ON THE COHOMOLOGY RINGS OF SMALL CATEGORIES

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Abstract. Let $\mathcal{C}$ be a small category and R a commutative ring with identity. The cohomology ring of $\mathcal{C}$ with coefficients in R is defined as the cohomology ring of the topological realization of its nerve. First we give an example showing that this ring modulo nilpotents is not finitely generated in general, even when the category is finite EI. Then we study the relationship between the cohomology ring of a category and those of its subcategories and extensions. The main results generalize certain theorems in group cohomology theory.

Keywords. Cohomology ring, EI-categories, finite generation, restriction, extension, generalized LHS spectral sequence.

1. Introduction

Let $\mathcal{C}$ be a small category and $\mathcal{A}b$ the category of abelian groups. We denote by $\mathcal{C}$-mod the abelian category of all covariant functors from $\mathcal{C}$ to $\mathcal{A}b$. The nth cohomology group of $\mathcal{C}$ with coefficients in a functor $F \in \mathcal{C}$-mod, $\mathrm{H}^n(\mathcal{C}; F)$, can be defined as the nth higher inverse limit $\varprojlim^n_{\mathcal{C}} F$ [1, 14, 26, 32]. If A is an abelian group and $\underline{A}$ is the corresponding constant functor which sends every object to A and every morphism to the identity, then $\mathrm{H}^n(\mathcal{C}; \underline{A}) \cong \mathrm{H}^n(|\mathcal{C}|, A)$, where $|\mathcal{C}|$ is the topological realization of $N\mathcal{C}$–the nerve of $\mathcal{C}$. We are particularly interested in the case where A is a commutative ring with identity, because then $\mathrm{H}^*(\mathcal{C}; \underline{A}) \cong \mathrm{H}^*(|\mathcal{C}|, A)$ will become a graded commutative ring. To this end, we will study the cohomology of $\mathcal{C}$ with coefficients in a functor $F : \mathcal{C} \to R$-mod for a commutative ring with identity R (not just the ring of integers $\mathbb{Z}$). We call $\mathrm{H}^*(\mathcal{C}; \underline{R}) \cong \mathrm{H}^*(|\mathcal{C}|, R)$ the cohomology ring of $\mathcal{C}$ (with coefficients in R).

For each small category $\mathcal{C}$ and a fixed ring R, one can define an associative R-algebra, called the category algebra $R\mathcal{C}$ of $\mathcal{C}$ (see Xu [32]), generalizing the notion of the group algebra of a group and the notion of the incidence algebra of a poset. If $\mathrm{Ob}\,\mathcal{C}$ is finite, Mitchell [22] showed that $R\mathcal{C}$-mod $\simeq (R\text{-mod})^{\mathcal{C}}$, which implies that

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every functor is an $R\mathcal{C}$-module, and vise versa. This equivalence allows us to use representation theory of finite-dimensional algebras to investigate homological properties of functor categories. Since all categories here in this paper are assumed to have finite object sets, we will just denote by $R\mathcal{C}$-mod the category of all covariant functors from $\mathcal{C}$ to R-mod. Upon this equivalence of categories, $\mathrm{H}^n(\mathcal{C};F)$ can be identified with $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R},F)$ and $\mathrm{H}^*(\mathcal{C};\underline{R}) \cong \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},\underline{R})$. For the reader's convenience, in the Appendix we will give more information about cohomology of small categories and, more generally, Ext and Tor over category algebras.

When a finite group G is regarded as a category with a single object (written as $\widehat{G}$), Evens [8] and Venkov [29] proved that $\mathrm{H}^*(\widehat{G};\underline{R}) \cong \mathrm{H}^*(G,R)$ is finitely generated when R is Noetherian. This is one of the fundamental theorems in group cohomology theory, and is the starting point of the theory of varieties for modules [2]. Other categories whose cohomology rings are finitely generated include finite posets and centric linking systems (when they exist) in the theory of p-local finite groups established by Broto, Levi and Oliver [5]. Motivated by these known results, one wants to know whether the finite generation property is true in greater generality. Since many categories considered in group representation and cohomology theory are the so-called *EI-categories* and are finite, it is natural to focus our investigation on the special class of finite EI-categories. By definition, a category is finite if $\mathrm{Mor}\,\mathcal{C}$ is a finite set and is EI if every endomorphism is an isomorphism. Given an EI-category, it's easy to see that the isomorphism classes of objects in it possesses a natural poset structure. Hence to some extent a finite EI-category may be regarded as an amalgam of a finite poset and several finite groups. In fact, Słomińska [27] showed that the classifying space of $\mathcal{C}$, $|\mathcal{C}|$, is homotopy equivalent to the homotopy colimit of some functor from a finite poset to the category of finite groupoids. Since cohomology rings of finite groups and finite posets are finitely generated, one wishes to generalize the finite generation property of cohomology rings to finite EI-categories. The finite generation property of such a cohomology ring would be useful in computing higher limits over a category. For example, in the theory of p-local finite groups [5], in order to know the existence and uniqueness of a centric linking system for a given fusion system $\mathcal{F}$ over a p-group S, one needs to calculate the higher limits of the central functor $\mathcal{Z}$ over a full subcategory $\mathcal{F}^c \subset \mathcal{F}$, which sends a subgroup of S to its center. A characterization of $\mathrm{H}^*(\mathcal{F}^c;\underline{\mathbb{Z}})$, where $\mathbb{Z}$ is the ring of integers, would be useful in computing $\varprojlim^*_{\mathcal{F}^c} \mathcal{Z}$ because the cohomology ring acts on it. The finite generation property of cohomology rings with coefficients in a field would also lead us to a theory of varieties for modules over category algebras. We note that since the category algebra $R\mathcal{C}$ for a finite EI-category is not a Hopf algebra in general, the finite generation property doesn't follow from the main theorem of Friedlander-Suslin [10].

It turns out that the cohomology ring of a finite EI-category is *not* finitely generated in general, even after modulo nilpotents. The reader can find an example in Section 2 of the present paper. However, it would be of great interest if the finite generation could be proved for a certain subclass of finite EI-categories that includes the important cases in group representations and cohomology (for example, one may want to consider the concrete categories [23] whose morphisms are monomorphic).

In Section 2, we compute the cohomology rings of some finite EI-categories. The examples exhibit the failure of the finite generation of the cohomology ring in general, even after modulo nilpotents. Then we continue to study the cohomology of finite EI-categories with the goal of extending some classical results in group cohomology. In Section 3, given a finite EI-category $\mathcal{C}$, we show that the cohomology ring of $\mathcal{C}$ modulo some nilpotents is isomorphic to a subring of the cohomology ring of a disjoint union of finite groups. Furthermore we prove that, for a fixed prime p, there exist certain subcategories such that the cohomology ring of $\mathcal{C}$ modulo nilpotents can be embedded into the cohomology ring of such a subcategory modulo nilpotents.

Theorem A *Let $\mathcal{C}$ be a finite EI-category and $\mathcal{D}$ a subcategory. Suppose p is a prime. If $\mathrm{Ob}\,\mathcal{D} = \{x \in \mathrm{Ob}\,\mathcal{C} : p \mid |\mathrm{Aut}_{\mathcal{C}}(x)|\}$ and $\mathrm{Aut}_{\mathcal{D}}(x)$ contains a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{C}}(x)$, then the map induced by the restriction $\mathrm{H}^*(\mathcal{C};\underline{\mathbb{F}_p})/\mathcal{N}_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{D};\underline{\mathbb{F}_p})/\mathcal{N}_{\mathcal{D}}$ is injective, where $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_{\mathcal{D}}$ are the ideals of nilpotents in $\mathrm{H}^*(\mathcal{C};\underline{\mathbb{F}_p})$ and $\mathrm{H}^*(\mathcal{D};\underline{\mathbb{F}_p})$, respectively.*

We note that for any finite EI-category $\mathcal{C}$, a subcategory $\mathcal{D}$ satisfying the conditions in Theorem A always exists. At last in Section 4, we consider the extensions of small categories and deduce a generalized LHS spectral sequence. We introduce the concepts of opposite extensions and subextensions.

Proposition B Given a functor $F : \mathcal{C} \to R$-mod, there are two spectral sequences associated with an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ as follows:

(1) a homology spectral sequence
$$E^2_{ij} = \mathrm{H}_i(\mathcal{C};\mathrm{H}_j(\mathcal{K};F)) \Rightarrow \mathrm{H}_{i+j}(\mathcal{E};F);$$
and

(2) a cohomology spectral sequence
$$E_2^{ij} = \mathrm{H}^i(\mathcal{C}^{op};\mathrm{H}^j(\mathcal{K}^{op};F)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{E}^{op},F).$$

Based on the second spectral sequence, we can find connections between the cohomology ring of a category and those of its extensions.

Theorem C *Suppose there is an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion such that the undercategory $\iota_{\mathcal{D}} \downarrow^y$ is contractible for every $y \in \mathrm{Ob}\,\mathcal{C}$, then $H^*(\mathcal{E};F) \cong \mathrm{H}^*(\mathcal{E}_{\mathcal{D}};F)$ for any contravariant functor $F : \mathcal{E} \to R$-mod. Here $\mathcal{E}_{\mathcal{D}} \subset \mathcal{E}$ is the subextension corresponding to $\mathcal{D}$.*

Suppose there is an opposite extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion such that overcategory $\iota_{\mathcal{D}} \downarrow_y$ is contractible for every $y \in \mathrm{Ob}\,\mathcal{C}$, then $H^*(\mathcal{E};F) \cong \mathrm{H}^*(\mathcal{E}_{\mathcal{D}};F)$ for any covariant functor $F : \mathcal{E} \to R$-mod. Here $\mathcal{E}_{\mathcal{D}} \subset \mathcal{E}$ is the opposite subextension corresponding to $\mathcal{D}$.

Conventions We use curly letters, such as $\mathcal{C}$, to denote small categories in this paper. Symbols like $\mathcal{D}$ and $\mathcal{E}$ are used to denote subcategories and extensions of $\mathcal{C}$, respectively. When G is a group, we use $\widehat{G}$ to denote the corresponding category with a single object. For instance, let x be an object of $\mathcal{C}$. We frequently refer to its automorphism group $\mathrm{Aut}_{\mathcal{C}}(x)$ via the corresponding subcategory $\widehat{\mathrm{Aut}_{\mathcal{C}}(x)} \subset \mathcal{C}$. For each $x \in \mathrm{Ob}\,\mathcal{C}$, we denote by $[x] \subset \mathrm{Ob}\,\mathcal{C}$ the set of all objects isomorphic to x in $\mathcal{C}$. We also use $\widehat{[x]} \subset \mathcal{C}$ to denote the groupoid which consists of all objects in $[x]$ and all isomorphisms among them. The symbol $\mathcal{A}$ is reserved for the subcategory of $\mathcal{C}$ which is the disjoint union of all these groupoids $\widehat{[x]}$, $[x] \subset \mathrm{Ob}\,\mathcal{C}$.

For each category $\mathcal{C}$, we use the corresponding blackboard bold letter $\mathbb{C}$ to denote the naturally constructed chain complex from its nerve. For simplicity, when R is understood, sometimes we omit R or $\underline{R}$ in the cohomology ring of a category $\mathcal{C}$ and will write it as $\mathrm{H}^*(\mathcal{C})$ instead of $\mathrm{H}^*(|\mathcal{C}|, R)$ or $\mathrm{H}^*(\mathcal{C}; \underline{R})$. A category $\mathcal{C}$ is said to be contractible if $|\mathcal{C}|$ is.

We mainly work with finite EI-categories in this paper. Given an EI-category $\mathcal{C}$, there is a natural poset structure defined on the set of isomorphism classes of objects in $\mathcal{C}$. For any $[x], [y] \subset \mathrm{Ob}\,\mathcal{C}$, we define $[x] \leq [y]$ if $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$. The resulting poset is written as $[\mathcal{C}]$, and there is a canonical functor $\mathcal{C} \to [\mathcal{C}]$. Based on the existence of the partial order on the set of isomorphism classes of objects, for any $x \in \mathrm{Ob}\,\mathcal{C}$, we can define a full subcategory $\mathcal{C}_{\leq x} \subset \mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{C}_{\leq x} = \{y \in \mathrm{Ob}\,\mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(y, x) \neq \emptyset\}$. We can similarly define full subcategories such as $\mathcal{C}_{<x}$, $\mathcal{C}_{\geq x}$ and $\mathcal{C}_{>x}$ et cetera.

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2. The cohomology ring modulo nilpotents

In this section, we first recall the main construction and theorem in Słomińska's paper [27]. Using her decomposition formula for classifying spaces of categories, we are able to determine the homotopy types of the classifying spaces of certain categories. Then we investigate the structures of the cohomology rings of these categories. Our main examples says that the cohomology ring of a finite EI-category modulo nilpotent is not necessarily finite generated.

The materials in this section are pretty much independent of the general cohomology theory of small categories. For those who want to know more about the general theory, please read the appendix.

2.1. **Decompositions of classifying spaces.** Let $\mathcal{C}$ be a finite EI-category. Let $\pi : \mathcal{C} \to [\mathcal{C}]$ be the natural functor from $\mathcal{C}$ to its underlying poset. According to Słomińska's main result in [27], one has a homotopy equivalence

$$|\mathcal{C}| \simeq \mathrm{hocolim}_{sd[\mathcal{C}]}|\tilde{\pi}|,$$

where sd means subdivision and $\tilde{\pi} : sd[\mathcal{C}] \to sCats$, the category of small categories, is a functor whose values are groupoids. For the reader's convenience, we recall the definition of $\tilde{\pi}$. Assume $v. = [c_0 \overset{v_1}{\to} c_1 \overset{v_2}{\to} \cdots \overset{v_n}{\to} c_n]$ is an object of $sd[\mathcal{C}]$. Then $\tilde{\pi}(v.)$ is a small category whose objects are of the form $x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_n}{\to} x_n$ such that $\pi(x_i) = c_i$ and $\pi(\alpha_i) = v_i$ and the morphisms are the $(n+1)$-tuples of isomorphisms in $\mathcal{C}$ which make the following diagram commutative:

$$\begin{array}{ccccccccc}
x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & \cdots & \longrightarrow & x_{n-1} & \xrightarrow{\alpha_n} & x_n \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_2} & & & & \downarrow{\scriptstyle g_{n-1}} & & \downarrow{\scriptstyle g_n} \\
x'_0 & \xrightarrow{\alpha'_1} & x'_1 & \xrightarrow{\alpha'_2} & \cdots & \longrightarrow & x'_{n-1} & \xrightarrow{\alpha_n} & x'_n
\end{array} .$$

2.2. **Categories with two objects.** We want to compute the cohomology rings of a special class of finite EI-categories of the following form

$$G \circlearrowright x \overset{G/H}{\rightrightarrows} y \circlearrowleft K ,$$

in which the automorphism groups are two finite groups G and H. The morphisms from x to y are given by the set of left cosets G/H for some subgroup $H \subset G$. We let G act on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ in the natural way and K act on it trivially. Then Słomińska's formula gives an explicit form of $|\mathcal{C}|$ up to homotopy equivalence. In fact, the homotopy colimit becomes the homotopy pushout of the following diagram

$$\begin{array}{ccc}
BH \times BK & \longrightarrow & BK \\
\downarrow & & \\
BG & &
\end{array} ,$$

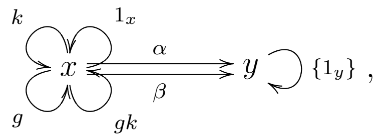
where the two maps are projections followed by inclusions. A short calculation will show that the homotopy pushout is the join of BG and BK, wedge sum the mapping cone of the inclusion $BH \hookrightarrow BG$. Hence $|\mathcal{C}| \simeq (BG * BK) \vee (BG/BH)$, where $*$ denotes the join of two spaces (see [13], [9]). Since the cup product in $\mathrm{H}^*(BG * BK)$ is trivial, up to nilpotents the structure of $\mathrm{H}^*(\mathcal{C})$ is completely determined by that of $\mathrm{H}^*(BG/BH)$.

As an example, when $G = H$ in the above category, the homotopy pushout is simply the join of BG and BK. Hence the cup product of any two elements of positive degree in $\mathrm{H}^*(|\mathcal{C}|)$ is trivial and $\mathrm{H}^*(|\mathcal{C}|)$ modulo nilpotents is the base ring. More generally, one may describe the chomology ring structure of the following category $\mathcal{C}_n$, $n > 1$,

$$\overset{G_1}{\circlearrowright} x_1 \xrightarrow{\alpha_1} \overset{G_2}{\circlearrowright} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \overset{G_n}{\circlearrowright} x_n ,$$

where $\{\alpha_i\}\cdot G_i = \{\alpha_i\} = G_{i+1}\cdot\{\alpha_i\}$ as sets for all i. If the structures of the cohomology rings of G_i are provided, then one can write down explicitly the structure of $\mathrm{H}^*(\mathcal{C})$. For those who are not familiar with the join construction, a direct computation of the cohomology ring of $\mathcal{C}_n$ (when $G_i \cong \mathbb{Z}_2$, the cyclic group of order 2) with coefficients in a field of characteristic 2 can be found in the Appendix.

Next we consider an explicitly constructed category $\mathcal{C}$ as follows.



where $g^2 = k^2 = 1_x, gk = kg, \alpha k = \alpha, \alpha g = \beta, \beta k = \beta$ and $\beta g = \alpha$. If we name $G = \mathrm{Aut}_{\mathcal{C}}(x)$ and $H = \{1_x, k\}$, up to homotopy equivalence $|\mathcal{C}|$ is the homotopy pushout of the following diagram

$$\begin{array}{ccc} BH \times B\{1_y\} & \longrightarrow & B\{1_y\} \\ \downarrow & & \\ BG & & . \end{array}$$

Hence the classifying space $|\mathcal{C}| \simeq BG/BH$. One can compute the cohomology of BG/BH via the relative cohomology ring $\mathrm{H}^*(BG, BH)$. In fact, $\mathrm{H}^*(BG, BH)$ can be identified with the (maximal) ideal of $\mathrm{H}^*(BG/BH)$ consisting of all positive degree elements. Let k be a field of characteristic 2. Then one can use the long exact sequence for relative cohomology, along with the well-known structures of the mod-2 cohomology rings of elementary 2-groups, to establish a ring isomorphism

$$\mathrm{H}^*(BG, BH) \cong \mathrm{H}^*(\mathbb{Z}_2) \otimes_k \tilde{\mathrm{H}}^*(\mathbb{Z}_2),$$

in which $\tilde{\mathrm{H}}^*(\mathbb{Z}_2)$ is the reduced cohomology ring. The ring on the right has no non-trivial nilpotents and it is not finitely generated since $\tilde{\mathrm{H}}^*(\mathbb{Z}_2)$ doesn't have a unit. Finally we comment that the non finite generation of the relative group cohomology ring was first shown by Blower [3]

3. The cohomology ring modulo nilpotents and the restriction

Although the cohomology ring of a finite EI-category modulo nilpotents is not finitely generated in general, it would be very useful if one could prove the finite generation for certain special classes of finite EI-categories in group representation and cohomology theory. This is still an ongoing research project.

In Sections 3 and 4, we try to generalize some classic results in group cohomology.

3.1. **Comparing the cohomology of a category with those of its subcategories.** Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a subcategory. Then the inclusion $\iota : \mathcal{D} \to \mathcal{C}$ naturally induces the restriction $\mathrm{H}^*(\mathcal{C}) \to \mathrm{H}^*(\mathcal{D})$. We want to compare the cohomology rings of $\mathcal{C}$ and of its various subcategories. In [5], [16], [12] and [32] the authors studied the case where $\mathcal{D} \subset \mathcal{C}$ is a full subcategory which has fewer objects, and showed under certain assumptions one can have $\mathrm{H}^*(\mathcal{C}) \cong \mathrm{H}^*(\mathcal{D})$. Here we investigate subcategories $\mathcal{D} \subset \mathcal{C}$ with the same set of objects but with fewer morphisms. Our approach is quite elementary.

Let's consider the subcategory $\mathcal{A} \subset \mathcal{C}$ consisting of all objects and exactly all isomorphisms in $\mathcal{C}$. Then the chain complexes associated with the nerves of these two categories give a short exact sequence of complexes

$$0 \to \mathbb{A} \xrightarrow{i} \mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{A} \to 0,$$

where i is the inclusion and π is the natural surjection. Note that for any abelian group A, $\mathrm{H}^*(\mathcal{C}; \underline{A}) = \mathrm{H}^*(\mathbb{C}, A)$ and $\mathrm{H}^*(\mathcal{A}; \underline{A}) = \mathrm{H}^*(\mathbb{A}, A) \cong \oplus_{[x]\subset \mathrm{Ob}\,\mathcal{C}} \mathrm{H}^*(\mathrm{Aut}_{\mathcal{C}}(x), A)$, where $[x]$ is the isomorphism class of an object $x \in \mathrm{Ob}\,\mathcal{C}$. Suppose $\mathcal{D} \subset \mathcal{C}$ is a subcategory such that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$ and $\mathbb{D}$ is the chain complex associated with the nerves of $\mathcal{D}$. Let $\mathcal{D} \cap \mathcal{A} \subset \mathcal{A}$ be the obvious subcategory consisting of objects and morphisms in both $\mathcal{D}$ and $\mathcal{A}$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{A} & \xrightarrow{i} & \mathbb{C} & \xrightarrow{\pi} & \mathbb{C}/\mathbb{A} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \kappa} & & \\
0 & \longrightarrow & \mathbb{A}_{\mathbb{D}} & \xrightarrow[i]{} & \mathbb{D} & \xrightarrow[\pi]{} & \mathbb{D}/\mathbb{A}_{\mathbb{D}} & \longrightarrow & 0,
\end{array}$$

where $\mathbb{A}_{\mathbb{D}}$ is $\mathbb{A} \cap \mathbb{D}$, the chain complex of $\mathcal{D} \cap \mathcal{A}$. Let A be an abelian group. This diagram gives rise to an infinite commutative diagram with a typical segment as follows

$$\begin{array}{ccccccccc}
\mathrm{H}^{n-1}(\mathbb{A}, A) & \longrightarrow & \mathrm{H}^n(\mathbb{C}/\mathbb{A}, A) & \xrightarrow{\pi^*} & \mathrm{H}^n(\mathbb{C}, A) & \xrightarrow{i^*} & \mathrm{H}^n(\mathbb{A}, A) & \longrightarrow & \mathrm{H}^{n+1}(\mathbb{C}/\mathbb{A}, A) \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \kappa^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \kappa^*} \\
\mathrm{H}^{n-1}(\mathbb{A}_{\mathbb{D}}, A) & \longrightarrow & \mathrm{H}^n(\mathbb{D}/\mathbb{A}_{\mathbb{D}}, A) & \xrightarrow[\pi^*]{} & \mathrm{H}^n(\mathbb{D}, A) & \xrightarrow[i^*]{} & \mathrm{H}^n(\mathbb{A}_{\mathbb{D}}, A) & \longrightarrow & \mathrm{H}^{n+1}(\mathbb{D}/\mathbb{A}_{\mathbb{D}}, A),
\end{array}$$

which induces a commutative diagram for each n

$$\begin{array}{ccc}
\mathrm{H}^n(\mathcal{C}; \underline{A})/I^n_{\mathcal{C}} & \xrightarrow{i'} & \mathrm{H}^n(\mathcal{A}; \underline{A}) \\
\downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota^*} \\
\mathrm{H}^n(\mathcal{D}; \underline{A})/I^n_{\mathcal{D}} & \xrightarrow[i']{} & \mathrm{H}^n(\mathcal{A}_{\mathcal{D}}; \underline{A}),
\end{array}$$

where ι' is induced by $\iota^* : \mathrm{H}^n(\mathcal{C}; \underline{A}) \to \mathrm{H}^n(\mathcal{D}; \underline{A})$, $I^n_{\mathcal{C}} = \pi^* \mathrm{H}^n(\mathbb{C}/\mathbb{A}, A)$ and $I^n_{\mathcal{D}} = \pi^* \mathrm{H}^n(\mathbb{D}/\mathbb{A}_{\mathbb{D}}, A)$ satisfying $\iota^*(I^n_{\mathcal{C}}) = \pi^*\kappa^*(\mathrm{H}^n(\mathbb{C}/\mathbb{A}, A)) \subset \pi^*(\mathrm{H}^n(\mathbb{D}/\mathbb{A}_{\mathbb{D}}, A)) = I^n_{\mathcal{D}}$. By the exactness of the long exact sequences, both i' are injective.

Remark 3.1.1. *When $\mathcal{C}$ is a finite group, $I^n_{\mathcal{C}}$ and $I^n_{\mathcal{D}}$ vanish and the map ι' coincides with the usual restriction map.*

We show, when $A = R$ is a ring, $I_{\mathcal{C}} = \oplus I^n_{\mathcal{C}}$ and $I_{\mathcal{D}} = \oplus I^n_{\mathcal{D}}$ are nilpotent ideals in $\mathrm{H}^*(\mathcal{C})$ and $\mathrm{H}^*(\mathcal{D})$, respectively. Thus we have a commutative diagram of rings, not just of groups, with injective horizontal homomorphisms.

Proposition 3.1.2. *Let $\mathcal{C}$ be a finite category. The image of* $\mathrm{H}^*(\mathbb{C}/\mathbb{A})$*, denoted by $I_{\mathcal{C}} = \oplus_{n\geq 1} I^n_{\mathcal{C}}$, is an ideal of* $\mathrm{H}^*(\mathcal{C})$ *consisting of nilpotents, and* $i^*(\mathrm{H}^*(\mathcal{C})) = i'(\mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}}) \cong \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}}$ *is a subalgebra of* $\mathrm{H}^*(\mathcal{A})$.

Proof. The first observation comes from the long exact sequence induced by

$$0 \to \mathbb{A} \xrightarrow{i} \mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{A} \to 0,$$

$$\cdots \to \mathrm{H}^{n-1}(\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}/\mathbb{A}) \xrightarrow{\pi^*} \mathrm{H}^n(\mathbb{C}) \xrightarrow{i^*} \mathrm{H}^n(\mathbb{A}) \to \mathrm{H}^{n+1}(\mathbb{C}/\mathbb{A}) \to \cdots .$$

It's a canonical result that i^* induces a ring homomorphism, still denoted by i^*. The image $I_{\mathcal{C}} = \pi^*(\mathrm{H}^*(\mathbb{C}/\mathbb{A})) \subset \oplus_{n\geq 1} \mathrm{H}^n(\mathbb{C})$ because $H^0(\mathbb{C}/\mathbb{A}) = 0$. It implies $i^*(\mathrm{H}^*(\mathcal{C}))$ contains the identity in $\mathrm{H}^*(\mathcal{A})$. Since $I_{\mathcal{C}}$ is the kernel of the ring homomorphism i^*, it's an ideal in $\mathrm{H}^*(\mathcal{C})$. Furthermore elements of $I_{\mathcal{C}}$ are nilpotent because the category $\mathcal{C}$ is finite. If k is the maximum length of chains of non-isomorphisms in $\mathcal{C}$, then for any $f \in I_{\mathcal{C}}$, we must have $f^{k+1} = 0$ by direct calculations using the definition of cup product. □

Note that a subalgebra of a finitely generated algebra doesn't have to be finitely generated.

In the commutative diagram we considered, we can artificially add an upwards map, which is the transfer, pictured as follows

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} & \xrightarrow{incl.} & \mathrm{H}^*(\mathcal{A}) \\ {\scriptstyle ?}\uparrow \downarrow {\scriptstyle \iota'} & & {\scriptstyle \mathrm{tr}}\uparrow \downarrow {\scriptstyle \iota^*} \\ \mathrm{H}^*(\mathcal{D})/I_{\mathcal{D}} & \xrightarrow[incl.]{} & \mathrm{H}^*(\mathcal{A}_{\mathcal{D}}). \end{array}$$

Although it's unclear whether or not the transfer restricts to a well-defined map $\mathrm{H}^*(\mathcal{D})/I_{\mathcal{D}} \to \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}}$, the composite $\mathrm{tr} \circ \iota^* : \mathrm{H}^*(\mathcal{A}) \to \mathrm{H}^*(\mathcal{A})$ is given by

$$\alpha = (\alpha_x)_{[x]\subset \mathrm{Ob}\,\mathcal{C}} \mapsto \mathrm{tr} \circ \iota^*(\alpha) = (n_x \alpha_x)_{[x]\subset \mathrm{Ob}\,\mathcal{C}},$$

by a standard result in group cohomology theory, where $\alpha = (\alpha_x)_{[x]\subset \mathrm{Ob}\,\mathcal{C}}$ is an element of $\mathrm{H}^*(\mathcal{A}) = \oplus_{[x]\subset \mathrm{Ob}\,\mathcal{C}} \mathrm{H}^*(\mathrm{Aut}_{\mathcal{C}}(x))$ such that for each $[x] \subset \mathrm{Ob}\,\mathcal{C}$ $\alpha_x \in \mathrm{H}^*(\mathrm{Aut}_{\mathcal{C}}(x))$ and $n_x := |\,\mathrm{Aut}_{\mathcal{C}}(x) : \mathrm{Aut}_{\mathcal{D}}(x)|$.

Corollary 3.1.3. *Let $n_{\mathcal{C}}$ be the least common multiple of the integers $\{|\,\mathrm{Aut}_{\mathcal{C}}(x)|\}$, where x runs over the set of isomorphism classes of objects whose automorphism groups have order not invertible in R. Then, for any $i > 0$ and $\alpha \in \mathrm{H}^i(\mathcal{C})/I_{\mathcal{C}}$, $n_{\mathcal{C}}\alpha = 0$.*

Remark 3.1.4. *If A is an abelian group and $\underline{A}$ is the corresponding constant functor, usually* $\mathrm{H}^*(\mathcal{C}; \underline{A})$ *doesn't have a ring structure. Hence the preceding proposition makes no sense in this case. However, the restriction* $\mathrm{H}^*(\mathcal{C}; \underline{A}) \to \mathrm{H}^*(\mathcal{A}; \underline{A})$ *is interesting in its own right.*

Let G be a finite group, k an algebraic closed field of characteristic p and $\mathcal{F}$ a fusion system of a block of the group algebra kG, see Linckelmann [20]. *Let $\underline{k^\times}$ be the constant functor from $\mathcal{F}$ to $\mathcal{A}b$, sending every object in $\mathcal{F}$ to the multiplicative group $k^\times$ of*

k. People are interested in the cohomology groups of the full subcategory $\mathcal{F}^c \subset \mathcal{F}$, consisting of $\mathcal{F}$-centric subgroups, and its quotient category (orbit category of $\mathcal{F}^c$) $\bar{\mathcal{F}}^c$. In particular, one wants to understand the restrictions $\mathrm{H}^2(\mathcal{F}^c;\mathcal{Z}) \to \mathrm{H}^2(\mathcal{A}_{\mathcal{F}^c};\mathcal{Z})$ and $\mathrm{H}^2(\bar{\mathcal{F}}^c;\underline{k^\times}) \to \mathrm{H}^2(\mathcal{A}_{\bar{\mathcal{F}}^c};\underline{k^\times})$, because they may provide important information for studying Alperin's Conjecture in modular representation theory of finite groups, see Linckelmann [18] *Section 4.1. There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^2(\bar{\mathcal{F}}^c;\underline{k^\times}) & \xrightarrow{\iota^*} & \mathrm{H}^2(\mathcal{A}_{\bar{\mathcal{F}}^c};\underline{k^\times}) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ \mathrm{H}^2(\mathcal{F}^c;\underline{k^\times}) & \xrightarrow{\iota^*} & \mathrm{H}^2(\mathcal{A}_{\mathcal{F}^c};\underline{k^\times}) \end{array}$$

where the maps are induced by the functors $\pi : \mathcal{F}^c \to \bar{\mathcal{F}}^c$, $\iota : \mathcal{A}_{\mathcal{F}^c} \to \mathcal{F}^c$ and $\iota : \mathcal{A}_{\bar{\mathcal{F}}^c} \to \bar{\mathcal{F}}^c$.

3.2. **Restriction to subcategories with fewer isomorphisms.** We assume that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{D}$ contains all non-isomorphisms in $\mathrm{Mor}(\mathcal{C})$. We show there exist subcategories $\mathcal{D}$ of $\mathcal{C}$ such that $\mathcal{D}$ shares the same objects and non-isomorphisms with $\mathcal{C}$, while for every $x \in \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{D}$ the automorphism group $\mathrm{Aut}_{\mathcal{D}}(x)$ is a designated subgroup (up to isomorphism) of $\mathrm{Aut}_{\mathcal{C}}(x)$.

Lemma 3.2.1. *Let $\mathcal{C}$ be a finite EI-category. For each $[y] \subset \mathrm{Ob}\,\mathcal{C}$ we fix a representative $x \in [y]$ and choose a subgroup $H_x \subset \mathrm{Aut}_{\mathcal{C}}(x)$. Then there exists a subcategory $\mathcal{D}$ such that $\mathcal{D}$ contains all objects and non-isomorphisms in $\mathcal{C}$ and such that $\mathrm{Aut}_{\mathcal{D}}(z)$ is isomorphic to H_x if $z \cong x$.*

Suppose for different choices of sets of representatives x' and corresponding subgroups $H_{x'} \subset \mathrm{Aut}_{\mathcal{C}}(x')$, $x' \in [y] \subset \mathrm{Ob}\,\mathcal{C}$, there is a $\mathcal{D}'$ with the same properties as $\mathcal{D}$. Then $\mathcal{D}' \cong \mathcal{D}$, if $H_x = gH_{x'}g^{-1}$ for some $g \in \mathrm{Is}_{\mathcal{C}}(x',x)$ whenever $[x]=[x']$.

Proof. Since we want to keep all objects and all non-isomorphisms, the key to finding such a subcategory is to find a set of isomorphisms in $\mathcal{C}$ such that they can form $\mathrm{Mor}(\mathcal{D})$ along with the non-isomorphisms. In fact, we need to construct for each isomorphism class $[x] \subset \mathrm{Ob}\,\mathcal{C}$ a subgroupoid of $[x]$ whose skeleton is isomorphic to the chosen $\widehat{H_x}$.

We proceed in two steps. First of all, we want to find a unique isomorphism $\alpha_{ij} \in \mathrm{Hom}_{\mathcal{C}}(x_i,x_j)$ for every pair of objects in $[x] = \{x_1,\cdots,x_i,\cdots,x_j,\cdots\}$ such that this set of isomorphisms is closed under composition, and such that $\alpha_{ii} = 1_{x_i}$ and $\alpha_{ij} = \alpha_{ji}^{-1}$. Let's do this by induction on the number of objects in $[x]$. When $[x]$ only has one or two objects, the construction is trivial. When there are three objects in $[x]$, say x_1,x_2,x_3, we can choose arbitrary $\alpha_{12} \in \mathrm{Hom}_{\mathcal{C}}(x_1,x_2)$, $\alpha_{13} \in \mathrm{Hom}_{\mathcal{C}}(x_1,x_3)$ and then define $\alpha_{23} = \alpha_{13}\alpha_{12}^{-1}$. Suppose we have constructed such sets for all isomorphism classes with less than n objects. For an isomorphism class of objects in $\mathcal{C}$ with n objects, we fix a set of compatible isomorphisms for the subgroupoid consisting of any $n-1$ objects, say $x_1,\cdots,x_{n-1}$. Then any choice of an isomorphism $\alpha_{1,n} \in \mathrm{Hom}_{\mathcal{C}}(x_1,x_n)$ can be used to get a compatible set of isomorphisms for $[x]$, by composing 1_{x_n}, $\alpha_{1,n}$ and its inverse with existing isomorphisms among $x_1,\cdots,x_{n-1}$.

Second of all, without loss of generality, for a fixed object $x_1 = x \in [x]$, we let $H_{x_1} = H_x$. Then for each $x_i \in [x]$, we define H_{x_i} to be $\alpha_{1i} H_{x_1} \alpha_{1i}^{-1}$. These groups of automorphisms, along with the compatible set of isomorphisms $\{\alpha_{ij}\}$ define a subgroupoid in $[x]$ whose skeleton is isomorphic to $\widehat{H_x}$.

Let x run over the set of isomorphism classes of objects in $\mathcal{C}$. Then we get a collection of isomorphisms from subgroupoids constructed above. These isomorphisms, together with all non-isomorphisms in $\mathcal{C}$, form the morphism set of a subcategory $\mathcal{D}$ satisfying our conditions.

Now let's prove the second half of the lemma. We want to show with the given property there exists a $g_y \in \mathrm{Aut}_{\mathcal{C}}(y)$ for each $y \in \mathrm{Ob}\,\mathcal{C}$ such that such that ${}^{g_y}\mathrm{Aut}_{\mathcal{D}'}(y) = \mathrm{Aut}_{\mathcal{D}}(y)$. If this is true, then we can define a functor $\tau : \mathcal{D}' \to \mathcal{D}$ by setting $\tau(y) = y$ for each object y, and $\tau(\alpha) = g_z^{-1} \alpha g_y$ if $\alpha \in \mathrm{Hom}_{\mathcal{D}'}(y, z)$. It is easy to verify that τ is an isomorphism of categories.

Let $\{x'\}$ be a different set of representatives of objects and $\{H_{x'}\}$ the corresponding chosen subgroups. We just proved there exists a $\mathcal{D}'$ satisfying conditions in the first part of our statement. Without loss of generality we can assume $\{x'\} = \{x\}$ and then use new symbols $\{H'_x\}$, instead of $\{H_{x'}\}$, for the set of subgroups different from $\{H_x\}$. Suppose $H_x = g_x H'_x g_x^{-1}$ for some $g_x \in \mathrm{Aut}_{\mathcal{C}}(x)$, any chosen x. We need to find such an element for every object in $\mathcal{C}$, not just the given representatives $\{x\}$. Let's take a compatible set of isomorphisms $\{\alpha^x_{ij}\}$ for each $[x] \subset \mathcal{D}$ and another compatible set $\{\beta^x_{ij}\}$ in $\mathcal{D}'$. Then for any $x_i, x_j \in [x]$, we must have $\mathrm{Aut}_{\mathcal{D}}(x_j) = \alpha^x_{ij}\,\mathrm{Aut}_{\mathcal{D}}(x_i)\alpha^x_{ji}$ and $\mathrm{Aut}_{\mathcal{D}'}(x_j) = \beta^x_{ij}\,\mathrm{Aut}_{\mathcal{D}'}(x_i)\beta^x_{ji}$. Assume $x = x_1$. Since $\mathrm{Aut}_{\mathcal{D}}(x) = H_x = g_x H'_x g_x^{-1} = g_x\,\mathrm{Aut}_{\mathcal{D}'}(x) g_x^{-1}$, for each $x_i \in [x]$ we can find $g_{x_i} = \alpha^x_{1i} g_x \beta^x_{i1} \in \mathrm{Aut}_{\mathcal{C}}(x_i)$ satisfying $g_{x_i}\,\mathrm{Aut}_{\mathcal{D}'}(x_i) g_{x_i}^{-1} = \mathrm{Aut}_{\mathcal{D}}(x_i)$. Hence we're done. □

In particular, for a fixed prime p we can choose $\mathrm{Aut}_{\mathcal{D}}(x)$ to be a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{C}}(x)$, where x runs over the set of isomorphism classes in $\mathrm{Ob}\,\mathcal{C}$. We'll use $\mathcal{C}_p$ to denote a representative of such subcategories, because they are isomorphic to each other.

Corollary 3.2.2. *Suppose p is a prime and $\mathcal{C}$ is a finite EI-category. Then there exists a unique subcategory $\mathcal{C}_p$ up to isomorphism such that $\mathrm{Ob}\,\mathcal{C}_p = \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}_p$ contains all non-isomorphisms in $\mathcal{C}$ and for each object x $\mathrm{Aut}_{\mathcal{C}_p}(x)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{C}}(x)$.*

One may call each $\mathcal{C}_p$ a Sylow p-subcategory of $\mathcal{C}$ since if $\mathcal{D}$ is a subcategory such that it shares all objects and all non-isomorphisms with $\mathcal{C}$ and such that $\mathrm{Aut}_{\mathcal{D}}(x)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{C}}(x)$, for each $x \in \mathrm{Ob}\,\mathcal{C}$, then $\mathcal{D}$ is contained in a $\mathcal{C}_p$. To some extent, $\mathcal{C}_p$ plays the role of a Sylow p-subgroup of a finite group. If we consider the chain complexes of categories $\mathcal{C}, \mathcal{A}, \mathcal{C}_p$ and $\mathcal{A}_p = \mathcal{C}_p \cap \mathcal{A}$, we'll have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{A} & \xrightarrow{i} & \mathbb{C} & \xrightarrow{\pi} & \mathbb{C}/\mathbb{A} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \kappa} & & \\
0 & \longrightarrow & \mathbb{A}_p & \xrightarrow[i]{} & \mathbb{C}_p & \xrightarrow[\pi]{} & \mathbb{C}_p/\mathbb{A}_p & \longrightarrow & 0,
\end{array}$$

and hence

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} & \xrightarrow{i'} & \mathrm{H}^*(\mathcal{A}) \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{H}^*(\mathcal{C}_p)/I_{\mathcal{C}_p} & \xrightarrow[i']{} & \mathrm{H}^*(\mathcal{A}_p). \end{array}$$

The next result follows from the standard fact that the restriction $\iota^* : \mathrm{H}^*(\mathcal{A}; \underline{\mathbb{F}_p}) \to \mathrm{H}^*(\mathcal{A}_p; \underline{\mathbb{F}_p})$ is injective.

Proposition 3.2.3. *Let $\mathcal{C}$ be a finite EI-category and p a prime. Then the map*

$$\iota' : \mathrm{H}^*(\mathcal{C}; \underline{\mathbb{F}_p})/I_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{C}_p; \underline{\mathbb{F}_p})/I_{\mathcal{C}_p}$$

is an injection, where $I_{\mathcal{C}} = \pi^* \mathrm{H}^*(\mathbb{C}/\mathbb{A})$ and $I_{\mathcal{C}_p} = \pi^* \mathrm{H}^*(\mathbb{C}_p/\mathbb{A}_p)$ are two nilpotent ideals.

At present, it's not clear that whether or not ι' is an isomorphism if the other restriction $\iota^* : \mathrm{H}^*(\mathcal{A}, \mathbb{F}_p) \to \mathrm{H}^*(\mathcal{A}_p, \mathbb{F}_p)$ is an isomorphism.

3.3. **Restriction to subcategories with fewer non-isomorphisms.** We assume that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{D}$ and $\mathcal{C}$ have the same set of isomorphisms. Then the commutative diagram of short exact sequences reads as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{A} & \xrightarrow{i} & \mathbb{C} & \xrightarrow{\pi} & \mathbb{C}/\mathbb{A} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \kappa} & & \\ 0 & \longrightarrow & \mathbb{A} & \xrightarrow[i]{} & \mathbb{D} & \xrightarrow[\pi]{} & \mathbb{D}/\mathbb{A} & \longrightarrow & 0. \end{array}$$

It induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*(\mathcal{C})/I^*_{\mathcal{C}} & \xrightarrow{i'} & \mathrm{H}^*(\mathcal{A}) \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{H}^*(\mathcal{D})/I^*_{\mathcal{D}} & \xrightarrow[i']{} & \mathrm{H}^*(\mathcal{A}), \end{array}$$

where i' is induced by i^*, ι' is induced by $\iota^* : \mathrm{H}^*(\mathcal{C}) \to \mathrm{H}^*(\mathcal{D})$, $I^*_{\mathcal{C}} = \pi^* \mathrm{H}^*(\mathbb{C}/\mathbb{A})$ and $I^*_{\mathcal{D}} = \pi^* \mathrm{H}^*(\mathbb{D}/\mathbb{A})$ are two nilpotent ideals satisfying $\iota^*(I^*_{\mathcal{C}}) = \pi^*\kappa^*(\mathrm{H}^*(\mathbb{C}/\mathbb{A})) \subset \pi^*(\mathrm{H}^*(\mathbb{D}/\mathbb{A}))$.

Proposition 3.3.1. *Let $\mathcal{C}$ be a finite EI-category. Assume there exists a subcategory $\mathcal{D} \subset \mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Aut}_{\mathcal{D}}(x) = \mathrm{Aut}_{\mathcal{C}}(x)$ for each $x \in \mathrm{Ob}\,\mathcal{C}$. Then $\iota' : \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{D})/I_{\mathcal{D}}$ is injective for any base ring R.*

Proof. It follows directly from Proposition 3.1.2 and the commutative diagram above it. ☐

As a simple example, one can verify that if the orders of all automorphism groups in $\mathcal{C}$ are invertible in the base ring R, then $\mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} = R$. If for some $x \in \mathrm{Ob}\,\mathcal{C}$,

$|\operatorname{Aut}_{\mathcal{C}}(x)|^{-1} \in R$, then the isomorphism class $[x]$ can be "dropped" due to the following result. Let $x \in \operatorname{Ob}\mathcal{C}$. We define $\mathcal{C}_x$ to be the full subcategory of $\mathcal{C}$ consisting of all objects but those isomorphic to x.

Corollary 3.3.2. *Let $\mathcal{C}$ be a finite EI-category. Suppose $x \in \operatorname{Ob}\mathcal{C}$ such that $|\operatorname{Aut}_{\mathcal{C}}(x)|$ is invertible in R. Then $\mathrm{H}^*(\mathcal{C})/\mathcal{N}_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{C}_x)/\mathcal{N}_{\mathcal{C}_x}$ is injective.*

Proof. The subcategory $\mathcal{D} = \mathcal{C}_x \sqcup \widehat{[x]}$ (a disjoint union) obviously satisfies the condition in our previous proposition. Thus we have an injective map $\iota' : \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{D})/I_{\mathcal{D}}$. Since $\mathrm{H}^*(\mathcal{D}) = \mathrm{H}^*(\mathcal{C}_x) \oplus \mathrm{H}^*(\widehat{[x]})$ and $\mathrm{H}^*(\widehat{[x]})$ vanishes at any positive degree, $pr \circ \iota' : \mathrm{H}^*(\mathcal{C})/I_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{C}_x)/I_{\mathcal{D}}$ is an injective homomorphism, where pr is the natural projection $\mathrm{H}^*(\mathcal{D})/I_{\mathcal{D}} \to \mathrm{H}^*(\mathcal{C}_x)/I_{\mathcal{D}}$. Hence the statement follows. □

Thus to some extent, when considering mod p cohomology one can focus on categories whose automorphism groups of objects are p-groups. Combining Propositions 3.2.3 and 3.3.1 and Corollary 3.3.2, we have the following main result.

Theorem 3.3.3. *Let $\mathcal{C}$ be a finite EI-category and $\mathcal{D}$ a subcategory. Suppose p is a prime. If $\operatorname{Ob}\mathcal{D} = \{x \in \operatorname{Ob}\mathcal{C} : p \mid |\operatorname{Aut}_{\mathcal{C}}(x)|\}$ and $\operatorname{Aut}_{\mathcal{D}}(x)$ contains a Sylow p-subgroup of $\operatorname{Aut}_{\mathcal{C}}(x)$, then the map induced by the restriction $\mathrm{H}^*(\mathcal{C}; \underline{\mathbb{F}_p})/\mathcal{N}_{\mathcal{C}} \to \mathrm{H}^*(\mathcal{D}; \underline{\mathbb{F}_p})/\mathcal{N}_{\mathcal{D}}$ is injective, where $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_{\mathcal{D}}$ are the ideals of nilpotents in $\mathrm{H}^*(\mathcal{C}; \underline{\mathbb{F}_p})$ and $\mathrm{H}^*(\mathcal{D}; \underline{\mathbb{F}_p})$, respectively.*

A nontrivial example can be found in the theory of p-local finite groups, see [4], [5] and [17] for a complete description of results stated in the next example.

Example 3.3.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group* [5] *with finite covering data (see* [17]*) (θ, Γ, H), where Γ is a p-group or a p'-group, $\theta : S \to \Gamma$ is a group homomorphism and $H \subset \Gamma$ is a subgroup. Define $S_H = \theta^{-1}(H)$. Then there exists a fusion system $\mathcal{F}_H$ over S_H (*[4] *Proposition 3.8) and a p-local finite group $(S_H, \mathcal{F}_H, \mathcal{L}_H)$ (*[4] *Theorem 3.9) such that $|\mathcal{L}_H|$ is a covering space of $|\mathcal{L}|$. Let $\Gamma = H = 1$. Then $S_1 = S$ and $\mathcal{L}_1 \subset \mathcal{L}$ satisfies the condition that, for any $P \in \operatorname{Ob}\mathcal{L} = \operatorname{Ob}\mathcal{L}_1$, $\operatorname{Aut}_{\mathcal{L}_1}(P)$ contains a Sylow p-subgroup of $\operatorname{Aut}_{\mathcal{L}}(P)$ (*[4] *Proposition 3.8 c). By the preceding theorem, we get an injection $\mathrm{H}^*(\mathcal{L})/\mathcal{N}_{\mathcal{L}} \to \mathrm{H}^*(\mathcal{L}_1)/\mathcal{N}_{\mathcal{L}_1}$.*

On the other hand, Levi-Ragnarsson [17] *showed that in this case one can construct a transfer map for p-local finite groups which composes with the restriction is the identity: $\mathrm{H}^*(\mathcal{L}) \to \mathrm{H}^*(\mathcal{L}_1) \to \mathrm{H}^*(\mathcal{L})$. Especially, it implies $\operatorname{Res} : \mathrm{H}^*(\mathcal{L}) \to \mathrm{H}^*(\mathcal{L}_1)$ is injective hence another proof of the injectivity of the induced map $\mathrm{H}^*(\mathcal{L})/\mathcal{N}_{\mathcal{L}} \to \mathrm{H}^*(\mathcal{L}_1)/\mathcal{N}_{\mathcal{L}_1}$.*

We conclude this section with a couple of final remarks. Let $\mathcal{C}$ be a small category. One can define the subdivision $S(\mathcal{C})$ of $\mathcal{C}$, which is homotopy equivalent to $\mathcal{C}$ (see for example [19]). The subdivision is a category with subobjects [24], so especially all morphisms are monomorphic. Thus when studying cohomology rings of small categories one can just focus on categories whose morphisms are monomorphic. We comment that in the theory of p-local finite groups [5], as well as in local representation theory [28], many categories have the property that either all morphisms are monomorphic or all of them are epimorphic.

4. Extensions of categories and the generalized LHS spectral sequences

In last section we considered certain subcategories of a category $\mathcal{C}$ and the relationship between the cohomology rings of those subcategories and of $\mathcal{C}$. In this section, we do the "converse" and consider the cohomology rings of extensions of $\mathcal{C}$, which have the same sets of objects but have "larger" morphism sets. At the beginning we go over some basic knowledge about extensions of categories. The reader is referred to Hoff [15] or Webb [30] for more information about extensions of categories.

Given a functor $\mu : \mathcal{E} \to \mathcal{C}$, Gabriel and Zisman [11] showed there exists a spectral sequence converging to the homology of $\mathcal{E}$. When the category $\mathcal{E}$ is an extension of $\mathcal{C}$, we deduce a spectral sequence, based on Gabriel and Zisman's, as a generalization of the LHS spectral sequence for group extensions. It will be used to compare cohomology rings of $\mathcal{E}$ and $\mathcal{C}$.

4.1. **Extensions of categories.** An extension $\mathcal{E}$ of a category $\mathcal{C}$ via a category $\mathcal{K}$, in the sense of Hoff [15], is a sequence of functors

$$\mathcal{K} \xrightarrow{\iota} \mathcal{E} \xrightarrow{\pi} \mathcal{C},$$

which has the following properties:

(1) $\mathrm{Ob}\,\mathcal{K} = \mathrm{Ob}\,\mathcal{E} = \mathrm{Ob}\,\mathcal{C}$, ι is injective and π is surjective on morphisms;
(2) if $\pi(\alpha) = \pi(\beta)$, for two morphisms $\alpha, \beta \in \mathrm{Mor}(\mathcal{E})$, if and only if there is a unique $g \in \mathrm{Mor}(\mathcal{K})$ such that $\beta = \iota(g)\alpha$;
(3) if $\alpha\iota(h)$ exists for $\alpha \in \mathrm{Mor}(\mathcal{E})$ and $h \in \mathrm{Mor}(\mathcal{K})$, then there exists a unique $h' \in \mathrm{Mor}(\mathcal{K})$ such that $\iota(h')\alpha = \alpha\iota(h)$;
(4) for any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, $K(y)$ acts regularly on $\pi^{-1}(\alpha)$.

Note that (3) and (4) can be deduced from (1) and (2) which are normally used to define an extension of a category. It's known by Hoff [15] that $\mathcal{K}$ is a disjoint union of the groups $\pi^{-1}(1_x)$ for all $1_x \in \mathrm{Mor}(\mathcal{C})$ (regarded as categories), and can be identified with a functor $\mathcal{K} : \mathcal{E} \to$ *Groups*. Usually from the context, one can easily see when we take $\mathcal{K}$ to be a category and when it is regarded as a functor.

An extension is *split* if it admits a functor $s : \mathcal{C} \to \mathcal{E}$ such that $\pi \circ s = 1_{\mathcal{C}}$. In this case, $\mathcal{E}$ is a Grothendieck construction [30].

For future reference, we define an *opposite extension* $\mathcal{E}$ of $\mathcal{C}$ via $\mathcal{K}$ to be a sequence of functors $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ such that the following sequence is an extension of $\mathcal{C}^{op}$

$$\mathcal{K}^{op} \to \mathcal{E}^{op} \to \mathcal{C}^{op}.$$

When it won't cause any confusion, we'll just say $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an opposite extension of $\mathcal{C}$.

Example 4.1.1. (1) *Let G be a finite group and p a prime dividing the order of G. A collection $\mathcal{C}$ of p-subgroups of G is a set of p-subgroups which is closed under conjugations in G. The transporter category* $\mathrm{Tr}_{\mathcal{C}}(G)$ *(see* [7]*) is an extension of the orbit category $\mathcal{O}_{\mathcal{C}}(G)$*

$$\mathcal{K}_s \to \mathrm{Tr}_{\mathcal{C}}(G) \to \mathcal{O}_{\mathcal{C}}(G),$$

where $\mathcal{K}_s(H) = H \subset \mathrm{Aut}_{\mathrm{Tr}_{\mathcal{C}}(G)}(H) = N_G(H)$*;*

(2) *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a p-local finite group* [5]*. Then* $\mathcal{Z} \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c$ *is an opposite extension of* $\mathcal{F}^c$*, where* $\mathcal{F}^c \subset \mathcal{F}$ *is a full subcategory consisting of all* $\mathcal{F}$*-centric subgroups and* $\mathcal{Z}(P) = Z(P)$ *is the so-called central functor.*

4.2. **Generalized Lyndon-Hochschild-Serre spectral sequence.** Gabriel and Zismann ([11] Appendix II Theorem 3.6) gave a homology spectral sequence for a functor between two categories $\pi : \mathcal{E} \to \mathcal{C}$

$$E^2_{ij} = \mathrm{H}_i(\mathcal{C}; \mathrm{H}_j(\pi\downarrow_?; F)) \Rightarrow \mathrm{H}_{i+j}(\mathcal{E}; F),$$

where $F \in R\mathcal{E}$-mod is a functor from $\mathcal{E}$ to R-mod.

Since we are more interested in cohomology, we also write out the cohomology version of their spectral sequence

$$E_2^{ij} = \mathrm{H}^i(\mathcal{C}^{op}; \mathrm{H}^j(\pi\downarrow_?; F)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{E}^{op}; F),$$

where $F \in R\mathcal{E}^{op}$-mod, or a *contravariant* functor $\mathcal{E} \to R$-mod. When $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an opposite extension, we prove the cohomology groups of the overcategory with coefficients in F, $\mathrm{H}^*(\pi\downarrow_y; F)$, can be identified with $\mathrm{H}^*(\mathcal{K}(y), F(y))$ by a formula of Jackowski-Słomińska ([16] Proposition 5.4), which says if a functor $\mu : \mathcal{C}_1 \to \mathcal{C}_2$ satisfies the condition that every *undercategory* $\mu\downarrow^y$, $y \in \mathrm{Ob}\,\mathcal{C}_2$, is contractible, then $\mathrm{H}^*(\mathcal{C}_2; F) \cong \mathrm{H}^*(\mathcal{C}_1; F \circ \mu)$ for any *contravariant* functor $F : \mathcal{C}_2 \to R$-mod. Similarly one can get a Jackowski-Słomińska formula for the cases when F is a covariant functor and for homology.

Note that if there is a covariant functor $\pi : \mathcal{E} \to \mathcal{C}$ then any functor from $\mathcal{C}$ to R-mod induces a functor from $\mathcal{E}$ to R-mod (restriction along π).

Lemma 4.2.1. *Let* $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ *be an extension. Then there exists a natural functor* $\iota : \widehat{\mathcal{K}(y)} \to \pi\downarrow_y$ *such that every undercategory associated with it is contractible. Hence*

(1) $\mathrm{H}_*(\pi\downarrow_?; F) \cong \mathrm{H}_*(\mathcal{K}(?), F(?))$ *as functors in* $R\mathcal{C}$*-mod for any* $F \in R\mathcal{C}$*-mod; and*

(2) $\mathrm{H}^*(\pi\downarrow_?; F) \cong \mathrm{H}^*(\mathcal{K}(?), F(?))$ *as functors in* $R\mathcal{C}^{op}$*-mod for any* $F \in R\mathcal{C}^{op}$*-mod.*

Proof. The category $\pi\downarrow_y$ has objects of the form (x, α), where $x \in \mathrm{Ob}\,\mathcal{E} = \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. From the definition of $\pi\downarrow_y$, it's easy to see the maximal objects are (y, g), $g \in \mathrm{Aut}_{\mathcal{C}}(y)$, which are isomorphic to each other and have automorphism groups isomorphic to $\mathcal{K}(y)$.

Next we take the full subcategory of $\pi\downarrow_y$, consisting of all maximal objects. This full subcategory is denoted by $[(y, 1_y)]$ and its skeleton is isomorphic to the group $\widehat{\mathcal{K}(y)}$. Using Quillen's Theorem A [25], we show the undercategories associated with $\iota : [(y, 1_y)] \hookrightarrow \pi\downarrow_y$ are contractible, and thus we can apply Jackowski-Słomińska's result we just mentioned to get the isomorphism of cohomology groups.

Fix an object $(x, \alpha) \in \pi\downarrow_y$. The undercategory $(\pi\downarrow_y)\downarrow^{(x,\alpha)}$ has objects of the form $(\beta, (y, g))$, where $\beta : (x, \alpha) \to (y, g)$ is an morphism in $\pi\downarrow_y$ satisfying $g\pi(\beta) = \alpha$. Since $\pi(\beta) = g^{-1}\alpha$, by the definition of a category extension, $\beta = g^{-1}\alpha k$ for a unique $k \in \mathcal{K}(x)$. From here we can deduce that $(\beta, (y, g)) \cong (\beta', (y, g'))$ for any (y, g') and

$\beta' : (x,\alpha) \to (y,g')$, and that $(\beta,(y,g)) \in (\pi\downarrow_y)\downarrow^{(x,\alpha)}$ has a trivial automorphism group. These imply $(\pi\downarrow_y)\downarrow^{(x,\alpha)}$ is equivalent to a point, and hence is contractible.

The isomorphism of homology and cohomology groups follows from the result of Jackowski and Słomińska ([16] Proposition 5.4) we quoted above. It's not hard to see the isomorphisms give rise to the desired isomorphisms of functors. □

Combining the above lemma and the spectral sequences of Gabriel-Zisman, one can write out the following spectral sequences for category extensions. For simplicity we use $\mathrm{H}^*(\mathcal{K};F)$ etc, instead of $\mathrm{H}^*(\mathcal{K}(?);F(?))$ etc, for the functors in Lemma 4.2.1.

Proposition 4.2.2. *Given a functor $F:\mathcal{C}\to R$-mod, there are two spectral sequences associated with an extension $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$ as follows:*

(1) *a homology spectral sequence*
$$E^2_{ij} = \mathrm{H}_i(\mathcal{C};\mathrm{H}_j(\mathcal{K};F)) \Rightarrow \mathrm{H}_{i+j}(\mathcal{E};F);$$
and

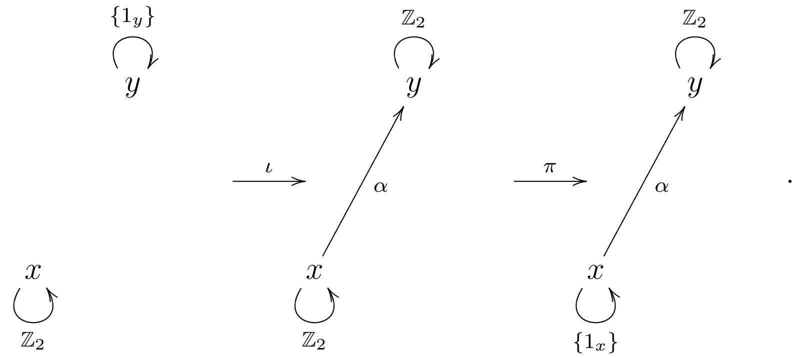
(2) *a cohomology spectral sequence*
$$E_2^{ij} = \mathrm{H}^i(\mathcal{C}^{op};\mathrm{H}^j(\mathcal{K}^{op};F)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{E}^{op},F).$$

Note that $\mathrm{H}^j(\mathcal{K}^{op};F)\cong\mathrm{H}^j(\mathcal{K};F)$.

Remark 4.2.3. *From these two spectral sequence, one can obtain two five term exact sequences*
$$\mathrm{H}_2(\mathcal{E};F)\to\mathrm{H}_2(\mathcal{C};F)\to\mathrm{H}_0(\mathcal{C};\mathrm{H}_1(\mathcal{K};F))\to\mathrm{H}_1(\mathcal{E};F)\to\mathrm{H}_1(\mathcal{C};F)\to 0,$$
and
$$0\to\mathrm{H}^1(\mathcal{C}^{op};F)\to\mathrm{H}^1(\mathcal{E}^{op};F)\to\mathrm{H}^0(\mathcal{C}^{op};\mathrm{H}^1(\mathcal{K}^{op};F))\to\mathrm{H}^2(\mathcal{C}^{op};F)\to\mathrm{H}^2(\mathcal{E}^{op};F),$$
where $F:\mathcal{C}\to R$-mod is a functor. When $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$ is a group extension then these two exact sequences are the usual five term sequence in group homology and cohomology.

Webb [31] *has produced the same five term exact sequences using other techniques.*

In general the finite generation of cohomology rings of both $\mathcal{K}$ and $\mathcal{C}$ doesn't guarantee the cohomology ring of $\mathcal{E}$ has the same property. One of the examples, $\mathcal{C}_2$ in Section 2 and the Appendix, used to demonstrate that the cohomology rings of EI-categories are not finitely generated, is an extension of a contractible category:



However, when $\mathcal{K}$ is cohomologically trivial, the cohomology rings of $\mathcal{E}$ and $\mathcal{C}$ are isomorphic.

Corollary 4.2.4. *Suppose $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an extension, and $|\mathcal{K}(x)|$ is invertible in R for every object x. Then for any $F \in R\mathcal{C}$-mod*

$$\mathrm{H}_*(\mathcal{E}; F) \cong \mathrm{H}_*(\mathcal{C}; \mathrm{H}_0(\mathcal{K}; F)) \cong \mathrm{H}_*(\mathcal{C}; \varinjlim_{\mathcal{K}} F).$$

$$\mathrm{H}^*(\mathcal{E}^{op}; F) \cong \mathrm{H}^*(\mathcal{C}^{op}; \mathrm{H}^0(\mathcal{K}^{op}; F)) \cong \mathrm{H}^*(\mathcal{C}^{op}; \varprojlim_{\mathcal{K}^{op}} F).$$

Since $\varprojlim_{\mathcal{K}^{op}} \underline{R} \cong \underline{R}$ in $R\mathcal{C}^{op}$-mod, we have $\mathrm{H}^*(\mathcal{C}; \underline{R}) \cong \mathrm{H}^*(\mathcal{C}^{op}; \underline{R}) \cong \mathrm{H}^*(\mathcal{E}^{op}; \underline{R}) \cong \mathrm{H}^*(\mathcal{E}; \underline{R})$ as algebras.

Proof. Under the assumption, the E_2 (resp. E^2) page of the cohomology (resp. homology) spectral sequence collapses to the horizontal axis. □

4.3. **Subextensions and reduction.** Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension. The generalized LHS spectral sequences establish connections between the cohomology and homology of $\mathcal{E}$ and $\mathcal{D}$. Since there is a natural correspondence between the subcategories of $\mathcal{C}$ and those of $\mathcal{E}$, one would like to exploit further connections between the homological properties of $\mathcal{C}$ and $\mathcal{E}$. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ and $\mathcal{E}_{\mathcal{D}}$ its "preimage" in $\mathcal{E}$. We show the undercategories (or overcategories) associated with the inclusions are equivalent, when $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an extension (or an opposite extension) of $\mathcal{C}$.

Definition 4.3.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension and $\mathcal{D} \subset \mathcal{C}$ a subcategory. The subextension of $\mathcal{D}$ in $\mathcal{E}$ via $\mathcal{K}|_{\mathcal{D}}$, named $\mathcal{E}_{\mathcal{D}}$, is a subcategory of $\mathcal{E}$ whose object set is the same as $\mathcal{D}$ and whose morphism set consists of morphisms in $\mathcal{E}$ which are preimages of morphisms in $\mathcal{D}$.*

If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ then $\mathcal{E}_{\mathcal{D}}$ is a full subcategory of $\mathcal{E}$. Given an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$, $\widehat{\mathrm{Aut}_{\mathcal{K}}(x)} \to \widehat{\mathrm{Aut}_{\mathcal{E}}(x)} \to \widehat{\mathrm{Aut}_{\mathcal{C}}(x)}$ is a subextension for any $x \in \mathrm{Ob}\,\mathcal{C}$.

Proposition 4.3.2. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ a sequence of functors and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ with the inclusion $\iota_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$. Then*

(1) *if $\mathcal{E}$ is an extension of $\mathcal{C}$, $\mathcal{K}|_{\mathcal{D}} \to \mathcal{E}_{\mathcal{D}} \to \mathcal{D}$ is the subextension and $\iota_{\mathcal{E}_{\mathcal{D}}} : \mathcal{E}_{\mathcal{D}} \to \mathcal{E}$ is the inclusion, then for any $y \in \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{E}$, the undercategory $\iota_{\mathcal{D}} \downarrow^y$ is isomorphic to a subcategory of the undercategory $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow^y$, which is equivalent to $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow^y$;*

(2) *if $\mathcal{E}$ is an opposite extension of $\mathcal{C}$, $\mathcal{K}|_{\mathcal{D}} \to \mathcal{E}_{\mathcal{D}} \to \mathcal{D}$ is the (opposite) subextension and $\iota_{\mathcal{E}_{\mathcal{D}}} : \mathcal{E}_{\mathcal{D}} \to \mathcal{E}$ is the inclusion, then for any $y \in \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{E}$, the overcategory $\iota_{\mathcal{D}} \downarrow_y$ is isomorphic to a subcategory of the overcategory $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y$, which is equivalent to $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y$.*

Proof. We'll prove (2). In $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y$, any two objects (x, α) and (x, β) are isomorphic if and only if $\pi(\alpha) = \pi(\beta)$. Let $\overline{\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y} \subset \iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y$ be the full subcategory consisting of one object from each isomorphism class of objects described above. Then $\overline{\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y}$ and $\iota_{\mathcal{E}_{\mathcal{D}}} \downarrow_y$ are equivalent. We prove the former is isomorphic to $\iota_{\mathcal{D}} \downarrow_y$.

There is a natural bijection between objects sets of these two categories $(x, \alpha) \to (x, \pi(\alpha))$ (π is surjective on morphisms). We show there is a bijection between the morphism sets and the bijections extend to a functor which gives an isomorphism between two categories. Any $(x,\alpha)\xrightarrow{\gamma}(z,\beta)$ in $\mathrm{Mor}(\overline{\iota_{\mathcal{E}_\mathcal{D}}\downarrow_y})$ gives rise to a morphism $(x,\pi(\alpha))\xrightarrow{\pi(\gamma)}(z,\pi(\beta))$ in $\iota_\mathcal{D}\downarrow_y$. On the other hand, a morphism $(x,\pi(\alpha))\xrightarrow{\pi(\gamma)}(z,\pi(\beta))$ in $\iota_\mathcal{D}\downarrow_y$ implies $\pi(\beta)\pi(\gamma)=\pi(\alpha)$, which means there exists a unique $g\in\mathcal{K}(x)$ such that $\beta\gamma=\alpha g$. Thus we have a uniquely defined morphism $(x,\alpha)\xrightarrow{g^{-1}}(x,\alpha g)\xrightarrow{\gamma}(z,\beta)=(x,\alpha)\xrightarrow{\gamma g^{-1}}(z,\beta)$ in $\mathrm{Mor}(\overline{\iota_{\mathcal{E}_\mathcal{D}}\downarrow_y})$. Note that a different γ' such that $\pi(\gamma')=\pi(\gamma)$ gives the same morphism $(x,\alpha)\xrightarrow{\gamma g^{-1}}(z,\beta)$, so the map from $\mathrm{Mor}(\iota_\mathcal{D}\downarrow_y)$ to $\mathrm{Mor}(\overline{\iota_{\mathcal{E}_\mathcal{D}}\downarrow_y})$ is well-defined. It's straightforward to check these two assignments on morphisms are mutually inverse to each other.

In order to show the bijections on objects and morphisms defining an isomorphism between categories, we need to verify they preserve composition and identity. We'll just prove the former and leave the proof of preserving identity to the reader. Suppose $(x,\alpha)\to(z,\beta)\to(w,\gamma)$ is a composite of two morphisms in $\overline{\iota_{\mathcal{E}_\mathcal{D}}\downarrow_y}$. Then our map naturally sends it to a composite of morphisms $(x,\pi(\alpha))\to(z,\pi(\beta))\to(w,\pi(\gamma))$. Conversely, if $(x,\pi(\alpha))\xrightarrow{\pi(u)}(z,\pi(\beta))\xrightarrow{\pi(v)}(w,\pi(\gamma))=(x,\pi(\alpha))\xrightarrow{\pi(vu)}(w,\pi(\gamma))$ is the composite of two morphisms in $\iota_\mathcal{D}\downarrow_y$, then we need to show the two morphisms $(x,\alpha)\xrightarrow{ug^{-1}}(z,\beta)\xrightarrow{vh^{-1}}(w,\gamma)=(x,\alpha)\xrightarrow{vh^{-1}ug^{-1}}(w,\gamma)$ and $(x,\alpha)\xrightarrow{vut^{-1}}(w,\gamma)$ are equal, where g,h,t are isomorphisms, described in the preceding paragraph. Since $\pi(vh^{-1}ug^{-1})=\pi(vut^{-1})$, there is a unique isomorphism s satisfying $vh^{-1}ug^{-1}=vut^{-1}s$. But then we have $\alpha=\gamma vut^{-1}=\gamma vh^{-1}ug^{-1}=\gamma vut^{-1}s$, and this forces $s=1$ because $\mathcal{K}^{op}$ acts freely on morphisms in $\mathrm{Mor}(\mathcal{E}^{op})$. Hence we get $vh^{-1}ug^{-1}=vut^{-1}$. □

The following corollary is a natural outcome of the proposition. A space X is said to be R-acyclic, if the reduced homology groups $\tilde{\mathrm{H}}^*(X,R)$ vanish.

Corollary 4.3.3. *Let $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$ be a sequence of functors and $\mathcal{D}\subset\mathcal{C}$ a full subcategory with the inclusion $\iota_\mathcal{D}:\mathcal{D}\to\mathcal{C}$. Then*

1. *if $\mathcal{E}$ is an extension of $\mathcal{C}$, then $\iota_\mathcal{D}\downarrow^y$ is contractible (or R-acyclic or connected) if and only if $\iota_{\mathcal{E}_\mathcal{D}}\downarrow^y$ is;*
2. *if $\mathcal{E}$ is an opposite extension of $\mathcal{C}$, then $\iota_\mathcal{D}\downarrow_y$ is contractible (or R-acyclic or connected) if and only if $\iota_{\mathcal{E}_\mathcal{D}}\downarrow_y$ is.*

Example 4.3.4. *Let $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$ be an extension with a unique maximal object x such that $\mathrm{Aut}_\mathcal{C}(x)$ acts freely and transitively on $\mathrm{Hom}_\mathcal{C}(y,x)$ for any $y\in\mathrm{Ob}\,\mathcal{C}$. Then it's easy to check that $\iota:\widehat{\mathrm{Aut}_\mathcal{C}(x)}\hookrightarrow\mathcal{C}$ induces a homotopy equivalence since all undercategories associated to it are contractible. Hence we know $\widehat{\mathrm{Aut}_\mathcal{E}(x)}\hookrightarrow\mathcal{E}$ is a homotopy equivalence as well.*

Since any category can be regarded as a trivial extension of itself, the following result is a generalization of Jackowski-Słomińska's formula we mentioned in the previous subsection.

Corollary 4.3.5. *Suppose there is an extension $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$. If $\iota_\mathcal{D}:\mathcal{D}\hookrightarrow\mathcal{C}$ is an inclusion such that $\iota_\mathcal{D}\downarrow^y$ is contractible for every $y\in\mathrm{Ob}\,\mathcal{C}$, then $H^*(\mathcal{E};F)\cong$*

$\mathrm{H}^*(\mathcal{E}_{\mathcal{D}};F)$ *for any contravariant functor* $F:\mathcal{E}\to R$*-mod, and* $\mathrm{H}_*(\mathcal{E};F)\cong\mathrm{H}_*(\mathcal{E}_{\mathcal{D}};F)$ *for any covariant functor* $F:\mathcal{E}\to R$*-mod. Here* $\mathcal{E}_{\mathcal{D}}$ *is the subextension corresponding to* $\mathcal{D}$.

Suppose there is an opposite extension $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$. *If* $\iota_{\mathcal{D}}:\mathcal{D}\hookrightarrow\mathcal{C}$ *is an inclusion such that* $\iota_{\mathcal{D}}\downarrow_y$ *is contractible for every* $y\in\mathrm{Ob}\,\mathcal{C}$, *then* $H^*(\mathcal{E};F)\cong\mathrm{H}^*(\mathcal{E}_{\mathcal{D}};F)$ *for any covariant functor* $F:\mathcal{E}\to R$*-mod. Here* $\mathcal{E}_{\mathcal{D}}$ *is the opposite subextension corresponding to* $\mathcal{D}$.

Proof. We prove the statements for cohomology. Since $\iota_{\mathcal{D}}\downarrow^y$ is contractible for every $y\in\mathrm{Ob}\,\mathcal{C}$, $\iota_{\mathcal{E}_{\mathcal{D}}}\downarrow^y$ is contractible for every $y\in\mathrm{Ob}\,\mathcal{E}$ as well by Proposition 4.3.2. That means if we take the dual version P'_* of the projective resolution P_*, described in 2.1, of the *contravariant* functor $\underline{R}:\mathcal{E}_{\mathcal{D}}\to R$-mod, the left Kan extension of $P'_*\to\underline{R}\to 0$ is still a projective resolution $K(P'_*)\to K(\underline{R})\cong\underline{R}\to 0$ of the contravariant functor $\underline{R}:\mathcal{E}\to R$-mod. Thus our results follows from the isomorphism of complexes of R-modules $\mathrm{Hom}_{R\mathcal{E}_{\mathcal{D}}}(P'_*,F\circ\iota_{\mathcal{E}_{\mathcal{D}}})\cong\mathrm{Hom}_{R\mathcal{E}}(K(P'_*),F)$.

When we have an opposite extension, using the same proposition we get $\iota_{\mathcal{E}_{\mathcal{D}}}\downarrow_y$ is contractible for every $y\in\mathrm{Ob}\,\mathcal{E}$. If $P_*\to\underline{R}\to 0$ is the projective resolutions of $\underline{R}$ as a covariant functor, then the left Kan extension of it, $K(P_*)\to K(\underline{R})\cong\underline{R}\to 0$, is a projective resolution of $\underline{R}$ as a covariant functor from $\mathcal{E}$ to R-mod. Hence we can obtain a similar isomorphism of complex using the adjunction of K and $\mathrm{Res}_{\iota_{\mathcal{E}_{\mathcal{D}}}}$, the restriction along $\iota_{\mathcal{E}_{\mathcal{D}}}$. □

As an example when $\mathcal{K}\to\mathcal{E}\to\mathcal{C}$ is an extension (or an opposite extension) and $\mathcal{C}$ has a unique maximal (or minimal) object x and $\mathrm{Aut}_{\mathcal{C}}(x)$ acts regularly on $\mathrm{Hom}_{\mathcal{C}}(y,x)$ (or $\mathrm{Hom}_{\mathcal{C}}(x,y)$) for any $y\in\mathrm{Ob}\,\mathcal{C}$, we have $\mathrm{H}^*(\mathcal{C};F)\cong\mathrm{H}^*(\mathrm{Aut}_{\mathcal{C}}(x),F(x))$ hence $\mathrm{H}^*(\mathcal{E};F)\cong\mathrm{H}^*(\mathrm{Aut}_{\mathcal{E}}(x),F(x))$ for any contravariant (or covariant) functor F.

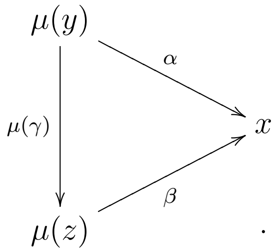
5. Appendix : Cohomology of small categories

This section intends to provide a self-contained introduction to the standard materials that the reader may want to know about the cohomology theory of small categories. Here we begin with a general setup, which is followed by a canonical resolution for computing the cohomology of small categories with coefficients in functors. Using this resolution, we calculate the cohomology of a family of categories named $\mathcal{C}_n$, where $n>1$ denotes the number of (non-isomorphic) objects.

Let R be a commutative ring with identity. In the introduction we mentioned that for any small category $\mathcal{C}$ one can define the category algebra $R\mathcal{C}$ to be a free R-module with basis the set of morphisms in $\mathcal{C}$ (see Xu [32]), in which multiplication is given by composition of basis elements. When $\mathrm{Ob}\,\mathcal{C}$ is finite, Mitchell [22] showed the category of left $R\mathcal{C}$-modules is equivalent to the category of covariant functors from $\mathcal{C}$ to R-mod, i.e. $R\mathcal{C}\text{-mod}\simeq(R\text{-mod})^{\mathcal{C}}$. The equivalence is given as follows. If $F\in(R\text{-mod})^{\mathcal{C}}$ is a functor, then we define an $R\mathcal{C}$-module to be the R-module $M_F=\oplus_{x\in\mathrm{Ob}\,\mathcal{C}}F(x)$ equipped with natural actions by morphisms in $\mathcal{C}$. Conversely if $M\in R\mathcal{C}$-mod, then we define a functor $F_M\in(R\text{-mod})^{\mathcal{C}}$ such that $F_M(x)=1_xM$ for each $x\in\mathrm{Ob}\,\mathcal{C}$. It's easy to verify that these two assignments are functors which are inverse to each other. Similarly, the category of right $R\mathcal{C}$-modules (same as $R\mathcal{C}^{op}$-mod) is equivalent to the category of contravariant functors from $\mathcal{C}$ to R-mod. For simplicity, we'll stick with the terms of covariant and contravariant functors.

These equivalences allow us to consider $\mathrm{Ext}^*_{R\mathcal{C}}(F_1, F_2) = \mathrm{Ext}^*_{(R\text{-mod})^{\mathcal{C}}}(F_1, F_2)$ and $\mathrm{Tor}^{R\mathcal{C}}_*(M_1, M_2) = \mathrm{Tor}^{(R\text{-mod})^{\mathcal{C}}}_*(M_1, M_2)$, where F_1, F_2 and M_2 are covariant functors and M_1 is a contravariant functor (for further information, see tom Dieck [6], Lück [21], Webb [30] or Xu [32]). For the reader's convenience, we point out that the tensor product of a contravariant functor M and a covariant functor N as an R-module is the following $M \otimes_{R\mathcal{C}} N$: $\sum_{x\in \mathrm{Ob}\,\mathcal{C}} M(x) \otimes_R N(x)/\sim$, where the relation is given by $m_x M(\alpha) \otimes_R n_y \sim m_x \otimes_R N(\alpha) n_y$ for any $m_x \in M(x), n_y \in N(y)$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(y, x)$.

The constant functor $\underline{R}$ plays an important role in the cohomology theory of small categories. In fact it should be regarded as the generalization of the trivial module of a group algebra. We define $\mathrm{H}^*(\mathcal{C}; F) := \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, F)$ and $\mathrm{H}_*(\mathcal{C}; F) := \mathrm{Tor}^{R\mathcal{C}}_*(F, \underline{R})$. In order to compute them, we need to construct a projective resolution of $\underline{R}$. There is a canonical projective resolution of $\underline{R}$ (see for example Grodal [12]) which can be defined using the so-called overcategories (see Mac Lane [23]) associated with the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Recall that a functor $\mu : \mathcal{D} \to \mathcal{C}$ can be used to produce an overcategory $\mu \downarrow_x$ for each $x \in \mathrm{Ob}\,\mathcal{C}$. This overcategory consists of objects of the form (y, α), where $y \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mu(y), x)$. A morphism between two objects (y, α) and (z, β) is given by some $\gamma \in \mathrm{Hom}_{\mathcal{D}}(y, z)$ which makes the following diagram commutative:



Dually, one can define an undercategory $\mu \downarrow^x$ for each object $x \in \mathrm{Ob}\,\mathcal{C}$. Now fix an object $x \in \mathrm{Ob}\,\mathcal{C}$. Then $\mathrm{Id}_{\mathcal{C}} \downarrow_x$ is always contractible, by Quillen's Theorem A [25], because it has a terminal object $(x, 1_x)$. For each integer $n \geq 0$ we can define a functor $\mathrm{Id}_{\mathcal{C}} \downarrow^n_?$: $\mathcal{C} \to R$-mod sending x to $\mathrm{Id}_{\mathcal{C}} \downarrow^n_x$—the free R-module spanned over the set of n-chains of morphisms in $\mathrm{Id}_{\mathcal{C}} \downarrow_x$. There is a natural way to assemble these functors into a sequence of functors, $\mathrm{Id}_{\mathcal{C}} \downarrow^*_? \to \underline{R} \to 0$, which evaluated at each $x \in \mathrm{Ob}\,\mathcal{C}$ is a complex for computing the reduced homology of $|\mathrm{Id}_{\mathcal{C}} \downarrow_x|$. Since every $|\mathrm{Id}_{\mathcal{C}} \downarrow_x|$ is contractible, the sequence is exact. In order to show it's in fact a projective resolution of $\underline{R}$, we need to prove each $\mathrm{Id}_{\mathcal{C}} \downarrow^*_?$ is a projective object in $R\mathcal{C}$-mod. This can be seen by rewriting the sequence in a slightly different form: $P_* \to \underline{R} \to 0$, where $P_n : \mathcal{C} \to R$-mod is defined as follows (see Oliver [24]). The new form of this resolution is less conceptual and is easier to use in practice. For any $x \in \mathrm{Ob}\,\mathcal{C}$, $P_n(x)$ is the free abelian group with a basis the set of all sequences $[x_0 \to x_1 \to \cdots \to x_n \to x]$ of morphisms in $\mathrm{Mor}(\mathcal{C})$ ending in x. For any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, $P_n(f)$ is defined by its action on basis elements: $[x_0 \to \cdots \to x_n \xrightarrow{\phi} x] \mapsto [x_0 \to \cdots \to x_n \xrightarrow{f\phi} y]$.

The boundary map $\sigma = \{\sigma_x\} : P_n \to P_{n-1}$ is given by setting

$$\sigma_x([x_0 \to x_1 \to \cdots \to x_n \to x]) = \sum_{i=0}^{n} (-1)^i [x_0 \to \cdots \to \hat{x}_i \to \cdots \to x_n \to x],$$

on base elements. Since

$$\mathrm{Hom}_{R\mathcal{C}}(P_n, F) \cong \prod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(\mathcal{C})} F(x_n),$$

where $N(\mathcal{C})$ is the nerve of $\mathcal{C}$, we know $\mathrm{Hom}_{R\mathcal{C}}(P_n, ?)$ is exact and hence P_n is projective for any $n \geq 0$. Thus $\mathrm{H}^*(\mathcal{C}; F)$ are the homology groups of the cochain complex $0 \to \mathrm{Hom}_{R\mathcal{C}}(P_*, F)$, or equivalently $0 \to \mathrm{Hom}_{R\mathcal{C}}(\mathrm{Id}_{\mathcal{C}} \downarrow_?^*, F)$. When $F = \underline{A}$ is a constant functor, one can easily see $\mathrm{H}^*(\mathcal{C}; \underline{A}) \cong \mathrm{H}^*(|\mathcal{C}|, A)$, because $0 \to \mathrm{Hom}_{R\mathcal{C}}(P_*, \underline{A})$ can be identified with a cochain complex used to compute $\mathrm{H}^*(|\mathcal{C}|, A)$.

5.1. **Two torsion groups.** Using Tor, one can define and compute the homology groups of $\mathcal{C}$ with coefficients in a *contravariant* functor F: $\mathrm{H}_*(\mathcal{C}; F) \cong \varinjlim^*_{\mathcal{C}} F \cong \mathrm{Tor}_*^{R\mathcal{C}}(F, \underline{R})$. As in cohomology, similarly we have $\mathrm{H}_*(\mathcal{C}; \underline{A}) \cong \mathrm{H}_*(|\mathcal{C}|, A)$ for any constant functor $\underline{A}$. One can use the same projective resolution we described above to prove $\mathrm{Tor}_*^{R\mathcal{C}}(\underline{A}, \underline{R}) \cong \mathrm{H}_*(|\mathcal{C}|, A)$. Here we only intend to give a taste of the concrete calculations of (co-)homology groups, and do not plan to recall the general theory of Ext and Tor. With this purpose in mind, we first discuss two special torsion groups. They will be used in our further examples at the end of the appendix.

A functor is *atomic* if it takes non-zero values at only one isomorphism class of objects in $\mathcal{C}$. For example, we can define an atomic bi-functor $S_{x,R} : \mathcal{C} \to R$-mod such that $S_{x,R}(y) = R$ if $y \cong x$ and $S_{x,R}(y) = 0$ otherwise. Using the projective resolution $P_* \cong \mathrm{Id}_{\mathcal{C}} \downarrow_?^*$ of $\underline{R}$, described above, one can show that $\mathrm{H}_*(\mathcal{C}; S_{x,R}) \cong \mathrm{Tor}_*^{R\mathcal{C}}(S_{x,R}, \underline{R})$ can be calculated as homology groups of the chain complex given by

$$\left\{ \coprod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(\mathcal{C})} S_{x,R}(x_n) \right\} = \left\{ \coprod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(\mathcal{C}),\ x_n \cong x} R \right\},$$

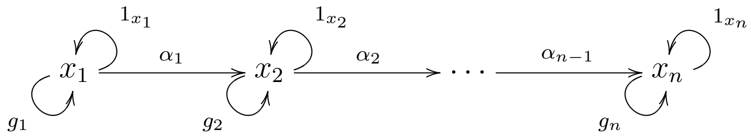
where chain maps are induced by the face maps for $N(\mathcal{C})$. We will encounter torsions like $\mathrm{Tor}_*^{R\mathcal{C}}(\underline{R}, S_{x,R})$ as well in the next subsection. This time they are the homology groups of the chain complex given by

$$\left\{ \coprod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(\mathcal{C})} S_{x,R}(x_0) \right\} = \left\{ \coprod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(\mathcal{C}),\ x_0 \cong x} R \right\},$$

where chain maps are induced by the face maps for $N(\mathcal{C})$.

5.2. **further examples of cohomology rings.** For each integer $n > 1$ we construct a category $\mathcal{C}_n$ which has exactly n objects and then examine the ring structure of

$\mathrm{H}^*(\mathcal{C}_n) := \mathrm{H}^*(\mathcal{C}_n;\underline{\mathbb{F}_2}) \cong \mathrm{H}^*(|\mathcal{C}_n|,\mathbb{F}_2)$, where $\mathbb{F}_2$ is a field of characteristic 2. Let $\mathcal{C}_n$, $n > 1$, be the following category



,

where $\alpha_i \cdot g_i = \alpha_i = g_{i+1} \cdot \alpha_i$ and $g_i^2 = 1_{x_i}$ for all i. We'll first calculate the homology of $\mathcal{C}_n$. Then the structure of $\mathrm{H}^*(\mathcal{C}_n;\underline{\mathbb{F}_2})$ can be easily obtained via the Universal Coefficient Theorem.

Since $\mathrm{H}^*(\mathcal{C}_n;\underline{\mathbb{F}_2}) \cong \mathrm{H}^*(|\mathcal{C}_n|,\mathbb{F}_2)$, one can just take the (normalized) chain complex associated to the nerve of $\mathcal{C}_n$ to compute its homology groups: $\cdots \to C_k \to \cdots \to C_1 \to C_0 \to 0$, where $C_0 = \oplus_i \mathbb{F}_2 x_i$ and C_k, $k \geq 1$, is an $\mathbb{F}_2$-vector space spanned by chains of k consecutive morphisms in $\mathcal{C}_n$. For example in C_2 of $\mathcal{C}_2$, the basis is $\{(g_1,g_1),(g_1,\alpha_1),(\alpha_1,g_2),(g_2,g_2)\}$.

In order to proceed, we need some auxiliary constructions. Suppose $\mathcal{C}$ is a finite EI-category. For each $x \in \mathrm{Ob}\,\mathcal{C}$, we'll be interested in two full subcategories of $\mathcal{C}$, $\mathcal{C}_{<x} \subset \mathcal{C}_{\leq x}$, (see conventions in the introduction) and the associated complexes $\mathbb{C}_{<x} \subset \mathbb{C}_{\leq x}$ and the corresponding quotient complex, written as $\{(\cdot,x\rangle_k\}_{k=0}^{\infty}$, which contains the linear combinations of chains ending at x. If $x \leq y \in \mathrm{Ob}\,\mathcal{C}$, then we also define $\mathcal{C}_{[x,y]}$ to be a full subcategory of $\mathcal{C}$ consisting of objects z such that both $\mathrm{Hom}_{\mathcal{C}}(x,z)$ and $\mathrm{Hom}_{\mathcal{C}}(z,y)$ are not empty. There are naturally defined full subcategories of $\mathcal{C}_{[x,y]}$: $\mathcal{C}_{(x,y]}$, $\mathcal{C}_{[x,y)}$ and $\mathcal{C}_{(x,y)}$. The quotient complexes of $\mathbb{C}_{[x,y]}$ by $\mathbb{C}_{(x,y]}$, $\mathbb{C}_{[x,y)}$ and the sum $\mathbb{C}_{(x,y]}+\mathbb{C}_{[x,y)}$ are named $\{\langle x,y]_k\}_{k=0}^{\infty}$, $\{[x,y\rangle_k\}_{k=0}^{\infty}$ and $\{\langle x,y\rangle_k\}_{k=0}^{\infty}$, respectively.

For the sake of simplicity, when it won't cause any confusion, we will use $(\cdot,x\rangle$, $[x,y]$ and $\langle x,y\rangle$ et cetera to denote the quotient complexes defined above.

Proposition 5.2.1. *Let $\mathcal{C}_n (n \geq 2)$ be the category defined above. Then*

$$\mathrm{H}^i(\mathcal{C}_n;\underline{\mathbb{F}_2}) = \begin{cases} \mathbb{F}_2, & \text{if } i = 0; \\ 0\ , & \text{if } 1 \leq i \leq 2n-2; \\ \mathbb{F}_2^{m(i,n)}, & \text{if } i \geq 2n-1. \end{cases}$$

Here $m(i,n) \in \mathbb{Z}$ is the number of chains of i consecutive morphisms in which every non-identity morphism in $\mathcal{C}_n$ appears at least once.

Proof. Suppose

$$\cdots \to C_n \to \cdots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$

is the normalized chain complex associated with the nerve of $\mathcal{C}_n$. Let σ be any chain in any C_i. A significant property of δ_i in this particular case is that all the non-zero summands of $\delta_i(\sigma)$ contain the same isomorphisms as those in σ itself. For this reason, we can classify chains of morphisms in C_* by the (ordered) lists of isomorphisms in them.

We say two chains in C_* are of the same type if, after removing non-isomorphisms in them, the resulting ordered lists (possibly empty) of isomorphisms are the same.

We'll denote the type of a chain by $T = \{t_1, t_2, \cdots, t_n\}$, where t_i is a non-negative integer and represents the number of g_i in the chain, for every i. The length of T is $|T| = t_1 + \cdots + t_n$. Due to our observation on δ_*, chains of morphisms of the same type T form a subcomplex of C_*, denoted by C_*^T, which is always finite because there are finitely many non-isomorphisms. Furthermore $C_* = \oplus_T C_*^T$, where T runs over the (infinite) set of all possible types. In particular, the type $\{0, \cdots, 0\}$ chains are chains of non-isomorphisms and if we demand C_0 to be of type $\{0, \cdots, 0\}$ then they form a natural subcomplex of C_*, which is the order complex for the *underlying poset* $[\mathcal{C}_n]$ of $\mathcal{C}_n$. In general, when a type T is given, all chains of type T have lengths greater than or equal to $|T|$ (that is, C_*^T begins at degree $|T|$).

Now we start to calculate $\mathrm{H}_*(\mathcal{C}_n)$. In order to compute the homology groups of C_*, we just have to know how to do it for the subcomplex formed by chains of each type. Fix a type T, it is not hard to see that C_*^T is isomorphic to the product of finitely many chain complexes of the forms $[x_1, y\rangle$, $\langle x, y\rangle$ and $\langle x, x_n]$ coming from the underlying poset $[\mathcal{C}_n]$ of $\mathcal{C}_n$ (since isomorphisms have been excluded and there is only one morphism between any two non-isomorphic objects.). Hence one can use the Künneth Theorem to compute the homology of each C_*^T if the homology groups of the three explicitly constructed quotient complexes are known.

Next we re-interpret the homology groups of the following chain complexes $[x, y\rangle$, $\langle x, y\rangle$ and $\langle x, y]$ for objects x, y belonging to a finite poset. Note that when $x = y$, these complexes are trivial.

Lemma 5.2.2. *Let $\mathcal{C}$ be a finite poset. For any two objects $x < y$, the homology groups of $[x, y\rangle$, $\langle x, y\rangle$ and $\langle x, y]$ are isomorphic to $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R}, \underline{R})$, $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R}, S_{x,R})$ and $\mathrm{Tor}_*^{R\mathcal{D}}(\underline{R}, S_{x,R})$, respectively, where $\mathcal{D}$ is the subposet $\mathcal{C}$ consisting of all objects z such that $x \le z \le y$.*

The groups $\mathrm{Tor}_i^{R\mathcal{D}}(\underline{R}, S_{y,R})$ and $\mathrm{Tor}_i^{R\mathcal{D}}(S_{x,R}, \underline{R})$ vanish unless $i = 0$, when they are equal to R. If there exists $x < z < y$ then $\mathrm{Tor}_i^{R\mathcal{D}}(S_{y,R}, S_{x,R}) = 0$. Otherwise there is only one non-zero torsion $\mathrm{Tor}_1^{R\mathcal{D}}(S_{y,R}, S_{x,R}) = R$.

Assuming the lemma is proved, we use it to calculate the ith homology of C_* when $i \ge 1$

$$\mathrm{H}_i(C_*) = \oplus_T \mathrm{H}_i(C_*^T).$$

By Künneth Theorem, we have

$$\begin{aligned} \mathrm{H}_i(C_*^T) &= \oplus_{i_1+\cdots+i_k=i} \mathrm{H}_{i_1}([\cdot,\cdot\rangle) \otimes \mathrm{H}_{i_2}(\langle\cdot,\cdot\rangle) \otimes \cdots \otimes \mathrm{H}_{i_k}(\langle\cdot,\cdot]) \\ &= \oplus_{i_2+\cdots+i_{k-1}=i} \mathrm{H}_{i_2}(\langle\cdot,\cdot\rangle) \otimes \cdots \otimes \mathrm{H}_{i_{k-1}}(\langle\cdot,\cdot\rangle), \end{aligned}$$

where the dots represent appropriate objects in $\mathcal{C}_n$.

The second equality is true because $\mathrm{H}_{i_1}([\cdot,\cdot\rangle)$ and $\mathrm{H}_{i_k}(\langle\cdot,\cdot])$ are non-zero if and only if $i_1 = i_k = 0$, when they equal R by the previous lemma. The same lemma also implies that $\mathrm{H}_{i_2}(\langle\cdot,\cdot\rangle) \otimes \cdots \otimes \mathrm{H}_{i_{k-1}}(\langle\cdot,\cdot\rangle)$ is non-zero if and only if $i_2 = \cdots = i_{k-1} = 1$ and each $\langle\cdot,\cdot\rangle$ is some $\langle x_s, x_{s+1}\rangle$, or equivalently every isomorphism in $\mathrm{Mor}(\mathcal{C})$ occurs at least once in T. When it's non-zero, the tensor product must be R. Thus $\mathrm{H}_i(C_*) = \oplus_T \mathrm{H}_i(C_*^T) = R^{m(i,n)}$, where $m(i, n)$ is the number of i-chains of all possible types in which every morphism in $\mathcal{C}_n$ appears at least once. □

Proof of the lemma. The homology groups of $[x,y\rangle$ are exactly $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R})$ by definition of $[x,y\rangle$ and discussion on $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R})$ in 2.2. All positive degree torsions are zero because $\underline{R} = R\,\mathrm{Hom}(x,?)$ is a representable functor. Similarly, we can prove $\mathrm{H}_*(\langle x,y]) \cong \mathrm{Tor}_*^{R\mathcal{D}}(\underline{R},S_{x,R})$ and all positive degree torsions are zero because $\underline{R} = R\,\mathrm{Hom}(?,x)$ is a contravariant representable functor. As for $\langle x,y\rangle$, we note that $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},S_{x,R})$ can be computed through the long exact sequence

$$\cdots \to \mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R}_{>x}) \to \mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R}) \to \mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},S_{x,R}) \to \cdots$$

coming from the following short exact sequence of functors: $0 \to \underline{R}_{>x} \to \underline{R} \to S_{x,R} \to 0$. Here $\underline{R}_{>x}$ is the maximal subfunctor of $\underline{R}$ (note that if there is no z such that $x < z < y$ then $\underline{R}_{>x} \cong S_{y,R}$). Since $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R}) \cong \mathrm{H}_*([x,y\rangle)$, $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R}_{>x}) \cong \mathrm{H}_*((x,y\rangle)$ and $0 \to (x,y\rangle \to [x,y\rangle \to \langle x,y\rangle \to 0$ also induces a long exact sequence, a Five Lemma argument shows that $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},S_{x,R}) \cong \mathrm{H}_*(\langle x,y\rangle)$.

Finally from the long exact sequence for $\mathrm{Tor}_*^{R\mathcal{D}}$ and the values of $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R}_{>x})$ and $\mathrm{Tor}_*^{R\mathcal{D}}(S_{y,R},\underline{R})$, we conclude that if there exists z such that $x < z < y$ then $\mathrm{Tor}_i^{R\mathcal{D}}(S_{y,R},S_{x,R}) = 0$. Otherwise there's only one torsion $\mathrm{Tor}_1^{R\mathcal{D}}(S_{y,R},S_{x,R}) = R$. □

Corollary 5.2.3. *For each $n > 1$, the cohomology ring $\mathrm{H}^*(\mathcal{C}_n;\underline{\mathbb{F}_2})$ is not finitely generated.*

Proof. Over the field $\mathbb{F}_2$ we always have $\mathrm{H}^*(\mathcal{C}_n) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{H}_*(\mathcal{C}_n),\mathbb{F}_2)$ in a natural way as vector spaces by the Universal Coefficient Theorem. It implies as a vector space $\mathrm{H}^*(\mathcal{C}_n)$ is infinite dimensional. It also means, for any $k \geq 0$, $\mathrm{H}^k(\mathcal{C}_n)$ is spanned by functions in $\mathrm{Hom}_{\mathbb{F}_2}(C_k,\mathbb{F}_2)$ which are dual to the base elements for $\mathrm{H}_k(\mathcal{C}_n)$. From the description of $\mathrm{H}_*(\mathcal{C}_n)$ we know what the generators of $\mathrm{H}^*(\mathcal{C}_n)$ are and thus we can compute the cup product of any two of these functions, which is always zero. Hence the statement is proved. □

Note that $\mathcal{C}_2$ is the smallest (in terms of number of objects and/or morphisms) non-contractible category that is neither a group nor a poset.

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