $R\,\mathrm{Aut}(x)$-module, and $x \in \mathrm{Ob}\,\mathcal{C}$. For consistency, we use $P_{x,V}$ for the projective cover of $S_{x,V}$, whose structure is determined by its value at the object x. More precisely, if $R\,\mathrm{Aut}(x) \cdot e$ is the projective cover of the simple $R\,\mathrm{Aut}(x)$-module V, then $R\mathcal{C} \cdot e$ is the projective cover of $S_{x,V}$. The simple modules are atomic in the sense we now define.

Definition 3.1.3. A functor $M : \mathcal{C} \to R$-mod is called atomic, concentrated on an isomorphism class of objects $[x] \subset \mathrm{Ob}\,\mathcal{C}$ if $M(y)(= 1_y \cdot M) \neq 0$ if and only if $y \cong x$.

For convenience, we just say M is concentrated on x, instead of $[x]$. We will call an $R\mathcal{C}$-module M atomic if the corresponding functor is. With the description of indecomposable projectives, we can show when the trivial module $\underline{R}$ is projective. This generalizes Lemma 2.5 of Symonds [34].

Proposition 3.1.4. *Let $\mathcal{C}$ be a finite EI-category. Then $\underline{R}$ is projective if and only if each connected component of $\mathcal{C}$ has a unique isomorphism class of minimal objects $[x]$, with the properties that for all y in the same connected component as x, $\mathrm{Aut}(x)$ has a single orbit on $\mathrm{Hom}(x, y)$, and $|\mathrm{Aut}(x)|$ is invertible in R.*

Proof. If $\underline{R}$ is projective then $\underline{R} \cong \bigoplus P_{x,V}$ for certain indecomposable projective modules $P_{x,V}$. The only V which can arise are $V = R$, and R must be projective as an $R\,\mathrm{Aut}(x)$-module, forcing $|\mathrm{Aut}(x)|$ to be invertible in R for the x which appear in the direct sum, as in the first proof.

Since $P_{y,R}(z) = 0$ unless $y \leqslant z$, $P_{x,R}$ must appear as a summand for each isomorphism class of minimal x. Now $P_{x,R}(z) = \mathrm{Hom}(x, z) \otimes_{\mathrm{Aut}(x)} R \cong R^n$, where n is the number of orbits of $\mathrm{Aut}(x)$ on $\mathrm{Hom}(x, z)$. For $P_{x,R}(z)$ to be a summand of R we must have $n = 1$. Finally, $\bigoplus_{\{\text{minimal } x\}} P_{x,R}$ at an object z is R^t where $t =$ number of isomorphism classes of minimal $[x]$ with $x \leqslant z$, so each component has a unique minimal x.

The other direction is easy. The conditions imply that $\underline{R} = \bigoplus_{\{\text{minimal } x\}} P_{x,R}$ and this is projective. □

For certain subcategories $\mathcal{D} \subset \mathcal{C}$, the restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}} : R\mathcal{C}$-mod $\to R\mathcal{D}$-mod may preserve projective modules.

Definition 3.1.5. Suppose $\mathcal{D} \subset \mathcal{C}$ is a full subcategory. We say $\mathcal{D}$ is an ideal in $\mathcal{C}$ if for any $x \in \mathrm{Ob}\,\mathcal{D}$ we have $\mathcal{C}_{\leqslant x} \subset \mathcal{D}$. Similarly, we say $\mathcal{D}$ is a co-ideal in $\mathcal{C}$ if for any $x \in \mathrm{Ob}\,\mathcal{D}$ we have $\mathcal{C}_{\geqslant x} \subset \mathcal{D}$.

Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. Then we can form a full subcategory of $\mathcal{C}$, named $\mathcal{C}\backslash\mathcal{D}$, which consists of all objects not belonging to $\mathcal{D}$. From the definitions it is easy to verify that if $\mathcal{D}$ is an ideal (respectively a co-ideal) then $\mathcal{C}\backslash\mathcal{D}$ is a co-ideal (respectively an ideal). Note that if $\mathcal{D} \subset \mathcal{C}$ is an ideal (respectively a co-ideal) then $R\mathcal{D}$ becomes a right ideal (respectively a left ideal) in $R\mathcal{C}$. If a full subcategory $\mathcal{D}$ forms an ideal (respectively a co-ideal) in $\mathcal{C}$, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projectives (respectively right projectives).

Lemma 3.1.6. *If $\mathcal{D}$ is an ideal in* $\mathrm{Ob}\,\mathcal{C}$*, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves left projective modules. If $\mathcal{D}$ is a co-ideal in* $\mathrm{Ob}\,\mathcal{C}$*, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves right projective modules.*

Proof. We only prove the first assertion by computing $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ explicitly. If $x \in \mathrm{Ob}\,\mathcal{D}$, then $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\,\mathrm{Hom}_{\mathcal{D}}(x,-)$. If $x \notin \mathrm{Ob}\,\mathcal{D}$, then by definition $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = 0$. Hence $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)$, and consequently $R\mathcal{C}$, are projective $R\mathcal{D}$-modules. In fact $R\mathcal{C}\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{D}$. □

For a complete description of the restrictions of projective modules, one can consult tom Dieck [10, I.11].

3.2. Relative projectivity

Suppose A is an R-subalgebra of an R-algebra B. Let M be a B-module. Then there is a natural epimorphism $\epsilon : B \otimes_A M \to M$ given by the multiplication $\epsilon(b \otimes m) = bm$. We shall only consider the case of a category algebra $R\mathcal{C}$ with a subalgebra $R\mathcal{D}$, for some subcategory $\mathcal{D}$ of $\mathcal{C}$. For consistency, we assume $\mathcal{C}$ is finite EI though in some definitions and results of this section the condition is not necessary.

Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be a subcategory and $M \in R\mathcal{D}$-mod. Then $M\uparrow^{\mathcal{C}}_{\mathcal{D}} \in R\mathcal{C}$-mod evaluated at any $y \in \mathrm{Ob}\,\mathcal{C}$ equals

$$M\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) \cong K(M)(y) = \varinjlim_{\iota\downarrow_y} M \cong \sum_{x \leqslant y} R\,\mathrm{Hom}_{\mathcal{C}}(x,y) \otimes_{R\mathcal{D}} M(x),$$

where K is the left Kan extension described in the paragraphs preceding Proposition 2.3.4.

Definition 3.2.1. Let M be an $R\mathcal{C}$-module. If the $R\mathcal{C}$-module epimorphism

$$\epsilon = \epsilon_M : M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{C} \otimes_{R\mathcal{D}} M \to M$$

is split, then we say M is projective relative to $\mathcal{D}$, or relatively $\mathcal{D}$-projective.

We have some equivalent descriptions of the relative projectivity of an $R\mathcal{C}$-module M.

Proposition 3.2.2. *Let $\mathcal{D} \subset \mathcal{C}$ be a subcategory and M an $R\mathcal{C}$-module. Then the following statements are equivalent*:

(1) *the canonical surjective map* $\epsilon : M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}} \to M$ *splits*;
(2) M *is a direct summand of* $M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}$;
(3) M *is a direct summand of* $N\uparrow_{\mathcal{D}}^{\mathcal{C}}$, *where* N *is an* $R\mathcal{D}$*-module*;
(4) *if* $0 \to M'' \to M' \to M \to 0$ *is an exact sequence of* $R\mathcal{C}$*-modules which splits as an exact sequence of* $R\mathcal{D}$*-modules, then it splits as an exact sequence of* $R\mathcal{C}$*-modules.*

Proof. The statements (1), (2) and (4) are proved to be equivalent in the context of Artin algebras, see for instance [2, Section VI, Proposition 3.6]. When $\mathcal{C}$ is a finite group, statement (3) is well-known to be equivalent to the others. Since the proof for category algebras is similar to that for group algebras, we omit it and refer the reader to Xu [39] for details. □

Suppose M and N are two $R\mathcal{C}$-modules. Then we write $M|N$ if M is isomorphic to a direct summand of N.

Proposition 3.2.3. *Let* $\mathcal{C}$ *be a category. Then*

(1) *if* $\mathcal{E} \subset \mathcal{D}$ *are subcategories of* $\mathcal{C}$ *and* M *is relatively* $\mathcal{E}$*-projective then* M *is relatively* $\mathcal{D}$*-projective*;
(2) *if* $\mathcal{E} \subset \mathcal{D}$ *are subcategories of* $\mathcal{C}$, N *is an* $R\mathcal{D}$*-module which is relatively* $\mathcal{E}$*-projective, and* M *is a direct summand of* $N\uparrow_{\mathcal{D}}^{\mathcal{C}}$, *then* M *is relatively* $\mathcal{E}$*-projective.*

Proof. Since M is a direct summand of $M\downarrow_{\mathcal{E}}^{\mathcal{C}}\uparrow_{\mathcal{E}}^{\mathcal{C}}$ which can be written as $(M\downarrow_{\mathcal{E}}^{\mathcal{C}}\uparrow_{\mathcal{E}}^{\mathcal{D}})\uparrow_{\mathcal{D}}^{\mathcal{C}}$, we have $M|N\uparrow_{\mathcal{D}}^{\mathcal{C}}$ for an $R\mathcal{D}$-module. So M is relatively $\mathcal{D}$-projective as stated in part (1).

From $N|N\downarrow_{\mathcal{E}}^{\mathcal{D}}\uparrow_{\mathcal{E}}^{\mathcal{D}}$, we get

$$M\big|N\uparrow_{\mathcal{D}}^{\mathcal{C}}\big|\big(N\downarrow_{\mathcal{E}}^{\mathcal{D}}\uparrow_{\mathcal{E}}^{\mathcal{D}}\big)\uparrow_{\mathcal{D}}^{\mathcal{C}} = \big(N\downarrow_{\mathcal{E}}^{\mathcal{D}}\big)\uparrow_{\mathcal{E}}^{\mathcal{C}}.$$

It means M is relatively $\mathcal{E}$-projective, which completes the proof for part (2). □

We need the following terminology before introducing our next two results.

Definition 3.2.4. Let $\mathcal{C}$ be a (finite) EI-category. For each $R\mathcal{C}$-module M, we define the M-minimal objects to be those $x \in \operatorname{Ob}\mathcal{C}$ which satisfy the condition that $M(y) = 0$ if $y \ncong x$ and $\operatorname{Hom}(y, x)$ non-empty. Similarly we can define M-maximal objects.

For example, the $\underline{R}$-minimal objects are the minimal objects of $\mathcal{C}$, and the $\underline{R}$-maximal objects are the maximal objects of $\mathcal{C}$. The $S_{x,V}$-minimal and $S_{x,V}$-maximal objects are the same: $y \in [x]$. We explain what is special about these M-minimal objects.

Lemma 3.2.5. *Let* M *be an* $R\mathcal{C}$*-module and* $\mathcal{D} \subset \mathcal{C}$ *a subcategory.*

(1) *If* M *is relatively* $\mathcal{D}$*-projective*: $M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}} \cong M \oplus M'$, *then* $\operatorname{Ob}\mathcal{D}$ *contains all* M*-minimal objects*;
(2) *If* M *is relatively* $\mathcal{D}$*-projective, then* $M(x)$ *is relatively* $\operatorname{Aut}_{\mathcal{D}}(x)$*-projective as an* $R\operatorname{Aut}_{\mathcal{C}}(x)$*-module for any* M*-minimal object* x.

Proof. If z is M-minimal and $z \notin \mathrm{Ob}\,\mathcal{D}$, then $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(z) = \sum_{y \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{\mathcal{C}}(y,z) \otimes_{R\mathcal{D}} M(y) = 0$, since z is M-minimal. Therefore M cannot be a direct summand of $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ which is a contradiction. Hence $\mathrm{Ob}\,\mathcal{D}$ contains all M-minimal objects.

In order to prove (2) we just evaluate the relation $M | M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ at x, and then the result follows. $\square$

We comment that $M(x)$ is not necessarily an indecomposable $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module even if M is indecomposable.

Corollary 3.2.6. *Let M be an indecomposable $R\mathcal{C}$-module that is relatively $\mathcal{D}$-projective for a subcategory $\mathcal{D} \subset \mathcal{C}$. Then $\mathcal{D}$ has a unique connected component relative to which M is projective.*

Proof. If $\mathcal{D}$ is a disjoint union of several connected components $\{\mathcal{D}_i\}_{i\in I}$, then from $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus_{i\in I} M\downarrow^{\mathcal{C}}_{\mathcal{D}_i}\uparrow^{\mathcal{C}}_{\mathcal{D}_i}$ and $M | M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ we know M is projective relative to some $\mathcal{D}_i$. Such a $\mathcal{D}_i$ has to be unique because it contains all the M-minimal objects by the preceding lemma. $\square$

Using (2) of Lemma 3.2.5, we may reveal some partial information about the structure of $\mathcal{D}$, relative to which M is projective. As an example if $\underline{R}$ is relatively $\mathcal{D}$-projective, then for any minimal object $x \in \mathrm{Ob}\,\mathcal{C}$, $\underline{R}(x) = R$ is relatively $\mathrm{Aut}_{\mathcal{D}}(x)$-projective as an $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. When $R = \mathbb{F}_p$ for some prime p dividing the order of $\mathrm{Aut}_{\mathcal{C}}(x)$, $\mathrm{Aut}_{\mathcal{D}}(x)$ has to contain a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{C}}(x)$, by a standard result from the theory of vertices and sources for group algebras.

3.3. Vertices and sources

If $\mathcal{D} \subset \mathcal{C}$ is a *full* subcategory and $M \in R\mathcal{C}$-mod is relatively $\mathcal{D}$-projective, we show $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M$ (without extra summands). Based on this fact, the $R\mathcal{C}$-modules can be parameterized using the set of full subcategories of $\mathcal{C}$. We will establish a theory of vertices and sources for indecomposable modules, which functions in a similar way as its counterpart in group representation theory.

Proposition 3.3.1. *If M is relatively $\mathcal{D}$-projective for a full subcategory $\mathcal{D} \subset \mathcal{C}$, then M is generated by its values on $\mathcal{D}$, that is, $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M$.*

Proof. Suppose $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} = M' \oplus M''$ for some $R\mathcal{C}$-module M', M'' with $M' \cong M$ and $M''(x) = 0$ for all $x \in \mathrm{Ob}\,\mathcal{D}$. Let is take $y \notin \mathrm{Ob}\,\mathcal{D}$ and consider $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = M'(y) \oplus M''(y)$. We claim $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = \sum_{y > x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{R\mathcal{C}}(x,y) \otimes_{R\mathcal{D}} M(x)$ equals $M'(y)$. In fact $M'(x) = 1_{\mathcal{D}} \otimes M(x)$ for all $x \in \mathrm{Ob}\,\mathcal{D}$, and given any $\alpha \in \mathrm{Hom}(x,y)$, $\alpha \cdot M'(x) \subset M'(y)$, which means $\alpha \otimes M(x) \subset M'(y)$. When x and α run over all possible choices, we get exactly $\sum_{y > x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{R\mathcal{C}}(x,y) \otimes_{R\mathcal{D}} M(x) \subset M'(y)$ which is indeed an equality since the converse direction inclusion is certainly true. Thus the statement is correct. $\square$

If M is relatively $\mathcal{D}$-projective ($\mathcal{D}$ full), then the natural surjection $\epsilon : M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \to M$ is an isomorphism. Let $y \in \mathrm{Ob}\,\mathcal{C}$. From $\epsilon_y : M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) \xrightarrow{\cong} M(y)$ we get

$$\begin{aligned}\epsilon_y\big(M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y)\big) &= \epsilon_y\Bigg(\sum_{x\in \operatorname{Ob}\mathcal{D}_{\leqslant y}} R\operatorname{Hom}_{\mathcal{C}}(x,y)\otimes_{R\mathcal{D}} M(x)\Bigg)\\ &= \sum_{x\in \operatorname{Ob}\mathcal{D}_{\leqslant y}} R\operatorname{Hom}_{\mathcal{C}}(x,y)\cdot M(x)\\ &= M(y).\end{aligned}$$

This explains why any relatively $\mathcal{D}$-projective $R\mathcal{C}$-module M is generated by its values on objects in $\mathcal{D}$. However, $\sum_{x\in \operatorname{Ob}\mathcal{D}_{\leqslant y}} R\operatorname{Hom}_{\mathcal{C}}(x,y)\cdot M(x) = M(y)$ for any $y \in \operatorname{Ob}\mathcal{C}\setminus \operatorname{Ob}\mathcal{D}$ does not guarantee M is relatively $\mathcal{D}$-projective. We can consider the category $x \rightrightarrows y$ with two non-isomorphisms and two trivial isomorphisms. The trivial module $\underline{R}$ is projective relative to the whole category, not $\{x\}$—the full subcategory with one object x, although both non-isomorphisms send $\underline{R}(x) = R$ isomorphically to $\underline{R}(y) = R$.

Theorem 3.3.2. *Let $\mathcal{D} \subset \mathcal{C}$ be a (connected) full subcategory and N an indecomposable $R\mathcal{D}$-module. Then the $R\mathcal{C}$-module $N\uparrow^{\mathcal{C}}_{\mathcal{D}}$ is indecomposable and moreover is relatively $\mathcal{D}$-projective.*

Proof. Suppose $N\uparrow^{\mathcal{C}}_{\mathcal{D}} = N_1 \oplus N_2$, where N_1, N_2 are both non-zero. Then $N = N\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}} = N_1\downarrow^{\mathcal{C}}_{\mathcal{D}} \oplus N_2\downarrow^{\mathcal{C}}_{\mathcal{D}}$, and since N is indecomposable we must have $N_1\downarrow^{\mathcal{C}}_{\mathcal{D}} = N$ and $N_2\downarrow^{\mathcal{C}}_{\mathcal{D}} = 0$ (or the other way around). Now that $N\uparrow^{\mathcal{C}}_{\mathcal{D}}$ is generated by its values on $\mathcal{D}$ implies $N_2 = 0$. Hence $N\uparrow^{\mathcal{C}}_{\mathcal{D}}$ is indecomposable, and its relative $\mathcal{D}$-projectivity follows from Definition 3.2.1. □

One can compare the above theorem with Green's indecomposability theorem in group representation theory (see for instance Alperin [1] or Benson [4]).

Definition 3.3.3. Let x be an object of an EI-category $\mathcal{C}$. Then we use $\{x\}$ to denote the full subcategory of $\mathcal{C}$ with a single object x. We use $\{[x]\}$ to denote the full subcategory of $\mathcal{C}$ consisting of all objects which are isomorphic to x.

Given an x, one can choose the full subcategory $\{[x]\}$ and use an indecomposable $R\{[x]\}$-module N to generate an $R\mathcal{C}$-module $N\uparrow^{\mathcal{C}}_{\{[x]\}}$. Then Theorem 3.3.2 asserts that such an induced module is indecomposable. This implies that $R\mathcal{C}$ is not of finite representation type if, for some $x \in \operatorname{Ob}\mathcal{C}$, $R\operatorname{Aut}_{\mathcal{C}}(x)$ is not.

Theorem 3.3.4. *Let M be an indecomposable $R\mathcal{C}$-module which is relatively $\mathcal{D}$-projective for a (connected) full subcategory $\mathcal{D} \subset \mathcal{C}$. Then $M\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is indecomposable.*

Proof. Suppose $M\downarrow^{\mathcal{C}}_{\mathcal{D}} = M_1 \oplus \cdots \oplus M_n$ is a direct sum of indecomposable $R\mathcal{D}$-modules. Then $M \cong M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} = M_1\uparrow^{\mathcal{C}}_{\mathcal{D}} \oplus \cdots \oplus M_n\uparrow^{\mathcal{C}}_{\mathcal{D}}$. Since M is indecomposable, we have $M | M_i\uparrow^{\mathcal{C}}_{\mathcal{D}}$ for some index i. This implies $M(x) | M_i\uparrow^{\mathcal{C}}_{\mathcal{D}}(x) = M_i(x)$ for all $x \in \operatorname{Ob}\mathcal{D}$, hence $M\downarrow^{\mathcal{C}}_{\mathcal{D}} = M_i$ is indecomposable. □

Theorems 3.3.2 and 3.3.4 also give us an equivalence of two module categories (a Green correspondence).

Definition 3.3.5. Let $\mathcal{D} \subset \mathcal{C}$ be a connected full subcategory. We define $R\mathcal{C}_{\mathcal{D}}$-mod to be the full subcategory of $R\mathcal{C}$-mod consisting of all relatively $\mathcal{D}$-projective $R\mathcal{C}$-modules.

For the sake of simplicity, we write $\mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M, N)$ for the set of morphisms between two modules $M, N \in R\mathcal{C}_{\mathcal{D}}$-mod.

Proposition 3.3.6. *The functor* $\downarrow^{\mathcal{C}}_{\mathcal{D}}: R\mathcal{C}_{\mathcal{D}}\text{-mod} \to R\mathcal{D}\text{-mod}$ *is an equivalence with* $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ *as its inverse.*

Proof. From the previous results we see the two functors are well-defined on objects, while $\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Id}_{\mathcal{C}_{\mathcal{D}}}$ and $\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Id}_{\mathcal{D}}$. Actions of the induction and the restriction on the morphisms are very clear. Furthermore on morphisms we have the following isomorphisms

$$\mathrm{Hom}_{R\mathcal{D}}\big(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, N\downarrow^{\mathcal{C}}_{\mathcal{D}}\big) \cong \mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}\big(M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}, N\big) \cong \mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M, N),$$

and

$$\mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}\big(M\uparrow^{\mathcal{C}}_{\mathcal{D}}, N\uparrow^{\mathcal{C}}_{\mathcal{D}}\big) \cong \mathrm{Hom}_{R\mathcal{D}}\big(M, N\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}}\big) \cong \mathrm{Hom}_{R\mathcal{D}}(M, N).$$

So $\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ and $\uparrow^{\mathcal{C}}_{\mathcal{D}}\downarrow^{\mathcal{C}}_{\mathcal{D}}$ are also identities on morphisms, because both M and N are generated by their values on $\mathcal{D}$. □

Now we are ready to develop the theory of vertices and sources for category algebras. The following result will be used as a stepping stone to define the vertex of an indecomposable module.

Proposition 3.3.7. *Let* $\mathcal{D}$ *and* $\mathcal{E}$ *be two ideals of* $\mathcal{C}$. *Suppose* M *is an* $R\mathcal{C}$*-module. Then* $M\downarrow_{\mathcal{D}}\uparrow^{\mathcal{C}}\downarrow_{\mathcal{E}}\uparrow^{\mathcal{C}} \cong M\downarrow_{\mathcal{D}\cap\mathcal{E}}\uparrow^{\mathcal{C}}$.

Proof. We need to consider the structure of ${}_{R\mathcal{E}}R\mathcal{C} \otimes_{R\mathcal{D}} M$. Since $\mathrm{Ob}\,\mathcal{E}$ forms an ideal in $\mathrm{Ob}\,\mathcal{C}$, we get $R\mathcal{C}\downarrow^{\mathcal{C}}_{\mathcal{E}} \cong R\mathcal{E}$ as an $R\mathcal{E}$-module. The only terms in this direct sum on which $\mathcal{D}$ is non-zero in the action from the right are the ones where x is in $\mathrm{Ob}\,\mathcal{D}$. Regarded as a right $R\mathcal{D}$-module, $R\mathcal{C}$ can be identified with $R\mathcal{D} = \bigoplus_{x\in\mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{\mathcal{C}}(x, -)$. So as an $R\mathcal{E}$–$R\mathcal{D}$-bimodule, $R\mathcal{C} \cong \bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})} R\,\mathrm{Hom}_{\mathcal{C}}(x, -)$. Thus

$$\begin{aligned} R\mathcal{C} \otimes_{R\mathcal{E}} R\mathcal{C} \otimes_{R\mathcal{D}} M &\cong R\mathcal{C} \otimes_{R\mathcal{E}} \Big\{ \bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})} R\,\mathrm{Hom}_{\mathcal{C}}(x, -) \Big\} \otimes_{R\mathcal{D}} M \\ &= R\mathcal{C} \otimes_{R\mathcal{E}} \Big\{ \bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})} R\,\mathrm{Hom}_{\mathcal{C}}(x, -) \Big\} \otimes_{R(\mathcal{D}\cap\mathcal{E})} M \\ &\cong R\mathcal{C} \otimes_{R\mathcal{E}} R\mathcal{E} \otimes_{R(\mathcal{D}\cap\mathcal{E})} M \\ &\cong R\mathcal{C} \otimes_{R(\mathcal{D}\cap\mathcal{E})} M. \qquad \square \end{aligned}$$

We note that the above argument does not work for an arbitrary pair of full subcategories relative to which M is projective.

Corollary 3.3.8. *Suppose $\mathcal{D}$ and $\mathcal{E}$ are two ideals of $\mathcal{C}$. Let M be an $R\mathcal{C}$-module, which is both relatively $\mathcal{D}$-projective and $\mathcal{E}$-projective. Then M is also relatively $\mathcal{D} \cap \mathcal{E}$-projective. Thus for any indecomposable $R\mathcal{C}$-module M, there exists the smallest ideal $\widetilde{\mathcal{V}_M}$ in $\mathcal{C}$, relative to which M is projective.*

Proof. We just need to check that $\mathcal{D} \cap \mathcal{E}$ forms an ideal in $\mathcal{C}$, and then the results follow from the above proposition. □

Obviously, $\widetilde{\mathcal{V}_M}$ has to be connected, because if $\widetilde{\mathcal{V}_M} = \mathcal{D}_1 \cup \mathcal{D}_2$, then $M{\downarrow}^{\mathcal{C}}_{\widetilde{\mathcal{V}_M}} = M{\downarrow}^{\mathcal{C}}_{\mathcal{D}_1} \oplus M{\downarrow}^{\mathcal{C}}_{\mathcal{D}_2}$, and M must be projective relative to one of its connected components, which contradicts with the minimality of $\widetilde{\mathcal{V}_M}$.

Before defining the vertex of an indecomposable module, we introduce some auxiliary notation.

Definition 3.3.9. For any $R\mathcal{C}$-module M we define the full subcategory of $\mathcal{C}$, $\mathcal{C}_M$ to be a category whose object set is

$$\operatorname{Ob}\mathcal{C}_M = \{y \mid [y] \geqslant [x], \text{ some } x \text{ with } M(x) \neq 0\}.$$

Similarly we define $\mathcal{C}^M$ to be the full subcategory whose object set is

$$\operatorname{Ob}\mathcal{C}^M = \{y \mid [y] \leqslant [x], \text{ some } x \text{ with } M(x) \neq 0\}.$$

In other words, $\mathcal{C}_M$ consists of all objects above M-minimal objects and $\mathcal{C}^M$ consists of all objects below M-maximal objects. In fact, $\mathcal{C}_M$ is a co-ideal in $\mathcal{C}$ generated by the M-minimal objects, and $\mathcal{C}^M$ is an ideal in $\mathcal{C}$ generated by the M-maximal objects. In particular, In particular, we have $\mathcal{C}_{S_{x,V}} = \mathcal{C}_{\geqslant x}$, $\mathcal{C}^{S_{x,V}} = \mathcal{C}_{\leqslant x}$ and $\mathcal{C}^{\underline{R}} = \mathcal{C}_{\underline{R}} = \mathcal{C}$.

Definition 3.3.10. The full subcategory $\mathcal{V}_M = \widetilde{\mathcal{V}_M} \cap \mathcal{C}_M$ is called the vertex of M.

We provide two alternative descriptions of the vertex of M.

Definition 3.3.11. Let $\mathcal{D} \subset \mathcal{C}$ be a subcategory. Then $\mathcal{D}$ is said to be convex if whenever there is a sequence of morphisms $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathcal{C}$ with $x, z \in \operatorname{Ob}\mathcal{D}$, then both α and β are in $\operatorname{Mor}(\mathcal{D})$.

Ideals and co-ideals in $\mathcal{C}$ are full convex subcategories. Let M be an indecomposable $R\mathcal{C}$-module. Then its vertex $\mathcal{V}_M$ is convex. Note that in general a convex subcategory $\mathcal{D}$ does not have to be full. Since intersection of two convex subcategories is still convex, it is natural to define the *convex hull* of a subcategory $\mathcal{D}$ of $\mathcal{C}$ as the smallest convex subcategory containing $\mathcal{D}$. This terminology will be used in the next two sections.

Proposition 3.3.12. *Let M be an indecomposable $R\mathcal{C}$-module and $\mathcal{D}$ a full* (*connected*) *subcategory of $\mathcal{C}$. Then the following statements are equivalent*:

(1) *$\mathcal{D}$ is the vertex of M*;

(2) *$\mathcal{D}$ is the smallest ideal in $\mathcal{C}_M$, relative to which M is projective*;
(3) *$\mathcal{D}$ is the smallest full convex subcategory of $\mathcal{C}$, relative to which M is projective.*

Proof. (1) $\Rightarrow$ (2): If $\mathcal{D} = \mathcal{V}_M$, then by definition $\mathcal{D}$ is full convex and M is relatively $\mathcal{D}$-projective. Suppose $\mathcal{E}$ is an ideal in $\mathcal{C}_M$, relative to which M is projective. We claim $\mathcal{D} \subset \mathcal{E}$. In fact, $\mathcal{D}$ is an ideal in $\mathcal{C}_M$, and so is $\mathcal{D} \cap \mathcal{E}$. We can naturally extend $\mathcal{D} \cap \mathcal{E}$ to an ideal $\widetilde{\mathcal{D} \cap \mathcal{E}} \subset \widetilde{\mathcal{V}_M}$ in $\mathcal{C}$, relative to which M is projective. But then by definition we have $\widetilde{\mathcal{V}_M} = \widetilde{\mathcal{D} \cap \mathcal{E}}$, which implies $\mathcal{D} = \widetilde{\mathcal{V}_M} \cap \mathcal{C}_M = \widetilde{\mathcal{D} \cap \mathcal{E}} \cap \mathcal{C}_M = \mathcal{D} \cap \mathcal{E}$.

(2) $\Rightarrow$ (3): Let $\mathcal{E}$ be a full convex subcategory for which M is relatively $\mathcal{E}$-projective. Then $\mathcal{E}$ contains all M-minimal objects, and thus $\mathcal{E} \cap \mathcal{C}_M$ must be an ideal in $\mathcal{C}_M$. Since as an $R\mathcal{C}_M$-module M is projective relative to $\mathcal{E} \cap \mathcal{C}_M$, we have $\mathcal{D} \subset \mathcal{E}$.

(3) $\Rightarrow$ (1): Let $\mathcal{E}$ be an ideal in $\mathcal{C}$, relative to which M is projective. Then $\mathcal{E} \cap \mathcal{C}_M$ is a full convex subcategory in $\mathcal{C}$, which means $\mathcal{D} \subset \mathcal{E} \cap \mathcal{C}_M$. We can take $\mathcal{E}$ to be $\widetilde{\mathcal{V}_M}$, and this results in an inclusion $\mathcal{D} \subset \mathcal{V}_M$, which can be shown to be an equality by extend $\mathcal{D}$ to an ideal in $\widetilde{\mathcal{V}_M}$. $\square$

Proposition 3.3.13. *Let $\mathcal{D}$ be a connected full subcategory of $\mathcal{C}$ and N an indecomposable $R\mathcal{D}$-module with vertex $\mathcal{V}_N \subset \mathcal{D}$. Then the indecomposable $R\mathcal{C}$-module $M = N{\uparrow}_{\mathcal{D}}^{\mathcal{C}}$ is relatively $\mathcal{V}_N$-projective. If $\mathcal{V}_N$ is a (connected and full) convex subcategory of $\mathcal{C}$, then $\mathcal{V}_M = \mathcal{V}_N$.*

If M is an indecomposable $R\mathcal{C}$-module whose vertex is $\mathcal{V}_M$, and $\mathcal{D}$ is a connected full subcategory containing $\mathcal{V}_M$, then $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}$ is an indecomposable $R\mathcal{D}$-module whose vertex is $\mathcal{V}_M$.

Proof. The first statement holds because of Theorem 3.3.2, and by Proposition 3.2.3(2) we know $\mathcal{V}_M \subset \mathcal{V}_N$. After we prove the second part, we can show $\mathcal{V}_M$ is exactly $\mathcal{V}_N$, if $\mathcal{V}_N$ is a convex subcategory of $\mathcal{C}$.

The second statement is true because of Theorem 3.3.4. Suppose $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}$ has the vertex $\mathcal{E}' \subset \mathcal{D}$. Since $M \cong (M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}){\downarrow}_{\mathcal{V}_M}^{\mathcal{D}}{\uparrow}_{\mathcal{V}_M}^{\mathcal{C}} \cong (M{\downarrow}_{\mathcal{V}_M}^{\mathcal{C}}{\uparrow}_{\mathcal{V}_M}^{\mathcal{D}}){\uparrow}_{\mathcal{D}}^{\mathcal{C}}$, we obtain $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}} \cong M{\downarrow}_{\mathcal{V}_M}^{\mathcal{C}}{\uparrow}_{\mathcal{V}_M}^{\mathcal{D}} = (M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}){\downarrow}_{\mathcal{V}_M}^{\mathcal{D}}{\uparrow}_{\mathcal{V}_M}^{\mathcal{D}}$. Hence $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}$ is relatively $\mathcal{V}_M$-projective and $\mathcal{E}' \subset \mathcal{V}_M$. But by the first part, $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}{\uparrow}_{\mathcal{D}}^{\mathcal{C}}$ has a vertex that is contained in $\mathcal{E}'$. Since $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}{\uparrow}_{\mathcal{D}}^{\mathcal{C}} \cong M$, we must have $\mathcal{V}_M \subset \mathcal{E}'$.

Now we go back to finish proving part 1. Let $\mathcal{V}_M$ be the vertex of $M = N{\uparrow}_{\mathcal{D}}^{\mathcal{C}}$, from part 2 and $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}} = N{\uparrow}_{\mathcal{D}}^{\mathcal{C}}{\downarrow}_{\mathcal{D}}^{\mathcal{C}}$ we get desired equality $\mathcal{V}_M = \mathcal{V}_N$ because $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}}$ still has vertex $\mathcal{V}_M$ by part 2, while $M{\downarrow}_{\mathcal{D}}^{\mathcal{C}} = (N{\uparrow}_{\mathcal{D}}^{\mathcal{C}}){\downarrow}_{\mathcal{D}}^{\mathcal{C}} = N$ has vertex $\mathcal{V}_N$. $\square$

Remark 3.3.14. In the first part of Proposition 3.3.13, if $\mathcal{V}_N$ is not a convex subcategory in $\mathcal{C}$, then it is not necessarily true that $\mathcal{V}_M = \mathcal{V}_N$, where $M = N{\uparrow}_{\mathcal{D}}^{\mathcal{C}}$. One can check Example 3.4.8 in the next section, where we have a pair of categories $\mathcal{D} \subset \mathcal{C}$. If we choose the $R\mathcal{D}$-module $N = \underline{R}$, then $\mathcal{V}_N = \mathcal{D}$, which is not convex in $\mathcal{C}$. The induced module $M = N{\uparrow}_{\mathcal{D}}^{\mathcal{C}} = \underline{R}{\uparrow}_{\mathcal{D}}^{\mathcal{C}}$ is isomorphic to the trivial $R\mathcal{C}$-module $\underline{R}$, which is relatively $\mathcal{V}_N$-projective and whose vertex is shown to be $\mathcal{V}_M = \mathcal{C}$.

Since $M{\downarrow}_{\mathcal{V}_M}^{\mathcal{C}}{\uparrow}_{\mathcal{V}_M}^{\mathcal{C}} \cong M$ and $M{\downarrow}_{\mathcal{V}_M}^{\mathcal{C}}$ is indecomposable, M is determined up to isomorphism by the indecomposable $R\mathcal{V}_M$-module $M{\downarrow}_{\mathcal{V}_M}^{\mathcal{C}}$.

Definition 3.3.15. Suppose M is an indecomposable $R\mathcal{C}$-module with the vertex $\mathcal{V}_M$. Then $M{\downarrow}_{\mathcal{V}_M}$ is called the source for M.

Recall that we denote by $\{[x]\}$ the full subcategory of $\mathcal{C}$ (equivalent to $\widehat{\mathrm{Aut}_{\mathcal{C}}(x)}$) consisting of objects in the isomorphism class $[x]$.

Proposition 3.3.16. *The vertex of the indecomposable projective module $P_{x,V}$ is $\{[x]\}$. The source for $P_{x,V}$ is $P_V = P_{x,V}{\downarrow}^{\mathcal{C}}_{\{[x]\}}$, the projective cover of V as an $R\{[x]\}$-module.*

Proof. Let $P_{x,V} = R\mathcal{C} \cdot e_{x,V}$ for some primitive idempotent $e_{x,V} \in R\,\mathrm{Aut}(x)$. Then we can easily check that $P_{x,V}{\downarrow}^{\mathcal{C}}_{\mathcal{C}_{\leqslant x}}{\uparrow}^{\mathcal{C}}_{\mathcal{C}_{\leqslant x}} = R\mathcal{C} \otimes_{R\mathcal{C}_{\leqslant x}} R\,\mathrm{Aut}(x) \cdot e_{x,V} = R\mathcal{C}e_{x,V} \otimes_{R\mathcal{C}_{\leqslant x}} 1_x \cong P_{x,V}$. Since $\mathcal{C}_{P_{x,V}} = \mathcal{C}_{\geqslant x}$, by definition the vertex of $P_{x,V}$ is $\mathcal{C}_{\leqslant x} \cap \mathcal{C}_{\geqslant x} = \{[x]\}$. □

Definition 3.3.17. A morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is irreducible if it is not a composite of two non-isomorphisms. The subset of $\mathrm{Hom}_{\mathcal{C}}(x, y)$, $x, y \in \mathrm{Ob}\,\mathcal{C}$, consisting of irreducible morphisms is denoted by $\mathrm{Irr}_{\mathcal{C}}(x, y)$.

Note that our irreducible morphisms are different from the irreducible morphisms in the representation theory of Artin algebras [2].

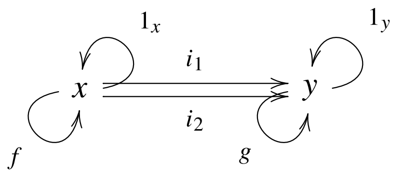
Proposition 3.3.18. *Let M be an indecomposable atomic module concentrated on $[x] \subset \mathrm{Ob}\,\mathcal{C}$. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}_{\geqslant x}$ whose object set consists of $[x]$ and those $y \ncong x$ which satisfy the condition that $\mathrm{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$. Then M is relatively $\mathcal{D}$-projective, and $\mathcal{V}_M$ is the convex hull of $\mathcal{D}$. The source for M is itself* (*but regarded as an $R\mathcal{V}_M$-module*).

Proof. It is easy to verify that $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}{\uparrow}^{\mathcal{C}}_{\mathcal{D}} \cong M$ and there is no proper full subcategory of $\mathcal{D}$ having the same property, because if $y \in \mathrm{Ob}\,\mathcal{C}_{\geqslant x}$, $\mathrm{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$ and $y \notin \mathrm{Ob}\,\mathcal{D}$ then $0 \neq R\,\mathrm{Irr}_{\mathcal{C}}(x, y) \otimes_{R\mathcal{D}} M(x) \subset R\,\mathrm{Hom}(x, y) \otimes_{R\mathcal{D}} M(x) \subset M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}{\uparrow}^{\mathcal{C}}_{\mathcal{D}}(y)$, which contradicts with the fact $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}{\uparrow}^{\mathcal{C}}_{\mathcal{D}}(y) \cong M(y) = 0$.

Since $\mathcal{D} \subset \mathcal{V}_M$, by Proposition 3.3.12(3) $\mathcal{V}_M$ is exactly the smallest full convex subcategory containing $\mathcal{D}$. □

The last example considers the vertex and source for an indecomposable module other than the indecomposable projective or atomic modules, and also discusses the representation type of $R\mathcal{C}$.

Example 3.3.19. Given a category $\mathcal{C}$



,

with $i_1 f = i_1$, $i_2 f = i_2$, $gi_1 = i_2$ and $gi_2 = i_1$. Let $R = \mathbb{F}_2$ be a field of characteristic 2. We consider the indecomposable module M such that $M(x) = \mathbb{F}_2$ and $M(y) = \mathbb{F}_2 \oplus \mathbb{F}_2$. The maps i_1, i_2 send $M(x) = \mathbb{F}_2$ to the first and the second component, respectively, of $M(y) = \mathbb{F}_2 \oplus \mathbb{F}_2$, and g interchanges the two entries of $\mathbb{F}_2 \oplus \mathbb{F}_2$ (it is easy to verify these define a functor $M : \mathcal{C} \to R$-mod, which is neither projective nor simple). The module M has vertex $\mathcal{V}_M = \{x\} \cong \widehat{\mathrm{Aut}_{\mathcal{C}}(x)}$ and source $M(x) = \mathbb{F}_2$, since $\mathbb{F}_2{\uparrow}^{\mathcal{C}}_{\mathcal{D}}(y) = (i_1 \otimes \mathbb{F}_2) \oplus (i_2 \otimes \mathbb{F}_2)$ as $\mathbb{F}_2\,\mathrm{Aut}(y)$-module.

We show $\mathbb{F}_2\mathcal{C}$ has infinite representation type by constructing infinitely many non-isomorphic modules whose vertices are $\mathcal{C}$. Define for any $n \in N$ an (indecomposable) $\mathbb{F}_2\mathcal{C}$-module M_n such that $M_n(x) = [\mathbb{F}_2 \operatorname{Aut}_{\mathcal{C}}(x)]^n$ and $M_n(y) = \mathbb{F}_2 i_1 + \mathbb{F}_2 i_2$. It is certainly not relatively $\{y\}$-projective, and is not $\{x\}$-projective when $n > 1$ because $[R\mathcal{C} \otimes_{R \operatorname{Aut}_{\mathcal{C}}(x)} M_n(x)](y)$ has dimension $2n$. If $n \neq m$ are both bigger than 1 then $M_n \ncong M_m$ since $M_n(x) \ncong M_m(x)$.

Finally we get to the applications of our theory. The following statement is actually an Eckmann–Shapiro type lemma.

Lemma 3.3.20. *Let M be an $R\mathcal{C}$-module which is relatively $\mathcal{D}$-projective for a full subcategory of $\mathcal{C}$. If $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module, then*

$$\operatorname{Ext}^*_{R\mathcal{C}}(M, N) \cong \operatorname{Ext}^*_{R\mathcal{D}}\big(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}\big).$$

In particular we get $\operatorname{Ext}^*_{R\mathcal{C}}(M, N) \cong \operatorname{Ext}^*_{R\mathcal{C}_M}(M{\downarrow}^{\mathcal{C}}_{\mathcal{C}_M}, N{\downarrow}^{\mathcal{C}}_{\mathcal{C}_M})$. *If $M = \underline{R}$ then we have*

$$\varprojlim\nolimits^*_{\mathcal{C}} N \cong \varprojlim\nolimits^*_{\mathcal{D}} N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}.$$

Proof. The proof is almost the same as the classic proof of the Eckmann–Shapiro Lemma $\operatorname{Ext}^*_{R\mathcal{C}}(M'{\uparrow}^{\mathcal{C}}_{\mathcal{D}}, N) \cong \operatorname{Ext}^*_{R\mathcal{D}}(M', N{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$, see Benson [5]. One just has to replace M' by $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ afterwards. □

We conclude the applications of vertices and sources with an isomorphism of cohomology rings. General theory for the correspondence between extensions of modules over a finite-dimensional algebra A and the groups $\operatorname{Ext}^*_A(-,-)$ can be found in Benson [4].

Proposition 3.3.21. *Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory, relative to which M is projective. Suppose R is a field or a complete discrete valuation ring and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module. Then there is a ring isomorphism between* $\operatorname{Ext}^*_{R\mathcal{C}}(M, M)$ *and* $\operatorname{Ext}^*_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$.

Proof. Let $0 \to M{\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to N_{n-1} \to \cdots \to N_0 \to M{\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to 0$ represent an element of $\operatorname{Ext}^n_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$ for some positive integer n. Then it gives rise to an element of $\operatorname{Ext}^n_{R\mathcal{C}}(M, M)$ vis induction $0 \to M \to N_{n-1}{\uparrow}^{\mathcal{C}}_{\mathcal{D}} \to \cdots \to N_0{\uparrow}^{\mathcal{C}}_{\mathcal{D}} \to M \to 0$, since $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}{\uparrow}^{\mathcal{C}}_{\mathcal{D}} \cong M$ and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module. But this element of $\operatorname{Ext}^n_{R\mathcal{C}}(M, M)$ restricts back to the given element of $\operatorname{Ext}^n_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$. Hence the composite of these two maps $\operatorname{Ext}^n_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \to \operatorname{Ext}^n_{R\mathcal{C}}(M, M) \to \operatorname{Ext}^n_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$ is the identity, which implies the first map $\operatorname{Ext}^n_{R\mathcal{D}}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}, M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \to \operatorname{Ext}^n_{R\mathcal{C}}(M, M)$ is injective. Since we know these two Ext groups are isomorphic (Lemma 3.3.20), this map has to be bijective. Now it is easy to check that this map respects the Yoneda splice, and thus defines a ring isomorphism. □

3.4. Structure of the vertex of the trivial module

Because of the special interests in the trivial module $\underline{R}$, we try to obtain a precise description of the vertex $\mathcal{V}_{\underline{R}}$. For the definition of left Kan extension, the reader is referred to the paragraphs preceding Proposition 2.3.4.

Proposition 3.4.1. *Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be the inclusion of a full subcategory. Then $\underline{R}$ is relatively $\mathcal{D}$-projective if and only if $\iota\downarrow_y$ is connected and non-empty for each $y \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. Let $\underline{R} = \underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}$ be the trivial $R\mathcal{D}$-module. Then $\underline{R}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) \cong K(\underline{R})(y) = \varinjlim_{\iota\downarrow_y} \underline{R}$ equals a direct sum of R over the connected components of $\iota\downarrow_y$. Hence the statement follows. □

The proposition is not true for subcategories which are not full in $\mathcal{C}$. A simple example will be a group G with a proper Sylow-p subgroup P, both of which are regraded as categories with a single object $*$. When $R = \mathbb{F}_p$, $\underline{R}$ is relatively $\hat{P}$-projective, while $\iota\downarrow_*$ has $[G : P]$ connected components which is not connected. Now we turn to an alternative characterization of the category $\mathcal{V}_{\underline{R}}$.

Definition 3.4.2. An object $x \in \mathrm{Ob}\,\mathcal{C}$ is weakly essential if the full subcategory $\mathcal{C}_{<x}$ is empty or has more than one component. The full subcategory of $\mathcal{C}$ consisting of all weakly essential objects is named $\mathrm{Wess}_0(\mathcal{C})$. There is a larger full subcategory $\mathrm{Wess}(\mathcal{C}) \supset \mathrm{Wess}_0(\mathcal{C})$ containing objects $x \in \mathrm{Ob}\,\mathcal{C}$ so that the full subcategory $\mathcal{C}_{<x}$ is not contractible.

When $\mathcal{C}$ is a certain subgroup poset of a group, Quillen [30,31] and Bouc [5] considered $\mathrm{Wess}(\mathcal{C})$, and Puig [29] (see also Thévenaz [35]) introduced the so-called *essential objects* of $\mathcal{C}$ which we do not need and are contained in $\mathrm{Wess}_0(\mathcal{C})$. It is Symonds [34] who generalized $\mathrm{Wess}(\mathcal{C})$ to arbitrary posets, and defined $\mathrm{Wess}_0(\mathcal{C})$ for posets. Obviously the minimal objects of any category $\mathcal{C}$ are weakly essential, contained in both $\mathrm{Wess}_0(\mathcal{C})$ and $\mathcal{V}_{\underline{R}}$.

Lemma 3.4.3. *Let $\mathcal{D}$ be a connected full subcategory of $\mathcal{C}$. Suppose $\underline{R}$ is relatively $\mathcal{D}$-projective. Then for any $y \notin \mathrm{Ob}\,\mathcal{D}$, $\mathcal{D}_{<y}$ is non-empty and connected.*

Proof. Obviously $\mathcal{D}_{<y}$ is non-empty by Proposition 3.4.1. If $\mathcal{D}_{<y}$ were disconnected, then we prove

$$\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = \sum_{x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}(x, y) \otimes \underline{R}(x)$$

is a direct sum of at least two non-zero summands. Hence a contradiction since $\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = R$.

If $\mathcal{D}_{<y}$ were disconnected, then $\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = \sum_{x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}(x, y) \otimes \underline{R}(x)$ contains two elements $\alpha \otimes 1$ and $\beta \otimes 1$, where $\alpha \in \mathrm{Hom}(x_1, y)$ and $\beta \in \mathrm{Hom}(x_2, y)$ for x_1, x_2 from different components of $\mathcal{D}_{<y}$. Let is assume x_1, x_2 minimal. Now, since $\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}(y) = R$ has rank 1, we have $r\alpha \otimes_{R\mathcal{D}} 1 = \beta \otimes_{R\mathcal{D}} 1$ for some $r \in R$. But it means that $r\alpha\gamma \otimes_{R\mathcal{D}} 1_{x_2} = \beta \otimes_{R\mathcal{D}} 1_{x_2}$ (or $r\alpha \otimes_{R\mathcal{D}} 1_{x_1} = \beta\gamma \otimes_{R\mathcal{D}} 1_{x_1}$) for some $\gamma \in R\,\mathrm{Hom}(x_2, x_1)$ (or in $R\,\mathrm{Hom}(x_1, x_2)$), which implies $\mathrm{Hom}(x_2, x_1)$ (or $\mathrm{Hom}(x_1, x_2)$) is non-empty. So x_1 and x_2 belong to the same connected component which is a contradiction. □

The above fact results in a corollary which is more convenient to use than Proposition 3.4.1 as a tool to narrow down the subcategory $\mathcal{V}_{\underline{R}}$.

Corollary 3.4.4. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, relative to which $\underline{R}$ is projective. Then* $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$. *In particular* $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{V}_{\underline{R}}$.

Proof. We show if $y \notin \operatorname{Ob}\mathcal{D}$, then $y \notin \mathrm{Wess}_0(\mathcal{C})$. First of all since $\mathcal{D}$ contains all minimal objects in $\mathcal{C}$, $\mathcal{C}_{<y}$ cannot be empty because $y \notin \operatorname{Ob}\mathcal{D}$. We claim $\mathcal{C}_{<y}$ is connected. Assume the opposite. Since $\mathcal{D}_{<y} \subset \mathcal{C}_{<y}$ and $\mathcal{C}_{<y}$ is disconnected, by Lemma 3.4.3 $\mathcal{D}_{<y}$ must lie in only one of the components of $\mathcal{C}_{<y}$. But then $\mathcal{C}_{<y}$ contains at least one minimal object x which does not belong to $\operatorname{Ob}\mathcal{D}_{<y}$. Actually, x is not in $\operatorname{Ob}\mathcal{D}$ either, because otherwise $\mathcal{D}_{<y}$ is disconnected. Now $x \notin \operatorname{Ob}\mathcal{D}$ contradicts with the fact that $\mathcal{D}$ contains all minimal objects. □

The following result extends Symonds [34] Proposition 3.10, saying that $\mathrm{Wess}_0(\mathcal{C})$ and $\mathcal{C}$ have the same numbers of connected components. Recall that there is a poset $P(\mathcal{C})$ associated to each EI-category $\mathcal{C}$. When $\mathcal{C}$ is finite we call the maximal length of chains of non-isomorphisms in $P(\mathcal{C})$ the length of $\mathcal{C}$, denoted by $l(\mathcal{C})$.

Lemma 3.4.5. *Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. If $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$, then $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D} \subset \mathcal{C}$ induce bijections on connected components.*

Proof. Note that for any EI-category $\mathcal{C}$ there is a one-one bijection between the connected components of $\mathcal{C}$ and those of its underlying poset $P(\mathcal{C})$, and thus one can mimic Symonds' proof for finite posets by doing induction on the length of a category (or its underlying poset). □

Corollary 3.4.6. *Let $\mathcal{C}$ be an EI-category and $\mathcal{D}$ a connected full subcategory, relative to which $\underline{R}$ is projective. For every $y \notin \mathrm{Wess}_0(\mathcal{C})$, $\mathrm{Wess}_0(\mathcal{C})_{<y}$, $\mathrm{Wess}(\mathcal{C})_{<y}$ and $\mathcal{D}_{<y}$ are all connected.*

Proof. By definition of $\mathrm{Wess}_0(\mathcal{C})$, $\mathcal{C}_{<y}$ is always connected. The results follow from the inclusions $\mathrm{Wess}_0(\mathcal{C}_{<y}) = \mathrm{Wess}_0(\mathcal{C})_{<y} \subset \mathcal{D}_{<y} \subset \mathcal{C}_{<y}$ and $\mathrm{Wess}_0(\mathcal{C}_{<y}) = \mathrm{Wess}_0(\mathcal{C})_{<y} \subset \mathrm{Wess}(\mathcal{C})_{<y} \subset \mathcal{C}_{<y}$, combined with Lemma 3.4.5 and the fact that $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$. □

We give a sufficient condition on the connectedness of overcategories and the relative projectivity of $\underline{R}$.

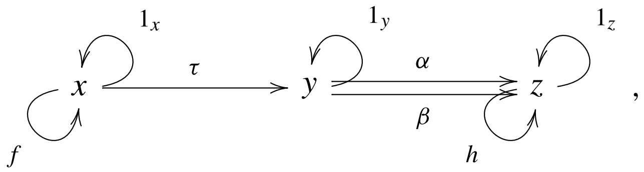
Proposition 3.4.7. *Let $\mathcal{C}$ be an EI-category and $\mathcal{D}$ a full subcategory containing $\mathrm{Wess}_0(\mathcal{C})$. Let $\iota : \mathcal{D} \to \mathcal{C}$ be the inclusion. Then every $\iota{\downarrow}_y$, $y \in \operatorname{Ob}\mathcal{C}$, is connected if for any pair of objects $x \in \operatorname{Ob}\mathcal{D}$ and $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, $\mathrm{Aut}_{\mathcal{C}}(x)$ acts transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$. When this is true, $\underline{R}$ is relatively $\mathcal{D}$-projective.*

Proof. When $y \in \operatorname{Ob}\mathcal{D}$, $\iota{\downarrow}_y$ is connected because there is only one isomorphism class of maximal objects, of the form (y, g) where $g \in \mathrm{Aut}(y)$. Now we assume $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$. The objects of $\iota{\downarrow}_y$ are of the form (x, α), where $x \in \operatorname{Ob}\mathcal{D}_{<y}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. If for every pair of objects $x \in \operatorname{Ob}\mathcal{D}$ and $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, $\mathrm{Aut}_{\mathcal{C}}(x)$ acts transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$, then $\iota{\downarrow}_y$ has the same underlying poset as $\mathcal{D}_{<y}$ because $(x, \alpha) \cong (x, \beta)$ for any two morphisms $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. This implies $\mathcal{D}_{<y}$ and $\iota{\downarrow}_y$ have the same number of connected components. Since $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$, we know $\mathcal{D}_{<y}$ (hence $\iota{\downarrow}_y$) is connected for any $y \in \operatorname{Ob}\mathcal{C}$ by Corollary 3.4.6.

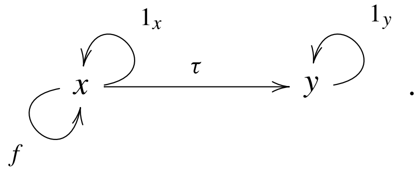
The second statement is a corollary of Proposition 3.4.1. □

The above proposition asserts that if $\mathcal{C}$ is a poset then $\mathrm{Wess}_0(\mathcal{C})$ is the smallest subposet, relative to which $\underline{R}$ is projective.

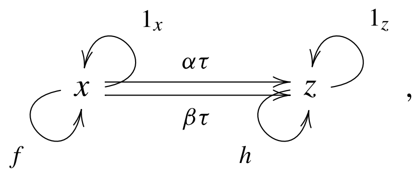
Example 3.4.8. Suppose $\mathcal{C}$ is the following category



with R arbitrary, $\mathrm{Aut}(x)$ acting trivially on $\mathrm{Hom}(x, y) = \{\tau\}$, $\mathrm{Aut}(z)$ interchanging α and β, and $\beta\tau \neq \alpha\tau$. From direct calculations, we can see that $\mathcal{V}_{\underline{R}} = \mathcal{C}$. The following category is $\mathrm{Wess}(\mathcal{C})$



We can check $\underline{R}$ is projective relative to the full subcategory $\mathcal{D}$



which is not convex and is the smallest full subcategory among all those relative to which $\underline{R}$ is projective. Comparing these categories, we get $\mathrm{Wess}(\mathcal{C}) \subset \mathcal{V}_{\underline{R}} = \mathcal{C}$, $\mathcal{D} \not\subset \mathrm{Wess}(\mathcal{C})$ and $\mathrm{Wess}(\mathcal{C}) \not\subset \mathcal{D}$. Note that $\mathrm{Wess}_0(\mathcal{C}) = \{x\}$ is contained in $\mathcal{V}_{\underline{R}}$, $\mathrm{Wess}(\mathcal{C})$ and $\mathcal{D}$.

3.5. Categories with subobjects

This part of the work grows out of our observation that if $\mathcal{C}$ is a finite poset then $\mathcal{V}_{\underline{R}}$ is determined by $\mathrm{Wess}_0(\mathcal{C})$. In this section, we prove the same result for the categories with subobjects, which were introduced and studied by Oliver [28]. A category with subobjects $(\mathcal{C}, \mathcal{I})$ is a pair of categories $\mathcal{I} \subset \mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{I} = \mathrm{Ob}\,\mathcal{C}$, and such that the following two conditions are satisfied:

(1) $|\mathrm{Hom}_{\mathcal{I}}(x, y)| \leqslant 1$ for any pair of objects x, y; and
(2) each morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ can be written in a unique way as a composite $\alpha = \alpha_0 \cdot f$, where $f \in \mathrm{Is}_{\mathcal{C}}(x, x')$ for some x', and $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x', y)$.

We note that Jackowski and Słomińska introduced the EI-categories with quotients in their paper [22], which is a concept dual to the EI-categories with subobjects in the sense that if $(\mathcal{C}, \mathcal{I})$ is a category with subobjects then $(\mathcal{C}^{op}, \mathcal{I}^{op})$ is a category with quotients, and vice versa. All results in this section have their counterparts for EI-categories with quotients.

Let $(\mathcal{C}, \mathcal{I})$ be a category with subobjects. We say $(\mathcal{C}, \mathcal{I})$ is a skeletal category with subobjects, if $\mathcal{C}$ is skeletal (see Mac Lane [26]). It is *not* true that if $(\mathcal{C}, \mathcal{I})$ is a category with subobjects, then the skeleton of $\mathcal{C}$ can be made into a category with subobjects. If we assume $\mathcal{C}$ is an EI-category, then we naturally have a definition of EI-categories with subobjects.

Lemma 3.5.1. *Let $(\mathcal{C},\mathcal{I})$ be a category with subobjects. The endomorphism group of each $x \in \mathrm{Ob}\,\mathcal{I}$ is precisely $\{1_x\}$, and the only isomorphisms in* $\mathrm{Mor}(\mathcal{I})$ *are $\{1_x \mid x \in \mathrm{Ob}\,\mathcal{I}\}$. If furthermore $\mathcal{C}$ is EI, $\mathcal{I}$ is a poset.*

Proof. It is easy to see $\mathrm{End}_{\mathcal{I}}(x) = \mathrm{Aut}_{\mathcal{I}}(x) = \{1_x\}$ since $|\mathrm{End}_{\mathcal{I}}(x)| \leqslant 1$ and $1_x \in \mathrm{End}_{\mathcal{I}}(x)$. Now we show $\mathrm{Is}_{\mathcal{I}}(x, y) = \emptyset$ if $x \cong y$ in $\mathcal{C}$ and $x \neq y$. If there were an $\alpha_0 \in \mathrm{Is}_{\mathcal{I}}(x, y)$, then we would have two distinct factorizations $\alpha_0 = 1_y\alpha_0 = \alpha_0 1_x$, a contradiction to the definition of a category with subobjects.

If $\mathcal{C}$ is EI and $\mathrm{Hom}_{\mathcal{I}}(x, y) \neq \emptyset$ for some $x \neq y$, we show $\mathrm{Hom}_{\mathcal{I}}(y, x) = \emptyset$. This implies $\mathcal{I}$ is a poset. Indeed if there exists $\alpha \in \mathrm{Hom}_{\mathcal{I}}(x, y)$ and $\beta \in \mathrm{Hom}_{\mathcal{I}}(y, x)$, then $\alpha\beta = 1_y$ and $\beta\alpha = 1_x$ in $\mathcal{I}$ (and $\mathcal{C}$). Hence α and β are isomorphisms, which is impossible because from above we know $\mathrm{Is}_{\mathcal{I}}(x, y) = \emptyset$ if $x \neq y$. □

The above result implies that if $(\mathcal{C},\mathcal{I})$ is a (not necessarily EI) category with subobjects, any non-empty set $\mathrm{Hom}_{\mathcal{I}}(x, y)$ with $x \neq y$ will consist of a non-isomorphism.

Proposition 3.5.2. *Let $(\mathcal{C},\mathcal{I})$ be a category with subobjects. If every isomorphism class of objects in $\mathcal{C}$ is finite, then $\mathcal{C}$ is an EI-category.*

Proof. Suppose there exists an $\alpha \in \mathrm{End}_{\mathcal{C}}(x) \setminus \mathrm{Aut}_{\mathcal{C}}(x)$ for some $x \in \mathrm{Ob}\,\mathcal{C}$. We show this assumption leads to a contradiction. By definition of a category with subobjects, $\alpha = \alpha_0 f$ for some $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x', x)$ and $f \in \mathrm{Is}_{\mathcal{C}}(x, x')$, where $x' \cong x$ in $\mathcal{C}$. We claim $x' \neq x$. If $x = x'$, then $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x, x) = \mathrm{End}_{\mathcal{I}}(x) = \{1_x\}$ by preceding lemma. But then $\alpha = f$ is an isomorphism, a contradiction to our assumption. From $x \neq x'$, we know $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x', x)$ is not an isomorphism.

Now since $x' \cong x$, there exists a $\beta \in \mathrm{End}_{\mathcal{C}}(x') \setminus \mathrm{Aut}_{\mathcal{C}}(x')$, and $\beta = \beta_0 g$ for some $\beta_0 \in \mathrm{Hom}_{\mathcal{I}}(x'', x')$ and $g \in \mathrm{Is}_{\mathcal{C}}(x', x'')$. As is shown in last paragraph, $x'' \neq x'$ and β_0 is not an isomorphism. In fact x'' cannot be x either, since if they were equal, we would have two morphisms in $\mathcal{I}$: $\beta_0 : x \to x'$ and $\alpha_0 : x' \to x$, which implies $\alpha_0\beta_0 = 1_x$ and $\beta_0\alpha_0 = 1_{x'}$. The two equalities assert that α_0 and β_0 are isomorphisms, inverse to each other, hence a contradiction to our assumptions on α_0 and β_0. Thus any two of x, x' and x'' are not equal, and we can find a third non-isomorphism $\gamma \in \mathrm{End}_{\mathcal{C}}(x'') \setminus \mathrm{Aut}_{\mathcal{C}}(x'')$ so that we can repeat what we have done for $\beta \in \mathrm{End}_{\mathcal{C}}(x') \setminus \mathrm{Aut}_{\mathcal{C}}(x')$. Gradually, we are going to produce an infinite list of isomorphic objects in $\mathcal{C}$, $x, x', x'', \ldots$, while any two of them are not equal. This leads to a contradiction since we assume $[x]$ is finite. Thus there is no such $\alpha \in \mathrm{End}_{\mathcal{C}}(x) \setminus \mathrm{Aut}_{\mathcal{C}}(x)$ for any $x \in \mathrm{Ob}\,\mathcal{C}$, and then $\mathrm{End}_{\mathcal{C}}(x) = \mathrm{Aut}_{\mathcal{C}}(x)$ for all $x \in \mathrm{Ob}\,\mathcal{C}$, or $\mathcal{C}$ is EI. □

Some easy but useful facts about EI-categories with subobjects.

Lemma 3.5.3. *Let $(\mathcal{C},\mathcal{I})$ be an EI-category with subobjects. Then*

(1) *all morphisms in $\mathcal{C}$ are monomorphisms*;
(2) *if $\mathcal{C}$ is a finite category with subobjects, then the number of objects $z \in [x]$ for which* $\mathrm{Hom}_{\mathcal{I}}(z, y) \neq \emptyset$ *equals* $|\mathrm{Hom}_{\mathcal{C}}(x, y)|/|\mathrm{Aut}_{\mathcal{C}}(x)|$*, for any $y \ncong x$ such that* $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$;
(3) *if $\mathcal{C}$ is skeletal* $\mathrm{Aut}_{\mathcal{C}}(x)$ *acts regularly on* $\mathrm{Hom}_{\mathcal{C}}(x, y)$*, for any pair of objects $x, y \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. Jackowski–Słomińska [22] proved the morphisms in any EI-category with quotients are epimorphism. Thus all the morphisms in $(\mathcal{C},\mathcal{I})$ are monomorphisms.

By (1), $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(x,y)$. Statement (2) follows directly from counting the number of the $\mathrm{Aut}_{\mathcal{C}}(x)$-orbits on $\mathrm{Hom}_{\mathcal{C}}(x,y)$.

Statement (3) is true by (2). □

Let $(\mathcal{C},\mathcal{I})$ be a category with subobjects. Suppose $\mathcal{D}\subset\mathcal{C}$ is a full subcategory satisfying the condition that if $x\in\mathrm{Ob}\,\mathcal{D}$ then $[x]\subset\mathrm{Ob}\,\mathcal{D}$. Then we can naturally make $\mathcal{D}$ into a category with subobjects $(\mathcal{D},\mathcal{I}\cap\mathcal{D})$, and can talk about full subcategories of a category with subobjects.

Lemma 3.5.4. *Let $(\mathcal{C},\mathcal{I})$ be an EI-category with subobjects, and $\iota:\mathcal{D}\hookrightarrow\mathcal{C}$ a full subcategory. Then for any $y\in\mathrm{Ob}\,\mathcal{C}\setminus\mathrm{Ob}\,\mathcal{D}$, the skeleton of $\iota{\downarrow}_y$ is isomorphic to a poset, which can be identified with the poset $\mathcal{I}_{<y}\cap\mathcal{D}$. If $\mathcal{D}$ has a unique minimal object, then $\mathcal{I}_{<y}\cap\mathcal{D}$, if not empty, has an initial object and is contractible. If $\mathcal{C}$ is skeletal, we have $\mathcal{I}_{<y}\cap\mathcal{D}\cong P(\mathcal{D}_{<y})$.*

Proof. If we fix an $x\in\mathrm{Ob}\,\mathcal{D}_{<y}$ then every object $(x',\alpha')\in\iota{\downarrow}_y$ with $x'\cong x$ is isomorphic to some (x_i,α_i), where $x_i\cong x$ and $\alpha_i\in\mathrm{Hom}_{\mathcal{I}}(x_i,y)$. Since $(x_i,\alpha_i)\cong(x_j,\alpha_j)$ if and only if $x_i=x_j$, the skeleton of $\iota{\downarrow}_y$ is isomorphic to the full subcategory consisting of objects $\{(x,\alpha)\mid x\in\mathrm{Ob}\,\mathcal{D}_{<y},\ \alpha\in\mathrm{Hom}_{\mathcal{I}}(x,y)\}$. Using the definition of a category with subobjects, it is easy to see the full subcategory is a poset, and is isomorphic to $\mathcal{I}_{<y}\cap\mathcal{D}$ by our assumption.

When $\mathcal{D}$ has a unique minimal object, so does $\mathcal{I}_{<y}\cap\mathcal{D}$. Hence it is contractible because in the poset the unique minimal object is indeed an initial object. If $\mathcal{C}$ is skeletal, every isomorphism class of objects contains only one object. So the identification $\mathcal{I}_{<y}\cap\mathcal{D}\cong P(\mathcal{D}_{<y})$ follows. □

Definition 3.5.5. Let $(\mathcal{C},\mathcal{I})$ be a finite category with subobjects. Then we denote the two full subcategories of $\mathcal{C}$ which share the same object sets with $\mathrm{Wess}_0(\mathcal{I})$ and $\mathrm{Wess}(\mathcal{I})$, respectively, by $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$ and $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$. Obviously $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}\subset\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$.

We comment here that in general $\mathrm{Wess}_0(\mathcal{C})\subsetneq\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$ and $\mathrm{Wess}(\mathcal{C})\subsetneq\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$. But when $(\mathcal{C},\mathcal{I})$ is skeletal, we do have $\mathrm{Wess}_0(\mathcal{C})=\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$ and $\mathrm{Wess}(\mathcal{C})=\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$. Our next two propositions show the importance of these two new full subcategories of $\mathcal{C}$.

Proposition 3.5.6. *Let $\mathcal{C}$ be a finite category with subobjects. Then $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$ is the smallest full subcategory among all full subcategories of $\mathcal{C}$, relative to which $\underline{R}$ is projective. Consequently, $\underline{R}$ is projective relative to $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$, and $\mathcal{V}_{\underline{R}}$ is the smallest convex subcategory* (*or ideal*) *of $\mathcal{C}$ that contains $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$.*

Proof. By Proposition 3.4.1 and Lemma 3.5.4, we know $\underline{R}$ is relatively $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$-projective. Hence if $\mathcal{D}$ contains $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$ then $\underline{R}$ is relatively $\mathcal{D}$-projective. Now suppose $\mathcal{D}$ is a full subcategory and $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}\not\subset\mathcal{D}$. Then by Lemma 3.5.4 again, there exists an overcategory associated with $\iota:\mathcal{D}\hookrightarrow\mathcal{C}$ which is disconnected. Thus $\underline{R}$ will not be relatively $\mathcal{D}$-projective, and we have proved that $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0}$ is the smallest full subcategory among all full subcategories of $\mathcal{C}$, relative to which $\underline{R}$ is projective. □

Our next result is a generalization of a Bouc's theorem [6] on finite posets.

Proposition 3.5.7. *Let $(\mathcal{C},\mathcal{I})$ be a finite category with subobjects. Given a full subcategory $\mathcal{D}$ with $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}} \subset \mathcal{D}$, the inclusions $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}} \subset \mathcal{D} \subset \mathcal{C}$ induce homotopy equivalences.*

Proof. Following Bouc's idea, we are going to use Quillen's Theorem A to prove $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}} \subset \mathcal{D}$ induces an equivalence. Let is take any object $y \in \mathrm{Ob}\,\mathcal{D}$. We want to show the overcategory category $\iota{\downarrow_y}$ is always contractible, where $\iota: \mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}} \hookrightarrow \mathcal{D}$ is the inclusion. If $y \in \mathrm{Ob}\,\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$, then this category has a terminal object $(y, 1_y)$, and hence is contractible. If $y \notin \mathrm{Ob}\,\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}$, we claim $\iota{\downarrow_y}$ is still contractible. By our assumption and Lemma 3.5.4, we know the skeleton of $\iota{\downarrow_y}$ is isomorphic to the poset $(\mathcal{I} \cap \mathcal{D})_{<y} \cap \mathrm{Wess}(\mathcal{I})$, because $(\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}}, \mathrm{Wess}(\mathcal{I})) \subset (\mathcal{D}, \mathcal{I} \cap \mathcal{D})$ is a full subcategory. Since $(\mathcal{I} \cap \mathcal{D})_{<y} \cap \mathrm{Wess}(\mathcal{I}) = \mathrm{Wess}(\mathcal{I})_{<y} = \mathrm{Wess}(\mathcal{I}_{<y}) \subset \mathcal{I}_{<y}$ and $\mathcal{I}_{<y}$ is contractible by definition, Bouc's original result on posets (see Bouc [6] or Benson [5, Proposition 6.6.5]) implies $\mathrm{Wess}(\mathcal{I})_{<y}$ is contractible. Hence so is $\iota{\downarrow_y}$ and we are done. □

The above two propositions result in the following reduction of higher limits for several types of categories with subobjects.

Corollary 3.5.8. *Let $(\mathcal{C},\mathcal{I})$ be a category with subobjects and $\mathcal{D}$ a full subcategory satisfying either one of the following conditions*:

(1) $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}} \subset \mathcal{D}$; *or*
(2) $\mathcal{C}^{\mathcal{I}}_{\mathrm{Wess}_0} \subset \mathcal{D}$ *and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module,*

then we have $\varprojlim^*_{\mathcal{C}} N \cong \varprojlim^*_{\mathcal{D}} (N{\downarrow^{\mathcal{C}}_{\mathcal{D}}})$. If $N = \underline{R}$, this isomorphism gives rise to a ring isomorphism.

Proof. If the condition in (1) is satisfied, then every overcategory associated to the inclusion $\iota: \mathcal{D} \hookrightarrow \mathcal{C}$ is contractible as is shown in the proof of Proposition 3.5.7. So we can use Proposition 2.3.4 to establish the isomorphisms.

If the conditions in (2) are satisfied, then we can use Proposition 3.5.6 and Lemma 3.3.20 to get the isomorphism. □

4. Resolutions and their applications

Let $\mathcal{C}$ be a finite EI-category and R a field or a complete discrete valuation ring. Given an $R\mathcal{C}$-module M, we study the structure of its projective cover and the minimal projective resolution. By a key property of the minimal projective resolution $\mathcal{P} \to M \to 0$, the Ext group $\mathrm{Ext}^n_{R\mathcal{C}}(M, N)$ is equal to $\mathrm{Hom}_{R\mathcal{C}}(P_n, N)$ provided N is semi-simple.

4.1. The minimal projective resolution of an $R\mathcal{C}$-module

We describe the projective cover of an $R\mathcal{C}$-module M in this section. The full subcategories $\mathcal{C}^M$ and $\mathcal{C}_M$, for any $R\mathcal{C}$-module M, are defined in 3.3.9.

Lemma 4.1.1. *Suppose M is an $R\mathcal{C}$-module. If $P_M \cong \bigoplus_{y,U} P_{y,U}$ is the projective cover of M, then every such y belongs to $\mathcal{C}_M$, which means $\mathcal{C}_{P_M} \subset \mathcal{C}_M$. Thus $\mathcal{C}_{M/\mathrm{Rad}\,M} \subset \mathcal{C}_M$.*

If $\mathcal{D}$ is an ideal of $\mathcal{C}$ with $\mathcal{D} \cap \mathcal{C}_M \neq \emptyset$, then $P_M{\downarrow^{\mathcal{C}}_{\mathcal{D}}}$ is the projective cover of $M{\downarrow^{\mathcal{C}}_{\mathcal{D}}}$. Particularly, if x is an M-minimal object, then $P_M(x)$ is the projective cover of $M(x)$.

Proof. Since M is generated by its values on objects in $\mathrm{Ob}\,\mathcal{C}_M$, so is P_M and hence every y belongs to $\mathcal{C}_M$. We have $\mathcal{C}_{M/\mathrm{Rad}\,M} \subset \mathcal{C}_M$ because $\mathcal{C}_{M/\mathrm{Rad}\,M} = \mathcal{C}_{P_M}$.

Now let $\mathcal{D}$ be an ideal in $\mathcal{C}$ with $\mathcal{D} \cap \mathcal{C}_M \neq \emptyset$. Then $P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ is a projective $R\mathcal{D}$-module which admits a surjection onto $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$. In order to show $P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ is the projective cover of $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$, we need to prove $(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \cong (M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$.

Since $\mathcal{D}$ is an ideal of $\mathcal{C}$, for any $R\mathcal{C}$-module N we have $N{\downarrow}^{\mathcal{C}}_{\mathcal{D}} = 1_{\mathcal{D}} \cdot N$ as an $R\mathcal{D}$-module. If P is a projective $R\mathcal{C}$-module, we can easily see $\mathrm{Rad}(P){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Rad}(P{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$ from the structure theorem of projective modules. Particularly, we get $\mathrm{Rad}(R\mathcal{C}){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Rad}(R\mathcal{D})$. Now we claim $\mathrm{Rad}(N){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Rad}(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$ is true for any $R\mathcal{C}$-module N. In fact it follows from some simple calculations: $(\mathrm{Rad}\,N){\downarrow}^{\mathcal{C}}_{\mathcal{D}} = 1_{\mathcal{D}} \cdot \mathrm{Rad}\,N = 1_{\mathcal{D}} \cdot \mathrm{Rad}(R\mathcal{C}) \cdot N = \mathrm{Rad}(R\mathcal{D}) \cdot N = \mathrm{Rad}(R\mathcal{D}) \cdot 1_{\mathcal{D}} \cdot N = \mathrm{Rad}(R\mathcal{D}) \cdot N{\downarrow}^{\mathcal{C}}_{\mathcal{D}} = \mathrm{Rad}(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$. We have the following short exact sequences

$$0 \to \mathrm{Rad}\,N \to N \to N/\mathrm{Rad}\,N \to 0,$$

and

$$0 \to \mathrm{Rad}\big(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}\big) \to N{\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to \big(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}\big)/\,\mathrm{Rad}\big(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}\big) \to 0,$$

of $R\mathcal{C}$-modules and $R\mathcal{D}$-modules, respectively. The first short exact sequence restricts to a short exact sequence of $R\mathcal{D}$-modules

$$0 \to (\mathrm{Rad}\,N){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to N{\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to (N/\mathrm{Rad}\,N){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \to 0.$$

Since $(\mathrm{Rad}\,N){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Rad}(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$, comparing the second and the third short exact sequences we obtain $(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(N{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \cong (N/\mathrm{Rad}\,N){\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ for any $R\mathcal{C}$-module N. If we choose N to be M and P_M, respectively, we have the following isomorphisms $(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \cong (M/\mathrm{Rad}\,M){\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ and $(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \cong (P_M/\mathrm{Rad}\,P_M){\downarrow}^{\mathcal{C}}_{\mathcal{D}}$, which imply $(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(P_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}) \cong (M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})/\mathrm{Rad}(M{\downarrow}^{\mathcal{C}}_{\mathcal{D}})$ because $(M/\mathrm{Rad}\,M){\downarrow}^{\mathcal{C}}_{\mathcal{D}} \cong (P_M/\mathrm{Rad}\,P_M){\downarrow}^{\mathcal{C}}_{\mathcal{D}}$.

Finally if x is an M-minimal object, then we take $\mathcal{D} = \mathcal{C}_{\leqslant x}$, and the last statement follows. □

With the above result we can go on to describe the minimal projective resolution of an arbitrary $R\mathcal{C}$-module. Given an $R\mathcal{C}$-module M and its projective cover P_M, from the previous lemma we know $P_M \cong \bigoplus_{y,U} P_{y,U}$ with $y \in \mathrm{Ob}\,\mathcal{C}_M$. When we look at the minimal resolution of M

$$\mathcal{P}_M\colon \quad \cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

we know P_0 is simply P_M and P_1 is the projective cover of K_0, the kernel of the map $P_0 = P_M \to M$. Since $\mathcal{C}_{K_0} \subset \mathcal{C}_{P_0} \subset \mathcal{C}_M$, we have $\mathcal{C}_{P_1} \subset \mathcal{C}_M$ too by the lemma. Hence we conclude the following result on the minimal projective resolution by repeating the same argument for every P_n.

Corollary 4.1.2. *Let M be an $R\mathcal{C}$-module and $\mathcal{P}_M$ its minimal projective resolution. Then $\mathcal{C}_{P_n} \subset \mathcal{C}_M$ for each module P_n in the projective resolution. Suppose $\mathcal{D}$ is an ideal of $\mathcal{C}$. Then $\mathcal{P}_M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$ is the minimal projective resolution of $M{\downarrow}^{\mathcal{C}}_{\mathcal{D}}$.*

If M is an atomic module whose support is $\{[x]\}$ for some $x \in \operatorname{Ob}\mathcal{C}$, then in the minimal projective resolution of M, $P_1 \cong \bigoplus_{y,U} P_{y,U}$ with $y \cong x$ or $\operatorname{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$ if $y \not\cong x$.

Proof. The first statement is a direct consequence of the preceding lemma and the discussion after it, and so we only prove the second part here. Indeed, it is easier to see this using a larger projective resolution of M. Suppose $\widetilde{\mathcal{P}_M}$ is constructed in a way such that $\widetilde{P}_0 = (R\mathcal{C} \cdot 1_x)^i$ for some positive integer i, and $\widetilde{P}_1$ is the projective cover of the kernel $\widetilde{K}_0$ of the map $\widetilde{P}_0 \to M \to 0$. Then $\widetilde{K}_0$ contains all non-isomorphisms in $\widetilde{P}_0$, and thus $\operatorname{Rad}(R\mathcal{C}) \cdot \widetilde{K}_0 = \operatorname{Rad}(\widetilde{K}_0)$ contains all reducible morphisms in $\widetilde{P}_0$. This implies $\widetilde{K}_0/\operatorname{Rad}\widetilde{K}_0$ is isomorphic to a direct sum of the form $\bigoplus_{y,U} S_{y,U}$, where $y \cong x$ or $y \not\cong x$ and $\operatorname{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$. Thus $\widetilde{P}_1$ is isomorphic to $\bigoplus_{y,U} P_{y,U}$, where $y \cong x$ or $y \not\cong x$ and $\operatorname{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$. Since $\mathcal{P}_M$ is the minimal projective resolution of M, $\mathcal{P}_M$ must be isomorphic to a direct summand of $\widetilde{\mathcal{P}_M}$. In particular, P_1 is isomorphic to a direct summand of $\widetilde{P}_1$. Hence the statement follows. □

Let M be an $R\mathcal{C}$-module. The support of M is defined to be the full subcategory of $\mathcal{C}$ consisting of objects x such that $M(x) \neq 0$.

Proposition 4.1.3. *Let M be an $R\mathcal{C}$-module which is relatively $\mathcal{D}$-projective for a full subcategory $\mathcal{D}$. Then $M/\operatorname{Rad}(M)$ has a support contained in $\mathcal{D}$. As a consequence, the projective cover P_M is relatively $\mathcal{D}$-projective.*

Proof. Since M is $\mathcal{D}$-projective, for any $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, we have

$$M(y) = \sum_{x \in \operatorname{Ob}\mathcal{D}_{\leqslant y}} R\operatorname{Hom}_{\mathcal{C}}(x, y) \cdot M(x) \subset \operatorname{Rad}(R\mathcal{C}) \cdot M = \operatorname{Rad}(M).$$

Hence $M/\operatorname{Rad}(M)$ is non-zero only on some objects in $\mathcal{D}$. This implies $P_M \cong \bigoplus_{y,U} P_{y,U}$ for some $P_{y,U}$ with $y \in \operatorname{Ob}\mathcal{D}$, and then P_M is $\mathcal{D}$-projective because $\mathcal{D}$ contains the vertex of every $P_{y,U}$. □

Note that if M is indecomposable, then M and P_M usually have different vertices. One can consider $P_{x,V} \to S_{x,V}$ when they are not equal. From the same example we can see the above proposition cannot be strengthened: M being $\mathcal{D}$-projective does not imply $M/\operatorname{Rad} M$ is $\mathcal{D}$-projective. Furthermore the kernel of the surjection $P_M \to M$ does not have to be $\mathcal{D}$-projective if M is. Thus given a $\mathcal{D}$-projective module M and its minimal resolution $\mathcal{P}_M \to M \to 0$, usually we cannot expect any $P_n, n \geqslant 2$, to be $\mathcal{D}$-projective, except $\mathcal{D} = \mathcal{C}_M$.

4.2. Applications

Proposition 4.2.1. *Given two $R\mathcal{C}$-modules M and N, we have*

$$\operatorname{Ext}^*_{R\mathcal{C}}(M, N) = \operatorname{Ext}^*_{R\mathcal{C}_M^N}\big(M\downarrow^{\mathcal{C}}_{\mathcal{C}_M^N}, N\downarrow^{\mathcal{C}}_{\mathcal{C}_M^N}\big),$$

where $\mathcal{C}_M^N = \mathcal{C}_M \cap \mathcal{C}^N$.

Proof. We take the minimal $R\mathcal{C}$-resolution of M

$$\mathcal{P}_M: \quad \cdots \to P_1 \to P_0 \to M \to 0.$$

It is supported on $\mathcal{C}_M$ hence is an $R\mathcal{C}_M$-resolution of $M\downarrow^{\mathcal{C}}_{\mathcal{C}_M}$. It is obvious that

$$\mathrm{Hom}_{R\mathcal{C}}(P_n, N) = \mathrm{Hom}_{R\mathcal{C}^N_M}\big(P_n\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}, N\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}\big),$$

for all n, which furthermore give rise to an isomorphism of cochain complexes

$$\big\{\mathrm{Hom}_{R\mathcal{C}}(\mathcal{P}_M, N)\big\} \cong \big\{\mathrm{Hom}_{R\mathcal{C}^N_M}\big(\mathcal{P}_M\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}, N\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}\big)\big\}.$$

If we can show $\mathcal{P}_M\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}$ is still a projective resolution of $M\downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}$ as an $R\mathcal{C}^N_M$-module then we are done. But this comes from Lemma 4.1.1 since $\mathcal{C}^N_M$ is an ideal in $\mathcal{C}_M$. □

Corollary 4.2.2. *Let $\mathcal{C}^y_x = \mathcal{C}_{\geqslant x} \cap \mathcal{C}_{\leqslant y}$. Then*

$$\mathrm{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W}) \cong \mathrm{Ext}^*_{R\mathcal{C}^y_x}(S_{x,V}, S_{y,W}).$$

In particular we have $\mathrm{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{x,W}) \cong \mathrm{Ext}^*_{R\,\mathrm{Aut}(x)}(V, W)$.

Note that in Corollary 4.2.2 if $\mathrm{Hom}(x, y) = \emptyset$ then $\mathcal{C}^y_x = \emptyset$ hence $\mathrm{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W})$ vanish. Let A be a finite-dimensional algebra and M, N two A-modules. There is a standard result by S. Eilenberg [13, Proposition 10] asserting that if N is semi-simple, then $\mathrm{Ext}^n_A(M, N) \cong \mathrm{Hom}_A(P_n, N)$, where $\cdots \to P_n \to \cdots \to P_0 \to M \to 0$ is the minimal projective resolution of M.

Proposition 4.2.3. *Let $S_{x,V}$ and $S_{y,W}$ be two simple $R\mathcal{C}$-modules. If* $\mathrm{Irr}_{\mathcal{C}}(x, y) = \emptyset$*, then* $\mathrm{Ext}^1_{R\mathcal{C}}(S_{x,V}, S_{y,W}) = 0$.

Proof. Suppose $\cdots \to P_1 \to P_0 \to S_{x,V} \to 0$ is the minimal projective resolution of $S_{x,V}$. Then since $S_{y,W}$ is simple, we have $\mathrm{Ext}^1_{R\mathcal{C}}(S_{x,V}, S_{y,W}) \cong \mathrm{Hom}(P_1, S_{y,W})$. But $P_1 \cong \bigoplus P_{z,V}$ for $z \cong x$ or $z \in \mathrm{Ob}\,\mathcal{C}_{\geqslant x}$ such that $\mathrm{Irr}(x, z) \neq \emptyset$. By our assumption and Corollary 4.1.2 the degree one Ext group has to be zero because any indecomposable projective module of the form $P_{y,W}$ cannot be a direct summand of P_1. □

Recall that there is a poset $P(\mathcal{C})$ associated to every EI-category $\mathcal{C}$. When $\mathcal{C}$ is finite we call the maximal length of chains of non-isomorphisms in $P(\mathcal{C})$ the length of $\mathcal{C}$, denoted by $l(\mathcal{C})$. Recall that the global dimension of a finite-dimensional algebra A is the projective dimension of $A/\mathrm{Rad}(A)$.

Theorem 4.2.4. *Let $\mathcal{C}$ be a finite EI-category. Then $R\mathcal{C}$ has finite global dimension if and only if for all $x \in \mathrm{Ob}\,\mathcal{C}$, $|\mathrm{Aut}(x)|^{-1} \in R$. In fact for any $R\mathcal{C}$-module M,* $\mathrm{proj.dim}(M) \leqslant l(\mathcal{C}_M)$. *Particularly* $\mathrm{gl.dim}(R\mathcal{C}) \leqslant l(\mathcal{C})$.

Proof. Since $\mathcal{C}$ is finite, by Theorem 3.1.2 $R\mathcal{C}$ has finitely many simple modules. Note that $R\mathcal{C}$ having finite global dimension is equivalent to the statement that every simple $R\mathcal{C}$-module has a finite projective resolution.

Suppose $|\mathrm{Aut}(x)|^{-1} \in R$ for all $x \in \mathrm{Ob}\,\mathcal{C}$. We do an induction on $l(\mathcal{C})$, the length of $\mathcal{C}$. Fix a simple $S_{y,W}$. It has a minimal projective resolution $\mathcal{P}$ written as

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to S_{y,W} \to 0.$$

Since $\mathcal{P}(y) \to S_{y,W}(y) = W \to 0$ is the minimal projective resolution of W and $|\mathrm{Aut}(y)|^{-1} \in R$, we have $P_0(y) \cong W$ since $R\,\mathrm{Aut}(y)$ is semi-simple. It implies that all $P_n, n > 0$, are supported on $\mathcal{C}_{>y}$ for $P_n(y) = 0$ when $n > 0$. Therefore if we take K_0 as the kernel of the map $P_0 \to S_{y,W}$, K_0 must be an $R\mathcal{C}$-module supported on $\mathcal{C}_{>y}$, and

$$\cdots \to P_n \to \cdots \to P_1 \to K_0 \to 0$$

becomes a minimal projective resolution of K_0. Because $\mathcal{C}_{>y}$ has smaller length than $\mathcal{C}$, the resolution of K_0 is finite. So is $\mathcal{P}$, the resolution of $S_{y,W}$.

On the other hand if any $S_{y,W}$ has a finite projective resolution $\mathcal{P}$, then $\mathcal{P}(y)$ is a finite resolution of the simple $R\,\mathrm{Aut}(y)$-module $S_{y,W}(y) = W$. Since projective $R\,\mathrm{Aut}(y)$-modules are the same as injective $R\,\mathrm{Aut}(x)$-modules, the finite exact sequence $\mathcal{P}(y) \to W \to 0$ splits. Hence each and every W is projective which means $R\,\mathrm{Aut}(y)$ is semi-simple, or equivalently $|\mathrm{Aut}(y)|^{-1} \in R$.

Under the circumstance if we consider an arbitrary $R\mathcal{C}$-module M, we can show $\mathrm{proj.dim}(M) \leqslant l(\mathcal{C}_M)$. Let

$$\mathcal{P}_M\colon \quad 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be the minimal projective resolution of M. Then P_0 is supported on $\mathcal{C}_M$. Just like what we have done above, the kernel of $P_0 \to M$ is supported on $\mathcal{C}_M \backslash \{\text{M-minimal objects}\} \subset \mathcal{C}_M$ so the support of P_1 is contained in $\mathcal{C}_M \backslash \{\text{M-minimal objects}\}$ as well. Inductively the size of $\mathcal{C}_{P_i}$ decreases strictly when i grows bigger and bigger. This means $\mathrm{proj.dim}(M) \leqslant l(\mathcal{C}_M)$. □

Abusing the notation, when R is understood we say $\mathcal{C}$ has finite global dimension if $R\mathcal{C}$ has finite global dimension.

Corollary 4.2.5. *Let $\mathcal{C}$ be a finite EI-category with the property that every morphism is an epimorphism. Then $R\mathcal{C}$ has finite global dimension if and only if $\underline{R}$ has finite projective dimension.*

Proof. If $R\mathcal{C}$ has finite global dimension then $\underline{R}$ certainly has a finite projective resolution. On the other hand if $\underline{R}$ has a finite minimal projective resolution $\mathcal{P} \to \underline{R} \to 0$ then by our assumption we can show, for any $x \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{P}(x) \to R \to 0$ is a finite projective resolution for the $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module R. The reason is that we can produce a (larger) projective resolution $\widetilde{\mathcal{P}}$ of $\underline{R}$ such that every $\widetilde{P_n}$ is a direct sum of some representable functors of the form $R\,\mathrm{Hom}_{\mathcal{C}}(z, -)$. Since $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely on $R\,\mathrm{Hom}_{\mathcal{C}}(z, x)$ whenever $\mathrm{Hom}_{\mathcal{C}}(z, x) \neq \emptyset$, any non-zero $\widetilde{P_n}(x)$ is a projective $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module, and so is $P_n(x)$ as a direct summand of $\widetilde{P_n}(x)$. In the end the finite projective resolution of the $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module R is split for all $x \in \mathrm{Ob}\,\mathcal{C}$ so R has to be projective, or equivalently $|\mathrm{Aut}_{\mathcal{C}}(x)|^{-1} \in R$ for all x. It implies $R\mathcal{C}$ has finite global dimension by the preceding proposition. □

We note that the finite categories with quotients (see Section 3.6) satisfy the condition in Corollary 4.2.5 that every morphism is an epimorphism.

Acknowledgments

I would like to thank my thesis advisor, Prof. Peter Webb, for leading me into this area, for numerous helpful conversations and comments and for much encouragement.

I wish to thank the referee for his or her useful comments which help remove a couple of errors and make this paper better presented.

References

[1] J. Alperin, Local Representation Theory, Cambridge Stud. Adv. Math., vol. 11, Cambridge University Press, 1986.
[2] M. Auslander, I. Reiten, O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, 1997.
[3] H.-J. Baues, G. Wirsching, Cohomology of small categories, J. Pure Appl. Algebra 38 (1985) 187–211.
[4] D. Benson, Representations and Cohomology I, Cambridge Stud. Adv. Math., vol. 30, Cambridge University Press, 1998.
[5] D. Benson, Representations and Cohomology II, Cambridge Stud. Adv. Math., vol. 31, Cambridge University Press, 1998.
[6] S. Bouc, Homologie de certains ensembles ordonnés, C. R. Acad. Sci. Paris Sér. I 299 (1984) 49–52.
[7] C. Broto, R. Levi, B. Oliver, Homotopy equivalence of p-completed classifying spaces of finite groups, Invent. Math. 151 (2003) 611–664.
[8] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003) 779–856.
[9] L. Budach, B. Graw, C. Meinel, S. Waack, Algebraic and Topological Properties of Finite Partially Ordered Sets, Teubner-Texte zur Mathematik, Band 109, Teubner Publishing House, 1987.
[10] T. tom Dieck, Transformation Groups, de Gruyter Stud. Math., vol. 8, Walter de Gruyter, 1987.
[11] W.G. Dwyer, Homology decompositions for classifying spaces of finite groups, Topology 36 (4) (1997) 783–804.
[12] W.G. Dwyer, D.M. Kan, A classification theorem for diagrams of simplicial sets, Topology 23 (1984) 139–155.
[13] S. Eilenberg, Homological dimension and syzygies, Ann. of Math. 64 (2) (1956) 328–336.
[14] L. Evens, The cohomology ring of a finite group, Trans. Amer. Math. Soc. 101 (2) (1961) 224–239.
[15] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergeb. Math. Grenzgeb., New Series, vol. 35, Springer-Verlag, New York, 1967.
[16] A.I. Generalov, Relative homological algebra, cohomology of categories, posets and coalgebras, in: M. Hazewinkel (Ed.), Handbook of Algebra, vol. 1, North-Holland, 1996.
[17] J. Grodal, Higher limits via subgroup complexes, Ann. of Math. 155 (2002) 405–457.
[18] P.J. Hilton, U. Stammbach, A Course in Homological Algebra, Grad. Texts in Math., vol. 4, Springer-Verlag, 1971.
[19] J. Hollender, R.M. Vogt, Modules of topological spaces, applications to homotopy limits and E_∞ structures, Arch. Math. 59 (1992) 115–129.
[20] S. Jackowski, J. McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, Topology 31 (1) (1992) 113–132.
[21] S. Jackowski, J. McClure, B. Oliver, Homotopy decomposition of self-maps of BG via G-actions, I and II, Ann. of Math. 135 (1992) 183–226, 227–270.
[22] S. Jackowski, J. Słomińska, G-functors, G-posets and homotopy decompositions of G-spaces, Fund. Math. 169 (2001) 249–287.
[23] W. Lück, Transformation Groups and Algebraic K-Theory, Lecture Notes in Math., vol. 1408, Springer-Verlag, 1989.
[24] M. Linckelmann, Fusion category algebras, J. Algebra 277 (2004) 222–235.
[25] B. Mitchell, Rings with several objects, Adv. Math. 8 (1972) 1–161.
[26] S. Mac Lane, Categories for the Working Mathematician, Grad. Texts in Math., vol. 5, Springer-Verlag, 1971.

[27] S. Mac Lane, Homology, Classics in Mathematics, Springer-Verlag, 1995.
[28] B. Oliver, Higher limits via Steinberg representations, Comm. Algebra 22 (4) (1994) 1381–1393.
[29] L. Puig, Structure locale dans les groupes finis, Bull. Soc. Math. France Suppl. Mém. 47 (1976).
[30] D. Quillen, Higher algebraic K-theory I, in: Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 85–147.
[31] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Adv. Math. 28 (1978) 101–128.
[32] M. Ronan, S. Smith, Sheaves on buildings and modular representations of Chevalley groups, J. Algebra 96 (1985) 319–346.
[33] J.-E. Roos, Sur les foncteurs dérivés de $\varprojlim$. Applications, C. R. Acad. Sci. Paris 252 (1961) 3702–3704.
[34] P. Symonds, The Bredon cohomology of subgroup complexes, J. Pure Appl. Algebra 199 (1–3) (2005) 261–298.
[35] J. Thévenaz, G-Algebras and Modular Representation Theory, Oxford Math. Monogr., Clarendon Press, Oxford, 1995.
[36] B.B. Venkov, Cohomology algebras for some classifying spaces, Dokl. Akad. Nauk SSSR 127 (1959) 943–944.
[37] R. Villarroel-Flores, P. Webb, Some split exact sequences in the cohomology of groups, Topology 41 (2002) 483–494.
[38] P. Webb, A pure mathematician's approach to the representation theory of finite groups, Lecture Notes, School of Mathematics, University of Minnesota, available online at http://www.math.umn.edu/~webb/.
[39] F. Xu, Homological properties of category algebras, PhD thesis, University of Minnesota, 2006.
[40] F. Xu, On the cohomology rings and extensions of small categories, preprint, 2007.