CORRELATION SPECTRUM OF MORSE-SMALE GRADIENT FLOWS

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ABSTRACT. In this note, we review our recent works devoted to the spectral analysis of Morse-Smale flows. Then we give applications to differential topology and to the spectral theory of Witten Laplacians.

1. INTRODUCTION

Let M be a smooth ($\mathcal{C}^\infty$), compact, oriented and boundaryless manifold of dimension $n \geq 1$. Given a smooth vector field V on M, its integration defines a flow $\varphi^t : M \to M$, and maybe one of the most basic question in dynamical systems is to understand the long time behaviour of such a flow. Let us formulate more precisely the meaning of the above statement. Given some smooth differential k–form $\psi_1 \in \Omega^k(M)$, one can ask if the pulled-back differential form $\varphi^{-t*}(\psi_1)$ has a weak limit in the sense of currents when times t goes to $+\infty$. For such a limit to exist, the dynamical system $(\varphi^t : M \mapsto M)_t$ under study must have some particular structure. In order to study the weak limit $\lim_{t\to+\infty} \varphi^{-t*}(\psi_1)$, it is natural to introduce the *correlation function* of the flow:

$$\boxed{\forall t \geq 0, \quad C_{\psi_1,\psi_2}(t) := \int_M \varphi^{-t*}(\psi_1) \wedge \psi_2,} \tag{1}$$

where ψ_1 is a k-form and ψ_2 a $(n-k)-$form.

Let us now observe that $\varphi^{-t*}(\psi_1)$ is the solution of the following *transport equation*:

$$\boxed{\partial_t \psi = -\mathcal{L}_V \psi, \quad \psi(t=0) = \psi_1,} \tag{2}$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field V. Recall that Cartan's formula allows us to write $\mathcal{L}_V$ under the following supersymmetric form:

$$\boxed{\mathcal{L}_V = (d + \iota_V)^2,} \tag{3}$$

that can be thought as an analogue of the formula for the Hodge–De Rham Laplacian[1] : $\Delta_g = (d + d^*)^2$. This formal analogy with Hodge theory will turn out to be central for applications to differential topology that will be described at the end of these proceedings. Equation (2) shows how the study of the limit of $(\varphi^t)_{t\to+\infty}$, which is nonlinear in nature, can be turned into a linear PDE problem[2]. More precisely, one may try to find out some appropriate Banach space $\mathcal{B}$ on which $-\mathcal{L}_V$ has good spectral properties. Then, we would prove some kind of convergence to equilibrium result like what one would do in the case of

[1]Here, d^* denotes the adjoint of d with respect to a Riemannian metric g.

[2]This is of course at the expense of working in infinite dimension.

the heat equation associated to Δ_g. To understand this formal analogy, replacing $\mathcal{L}_V$ in the transport equation (2) by the Laplacian Δ_g yields the heat equation : $\partial_t\psi = -\Delta_g\psi, \quad \psi(t = 0) = \psi_1$. However, unlike Hodge theory, one cannot work with L^2 spaces since it does not give interesting spectral properties for $\mathcal{L}_V$. Still, a lot of progresses have been made towards this question in the last fifteen years, and the purpose of this report is to present these problems in the a priori simple framework of Morse-Smale gradient flows. We refer the reader to the introduction of our articles [8, 10, 11] for a brief overview of the literature.

There are many ways to construct appropriate spaces adapted to the dynamical properties of the flow and all of them give in the end the same objects. Equivalently, we will get the same eigenvalues and the same eigenmodes. Here, we choose to adopt a microlocal approach to this problem and it is most likely that we could get similar results by following other strategies such as the one developped by Liverani et al. [3, 23, 5]. This microlocal point of view was introduced for dynamical systems with hyperbolic behaviour by Baladi, Dyatlov, Faure, Sjöstrand, Tsujii, Zworski, etc [1, 30, 16, 31, 14]. In the microlocal approach, we start by the observation that the principal symbol of $-i\mathcal{L}_V$ is given by the Hamiltonian function

$$\forall(x,\xi) \in T^*M,\ H_V(x,\xi) := \xi(V(x)). \tag{4}$$

The dynamical properties of the corresponding Hamiltonian flow acting on cotangent space T^*M, denoted by :

$$\Phi^t(x,\xi) := \left(\varphi^t(x), \left(d\varphi^t(x)^T\right)^{-1}\xi\right). \tag{5}$$

must be studied in order to construct the appropriate anisotropic Sobolev spaces of currents adapted to the dynamics. A crucial feature of this flow is the hyperbolicity at the critical points. Moreover, a particular role will be played by the stable and unstable sets of the Hamiltonian flow in T^*M which are conical Lagrangians in T^*M.

2. A brief reminder on Morse-Smale gradient flows

Let us now focus on the particular case of gradient flows. For that purpose, we fix a smooth ($\mathcal{C}^\infty$) function of Morse type. In other words, f has only finitely many critical points, all of them being nondegenerate. We denote by $\text{Crit}(f)$ the set of critical points.

2.1. **Definition and first properties.**

Let g be a smooth Riemannian metric. Then we define *the gradient of f with respect to the metric g* as the following vector field :

$$\forall(x,v) \in TM,\ d_xf(v) = \langle V_f(x), v\rangle_{g(x)}. \tag{6}$$

Such a vector field generates a complete flow on M that we denote by $(\varphi_f^t)_{t\in\mathbb{R}}$ and it is called the gradient flow. The nonwandering set of this flow is equal to the set of critical points of f [26]. The critical points being non degenerate, we say that the nonwandering set of the flow is hyperbolic. Hence, given a in $\text{Crit}(f)$, one can define its stable manifold (resp. unstable) as :

$$W^{s/u}(a) := \left\{x \in M :\ \lim_{t\to+/-\infty}\varphi_f^t(x) = a\right\}.$$

It can be proved that these are embedded submanifolds inside M [28, 32, 22]. However, as we will later see, the submanifolds $W^{s/u}(a)_{a\in\text{Crit}(f)}$ are not necessarily properly embedded. We set $0\leq r\leq n$ (resp. $n-r$) to be the dimension of $W^s(a)$ (resp. $W^u(a)$), and we note that r is also the Morse index of the critical point a. Observe also that $W^u(a)\cap W^s(a)=\{a\}$. A notable feature of these submanifolds is that they form a partition of M [29], i.e.

$$M=\bigcup_{a\in\text{Crit}(f)} W^s(a),\ \text{and}\ \forall a\neq b,\ W^s(a)\cap W^s(b)=\emptyset.$$

The same of course holds for the unstable manifolds once we observe that unstable manifolds of V_f are stable manifolds of V_{-f}. This "cellular" decomposition plays an important role in the applications to topology as was observed by Thom [29]. For applications to topology, Smale introduced another requirement that, for every critical points a and b in $\text{Crit}(f)$, the submanifolds *$W^s(a)$ and $W^u(b)$ intersect transversally*[3]. This assumption turns out to be crucial in our analysis and it can be formulated in an equivalent manner by saying that the forward and backward trapped set of the Hamiltonian flow Φ^t defined by (5) intersect only along the zero section $\underline{0}\subset T^*M$. The Morse function being fixed, this transversality assumption is satisfied by an open and dense subset of metrics [19]. Once these properties are verified, we say that the flow φ_f^t is a *Morse-Smale gradient flow.*

2.2. **Correlation function of a gradient flow.** Let us now come back to the study of the correlation functions

$$C_{\psi_1,\psi_2}(t):=\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2$$

of the gradient flow. In that direction, Laudenbach and Harvey-Lawson showed that the following holds [2, 22, 20]:

Theorem 2.1 (Laudenbach, Harvey-Lawson)**.** *Let f be a smooth Morse function. Then, there exists an "adapted" Morse-Smale metric g such that :*

- *(Laudenbach) for every a in $\text{Crit}(f)$, $W^u(a)$ and $W^s(a)$ define integration currents in the sense of De Rham that we denote by $[W^u(a)]$ and $[W^s(a)]$,*
- *(Harvey-Lawson) for every $0\leq k\leq n$ and for every (ψ_1,ψ_2) in $\Omega^k(M)\times\Omega^{n-k}(M)$,*

$$\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2 \longrightarrow \sum_{a:\dim W^s(a)=k}\left(\int_{W^s(a)}\psi_1\right)\left(\int_{W^u(a)}\psi_2\right),\quad \text{as}\quad t\to+\infty. \tag{7}$$

The second part of the Theorem can also be reformulated in terms of weak limits in the space of currents as follows :

$$\varphi^{-t*}(\psi_1)\rightharpoonup \sum_{a:\dim W^s(a)=k}\left(\int_{W^s(a)}\psi_1\right)[W^u(a)].$$

[3]Note that if $\dim W^s(a)+\dim W^u(b)<n$ then transversality means that the intersection is empty

By an "adapted" metric, we mean that the metric is euclidean in a Morse chart [22] near each critical point. In particular, it means that g is flat near $\text{Crit}(f)$. However, Minervini showed [24] similar results on integration currents and the convergence of correlators under relaxed assumptions on the metric g. The main difficulty regarding the first part of the Theorem is that we can easily integrate a differential form whose support is included in a compact part of $W^u(a)$ but it is not clear that we can integrate a form whose support intersects $\partial W^u(a) := \overline{W^u(a)} \setminus W^u(a)$. To justify this point, one needs to analyse carefully the structure of $W^u(a)$ near its boundary and this is where the "adapted" condition comes in. Indeed, Laudenbach proves something more precise, namely that $\overline{W^u(a)}$ is a submanifold with conical singularities. This in particular implies that $\overline{W^u(a)}$ defines a current of finite mass in the sense of geometric measure theory. Observe now a remarkable thing about the second part of the Theorem : $\varphi_f^{-t*}(\psi_1)$ converges weakly to a limit current which can be decomposed as $\sum_{a\in\text{Crit}(f)}\left(\int_{W^s(a)}\psi_1\right)[W^u(a)]$ in the basis $(W^u(a))_{a\in\text{Crit}(f)}$ of unstable currents. This is highly reminiscent of Thom's partition of M as a union of unstable manifolds. In other words, if we study the convergence to equilibrium to solutions ψ of the transport equation (2), then we recover at the limit some linear combination of currents which appear in the cellular decomposition of the manifold. In particular, we can deduce from this Theorem classical results from differential topology such as the finiteness of Betti numbers or the Morse inequalities [20]. At the end of this lecture, we will explain how to recover these topological results from a spectral perspective.

2.3. **Lyapunov exponents and linearization assumptions.** In order to state our results, we need to introduce two more definitions. First of all, for every a in $\text{Crit}(f)$, we define $L_f(a)$ as the unique matrix verifying

$$\forall \xi,\eta\in T_aM,\ d_a^2f(\xi,\eta)=g_a(L_f(a)\xi,\eta). \tag{8}$$

As a is a nondegenerate critical point, the matrix $L_f(a)$ is invertible and symmetric with respect to g_a. Its eigenvalues are the *Lyapunov exponents* of the critical point a and we denote them by

$$\chi_1(a)\leq\ldots\leq\chi_r(a)<0<\chi_{r+1}(a)\leq\ldots\leq\chi_n(a),$$

where r is the index of the critical point a. For $l\geq 0$, the flow φ_f^t is said to be $\mathcal{C}^l$-linearizable if, for every critical point a of f, there exists a $\mathcal{C}^l$-chart near a such that the flow can be written locally, for t small enough,

$$\varphi_f^t(x_1,\ldots,x_n)=\left(e^{t\chi_1(a)}x_1,\ldots,e^{t\chi_n(a)}x_n\right). \tag{9}$$

Actually, thanks to the Hartman-Grobman Theorem, we can always find a $\mathcal{C}^0$-linearizing chart. The Sternberg-Chen Theorem [25] states that the chart can be chosen of class $\mathcal{C}^l$ as soon as a certain (finite) number of nonresonance assumptions are satisfied by the Lyapunov exponents. We emphasize that the metrics of Laudenbach and Harvey-Lawson generate by construction $\mathcal{C}^\infty$-linearizable flows with all the Lyapunov exponents equal to ±1.

3. Statement of the main results

Our first main result is the following refinement of Theorem 2.1 [8, 9]:

Theorem 3.1. *Suppose that φ_f^t is a $\mathcal{C}^1$-linearizable Morse-Smale gradient flow. Fix $0 \leq k \leq n$. Then, for every $a \in \text{Crit}(f)$ of index k, there exists a pair of currents (U_a, S_a) in*[a] *$\mathcal{D}'^{,k}(M) \times \mathcal{D}'^{,n-k}(M)$ such that the support of U_a is equal to $\overline{W^u(a)}$ and such that*

$$\mathcal{L}_{V_f}(U_a) = 0 \quad \text{and} \quad U_a = [W^u(a)] \quad \text{on} \quad M - \partial W^u(a).$$

Moreover, for every

$$0 < \chi < \min \{|\chi_j(a)| : 1 \leq j \leq n,\ a \in \text{Crit}(f)\},$$

one has, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$,

$$\int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2 = \sum_{a:\dim W^u(a)=n-k} \left(\int_M \psi_1 \wedge S_a \right) \left(\int_M U_a \wedge \psi_2 \right) + \mathcal{O}_{\psi_1,\psi_2}(e^{-\chi t}).$$

[a] $\mathcal{D}'^{,k}(M)$ denotes the space of currents of degree k.

The proof we gave of this result is of purely spectral nature and is completely independent of the Theorem by Laudenbach and Harvey–Lawson. Note that our proof yields an exponential rate of convergence towards equilibrium under rather general assumptions on the metric. Our Theorem also establishes the existence of the extension to M of the germ of current $[W^u(a)]$. However, we emphasize that the main drawback compared to Theorem 2.1 is that the extended currents are not a priori of *finite mass* while the construction from [19, 22] allows to establish that U_a is a standard current of integration. It is plausible that the rate of convergence in this Theorem could be recovered by techniques from geometric measure theory à la Federer but we are not aware of such proof in the literature. Yet, we emphasize that this result is just the first term of an asymptotic expansion that our analysis allows to compute at any order. To state a general statement, we introduce the following notation:

$$|\chi(a)| = (|\chi_1(a)|, \ldots, |\chi_n(a)|).$$

Then we have [8]:

Theorem 3.2. *Suppose that φ_f^t is a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let $0 \leq k \leq n$.*

Then, for every a in $\text{Crit}(f)$ and for every α in $\mathbb{Z}_+^n$, there exists a continuous linear map:

$$\pi_{a,k}^{(\alpha)} : \Omega^k(M) \to \mathcal{D}'^k(M),$$

such that, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$ and for every $\chi > 0$, one has

$$\int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2 = \sum_{a\in\text{Crit}(f)} \sum_{\alpha\in\mathbb{Z}_+^n:\alpha.|\chi(a)|\leq\chi} e^{-t\alpha.|\chi(a)|} \int_M \pi_{a,k}^{(\alpha)}(\psi_1) \wedge \psi_2 + \mathcal{O}_{\psi_1,\psi_2}(e^{-\chi t}),$$

as t tends to $+\infty$. Moreover, for every a in $\text{Crit}(f)$ and for every α in $\mathbb{Z}_+^n$, one has

- $0\leq \text{rk}(\pi_{a,k}^{(\alpha)})\leq 2^n$,
- *for every* ψ_1 *in* $\Omega^k(M)$*, the support of* $\pi_{a,k}^{(\alpha)}(\psi_1)$ *is contained in* $\overline{W^u(a)}$*,*
- $\text{rk}(\pi_{a,k}^{(0)})=\delta_{k,\dim(W^s(a))}$,
- *for every* α *in* $(\mathbb{Z}_+^*)^n$, $\text{rk}(\pi_{a,k}^{(\alpha)})=\frac{n!}{k!(n-k)!}$.

The assumption on the rational independence of the Lyapunov exponents allows us to state the result in a simpler manner but our method allows in fact to deal with $\mathcal{C}^1$-linearizable flows at the expense of having polynomial factors in the asymptotic expansion [9]. In the terminology of dynamical systems theory, this Theorem shows that the Pollicott-Ruelle resonances of a gradient flow are of the form $-\alpha.|\chi(a)|$ with α a multi-index in $\mathbb{Z}_+^n$. If we are only interested in observables ψ_1 and ψ_2 supported near a critical point, we will verify below that this result can be obtained as an application of the Taylor formula – see paragraph 4.1 below. Here, the main point is that this is a result on the *global dynamics* of the gradient flow and not necessarily on the local dynamics near a critical point. We emphasize that, in the case of the height function on the 2-sphere endowed with its canonical metric, a similar result was obtained by Frenkel, Losev and Nekrasov via Witten Laplacian methods [17]. We shall come back to this issue later in this note. Finally, even if we do not describe this here, our analysis extends to more general Morse-Smale flows that may have closed orbits and that we couple with a flat connection [10, 11].

4. About the proofs

In this review, we shall focus for the sake of simplicity on the case $k=0$ and just outline the main ideas. We will hide many technical issues and refer to the original papers– see [8, 10, 11] for details. Moreover, we will suppose that the flow is smoothly-linearizable. The extension to the $\mathcal{C}^1$-linearizable case can be found in [9].

4.1. **A preliminary calculation.** We first localize the study of the dynamics near critical points since this is the first natural places to look at for gradient flows. Let us start by proving Theorem 3.2 near a critical point a whose index will be denoted by r. We choose some neighborhood U of a on which there exists a smooth chart where the dynamics is linearized as in equation (9). Then, we fix two test forms $\psi_1(x)\in\Omega_c^0(U)$ and $\psi_2(x,dx)\in\Omega_c^n(U)$ compactly supported in U. Then, we write

$$\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2=\int_{\mathbb{R}^n}\tilde{\psi}_1(e^{-t\chi_1(a)}x_1,\ldots,e^{-t\chi_n(a)}x_n)\tilde{\psi}_2(x_1,\ldots,x_n)dx_1\wedge\ldots\wedge dx_n$$

where $(\tilde{\psi}_1,\tilde{\psi}_2)$ denotes the test forms in the linearizing chart. Then, we make the following change of variables $(x_1,\ldots,x_n)\mapsto(e^{t\chi_1(a)}x_1,\ldots,e^{t\chi_r(a)}x_r,x_{r+1},\ldots,x_n)$ inside the integral :

$$\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2=e^{t\sum_{j=1}^r\chi_j(a)}\int_{\mathbb{R}^n}\tilde{\psi}_1(x_1,\ldots,x_r,e^{-t\chi_{r+1}(a)}x_{r+1},\ldots,e^{-t\chi_n(a)}x_n)$$

$$\times\tilde{\psi}_2(e^{t\chi_1(a)}x_1,\ldots,e^{t\chi_r(a)}x_r,x_{r+1},\ldots x_n)dx_1\wedge\ldots\wedge dx_n.$$

Writing down the Taylor formula, we obtain the following formal asymptotic expansion :

$$\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2 \sim e^{t\sum_{j=1}^r \chi_j(a)} \sum_{(\alpha,\beta)\in\mathbb{Z}_+^r\times\mathbb{Z}_+^{n-r}} C_{\alpha,\beta} e^{-t(\alpha,\beta).|\chi(a)|}$$
$$\times \left\langle x^{(\alpha,0)}\delta_0^{(\beta)}(x_{r+1},\dots,x_n),\tilde{\psi}_1\right\rangle \left\langle x^{(0,\beta)}\delta_0^{(\alpha)}(x_1,\dots,x_r),\tilde{\psi}_2\right\rangle,$$

where $C_{\alpha,\beta}$ are universal constants i.e. independent of (ψ_1,ψ_2). Hence, for every α in $\mathbb{Z}_+^n$, one has, in a neighborhood of a in $\text{Crit}(f)$, a germ of eigendistribution $u_{\alpha,a}$ that can be written in local coordinates as :

$$\boxed{u_{\alpha,a}(x_1,\dots,x_r) := \delta_0^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)x_{r+1}^{\alpha_{r+1}}\dots x_n^{\alpha_n}.} \tag{10}$$

The distribution $u_{\alpha,a}$ satisfies the following equation in a neighborhood of a :

$$\boxed{\mathcal{L}_{V_f}(u_{\alpha,a}) = -\left(\alpha.|\chi(a)| + \sum_{j=1}^r \chi_j(a)\right)u_{\alpha,a}.} \tag{11}$$

The above differential equation should be understood in the weak sense where both sides are distributions in $\mathcal{D}'(U)$. Note that the eigenvalue 0 only shows up at critical points of index 0 (i.e. local minima). Similarly, in degree k, the eigenvalue 0 shows up at critical points of index k.

The strategy of the proof is then as follows:

(1) extend the germ of eigenmode into a globally defined generalized eigenmode,
(2) use these extended eigenmodes to write down the asymptotic expansion of the *global* correlation function.

For the first point, it is natural to use the integrated version of equation (11),

$$\varphi_f^{-t*}u_{\alpha,a} = e^{-t(\alpha.|\chi(a)|+\sum_{j=1}^r|\chi_j(a)|)}u_{\alpha,a},$$

in order to extend the local germ of distribution $u_{\alpha,a}$ into a distribution defined over the open set $M\setminus\partial W^u(a)$ where $\partial W^u(a) = \overline{W^u(a)} - W^u(a)$. The eigenvalue equation allows to propagate the germ of distribution from the neighborhood U of a to $M\setminus\partial W^u(a)$. This new distribution still satisfies the eigenvalue equation (11) and we need to extend it into a distribution globally defined over M. This is related to the problem of renormalization in quantum field theory [7] and also naturally appears in the proofs of Laudenbach and Harvey-Lawson. The analogy between the construction of eigenmodes by distributional extension and Epstein–Glaser renormalization was first noted by Frenkel–Losev–Nekrasov [17]. However, our approach to this problem is of completely different nature and it is based on spectral theory. More precisely, for every $\chi>0$, we construct in a first stage an anisotropic Sobolev space $\mathcal{H}^{m_\chi}(M)$ containing our germs of distributions and for which the operator $-\mathcal{L}_{V_f}$ has a discrete spectrum in the half plane $\{\text{Re}(z) > -\chi\}$. Then, we use the spectral projector to prove both points (1) and (2).

4.2. **Hamiltonian dynamics and anisotropic Sobolev spaces.** Our spectral construction is very much inspired by the microlocal approach developped by Faure and Sjöstrand to study the correlation spectrum of Anosov flows [16], e.g. geodesic flows on negatively curved manifolds. We briefly describe the general strategy. For a given function $m(x,\xi)$ in $S^0(T^*M)$, we define the following Sobolev space of variable order :

$$\mathcal{H}^m(M) := \text{Op}\left((1+\|\xi\|_x^2)^{\frac{m(x,\xi)}{2}}\right)^{-1} L^2(M).$$

Studying the operator $-\mathcal{L}_{V_f}$ on that space is equivalent to study the *non selfadjoint* operator

$$\hat{H}_{V_f} := \text{Op}\left((1+\|\xi\|_x^2)^{\frac{m(x,\xi)}{2}}\right) \circ \left(\frac{1}{i}\mathcal{L}_{V_f}\right) \circ \text{Op}\left((1+\|\xi\|_x^2)^{\frac{m(x,\xi)}{2}}\right)^{-1}$$

on $L^2(M)$. An application of the rules from pseudodifferential calculus shows that this operator can be rewritten

$$\hat{H}_{V_f} = \text{Op}\left(H_{V_f} + iX_{H_{V_f}}.\left(\frac{m(x,\xi)}{2}\ln(1+\|\xi\|_x^2)\right)\right) + \mathcal{O}(\Psi^0(M)) + \mathcal{O}_m(\Psi^{-1+0}(M)),$$

where H_{V_f} is the Hamiltonian defined from the symbol of V_f by (4). We denote by $X_{H_{V_f}}$ the corresponding Hamiltonian vector field whose dynamics (5) lifts the gradient flow. Hence, if, for every $c>0$, we manage to find a function $m(x,\xi)$ such that, for $\|\xi\|_x$ large enough,

$$X_{H_{V_f}}.\left(\frac{m(x,\xi)}{2}\ln(1+\|\xi\|_x^2)\right) \leq -c, \tag{12}$$

then the imaginary part of the symbol of the operator will be "elliptic" in a region $\|\xi\|_x \geq R$ with $R>0$ large enough. Using Fredholm theory, we can then invert the operator "modulo a compact operator" and deduce that the operator

$$-\mathcal{L}_{V_f} : \mathcal{H}^m(M) \to \mathcal{H}^m(M)$$

has discrete spectrum in the region $\{\text{Re}(z) > -\chi\}$, as soon as $c>0$ is chosen large enough in (12). In other words, if we follow the strategy of Faure-Sjöstrand, the main difficulty lies in the construction of a function m satisfying (12), which is a purely dynamical question on some Hamiltonian system. In order to understand how to construct such a function m, we write

$$X_{H_{V_f}}.\left(\frac{m(x,\xi)}{2}\ln(1+\|\xi\|_x^2)\right) = X_{H_{V_f}}(m) \times \frac{1}{2}\ln(1+\|\xi\|_x^2) + m(x,\xi)\frac{X_{H_{V_f}}.(\|\xi\|_x^2)}{2(1+\|\xi\|_x^2)}. \tag{13}$$

We can already remark that the second term on the right hand side is bounded. Hence, one has to impose $X_{H_{V_f}}(m) \leq 0$ in order to be able to ensure that inequality (12) holds for $\|\xi\|$ large enough. We can also note that $-X_{H_{V_f}}(f) \leq 0$. Hence, if we set $m(x,\xi) = -f(x) + m_0(x,\xi)$ with $X_{H_{V_f}}(m_0) \leq 0$, then the inequality will be satisfied away from the critical points. Near a critical point a, we use the hyperbolicity of the flow to show that $\frac{X_{H_{V_f}}.(\|\xi\|_x^2)}{2(1+\|\xi\|_x^2)} \geq c_0 > 0$ (resp. $\leq -c_0 < 0$) along the unstable (resp. stable) direction $N^*(W^u(a))$ (resp. $N^*(W^s(a))$). In particular, if we choose $m_0(x,\xi) \ll 0$ along the unstable direction and $m_0(x,\xi) \gg 0$ along the stable one, then inequality (12) will be proved in this region of phase space. To summarize, it is sufficient to construct a function $m_0(x,\xi)$ in $S^0(T^*M)$ meeting the following requirements :

- $X_{H_{V_f}}(m_0) \leq 0$,
- near the critical points, $m_0(x,\xi) \ll 0$ along the unstable direction and $m_0(x,\xi) \gg 0$ along the stable one,
- still near critical points but away from the stable and unstable directions, $X_{H_{V_f}}(m_0) \leq -c_1 < 0$.

If we are able to gather all these ingredients, then we will be able to apply the strategy of Faure and Sjöstrand described above. This is at this precise stage of the proof that we need to understand the topological and dynamical properties of the unstable manifolds. In particular, we prove the following Theorem which is almost sufficient to make the proof works [8, 10]:

Theorem 4.1. *Let φ_f^t be a $\mathcal{C}^1$-linearizable Morse-Smale gradient flow. Then,*

(1) *Then the set*

$$\Sigma := \left(\bigcup_{a \in \text{Crit}(f)} N^*(W^u(a)) \right) \cap S^*M$$

is compact. Equivalently, the union of Lagrangians $\bigcup_{a\in\text{Crit}(f)} N^*(W^u(a))$ is a closed, conical subset in T^*M.

(2) *For every $\epsilon > 0$, there exists an ϵ-neighborhood O of Σ in S^*M such that, for every $t \geq 0$,*

$$\tilde{\Phi}^t_{V_f}(O) \subset O,$$

where $\tilde{\Phi}^t_{V_f}$ is the flow induced by the Hamiltonian H_{V_f} on S^*M.

The proof of this result is a microlocal extension of the seminal works of Smale [28]. This Theorem allows us to avoid the delicate construction of Laudenbach in [2] – see also [19]. This is at the expense of having a much less precise information on the differentiable structure of $\overline{W^u(a)}$.

4.3. **Construction of the generalized eigenmodes.** Suppose that we have proved that the spectrum of the operator $-\mathcal{L}_{V_f}$ (acting on $\mathcal{H}^m(M)$) is discrete with finite multiplicity in the region $\text{Re}(z) > -\chi$. For every z_0 in that complex half-plane, if z_0 is an eigenvalue then we can define a spectral projector :

$$\Pi_{z_0} := \frac{1}{2i\pi} \int_{\Gamma_{z_0}} \frac{dz}{\left(z + \mathcal{L}_{V_f}\right)},$$

where Γ_{z_0} is a small circle surrounding a disk containing only z_0 as eigenvalue in its interior. In case where z_0 is not an eigenvalue, this defines the zero operator. For any critical point a

and any multiindex $\alpha \in \mathbb{N}^n$, we can extend the distribution $u_{\alpha,a}$ defined by (10) as follows. We fix a small cutoff function $\theta_{\alpha,a}$ near a and we set

$$\boxed{\mathbf{u}_{\alpha,a} := \Pi_{z_0}(\theta_{\alpha,a} u_{\alpha,a}),}$$

where $z_0 := -\left(\alpha.|\chi(a)| + \sum_{j=1}^r \chi_j(a)\right)$. Using the spectral projector replaces the Epstein–Glaser distributional extension argument used in [17]. Using [16, Th. 1.5], we can verify that this extension is independent of the choice of order function m used to define the anisotropic Sobolev space $\mathcal{H}^m(M)$. Moreover, we can show that $\mathbf{u}_{\alpha,a}$ coincides with $u_{\alpha,a}$ near the critical point and that the family $(\mathbf{u}_{\alpha,a})_{\alpha,a}$ of distributions constructed in that manner are linearly independent. They verify

$$-\mathcal{L}_{V_f}\mathbf{u}_{\alpha,a} = -\left(\alpha.|\chi(a)| + \sum_{j=1}^r \chi_j(a)\right)\mathbf{u}_{\alpha,a} \text{ on } M - \partial W^u(a)$$

but they only verify *a priori* that

$$\left(\mathcal{L}_{V_f} - \left(\alpha.|\chi(a)| + \sum_{j=1}^r \chi_j(a)\right)\right)^N \mathbf{u}_{\alpha,a} = 0 \text{ on } M,$$

for a large enough N depending on $a \in \text{Crit}(f)$ and on $\alpha \in \mathbb{N}^n$. Hence, via this spectral procedure, we have extended the germs of invariant distributions and the same analysis of course holds in any degree k. This spectral definition allows to bypass the analysis made in [2, 22] but, again, the difficulty has been displaced in the construction of a proper spectral framework, and our construction gives a rather imprecise statement on the regularity of these extensions. The microlocal character of the construction allows us to show that the wave front set $WF(\mathbf{u}_{\alpha,a})$ of the eigencurrent $\mathbf{u}_{\alpha,a}$ is contained in the union of Lagrangians $\bigcup_{a\in\text{Crit}(f)} N^*(W^u(a))$ [9, subsection 7.1].

4.4. **Conclusion.** It now remains to prove that these distributions allow to write down the full asymptotic expansion of the correlation function. This can be achieved by verifying that they generate all the generalized eigenmodes of the operator $-\mathcal{L}_{V_f}$. Indeed, recall that $\varphi_f^{-t*}(\psi_1)$ is solution to the transport equation (2), i.e. φ_f^{-t*} is formally equal to $e^{-t\mathcal{L}_{V_f}}$. To prove this generation result, we fix a generalized eigenmode u_0 and $p \geq 1$ minimal such that

$$\left(\mathcal{L}_{V_f} - z_0\right)^p u_0 = 0,$$

for a certain z_0 verifying $\text{Im}(z) > -\chi$. We associate to this current u_0 the family

$$u_0,\ u_1 := \left(\frac{1}{i}\mathcal{L}_{V_f} - z_0\right)u_0, \ldots,\ u_{p-1} := \left(\frac{1}{i}\mathcal{L}_{V_f} - z_0\right)^{p-1} u_0,$$

and we conclude by showing that each of the u_i can be expressed as a linear combination of the $\mathbf{u}_\alpha$. Without getting into the details, let us point out the main ingredients: (1) the gradient dynamics, (2) a theorem due to Schwartz on distributions carried by submanifolds [27, p. 102] and (3) the fact that the microsingularities of u_j are contained in the conormals of the unstable manifolds combined with a result from [7]. We omit this step and we refer the reader to [8, 11] for more details, especially regarding the possibility of having Jordan blocks. For instance, we show that, in every degree k,

$$C^k(V_f) := \mathrm{Ker}\left(\mathcal{L}_{V_f}^{(k)}\right) = \mathrm{Ker}\left(\mathcal{L}_{V_f}^{(k)}\right)^2. \tag{14}$$

5. Application to topology and to the Witten Laplacian

The works of Thom [29] and Smale [28] have shown that studying the dynamical properties of gradient flows has strong relations with topology. We would like to conclude this note by describing this problem via our spectral approach. First of all, note that, as d commutes with $\mathcal{L}_{V_f}$, we can define a natural complex :

$$0 \xrightarrow{d} C^0(V_f) \xrightarrow{d} C^1(V_f) \xrightarrow{d} \dots \xrightarrow{d} C^n(V_f) \xrightarrow{d} 0.$$

Our proof in [8] shows that the spaces $C^k(V_f)$ have dimension equal to the number $c_k(f)$ of critical points of index k i.e. the critical points whose stable manifold has dimension k. Recall that the De Rham complex is defined as follows:

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Introduce now the spectral projector associated to the eigenvalue 0:

$$\Pi_0^{(k)} := \frac{1}{2i\pi}\int_{\Gamma_0} \frac{dz}{\left(z + \mathcal{L}_{V_f}^{(k)}\right)}.$$

This finite rank operator is given by our spectral analysis in every degree and it induces a linear map from $\Omega^k(M)$ to $C^k(V_f)$. Omitting a few technical details, we will verify that this operator induces a chain homotopy equation between the two complexes. In fact, for every ψ in $\Omega^\bullet(M)$, one has

$$\begin{aligned}
\psi &= \Pi_0(\psi) + (\mathrm{Id} - \Pi_0)(\psi) \\
&= \Pi_0(\psi) + \left(d \circ \iota_{V_f} + \iota_{V_f} \circ d\right) \circ \mathcal{L}_{V_f}^{-1}(\mathrm{Id} - \Pi_0)(\psi) \\
&= \Pi_0(\psi) + d \circ \iota_{V_f} \circ \mathcal{L}_{V_f}^{-1}(\mathrm{Id} - \Pi_0)(\psi) + \iota_{V_f} \circ \mathcal{L}_{V_f}^{-1}(\mathrm{Id} - \Pi_0)\, d(\psi).
\end{aligned}$$

If we set $R_f := \iota_{V_f} \circ \mathcal{L}_{V_f}^{-1} \circ (\mathrm{Id} - \Pi_0)$, then we find the expected chain homotopy equation :

$$\boxed{\psi = \Pi_0(\psi) + dR_f(\psi) + R_f d(\psi).}$$

It is then classical, by making use of the elliptic properties of d, to deduce from the chain homotopy equation that *the two complexes* $(C^\bullet(V_f), d)$ *and* $(\Omega^\bullet(M), d)$ *are quasi-isomorphic* [8]. This above argument is rather robust and we showed how to apply it to more general flows such as Morse-Smale flows (not necessarily of gradient type) and Anosov flows [12]. A direct consequence of this observation is that we can write down Morse inequalities for such flows using only linear algebra [22]. For certain nonsingular Morse-Smale flows [12], we also showed that this correlation spectrum carries more topological contents such as the Reidemeister torsion [18]. Finally, still regarding applications to topology, we can mention the recent results of Dyatlov and Zworski in the case of contact Anosov flows in dimension 3 [15]. They expressed the dimension of $C^k(V)\cap\mathrm{Ker}(\iota_V)$ in terms of the Betti numbers of the underlying manifold. For the sake of comparison, note that one has $C^k(V_f)\cap\mathrm{Ker}(\iota_{V_f})=C^k(V_f)$ for gradient flows [8].

The cohomological complex $(C^\bullet(V_f),d)$ is known in the literature as the Thom-Smale-Witten complex or simply the Morse complex. It is often defined in algebraic terms following the works of Witten [33]. Our analysis shows that this complex can be realized in terms of currents carried by unstable manifolds as was already observed in [2, 20]. It also gives a spectral interpretation of the Morse complex, and it can be viewed as a kind of semiclassical limit of the twisted De Rham complex introduced by Witten [33] and Helffer-Sjöstrand [21]. Recall that Witten introduced the following semiclassical deformation of the coboundary operator:

$$d_{f,\hbar}:=e^{-\frac{f}{\hbar}}de^{\frac{f}{\hbar}}=d+\frac{df}{\hbar}\wedge:\Omega^\bullet(M)\to\Omega^{\bullet+1}(M).$$

To this operator, he associated an elliptic operator which is now referred as the Witten Laplacian

$$\boxed{W_{f,\hbar}=\frac{\hbar}{2}\left(d_{f,\hbar}d_{f,\hbar}^*+d_{f,\hbar}^*d_{f,\hbar}\right)=\frac{\hbar}{2}\left(d_{f,\hbar}+d_{f,\hbar}^*\right)^2,}$$

where $d_{f,\hbar}^*=d^*+\iota_{V_f}/\hbar$ is the adjoint of $d_{f,\hbar}$ with respect to the Riemannian metric g. In order to extract topological informations from this operator, one has to look at the small eigenvalues (and their eigenmodes) of this operator and to prove that the dimension of the corresponding eigenspaces is given by the number of critical points in every degree [21]. This can be achieved via semiclassical techniques developped for the study of Schrödinger operators.

In order to relate this to our approach, we can make the following classical observation :

$$e^{\frac{f}{\hbar}}W_{f,\hbar}e^{-\frac{f}{\hbar}}=\frac{\hbar}{2}\left(d+d_{2f,\hbar}^*\right)^2=\frac{\hbar\Delta_g}{2}+\mathcal{L}_{V_f}, \tag{15}$$

where Δ_g is the Laplace Beltrami operator. In other terms, up to conjugation, the Witten Laplacian is a stochastic perturbation of the operator $\mathcal{L}_{V_f}$ whose spectrum has just been described. This remark is at the heart of the construction from [17] who computed the spectrum of the Witten Laplacian for the height function on the 2-sphere and who showed how to take the limit $\hbar\to0^+$. In the case of Anosov vector fields, it was proved by Dyatlov and Zworski that the correlation spectrum is stable under this kind of stochastic perturbations [13]. In [9], we show that this remains true for general Morse-Smale gradient flows, meaning that the spectrum (eigenvalues and spectral projectors) of the Witten Laplacian converges to the correlation spectrum of the gradient flow. As an illustration of our results, let us mention the following:

Theorem 5.1 (Semiclassical versus dynamical convergence)**.** *Let φ_f^t be a $\mathcal{C}^1$-linearizable Morse-Smale gradient flow.*

Then, there exists $\epsilon_0 > 0$ small enough such that, for every $0 \leq k \leq n$, for every $0 < \epsilon \leq \epsilon_0$ and for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$,

$$\lim_{\hbar \to 0^+} \int_M \mathbf{1}_{[0,\epsilon]} \left(W_{f,\hbar}^{(k)} \right) \left(e^{-\frac{f}{\hbar}} \psi_1 \right) \wedge \left(e^{\frac{f}{\hbar}} \psi_2 \right) = \lim_{t \to +\infty} \int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2,$$

where $\mathbf{1}_{[0,\epsilon]} \left(W_{f,\hbar}^{(k)} \right)$ is the spectral projector on $[0, \varepsilon]$ for the self–adjoint elliptic operator $W_{f,\hbar}^{(k)}$.

Recall that the limit of the right-hand side was given by Theorem 2.1. In that sense, this Theorem illustrates the relation between Laudenbach-Harvey-Lawson approach to Morse theory via currents and the Witten-Helffer-Sjöstrand one via semiclassical analysis. This was made possible by providing a convenient spectral framework for the operator $-\mathcal{L}_{V_f}$. As far as we know, such a result cannot be obtained (at least directly) from the methods in [21]. Thus, even if this spectral approach does not allow to recover the full strength of the Helffer-Sjöstrand analysis (e.g. exponential decay of the small Witten eigenvalues), it still provides new properties related to the asymptotics of the Witten Laplacian. We believe that this may have other applications and we showed for instance how to use this point of view to give a new proof of a conjecture due to Fukaya on Witten's deformation of the wedge product – see [9] for details.

6. Analogy with renormalization in quantum field theory.

At the heart of our argument was the extension of certain germs of currents via spectral technics. Let us show, by some simple example, how one can construct general eigenstates by regularization of divergent integrals rather than by spectral techniques. Note that this kind of approach would require to have a nice enough description of $\partial W^u(a)$ for every critical point a which may be a subtle issue related to the works of Laudenbach [2]. Still, in the case where $\partial W^u(a)$ is a point, we can describe what it would give. This method is similar to Epstein–Glaser renormalization in quantum field theory.

Let us consider the canonical sphere $(\mathbb{S}^n, g_{\text{Can}})$ and we let f be the usual height function whose critical points are the south pole S and the north pole N. We are given two charts $\phi : \mathbb{S}^n \mapsto \mathbb{R}^n$ near the south pole and $\tilde{\phi} : \mathbb{S}^n \mapsto \mathbb{R}^n$ near the north pole. In stereographic chart $(x_1, \ldots, x_n)$ near the south pole, the gradient vector field reads $\phi_* V_f = \sum_{i=1}^n x_i \partial_{x_i}$. Let us show how to construct a global eigenmode for the eigenvalue $k \in \mathbb{N}$. In local coordinates $(x_1, \ldots, x_n)$ near the south pole S, a natural candidate for eigenfunction is the polynomial germ

$$u \circ \phi^{-1} = \prod_{i=1}^n x_i^{\alpha_i}$$

where $\alpha \in \mathbb{N}^n$ is a multiindex which satisfies $\sum_{i=1}^n \alpha_i = k$ and $\mathcal{L}_{V_f} u = ku$ near S. Since the chart $(x_1, \ldots, x_n)$ covers $\mathbb{S}^2 \setminus N$, the germ u extends as a smooth function u on $\mathbb{S}^2 \setminus N$ which solves $\mathcal{L}_{V_f} u = ku$.

Then in stereographic chart near the north pole N, $\tilde{\phi}_* V_f = -\sum_{i=1}^n x_i \partial_{x_i}$ and

$$\tilde{u} = u \circ \tilde{\phi}^{-1} = \frac{\prod_{i=1}^n x_i^{\alpha_i}}{\left(\sum_{i=1}^n x_i^2\right)^k}$$

since the transition map between charts reads $x_i \mapsto \frac{x_i}{x_1^2 + \cdots + x_n^2}$ for all $i \in \{1, \ldots, n\}$. Therefore $u \in C^\infty(\mathbb{S}^2 \setminus N)$ and has a singularity at N which can be measured by scaling. Consider the action of the following simple dynamical system :

$$\varphi^t : (x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto (e^t x_1, \ldots, e^t x_n) \in \mathbb{R}^n$$

which acts on $\tilde{u}$ by pull–back: $\varphi^{-t*}\tilde{u}$. Then one can show that $\tilde{u} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is *homogeneous of order $-k$ under scaling* hence weakly homogeneous of the same order in the sense of [7, definition 3.4 p. 828].

In [7, Theorem 5.1 p. 844], it is proved that if a distribution $\tilde{u}$ defined on some manifold M minus some submanifold X (here $X = \{N\}$) is weakly homogeneous of degree $-k$ such that $-k + \operatorname{codim}(X) > 0$ then there is a unique extension $\overline{u}$ which is an eigenmode $\mathcal{L}_{V_f}\overline{u} = k\overline{u}$.

Otherwise, if $-k + \operatorname{codim}(X) \leqslant 0$, the extension involves a renormalization as follows. For all test form $\psi(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge dx_n \in \Omega_c^n(\mathbb{R}^n)$:

$$\langle \overline{u}, \psi \rangle = \lim_{\varepsilon \to 0^+} \left(\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \tilde{u}\psi d^n x - \sum_{|\beta| \leqslant k-n} \left(\int_{\varepsilon \leqslant |x| \leqslant 1} \tilde{u} \frac{x^\beta}{\beta!} d^n x \right) \left(\partial_x^\beta \psi \right)(0) \right)$$

where $\overline{u} \in \mathcal{D}'(\mathbb{R}^n)$ defines a distributional extension of $\tilde{u} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. This can be written in purely current theoretic terms as :

$$\boxed{\overline{u} = \lim_{\varepsilon \to 0^+} \left(\tilde{u} 1_{\{|x| \geqslant \varepsilon\}} - \sum_{|\beta| \leqslant k-n} c_\beta(\varepsilon) \partial_x^\beta \delta \right)}$$

where $1_{\{|x| \geqslant \varepsilon\}}$ is the indicator function of $\{|x| \geqslant \varepsilon\}$, $c_\beta(\varepsilon) = (-1)^{|\beta|} \int_{\varepsilon \leqslant |x| \leqslant 1} u \frac{x^\beta}{\beta!} d^n x$ and $c_\beta(\varepsilon)\partial_x^\beta \delta$ is a local counterterm supported at $0 \in \mathbb{R}^n$ which can be singular when $\varepsilon \to 0$. This renormalization is analogous to Epstein–Glaser renormalization used in quantum field theory [4].

Once we have extended the function $u \in C^\infty(\mathbb{S}^n \setminus N)$ to a distribution $\overline{u}$ on $\mathbb{S}^n$, one may wonder if the extension $\overline{u}$ still satisfies the eigenvalue equation $\mathcal{L}_{V_f}\overline{u} = k\overline{u}$. In other words, *does renormalization preserve symmetries* ? The extension $\overline{u}$ satisfies the following residue formula [6, Thm 8.3.7 p. 182] :

$$\boxed{\mathcal{L}_{V_f}\overline{u} - k\overline{u} = \sum_{|\beta| \leqslant k-n} \left(\int_{\partial B(N,1)} u\omega \frac{x^\beta}{\beta!} \right) \partial_x^\beta \delta}$$

where the integral is over the $(n-1)$ sphere around N and $\omega = \sum_{i=1}^n (-1)^i x^i dx^1 \wedge \hat{dx^i} \wedge dx^n$. So one could try to subtract from $\overline{u}$ some distributions supported at N, if these subtractions fail to turn $\overline{u}$ into an eigenfunction of $\mathcal{L}_{V_f}$ with eigenvalue k, then we are in a situation where the generalized eigenspaces of the vector field $\mathcal{L}_{V_f}$ have Jordan blocks which is called *logarithmic mixing* in [17].

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