TOPOLOGY OF POLLICOTT-RUELLE RESONANT STATES

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Abstract. We prove that the twisted De Rham cohomology of a flat vector bundle over some smooth manifold is isomorphic to the cohomology of invariant Pollicott–Ruelle resonant states associated with Anosov and Morse–Smale flows. As a consequence, we obtain generalized Morse inequalities for such flows. In the case of Morse–Smale flows, we obtain a trace formula for the Pollicott-Ruelle resonances lying on the imaginary axis which involves the twisted Fuller measures used by Fried in his work on Reidemeister torsion. In particular, when V is a nonsingular Morse-Smale flow, we show that the Reidemeister torsion can be recovered from the resonances lying on the imaginary axis.

1. Introduction

Consider M a smooth ($\mathcal{C}^\infty$) closed and oriented manifold which is of dimension $n \geq 1$. We say that $f \in \mathcal{C}^\infty(M, \mathbb{R})$ is a Morse function if it has finitely many critical points all of them being nondegenerate. Denote by $c_k(f)$ the number of critical points of index k and by $b_k(M)$ the k-th Betti number of the manifold. A famous result of Morse states that the following inequalities hold [38]:

$$\forall 0 \leq k \leq n, \quad \sum_{j=0}^{k} (-1)^{k-j} b_j(M) \leq \sum_{j=0}^{k} (-1)^{k-j} c_j(f),$$

with equality when $k = n$. These inequalities are usually called Morse inequalities. Later on, Thom [54] and Smale [52] observed that, if we are given a Riemannian metric on M, then we can define a cell decomposition of the manifold by considering the stable manifolds associated with each critical point by the induced gradient flow. Then, $c_j(f)$ is thought as the number of stable manifolds of dimension j and Smale gave a new proof of these inequalities [52] which had in some sense a more dynamical flavour. This dynamical approach to Morse inequalities was then pursued by many people and generalized to much more general dynamical systems. Among others, we refer the reader to the books of Conley [12] and Franks [26] for examples of such generalizations. This dynamical approach to differential topology also leads to dynamical interpretation of other topological invariants such as the Reidemeister torsion by Fried [27]. Inspired by the theory of currents, but still from a dynamical perspective, we can also mention the works of Laudenbach [35] and Harvey–Lawson [31] who showed how to realize the cell decomposition of Thom and Smale in terms of De Rham currents.

More than fifty years after Morse's seminal work, Witten introduced in the context of Hodge theory another approach to Morse inequalities [58]. This point of view was further developped by Helffer and Sjöstrand using tools from semiclassical analysis [32] and brought a new spectral perspective on these results. In particular, the coefficients $c_j(f)$ are interpreted by Witten as the number of small eigenvalues (counted with multiplicity) of a certain deformation of the Hodge Laplacian acting on forms of degree j. This also lead to many developments in analysis and topology that would be again hard to describe in details. We can for instance quote the works of Bismut-Zhang [9] and Burghelea-Friedlander-Kappeler [10] who developped the relationship between Witten's approach and Reidemeister torsion.

If we come back to the dynamical perspective, a spectral approach slightly different from the ones arising from Hodge-Witten theory was recently developped by several authors to study dynamical systems with hyperbolic behaviour. We shall describe these results more precisely below. In particular, this theory applies to the dynamical systems studied by Smale [53]: Anosov flows [11, 22, 18, 24], Axiom A flows [17] and Morse-Smale flows [13, 14]. These different works were initially motivated by the study of the correlation function in dynamical systems and by the analytic properties of the so-called Ruelle zeta function [48]. In the present work, we aim at showing how these results can be also used to obtain classical results from differential topology in a *self-contained* manner which is somewhat intermediate between the spectral approach of Witten and the more dynamical one of Thom and Smale. As a by-product, it will also unveil the topological informations contained in the correlation spectrum (also called Pollicott-Ruelle spectrum), resonances on the imaginary axis in particular, studied in the above works.

2. Statement of the main results

2.1. Dynamical framework. Let M be a smooth, compact, oriented, boundaryless manifold of dimension $n\geq 1$ and let $\mathbf{p}:\mathcal{E}\rightarrow M$ be a smooth complex vector bundle of rank N. Suppose that $\mathcal{E}$ is endowed with a flat connection $\nabla:\Omega^0(M,\mathcal{E})\rightarrow\Omega^1(M,\mathcal{E})$ [36, Ch. 12]. This allows us to define an exterior derivative $d^\nabla:\Omega^\bullet(M,\mathcal{E})\rightarrow\Omega^{\bullet+1}(M,\mathcal{E})$ which satisfies $d^\nabla\circ d^\nabla=0$. In particular, one can introduce the twisted De Rham complex $(\Omega^*(M,\mathcal{E}),d^\nabla)$ associated with d^∇:

$$0\xrightarrow{d^\nabla}\Omega^0(M,\mathcal{E})\xrightarrow{d^\nabla}\Omega^1(M,\mathcal{E})\xrightarrow{d^\nabla}\ldots\xrightarrow{d^\nabla}\Omega^n(M,\mathcal{E})\xrightarrow{d^\nabla}0.$$

The kth-cohomology group of this complex is denoted by $\mathbf{H}^k(M,\mathcal{E})$. This kind of complex appears for instance naturally in Hodge theory [6, 37].

Fix now V a smooth vector field on M and denote by φ^t the induced flow on M. Using the flat connection ∇, one can lift this flow into a flow Φ_k^t on the vector bundle $\mathbf{p}_k:\Lambda^k(T^*M)\otimes\mathcal{E}\mapsto M$ for every $0\leq k\leq n$. For every $t\in\mathbb{R}$, one has $\mathbf{p}_k\circ\Phi_k^t=\varphi^t\circ\mathbf{p}_k$. If we define the corresponding Lie derivative on $\mathcal{E}$

$$\mathcal{L}_{V,\nabla}^{(k)}=\left(d^\nabla+\iota_V\right)^2:\Omega^k(M,\mathcal{E})\rightarrow\Omega^k(M,\mathcal{E}),\tag{1}$$

where ι_V is the contraction by the vector field V, then, for any section ψ_0 in $\Omega^k(M,\mathcal{E})$ and for any $t\geq 0$, $\Phi_k^{-t*}(\psi_0)$ is the solution of the following partial differential equation:

$$\partial_t\psi=-\mathcal{L}_{V,\nabla}^{(k)}\psi,\qquad \psi(t=0)=\psi_0\in\Omega^k(M,\mathcal{E}).\tag{2}$$

As a tool to describe the long time behaviour of this equation, it is natural to form the following function

$$\forall(\psi_1,\psi_2)\in\Omega^{n-k}(M,\mathcal{E}')\times\Omega^k(M,\mathcal{E}),\ C_{\psi_1,\psi_2}(t):=\int_M\psi_1\wedge\Phi_k^{-t*}(\psi_2),\tag{3}$$

which we will call the *correlation function.*

2.2. **Pollicott-Ruelle resonances.** Set $\mathcal{E}'$ to be the dual bundle to $\mathcal{E}$. Following the works of Pollicott [45] and Ruelle [49], it is sometimes simpler to consider first the Laplace transform of this quantity,

$$\forall(\psi_1,\psi_2)\in\Omega^{n-k}(M,\mathcal{E}')\times\Omega^k(M,\mathcal{E}),\ \hat{C}_{\psi_1,\psi_2}(z):=\int_0^{+\infty}e^{-zt}\left(\int_M\psi_1\wedge\Phi_k^{-t*}(\psi_2)\right)dt,\tag{4}$$

which is well defined for $\operatorname{Re}(z)>0$ large enough. In the above references, Pollicott and Ruelle showed that, for Axiom A vector fields [53] and for ψ_1 and ψ_2 compactly supported near the basic sets of the flow, this function admits a meromorphic extension to some half plane $\operatorname{Re}(z)>-\delta$ with $\delta>0$ small enough. The poles and residues of this meromorphic continuation describe in some sense the fine structure of the flow $\Phi_k^{-t*}=e^{-t\mathcal{L}_{V,\nabla}^{(k)}}$ as $t\to+\infty$. More recently, it was proved that, for Anosov vector fields, the Laplace transformed correlators have meromorphic extensions to the entire complex plane without any restrictions on the supports of ψ_1 and ψ_2 [11]. A different proof based on microlocal techniques was given in [22] – see also [18, 17] for related results[1]. In [13, 14], we proved this meromorphic extension in the case of Morse-Smale vector fields which are $\mathcal{C}^1$-linearizable – see appendix A for the precise definition.

Fix now $0\leq k\leq n$. To summarize, these recent developments showed that, for vector fields V which are either Anosov or ($\mathcal{C}^1$-linearizable) Morse-Smale, there exists a minimal discrete[2] set $\mathcal{R}_k(V,\nabla)\subset\mathbb{C}$ such that, given any $(\psi_1,\psi_2)\in\Omega^{n-k}(M,\mathcal{E}')\times\Omega^k(M,\mathcal{E})$, the map $z\mapsto\hat{C}_{\psi_1,\psi_2}(z)$ has a meromorphic extension whose poles are contained inside $\mathcal{R}_k(V,\nabla)$. These poles are called the *Pollicott-Ruelle resonances.* Moreover, given any such $z_0\in\mathcal{R}_k(V,\nabla)$, there exists an integer $m_k(z_0)$ and a linear map of *finite rank*

$$\pi_{z_0}^{(k)}:\Omega^k(M,\mathcal{E})\to\mathcal{D}'^k(M,\mathcal{E})\tag{5}$$

such that, given any $(\psi_1,\psi_2)\in\Omega^{n-k}(M,\mathcal{E}')\times\Omega^k(M,\mathcal{E})$, one has, in a small neighborhood of z_0,

$$\hat{C}_{\psi_1,\psi_2}(z)=\sum_{l=1}^{m_k(z_0)}(-1)^{l-1}\frac{\left\langle(\mathcal{L}_{V,\nabla}^{(k)}+z_0)^{l-1}\pi_{z_0}^{(k)}(\psi_2),\psi_1\right\rangle}{(z-z_0)^l}+R_{\psi_1,\psi_2}(z),$$

where $R_{\psi_1,\psi_2}(z)$ is holomorphic. Here, we use the convention that $\mathcal{D}'^k(M,\mathcal{E})$ represents the currents of degree k with values in $\mathcal{E}$. Elements inside the range of $\pi_{z_0}^{(k)}$ are called *Pollicott-Ruelle resonant states* and, as we shall explain it below, they can be interpreted as the generalized eigenvectors of the operator $-\mathcal{L}_{V,\nabla}^{(k)}$ acting on some appropriate Sobolev

[1]We refer to section 3 for a more detailed account.

[2]We mean that it has no accumulation point. In particular, it is at most countable.

space. For any resonance z_0, we denote by $C^k_{V,\nabla}(z_0)$ the range of the operator $\pi^{(k)}_{z_0}$ which is a finite dimensional space.

2.3. Realization of De Rham cohomology via Pollicott-Ruelle resonant states.

From the spectral interpretation of $C^\bullet_{V,\nabla}(z_0)$ – see section 3 for details, one can deduce that, for any[3] $z_0 \in \mathbb{C}$,

$$0 \xrightarrow{d^\nabla} C^0_{V,\nabla}(z_0) \xrightarrow{d^\nabla} C^1_{V,\nabla}(z_0) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} C^n_{V,\nabla}(z_0) \xrightarrow{d^\nabla} 0 \tag{6}$$

defines a cohomological complex. We will denote by $\mathbf{H}^k(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ the cohomology of that complex. Our first main result states that the cohomology of that complex is isomorphic to the one of the twisted De Rham complex when $z_0 = 0$:

Theorem 2.1. *Let $\mathcal{E} \to M$ be a smooth complex vector bundle of dimension N which is endowed with a flat connection ∇. Suppose that V is a vector field which is either Morse-Smale and $\mathcal{C}^1$-linearizable or Anosov. Then, for every $0 \leq k \leq n$, the maps*

$$\pi^{(k)}_0 : \Omega^k(M, \mathcal{E}) \to C^k_{V,\nabla}(0)$$

induce isomorphisms between $\mathbf{H}^k(M, \mathcal{E})$ and $\mathbf{H}^k(C^\bullet_{V,\nabla}(0), d^\nabla)$.

In the statement of our Theorem, the assumption that the vector field is either Anosov or Morse-Smale comes from the fact that we have a good spectral theory for such systems thanks to [11, 22, 18, 14]. Yet, as we shall see, our proof is rather robust and this result should hold for any type of vector fields which has a good spectral behaviour yielding in particular an isolated eigenvalue at $z = 0$ of finite multiplicity. Thanks to a classical result of Peixoto [44] and to the Sternberg-Chen Theorem [40, 56], the assumption of being a $\mathcal{C}^1$-linearizable Morse-Smale vector field is in fact generic when $\dim(M) = 2$ – see appendix A. In the case of Morse-Smale flows [15], we described explicitely a family of twisted currents generating the Pollicott-Ruelle resonant states and whose support are given by the stable manifolds. In a work related to the Anosov case [19], Dyatlov and Zworski showed how to identify the Pollicott-Ruelle resonant states $u \in \mathcal{D}'^k(M, \mathbb{C})$ satisfying $\iota_V(u) = 0$ with the De Rham cohomology of X when V is the geodesic vector field on $M := SX$ (with X negatively curved and $\dim(X) = 2$).

In the case of Morse-Smale *gradient* flows, this theorem was already proved in [13] under the assumption that $\mathcal{E}$ is the trivial bundle $M \times \mathbb{C}$ and it provided a new (spectral) proof of earlier results due to Laudenbach [35] and Harvey-Lawson [30, 31]. In a subsequent work [16], we also showed how this spectral approach could be interpreted as the semi-classical limit of Witten's analytical approach to Morse theory [58, 32]. For other flows with closed orbits, this result seems to be new and it gives a **spectral realization of the twisted De Rham cohomology as the cohomology of Pollicott-Ruelle resonant states**.

[3] Whenever z_0 is not a resonance, we use the convention $C^k_{V,\nabla}(z_0) = \{0\}$.

We were not able to locate a place in the literature where such an isomorphism is obtained for general Morse-Smale or Anosov vector fields via topological methods *à la* Smale or via analytical methods *à la* Hodge-Witten. In that latter context, recall that the strategy is to perform an elliptic (or hypoelliptic) deformation of the Lie derivative in order to interpolate it with a Laplace Beltrami operator Δ_g and to derive topological properties from this deformation. For gradient flows, the interpolation is achieved via the Witten Laplacian [58, 32], while, for geodesic flows, this role is played by the hypoelliptic Laplacian introduced by Bismut [7, 8]. For instance, in the case of gradient flows associated with a Morse function f, interpolation is obtained by conjugating the exterior derivative in the definition of Δ_g with a factor $e^{f/\hbar}$ and this allows to prove the isomorphism between de Rham cohomology and Morse one by considering the semiclassical limit $\hbar \to 0^+$ [32]. The main difference with [58, 32, 9, 7, 8] is that we are aiming at studying more general flows which are not necessarly gradient or geodesic flows. In particular, it is not clear how to associate them with a canonical (hypo-)elliptic operator without modifying the flow itself – see Remark 2.10 below. Instead of going through such a deformation, the novelty of our approach is to consider directly the spectrum of the Lie derivative and to derive the topological content from it. In that manner, we encompass much larger families of flows but this is of course at the expense of developping an appropriate spectral framework for the Lie derivative.

2.4. **Generalized Morse inequalities.** A direct consequence of Theorem 2.1 is that we get immediatly Morse inequalities for such hyperbolic flows. One of the advantage compared with the classical approach [52, 26] is that, besides the standard case of Morse-Smale flows, our analytical interpretation also allows to encompass the case of Anosov flows. Let us now formulate this generalization of the Morse-Smale inequalities from Theorem 2.1. In the following, we set $b_j(M,\mathcal{E}) = \dim \mathbf{H}^j(M,\mathcal{E})$. By standard arguments on exact sequences – see e.g. [13, Par. 8.3] for a brief reminder, one can verify that the following holds:

Corollary 2.2 (Spectral Morse inequalities)**.** *Suppose the assumptions of Theorem 2.1 are satisfied. Then, the following holds:*

$$\forall 0 \leq k \leq n,\ \sum_{j=0}^{k}(-1)^{k-j}\dim C^j_{V,\nabla}(0) \geq \sum_{j=0}^{k}(-1)^{k-j} b_j(M,\mathcal{E}),$$

with equality in the case $k = n$.

Recall that, for $k = n$, one recovers the Euler characteristic $\chi(M,\mathcal{E})$. Compared with the case of the Laplace-Beltrami in Hodge theory [6, 37], the dimension of the generalized kernel is only bounded from below by the Betti numbers. In particular, it may happen that this is not equal. As a first illustration of this result, we can note that this corollary gives nontrivial topological constraints on the space of resonant states associated with the eigenvalue $z_0 = 0$. In the Anosov case, the only other constraints we are aware of are the ones of Dyatlov and Zworski in the 3-dimensional contact case [19].

Let us now apply this result in the more general framework of Morse-Smale vector fields. By definition [52], the nonwandering set of a Morse-Smale flow is composed of finitely many critical points and closed orbits, each of them being hyperbolic. Closed orbits Λ can be divided into two categories: untwisted (if the corresponding stable manifold [4] $W^s(\Lambda)$ is orientable) and twisted (otherwise). We define the twisting index Δ_Λ of a closed orbit Λ to be equal to $+1$ in the untwisted case and to -1 in the twisted case. Suppose now for simplicity of exposition that ∇ preserves a Hermitian structure on $\mathcal{E}$ i.e. parallel transport by the flat connection ∇ preserves the fiber metric of $\mathcal{E}$. It implies that, for every closed orbit, the monodromy matrix $M_{\mathcal{E}}(\Lambda)$ for the parallel transport is a unitary matrix. In particular, its spectrum is included in $\mathbb{S}^1$. For every closed orbit Λ, we then define m_Λ to be the multiplicity of Δ_Λ as an eigenvalue of $M_{\mathcal{E}}(\Lambda)$ (this multiplicity can vanish and depends on ∇). Using these conventions, we can state the following :

Corollary 2.3 (Generalized Morse-Smale inequalities)**.** *Let $\mathcal{E} \to M$ be a smooth, complex, hermitian vector bundle of rank N endowed with a flat unitary connection ∇. Suppose that V is a Morse-Smale vector field. Denote by $c_k(V)$ the number of critical points of index*[5] *k. Then, the following holds, for every $0 \leq k \leq n$,*

$$\boxed{\sum_{\Lambda:\dim W^s(\Lambda)=k+1} m_\Lambda + N\sum_{j=0}^{k}(-1)^{k-j}c_k(V) \geq \sum_{j=0}^{k}(-1)^{k-j}b_j(M,\mathcal{E}),}$$

with equality in the case $k = n$.

When V is a gradient flow, we recover the classical Morse inequalities [38]. If $\mathcal{E} = M \times \mathbb{C}$ is the trivial bundle, then $m_\Lambda = 1$ for untwisted orbits and $m_\Lambda = 0$ otherwise. In other words, the sum over the closed orbits is exactly the number of untwisted closed orbits whose stable manifold has dimension $k+1$. Hence, for general Morse-Smale flows, this inequality is slightly stronger than the original result of Smale [52] whose upper bound involved also the number of twisted closed orbits. This stronger version of Morse-Smale inequalities was in fact proved by Franks in [26, Ch. 8] by a completely different approach than ours. Even if they can probably be recovered by using the more topological arguments of Smale [52], Franks [26] and Fried [27], we emphasize that the Morse type inequalities from Corollary 2.3 does not seem to appear in the literature for more general vector bundles.

We will give the proof of this result in paragraph 4.4 and the main additional ingredient compared with Corollary 2.2 is that we make use of the complete description of the Pollicott-Ruelle resonances for Morse-Smale flows that we gave in [15]. The fact that we require our flat connection to preserve a Hermitian structure is not optimal but makes the statement simpler to state. Recall in fact that our description of the fine structure of Pollicott-Ruelle resonances holds under weaker assumptions – see [15, Sec. 4] for related discussion. We emphasize that this assumption is often made in Hodge theory [6, 37] especially for problems related to analytic torsion even if it is not completely necessary

[4]See the appendix of [14] for a brief reminder.

[5]This means that the stable manifold of the point is of dimension k.

there too [39]. Compared to Theorem 2.1 and Corollary 2.2, we do not need to make the linearization assumption in that statement.

2.5. **Koszul homological complexes.** As for the case of the coboundary operator, we can verify from the spectral definition of the spaces $C^{\bullet}_{V,\nabla}(0)$ that the sequence of linear maps :

$$0 \xrightarrow{\iota_V} C^n_{V,\nabla}(0) \xrightarrow{\iota_V} C^{n-1}_{V,\nabla}(0) \xrightarrow{\iota_V} \dots \xrightarrow{\iota_V} C^0_{V,\nabla}(0) \xrightarrow{\iota_V} 0, \tag{7}$$

defines a complex. We shall refer to this complex as the Morse-Smale-Koszul complex and the next Theorem computes its homology in the case of Morse-Smale flows:

Theorem 2.4. *Let $\mathcal{E} \to M$ be a smooth, complex, hermitian vector bundle of rank N endowed with a flat unitary connection ∇. Suppose that V is a vector field which is Morse-Smale and $\mathcal{C}^\infty$-diagonalizable.*
Then, the dimension of the homology of degree k of $(C^{\bullet}_{V,\nabla}, \iota_V)$ is equal to $Nc_k(V)$, where $c_k(V)$ is the number of critical points such that $\dim W^s(\Lambda) = k$.

We will prove this Theorem in section 5 and it generalizes the results of [13, Sect. 8.6] which were only valid for gradient flows and for the trivial bundle $M \times \mathbb{C}$. The assumption of being $\mathcal{C}^\infty$ diagonalizable is defined in appendix A and it holds as soon as certain (generic) nonresonance assumptions are satisfied by the Lyapunov exponents. This hypothesis appeared in our previous work [15] in order to simplify the exposition and it could probably be removed. Yet, this would be at the expense of some extra (probably technical) work that was beyond the scope of [15] and of the present article.

Remark 2.5. The complexes appearing in Theorems 2.1 and 2.4 may sound reminiscent to equivariant cohomology as it appears for instance in [5]. In our case, the (noncompact) group would be $\mathbb{R}$ and the action would be given by the action of the flow induced by $\mathcal{L}_{V,\nabla}$. Following [5], one would then consider smooth forms $\psi \in \Omega^{\bullet}(M, \mathcal{E})$ such that $\mathcal{L}_{V,\nabla}\psi = 0$ and the complex induced by the operator $d^\nabla + \iota_V$. Instead of doing this, we rather consider currents which appear naturally in the correlation function of the flow, i.e. elements of $C^{\bullet}_{V,\nabla}$ which are in general not smooth [15]. Moreover, these currents are not necessarly cancelled by $\mathcal{L}_{V,\nabla}$ but only by a certain power of this operator due to the potential presence of Jordan blocks. In particular, we do not have a priori $(d^\nabla + \iota_V)^2\psi = 0$ for elements in $C^{\bullet}_{V,\nabla}$ which prevents us from defining the corresponding equivariant cohomology as in [5]. Yet, in the case of Morse-Smale gradient flows and of the trivial coboundary operator $d^\nabla = d$, we know from [13] that there is no Jordan blocks in the kernel of the operator. Hence, we can compare the equivariant cohomology on $C^{\bullet}_V$ and on $\Omega^{\bullet}(M)$ and verify that it leads to different things. Using Theorems 2.1 and 2.4, one can verify that the equivariant cohomology on $C^{\bullet}_V$ would be the standard Morse cohomology. On the other hand, if we consider the equivariant cohomology on smooth forms for Morse-Smale gradient flows, then we would find that it is of dimension $b_0(M)$ where $b_0(M)$ is the number of connected components of M.

2.6. **Resonances on the imaginary axis and Reidemeister torsion.** We conclude by relating the Pollicott-Ruelle resonances with the results of Fried in [27, p. 44–53]. Recall that, in this reference, Fried introduced the *twisted Fuller measure* of (V,∇) :

Definition 2.6 (Twisted Fuller measure)**.** In the notations of the previous paragraphs, the *twisted Fuller measure* $\mu_{V,\nabla}$ is a distribution in $\mathcal{D}'(\mathbb{R}_+^*)$ defined by the formula :

$$\begin{aligned}\mu_{V,\nabla}(t) \quad &:= \quad -\frac{N}{t}\sum_{\Lambda\text{ fixed point}}\frac{\det\left(\mathrm{Id}-d_\Lambda\varphi^t\right)}{|\det\left(\mathrm{Id}-d_\Lambda\varphi^t\right)|}\\ &+ \quad \frac{1}{t}\sum_{\Lambda\text{ closed orbit}}\mathcal{P}_\Lambda\sum_{m\geq 1}\frac{\det\left(\mathrm{Id}-P_\Lambda^m\right)}{|\det\left(\mathrm{Id}-P_\Lambda^m\right)|}\mathrm{Tr}\left(M_{\mathcal{E}}(\Lambda)^m\right)\delta(t-m\mathcal{P}_\Lambda),\end{aligned}$$

where P_Λ is a linearized Poincaré map associated with the closed orbit Λ and where $\mathcal{P}_\Lambda$ is the minimal period of Λ.

Recall from the Lefschetz formula that the first term can be expressed in terms of the Euler characteristic of $(M,\mathcal{E})$. In fact, Fried only defined this quantity when there are no critical points and he observed [27, p. 49] that $t\mu_{V,\nabla}(t)$ coincides with the distributional traces of Guillemin and Sternberg [29, Ch. VI] – see also [18] for a brief and self-contained account. We refer to the recent survey of Gouëzel [28, Sect. 2.4] for a brief account on the role of these distributional traces in the study of transfer operators and Ruelle zeta functions [48]. Here, we adopt a slightly more general definition than Fried's one which encompass the case of critical points. Again, our definition coincides with the distributional traces of [29].

Recall that we denote by $C^k_{V,\nabla}(z_0)$ the range of the spectral projector $\pi^{(k)}_{z_0}$ on the eigenspace of eigenvalue z_0 acting on k–forms. Our last Theorem relates these twisted Fuller measures with the *correlation spectrum lying on the imaginary axis* $i\mathbb{R}$:

Theorem 2.7 (Spectral interpretation of Fuller measures)**.** *Let $\mathcal{E}\to M$ be a smooth, complex, hermitian vector bundle of rank N endowed with a flat unitary connection ∇. Suppose that V is a vector field which is Morse-Smale and $\mathcal{C}^\infty$-diagonalizable. Then, one has*

$$\mu_{V,\nabla}(t)=\sum_{k=0}^n(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}\dim\left(C^k_{V,\nabla}(z_0)\cap\mathrm{Ker}(\iota_V)\right)\frac{e^{tz_0}}{t}$$

in the sense of distributions in $\mathbb{R}_+^*$.

This Theorem can be compared with the recent results of Jin and Zworski [34, Th. 1] who proved a local trace formula for the Pollicott-Ruelle resonances of Anosov flows verifying $\mathrm{Re}(z)>-A$ in terms of certain distributional traces. Compared with this reference, we deal with Morse-Smale flows and our Theorem gives a trace formula for the resonances verifying $\mathrm{Re}(z)=0$. We will prove this Theorem in section 6.4.

Let us briefly recall the topological content of these twisted Fuller measures and as a consequence of the Pollicott-Ruelle spectrum lying on the imaginary axis. Besides its first term which can be expressed as the Euler characteristic, Fried observed that the twisted Fuller measure shares a lot of properties with some topological invariants such as the Reidemeister torsion [27, p. 46-49]. In fact, we can observe that the coefficients defining $\mu_{V,\nabla}(t)$ are of purely topological nature even if the support of the measure depends on the parametrization of the flow and hence is not topological. In order to get rid of these nontopological informations, Fried evaluated the "total mass" of these Fuller measures and showed that in certain cases this defines indeed a topological invariant. For that purpose, one can introduce the following zeta function :

$$\zeta^\flat_{V,\nabla}(s,z) := \frac{1}{\Gamma(s)}\int_0^{+\infty} e^{-tz}\mu_{V,\nabla}(t)t^s dt$$

and following Fried [27, p. 51], we define the corresponding **torsion function**

$$Z_{V,\nabla}(z) = \exp\left(-\frac{d}{ds}\zeta^\flat_{V,\nabla}(s,z)|_{s=0}\right)$$

as the exponential of $-\partial_s\zeta^\flat_{V,\nabla}(0,z)$. We emphasize that Fried directly used the Laplace transform to define his twisted zeta function as he only considered nonsingular vector fields. Here, due to the fact that we allow critical points, we have to make this extra regularization involving a Mellin transform. In our slightly more general framework, the terminology "torsion function" may sound a little bit abusive as we do not recover the Reidemeister torsion in general. Computing the derivative of $\zeta^\flat_{V,\nabla}(s,z)$ w.r.t. s at $s=0$, one can verify (see paragraph 6.5) that, for $\mathrm{Re}(z)$ large enough,

$$\partial_s\zeta^\flat_{V,\nabla}(0,z) = \chi(M,\mathcal{E})\ln z + \sum_{\Lambda \text{ closed orbit}} (-1)^{\dim W^u(\Lambda)} \ln\det\left(\mathrm{Id} - e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right).$$

Therefore, the torsion function $Z_{V,\nabla}(z)$ satisfies the identity :

$$\boxed{Z_{V,\nabla}(z) = z^{-\chi(M,\mathcal{E})}\prod_{\Lambda \text{ closed orbit}} \det\left(\mathrm{Id} - e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)^{-(-1)^{\dim W^u(\Lambda)}}}$$

for $\mathrm{Re}(z)$ large enough. Equivalently, the right hand side of the above equality is a weighted Ruelle zeta function [28, equation (13) p. 19]. If the vector field is nonsingular which means the Euler characteristic term vanishes and if $m_\Lambda = 0$ for every closed orbit[6], then $Z_{V,\nabla}$ converges as $z \to 0^+$ and Fried showed that the modulus of the limit is equal to the Reidemeister torsion [27, Sect. 3] – see also [50] for extension of these results.

Following the works of Ray–Singer on analytic torsion [46], we now introduce the following:

[6]In particular, $(\mathcal{E},\nabla)$ is acyclic.

Definition 2.8 (Spectral zeta determinant of the nonzero resonances on the imaginary axis)**.** We define the *spectral zeta function*:

$$\zeta_{RS}(s,z) := \sum_{k=0}^{n}(-1)^{n-k}k \sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}^*} \dim\left(C^k_{V,\nabla}(z_0)\right)(z-z_0)^{-s}$$

and the corresponding spectral zeta determinant[7]

$$\boxed{Z_{RS}(z) := \exp\left(-\frac{d}{ds}\zeta_{RS}(s,z)|_{s=0}\right)}.$$

Then, for $\mathrm{Re}(z)$ large enough, as a consequence of Theorem 2.7 and of standard arguments from Hodge theory (see paragraph 6.5), both the flat zeta function $\zeta^\flat_{V,\nabla}(s,z)$ and the *spectral zeta function* $\zeta_{RS}(s,z)$ admit analytic continuations in $s\in\mathbb{C}$ and they are related by the equation :

$$\boxed{\zeta^\flat_{V,\nabla}(s,z) = -\left(\chi(M,\mathcal{E}) + \sum_{\Lambda \text{ closed orbit}} (-1)^{\dim\, W^u(\Lambda)} m_\Lambda\right) z^{-s} + \zeta_{RS}(s,z).}$$

Hence, as a direct corollary of our Theorem and of Fried's Theorem, we have:

Corollary 2.9. *Suppose that the assumptions of Theorem 2.7 are satisfied. Then, one has:*

$$\boxed{Z_{RS}(z) = \prod_{\Lambda \text{ closed orbit}} \left(z^{-m_\Lambda}\det\left(\mathrm{Id}-e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)\right)^{-(-1)^{\dim\, W^u(\Lambda)}}}.$$

In particular if V is non singular and $m_\Lambda = 0$ for every periodic orbit Λ then $|Z_{RS}(0)|$ coincides with the ***Reidemeister torsion****.*

Informally, the above discussion means that we can recover the Euler characteristic and the Reidemeister torsion (under proper assumptions) from the Pollicott-Ruelle spectrum lying on the imaginary axis. Note that, the assumption in the second part of the corollary exactly means that 0 does not lie in the Pollicott-Ruelle spectrum [15]. These different results illustrate that the Pollicott-Ruelle spectrum on the imaginary axis has a nice topological interpretation in the case of Morse-Smale flows. More specifically, the topological content seems to be contained in the first band of resonances in the terminology of Faure and Tsujii [23, 24]. Recall that, for Anosov contact flows, they showed that the Pollicott-Ruelle spectrum exhibits a band structure. We proved in [15] that the same band structure remains true in the case of Morse-Smale flows and that the first band of resonances is in that case contained inside the imaginary axis. Hence, the main observation of these results is that, at least in certain acyclic cases, **we can keep track of topological invariants inside the first band of the correlation spectrum and not only inside the kernel** which is here reduced to $\{0\}$.

[7]This regularizes the infinite product $\prod_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}^*}(z-z_0)^{(-1)^k k \dim\left(C^k_{V,\nabla}(\lambda)\right)}$ of nonzero resonances on the imaginary axis.

In section 6, we will be even more precise on the topological features of the first band of resonant states for Morse-Smale flows. In fact, for every $z_0 \in i\mathbb{R}^*$, $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ defines an acyclic cohomological complex of finite dimensional spaces – see paragraph 4.2. Thus, once we have specified a preferred basis of that space, we can compute the torsion of that complex [27, 37] which is a certain determinant related to the coboundary operator. In the case of Morse-Smale flows, such a preferred basis of Pollicott-Ruelle resonant states can be naturally introduced following our previous work [15] – see paragraphs 5.1.1 and 5.1.2 for a brief reminder. Then, we will introduce the infinite dimensional space

$$C^\bullet_{V,\nabla}(i\mathbb{R}^*) = \bigoplus_{z_0 \in \mathcal{R}_\bullet(V,\nabla)\cap i\mathbb{R}^*} C^\bullet_{V,\nabla}(z_0),$$

which still defines an acyclic complex. We will define a notion of regularized torsion of this infinite dimensional "complex" by regularizing the infinite product of torsions of every finite dimensional complex of Pollicott-Ruelle resonant states associated to the spectrum on the imaginary axis. In paragraph 6.4, we will see that this regularized torsion is related to the behaviour of the torsion function $Z_{V,\nabla}(z)$ at $z = 0$ and to the spectral zeta function ζ_{RS}. Thus, it is also linked to the Reidemeister torsion whenever V has no critical points and where m_Λ is equal to 0 for every closed orbit.

Remark 2.10. After publication of this preprint, Shen and Yu used the methods of Bismut and Zhang in the case of general Morse-Smale flows and related these flows to certain torsion functions in that manner [51]. Besides the Bismut-Zhang Theorem [9], a key ingredient of their approach is a topological procedure due to Franks [25, 26] which allows to perform surgery operations on a Morse-Smale flow in order to transform it into a gradient flow by keeping track of the topological content at each step of the construction. After these transformations, the flow became amenable to the techniques from [9]. Note also that these surgery operations were already used by Franks to give an alternative proof of the Morse-Smale inequalities based on the classical Morse inequalities for gradient flows [26]. We emphasize that the methods of the present work do not rely on any surgery operations both for the Morse-Smale inequalities and for the results in the last two paragraphs. In particular, they allow to deal directly with the flow under consideration with no need to transform it into another flow in order to unveil its topological content.

2.7. **Conventions.** All along the article, M will denote a smooth, closed and oriented manifold of dimension $n \geq 1$. We shall denote by V a smooth vector field on M and by $\mathcal{E} \to M$ a complex vector bundle of dimension N which is endowed with a flat connection ∇. Except mention of the contrary, $\mathcal{E}$ is not necessarily endowed with an Hermitian structure compatible with ∇.

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3. Spectral theory of Anosov and Morse-Smale vector fields

We begin with a brief account on the existence of Pollicott-Ruelle resonant states following the microlocal approach of Faure and Sjöstrand [22] which was further developped by Dyatlov and Zworski in [18] – see also [13] for extensions of these results to the Morse-Smale case. In the Anosov case, another approach would be to use the earlier results of Butterley and Liverani via spaces of anisotropic Hölder distributions [11]. We could probably also proceed via coherent states like in the works of Faure and Tsujii [55, 24]. We also refer to [4] for a recent account by Baladi concerning the related case of hyperbolic diffeomorphisms.

For Anosov vector fields or $\mathcal{C}^1$-linearizable Morse-Smale vector fields (see appendices A and B for a brief reminder), one can establish the existence of Pollicott-Ruelle resonances. More precisely, one can show that, for every $T_0 > 0$, there exists some $s > 0$ large enough such that[8] for every $0 \leq k \leq n$,

$$-\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^{sm+n-k}(M, \mathcal{E}) \to \mathcal{H}_k^{sm+n-k}(M, \mathcal{E})$$

has a **discrete spectrum** on the half plane $\text{Im}(z) > -T_0$, consisting of eigenvalues of finite algebraic multiplicity. We refer to [22, Th. 1.4] for a more precise statement. This result essentially follows from the fact that Faure and Sjöstrand proved that $(\mathcal{L}_{V,\nabla} + z)$ is a Fredholm depending analytically on z. The eigenvalues are the so-called **Pollicott-Ruelle resonances** and the corresponding generalized eigenmodes are called **resonant states.** These objects correspond exactly to the poles and the residues of the function $\hat{C}_{\psi_1,\psi_2}(z)$ defined in (4).

Remark 3.1. In [22, Sect. 3], the authors only treated the case $k = 0$ and $\mathcal{E} = M \times \mathbb{C}$. Yet, their proof can be adapted almost verbatim to our framework except that we have to deal with pseudodifferential operators with values in $\Lambda^k(T^*M) \otimes \mathcal{E}$ – see [6, Part I.3] [18, App. C] for a brief review. As was already observed in [18], the main point to extend directly Faure-Sjöstrand's proof to more general bundles is that the (pseudodifferential) operators under consideration have a *scalar symbol.* In fact, given any local basis $(e_j)_{j=1,\ldots J_k}$ of $\Lambda^k(T^*M) \otimes \mathcal{E}$ and any family $(u_j)_{j=1,\ldots J_k}$ of smooth functions $\mathcal{C}^\infty(M)$, one has

$$\mathcal{L}_{V,\nabla}^{(k)} \left(\sum_{j=1}^{J_k} u_j e_j \right) = \sum_{j=1}^{J_k} \mathcal{L}_V(u_j) e_j + \sum_{j=1}^{J_k} \mathcal{L}_{V,\nabla}^{(k)}(e_j) u_j,$$

[8] Here, $\mathcal{H}_k^{sm+n-k}(M, \mathcal{E})$ denotes an anisotropic Sobolev space whose construction is briefly recalled in appendix B. In particular, m is an order function belonging to $S^0(T^*M)$.

where the second part of the sum in the right-hand side is a lower order term (of order 0). In other words, the principal symbol of $\mathcal{L}_{V,\nabla}^{(k)}$ is $\xi(V(x))\mathbf{Id}_{\Lambda^k(T^*M)\otimes\mathcal{E}}$. This diagonal form allows to adapt the proofs of [22] to this vector bundle framework.

We conclude this preliminary paragraph with some properties of this spectrum:

- Faure and Sjöstrand implicitely show [22, Lemma 3.3] that, for every z in $\mathbb{C}$ satisfying $\mathrm{Im} z > C_0$ (for some $C_0 > 0$ large enough), one has

$$\left\| \left(\mathcal{L}_{V,\nabla}^{(k)} + z \right)^{-1} \right\|_{\mathcal{H}_k^m(M,\mathcal{E}) \to \mathcal{H}_k^m(M,\mathcal{E})} \leq \frac{1}{\mathrm{Re}(z) - C_0}. \tag{8}$$

  In particular, combining with the Hille-Yosida Theorem [20, Cor. 3.6, p. 76], it implies that

$$e^{-t\mathcal{L}_{V,\nabla}^{(k)}} = (\Phi_k^{-t})^* : \mathcal{H}_k^m(M,\mathcal{E}) \to \mathcal{H}_k^m(M,\mathcal{E}), \tag{9}$$

  generates a strongly continuous semigroup which is defined for every $t \geq 0$ and whose norm is bounded by e^{tC_0}.
- As in [22, Th. 1.5], we can show that the eigenvalues (counted with their algebraic multiplicity) and the eigenspaces of $-\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^{sm+n-k}(M,\mathcal{E}) \to \mathcal{H}_k^{sm+n-k}(M,\mathcal{E})$ are in fact independent of the choice of escape function and of the parameter s.
- By duality, the same spectral properties holds for the dual operator

$$(-\mathcal{L}_{V,\nabla}^{(k)})^\dagger = -\mathcal{L}_{-V,\nabla^\dagger}^{(n-k)} : \mathcal{H}_{n-k}^{-m}(M,\mathcal{E}') \to \mathcal{H}_{n-k}^{-m}(M,\mathcal{E}'), \tag{10}$$

  with $\nabla^\dagger$ the dual connection [14, Sect. 5].
- Given any z_0 in $\mathbb{C}$, the corresponding spectral projector $\pi_\lambda^{(k)}$ is given by [33, Appendix]:

$$\pi_{z_0}^{(k)} := \frac{1}{2i\pi}\int_{\gamma_{z_0}} (z + \mathcal{L}_{V,\nabla}^{(k)})^{-1} dz : \mathcal{H}_k^{sm+n-k}(M,\mathcal{E}) \to \mathcal{H}_k^{sm+n-k}(M,\mathcal{E}), \tag{11}$$

  where γ_{z_0} is a small contour around z_0 which contains at most the eigenvalue z_0 in its interior.
- Given any z_0 in $\mathbb{C}$ with $\mathrm{Re}(\lambda) > -T_0$, there exists $m_k(z_0) \geq 1$ such that, in a small neighborhood of z_0, one has

$$(z+\mathcal{L}_{V,\nabla}^{(k)})^{-1} = \sum_{j=1}^{m_k(z_0)} (-1)^{j-1} \frac{(\mathcal{L}_{V,\nabla}^{(k)} + z_0)^{j-1}\pi_{z_0}^{(k)}}{(z-z_0)^j} + R_{z_0,k}(z) : \mathcal{H}_k^{sm+n-k}(M,\mathcal{E}) \to \mathcal{H}_k^{sm+n-k}(M,\mathcal{E}), \tag{12}$$

  with $R_{z_0,k}(z)$ an holomorphic function.

The last two statements allow to make the connection between this spectral framework and the functions $\hat{C}_{\psi_1,\psi_2}(z)$ appearing in the introduction.

4. Morse-Smale complex and generalized Morse inequalities

In this section, we will give the proof of Theorem 2.1 and draw some consequences in terms of Morse-Smale inequalities and of spectral trace asymptotics. In all this section, V is a smooth vector field which is either Anosov or $\mathcal{C}^1$-linearizable Morse-Smale. Moreover, ∇ is a flat connection on $\mathcal{E}$. Except mention of the contrary, we will not suppose that ∇ preserves some hermitian structure on $\mathcal{E}$ (or equivalently that $\mathcal{E}$ is endowed with a flat unitary connection ∇). Recall that preserving an hermitian structure would mean that, for any (ψ_1,ψ_2) in $\Omega^0(M,\mathcal{E})\times\Omega^0(M,\mathcal{E})$, one has

$$d(\langle\psi_1,\psi_2\rangle_{\mathcal{E}})=\langle\nabla\psi_1,\psi_2\rangle_{\mathcal{E}}+\langle\psi_1,\nabla\psi_2\rangle_{\mathcal{E}}.$$

This last property ensures that, for any ψ_1 in $\Omega^{k-1}(M)$ and for any ψ_2 in $\Omega^{n-k}(M)$,

$$\int_M\langle d^\nabla\psi_1,\psi_2\rangle_{\mathcal{E}}=(-1)^k\int_M\langle\psi_1,d^\nabla\psi_2\rangle_{\mathcal{E}}. \tag{13}$$

It also imposes that the monodromy matrices for the parallel transport are unitary – see [15, Sect. 4].

4.1. De Rham cohomology.

We start with a brief reminder on de Rham cohomology [47, 6]. An element ω in $\Omega^*(M,\mathcal{E})$ such that $d^\nabla\omega=0$ is called a **cocycle** while an element ω which is equal to $d^\nabla\alpha$ for some $\alpha\in\Omega^*(M,\mathcal{E})$ is called a **coboundary**. We define then

$$Z^k(M,\mathcal{E})=\text{Ker}(d^\nabla)\cap\Omega^k(M,\mathcal{E}),\ \text{ and } B^k(M,\mathcal{E})=\text{Ran}(d^\nabla)\cap\Omega^k(M,\mathcal{E}).$$

Obviously, $B^k(M,\mathcal{E})\subset Z^k(M,\mathcal{E})$, and the quotient space $\mathbf{H}^k(M,\mathcal{E})=Z^k(M,\mathcal{E})/B^k(M,\mathcal{E})$ is called the **k-th de Rham cohomology**. The complex is said to be **acyclic** if all the cohomology is reduced to $\{0\}$.

The coboundary operator d^∇ can be extended into a map acting on the space of currents $\mathcal{D}'^{,*}(M,\mathcal{E})$. This allows to define another cohomological complex $(\mathcal{D}'^{,*}(M,\mathcal{E}),d^\nabla)$:

$$0\xrightarrow{d^\nabla}\mathcal{D}'^{,0}(M,\mathcal{E})\xrightarrow{d^\nabla}\mathcal{D}'^{,1}(M,\mathcal{E})\xrightarrow{d^\nabla}\dots\xrightarrow{d^\nabla}\mathcal{D}'^{,n}(M,\mathcal{E})\xrightarrow{d^\nabla}0,$$

where we recall that $\mathcal{D}'^{,k}(M,\mathcal{E})$ is the topological dual of $\Omega^{n-k}(M,\mathcal{E})$. One can similarly define the k-th cohomology of that complex. A remarkable result of de Rham is that these two cohomologies coincide [47, Ch. 4]:

Theorem 4.1 (de Rham)**.** *Let u be an element in $\mathcal{D}'^{,k}(M,\mathcal{E})$ satisfying $d^\nabla u=0$.*

1. *There exists ω in $\Omega^k(M,\mathcal{E})$ such that $u-\omega$ belongs to $d^\nabla\left(\mathcal{D}'^{,k-1}(M,\mathcal{E})\right)$.*
2. *If $u=d^\nabla v$ with (u,v) in $\Omega^k(M,\mathcal{E})\times\mathcal{D}'^{,k-1}(M,\mathcal{E})$, then there exists ω in $\Omega^{k-1}(M,\mathcal{E})$ such that $u=d^\nabla\omega$.*

Remark 4.2. We will in fact need something slightly more precise involving the anisotropic Sobolev spaces we have introduced – see also [19, Lemma 2.1] for related statements in terms of wavefront sets. More precisely, if u belongs to $\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})$, then $u-\omega$ belongs to $d^\nabla\mathcal{H}_{k-1}^{sm+n-k+1}(M,\mathcal{E})$. This observation will be crucial in our proof of Theorem 2.1. We refer to Appendix C for more details.

4.2. **Chain homotopy equation.** Thanks to (31) and to the fact that $d^\nabla\circ\mathcal{L}_{V,\nabla}^{(k)}=\mathcal{L}_{V,\nabla}^{(k+1)}\circ d^\nabla$, we can define, for every $z_0\in\mathbb{C}$ the cohomological complex $(\mathrm{Ker}(\mathcal{L}_{V,\nabla}^{(*)}+z_0)^{m_k(z_0)},d^\nabla)$

$$0\xrightarrow{d^\nabla}\mathrm{Ker}(\mathcal{L}_{V,\nabla}^{(0)}+z_0)^{m_0(0)}\xrightarrow{d^\nabla}\mathrm{Ker}(\mathcal{L}_{V,\nabla}^{(1)}+z_0)^{m_1(0)}\xrightarrow{d^\nabla}\dots\xrightarrow{d^\nabla}\mathrm{Ker}(\mathcal{L}_{V,\nabla}^{(n)}+z_0)^{m_n(0)}\xrightarrow{d^\nabla}0.$$

By construction, this cohomological complex coincide with the complex $(C^\bullet_{V,\nabla}(z_0),d^\nabla)$ defined by (6). Observe that, for $z_0\neq 0$, $(C^\bullet_{V,\nabla}(z_0),d^\nabla)$ is **acyclic** and we shall come back to this case later on. Before that, we focus on the case $z_0=0$ and we start with a few preliminary observations. First, as $(\mathcal{L}_{V,\nabla}+z)$ commutes with d^∇ and ι_V, we can verify from (11) that, for every $0\leq k\leq n$,

$$d^\nabla\circ\pi_0^{(k)}=\pi_0^{(k+1)}\circ d^\nabla\text{ and }\iota_V\circ\pi_0^{(k+1)}=\pi_0^{(k)}\circ\iota_V. \tag{14}$$

Hence, one has

$$\mathcal{L}_{V,\nabla}^{(k)}\circ\pi_0^{(k)}=\pi_0^{(k)}\circ\mathcal{L}_{V,\nabla}^{(k)}. \tag{15}$$

According to [33, Appendix A.18], the operator $\mathcal{L}_{V,\nabla}^{(k)}$ is invertible on the space

$$\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})\cap\mathrm{Ker}\pi_0^{(k)}=(\mathrm{Id}-\pi_0^{(k)})\mathcal{H}_k^{sm+n-k}(M,\mathcal{E}).$$

The same holds in degree $k+1$. Hence, thanks to the relation

$$d^\nabla\circ\mathcal{L}_{V,\nabla}^{(k)}=\mathcal{L}_{V,\nabla}^{(k+1)}\circ d^\nabla:\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})\cap\mathrm{Ker}\pi_0^{(k)}\to\mathcal{H}_{k+1}^{sm+n-(k+1)}(M,\mathcal{E})\cap\mathrm{Ker}\pi_0^{(k+1)},$$

we can deduce that

$$(\mathcal{L}_{V,\nabla}^{(k+1)})^{-1}\circ d^\nabla\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)})=d^\nabla\circ(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)}). \tag{16}$$

Now that we have settled these equalities, we write, for any ψ_1 in $\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})$,

$$\begin{aligned}\psi_1&=\pi_0^{(k)}(\psi_1)+(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)})(\psi_1)\\&=\pi_0^{(k)}(\psi_1)+(d^\nabla\circ\iota_V+\iota_V\circ d^\nabla)\circ(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)})(\psi_1)\\&=\pi_0^{(k)}(\psi_1)+d^\nabla\circ\left(\iota_V\circ(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)})\right)(\psi_1)\\&\quad+\left(\iota_V\circ(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_{k+1}^{sm+n-(k+1)}(M,\mathcal{E})}-\pi_0^{(k+1)})\right)\circ d^\nabla(\psi_1),\end{aligned}$$

where the last equality follows from (14) and (16). Thus, if we set

$$R^{(k)}:=\iota_V^{(k)}\circ(\mathcal{L}_V^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)}):\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})\to\mathcal{H}_{k-1}^{sm+n-k}(M,\mathcal{E}), \tag{17}$$

we obtain the following **chain homotopy equation**:

$$\forall\psi_1\in\mathcal{H}_k^{sm+n-k}(M,\mathcal{E}),\ \psi_1=\pi_0^{(k)}(\psi_1)+d^\nabla\circ R^{(k)}(\psi_1)+R^{(k+1)}\circ d^\nabla(\psi_1). \tag{18}$$

Remark 4.3. Note that the map R will play a crucial in our calculations related to the torsion. By a similar argument as the one used to prove (16), we can verify that

$$(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ\iota_V\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)})=\iota_V\circ(\mathcal{L}_{V,\nabla}^{(k)})^{-1}\circ(\mathrm{Id}_{\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})}-\pi_0^{(k)}).$$

In particular, for every $z_0 \neq 0$, R is a **chain contraction map** for the acyclic complex $(C^{\bullet}_{V,\nabla}(z_0), d^{\nabla})$ [37], in the sense that $\mathrm{Id} = (d^{\nabla} + R)^2$ and $R^2 = 0$ for that complex.

4.3. **Proof of Theorem 2.1.** We can now give the proof of Theorem 2.1. Thanks to (14), the induced maps on the cohomology are well defined. Suppose now that ψ is a cocycle in $\Omega^k(M, \mathcal{E})$ which verifies $\pi_0^{(k)}(\psi) = 0$. Thanks to (18), we deduce that

$$\psi = d^{\nabla} \circ R^{(k)}(\psi).$$

From the second part of de Rham's Theorem, we deduce that there exists $\omega \in \Omega^{k-1}(M, \mathcal{E})$ such that $\psi = d\omega$. Hence, ψ is a coboundary. This shows the injectivity of the map. Consider now surjectivity and fix ψ a cocycle in $\mathrm{Ker}(\mathcal{L}_V^{(k)})^{m_k(0)}$. From the first of de Rham's Theorem (and more precisely Remark 4.2), there exists a cocycle ω in $\Omega^k(M, \mathcal{E})$ such that $\psi - \omega$ belongs to $d^{\nabla}\mathcal{H}_{k-1}^{sm+n-(k-1)}(M, \mathcal{E})$. Using (18) one more time, we can write

$$\omega = \pi_0^{(k)}(\omega) + d^{\nabla} \circ R^{(k)}(\omega).$$

Hence, we deduce that $\psi - \pi_0^{(k)}(\omega)$ is in the range of $\mathcal{H}_{k-1}^{sm+n-k}(M, \mathcal{E})$ under d^{∇}. Using the second part of de Rham's Theorem, there exists $\tilde{\omega}$ in $\mathcal{H}_{k-1}^{sm+n-k}(M, \mathcal{E})$ such that

$$\psi = \pi_0^{(k)}(\omega) + d^{\nabla}\tilde{\omega}.$$

Recall that the Pollicott-Ruelle spectrum is independent of the choice of the anisotropic Sobolev space [22, Th. 1.5]. In particular, the resonant state associated with $z_0 = 0$ are the same for the spaces $\mathcal{H}_{\bullet}^{sm+n-\bullet}$ and $\mathcal{H}_{\bullet}^{sm+n-\bullet-1}$. Hence, applying the spectral projector $\pi_0^{(k)}$ and implementing relation (14), we find that

$$\psi = \pi_0^{(k)}(\omega) + d^{\nabla} \circ \pi_0^{(k-1)}(\tilde{\omega}),$$

which concludes the proof of the surjectivity.

Remark 4.4. Even if it is not the simplest manner to prove this, we note that a direct consequence of Theorem 2.1 is the finite dimension of the twisted de Rham cohomology.

4.4. **Morse type inequalities.** In this paragraph, we will explain how to prove Corollary 2.3. The proof of Corollary 2.2 follows from a standard argument on finite dimensional cohomological complexes and we omit it – see [13, Sect. 7] for a brief reminder. In the case of Corollary 2.3, we remind that, in the case where V is a Morse-Smale vector field which is $\mathcal{C}^{\infty}$-diagonalizable, the main result of [15] (see paragraphs 5.1.1 and 5.1.2 for a brief reminder) implies that the dimension of $C^k_{\nabla,V}(0)$ is equal to

$$Nc_k(V) + \sum_{\Lambda:\dim W^s(\Lambda)=k} m_{\Lambda} + \sum_{\Lambda:\dim W^s(\Lambda)=k-1} m_{\Lambda}.$$

Thus, under the additional assumption that V is $\mathcal{C}^{\infty}$-diagonalizable, we can conclude the proof of the Corollary 2.3 thanks to Corollary 2.2. Fix now a general Morse-Smale vector field V_0 on M. Knowing that the set of Morse-Smale vector fields is open inside the set of smooth vector fields, we can deduce that, for every V in a small neighborhood of V_0, the vector field is still Morse-Smale. Thanks to the Sternberg-Chen Theorem [40, 56] (see also [14, Appendix] for a brief reminder), we know that the assumption of being $\mathcal{C}^\infty$-diagonalizable is satisfied as soon as all the eigenvalues of the linearized systems satisfies certain nonresonance assumptions and as soon as they are distinct. In particular, all these conditions are satisfied by a dense subset of vector fields and we can find V arbitrarly close to V_0 which is Morse-Smale and $\mathcal{C}^\infty$-diagonalizable. In order to conclude, we observe that, for a small enough perturbation, we do not modify the number of critical elements of the flow of index k and every closed orbit stay in the same homotopy class. In particular, all the terms in the sum defining the upper bound of corollary 2.3 are equal for V and V_0.

4.5. **Trace asymptotics and spectral determinants.** Let us now draw some nice spectral interpretation of these results. Fix U a bounded open set in $\mathbb{C}$. We define

$$\Pi_U^{(k)} = \sum_{z_0 \in U} \pi_{z_0}^{(k)}.$$

Note that, as the spectrum is discrete, this defines a finite sum. We now write

$$C_{V,\nabla}^{\text{even}}(z_0) = \bigoplus_{k=0 \mod 2} C_{V,\nabla}^k(z_0) \quad \text{and} \quad C_{V,\nabla}^{\text{odd}}(z_0) = \bigoplus_{k=1 \mod 2} C_{V,\nabla}^k(z_0).$$

Thanks to Remark 4.3, we can note that $Q = d^\nabla + R$ defines an operator which exchanges the chiralities and that, for $z_0 \neq 0$, Q is an isomorphism on $C_{V,\nabla}^{\text{even}}(z_0) \oplus C_{V,\nabla}^{\text{odd}}(z_0)$ (as $Q^2 = \mathrm{Id}$). In particular,

$$\forall z_0 \neq 0,\ \dim C_{V,\nabla}^{\text{even}}(z_0) = \dim C_{V,\nabla}^{\text{odd}}(z_0).$$

Aguing as in [13, Sect. 6], we can then derive from the case of equality in Corollary 2.2:

Corollary 4.5. *Suppose the assumptions of Theorem 2.1 are satisfied. Then, for every bounded open set U in $\mathbb{C}$ containing 0 and for every $z \in \mathbb{C}^*$,*

$$\prod_{k=0}^{n} \det\left(\Pi_U^{(k)}\left(z + \mathcal{L}_{V,\nabla}^{(k)}\right)\Pi_U^{(k)}\right)^{(-1)^k} = z^{\chi(M,\mathcal{E})},$$

and, for all $t > 0$,

$$\sum_{k=0}^{n} (-1)^k \operatorname{Tr}\left(\Pi_U^{(k)} e^{-t\mathcal{L}_{V,\nabla}^{(k)}} \Pi_U^{(k)}\right) = \chi(M,\mathcal{E}).$$

4.6. **Poincaré duality.** Consider now the dual complex associated with the vector field $-V$ and the connection $\nabla^\dagger$, i.e.

$$0 \xrightarrow{d^{\nabla^\dagger}} \operatorname{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(0)})^{m_d(0)} \xrightarrow{d^{\nabla^\dagger}} \operatorname{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(1)})^{m_{n-1}(0)} \xrightarrow{d^{\nabla^\dagger}} \ldots \xrightarrow{d^{\nabla^\dagger}} \operatorname{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(n)})^{m_0(0)} \xrightarrow{d^{\nabla^\dagger}} 0.$$

From (10) and from section 3, we know that the operator $-\mathcal{L}_{-V,\nabla^\dagger}^{(k)}$ can be identified with the dual of $-\mathcal{L}_{V,\nabla}^{(n-k)}$. These two complexes are dual to each other via the duality bracket between

$\mathcal{H}_k^{m+k}(M,\mathcal{E})$ and $\mathcal{H}_{n-k}^{-(m+k)}(M,\mathcal{E}')$. In other words, we have, for every ψ_1 in $\text{Ker}(\mathcal{L}_{V,\nabla}^{(k)})^{m_k(0)}$ and every ψ_2 in $\text{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(n-k)})^{m_k(0)}$,

$$\langle\psi_1,\psi_2\rangle = \langle\psi_1,\psi_2\rangle_{\mathcal{H}_k^{m+k}(M,\mathcal{E})\times\mathcal{H}_{n-k}^{-(m+k)}(M,\mathcal{E}')} = \int_M \psi_2\wedge\psi_1.$$

We can then define a Poincaré map:

$$\mathcal{P}_0^{(k)} : \psi\in\text{Ker}(\mathcal{L}_{V,\nabla}^{(k)})^{m_k(0)}\mapsto\langle\psi,.\rangle\in\left(\text{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(n-k)})^{m_k(0)}\right)'.$$

By a classical argument for complex which are dual to each other – see [13, Paragraph 7.5] for a brief reminder, we can then deduce that $\mathcal{P}_0^{(k)}$ induces an isomorphism between the k-th cohomological group of $(\text{Ker}(\mathcal{L}_{V,\nabla}^{(*)})^{m_*(0)}, d^\nabla)$ and the dual of the $(n-k)$-th cohomological group of $(\text{Ker}(\mathcal{L}_{-V,\nabla^\dagger}^{(*)})^{m_*(0)}, d^{\nabla^\dagger})$. In particular, as a by-product of this discussion and of Theorem 2.1, one can recover the well-known Poincaré duality:

$$\forall 0\leq k\leq n,\ b_k(M,\mathcal{E}) = b_{n-k}(M,\mathcal{E}').$$

Note that this result could be obtained more directly by standard arguments of Hodge theory.

5. Koszul homological complex

As a warm-up for the calculation of the spectral torsion, we start with some consideration on the following homological complex:

$$0\xrightarrow{\iota_V} C_{V,\nabla}^n(0)\xrightarrow{\iota_V} C_{V,\nabla}^{n-1}(0)\xrightarrow{\iota_V}\ldots\xrightarrow{\iota_V} C_{V,\nabla}^0(0)\xrightarrow{\iota_V} 0. \tag{19}$$

This complex will be refered as the **Morse-Smale-Koszul** complex and we will first prove Theorem 2.4. Namely, we show that, in the case of Morse-Smale vector fields, its homology counts the number of critical points in each degree.

In all this section, we will suppose that V is a Morse-Smale vector field which is $\mathcal{C}^\infty$-diagonalizable and that ∇ presereves an Hermitian structure on $\mathcal{E}$.

5.1. **Spectrum of Morse-Smale flows on the imaginary axis.** Let us first use our assumptions that the flow is smoothly diagonalizable in order to put it into a normal form near the critical elements Λ of the vector field V, namely its closed orbits and its critical points. Thanks to our results from [15, Sect. 7], it also allows to write down the local expression of the Pollicott-Ruelle resonant states in this system of normal coordinates.

5.1.1. *Critical points.* Suppose that Λ is a critical point of V of index k, i.e. such that $\dim W^s(\Lambda)=k$. Thanks to the fact that the flow is supposed to be smoothly diagonalizable, we can find a system of smooth coordinates $(x,y)\in\mathbb{R}^k\times\mathbb{R}^{n-k}$ near Λ such that the flow φ^t induced by V can be written as follows:

$$\varphi^t(x,y) = \left(e^{tA_s^\Lambda}x, e^{tA_u^\Lambda}y\right),$$

where A_s^Λ (resp. A_u^Λ) belongs to $M_k(\mathbb{R})$ (resp. $M_{n-k}(\mathbb{R})$), is diagonalizable (in $\mathbb{C}$) and has all its eigenvalues on the half plane $\mathrm{Re}(z) < 0$ (resp. $\mathrm{Re}(z) > 0$). In particular, the local stable manifold of Λ corresponds to the set $y = 0$ while the unstable manifold is given by $x = 0$. Near Λ, we can also construct a moving frame $(\mathbf{c}_1^\Lambda, \dots, \mathbf{c}_N^\Lambda)$ of $\mathcal{E}$ such that $\nabla c_j^\Lambda = 0$ for every $0 \leq j \leq N$ – see e.g. [36, Th. 12.25]. According to [15, Sect. 7], one can associate to every such critical element of V a family of N Pollicott-Ruelle resonant states of degree k associated with the resonance $z_0 = 0$. We denote them by

$$U_1^\Lambda, \dots, U_N^\Lambda,$$

and their local form near Λ is given, for every $1 \leq j \leq N$ by

$$U_j^\Lambda(x, y, dx, dy) = \delta_0(x_1, \dots, x_k) dx_1 \wedge \dots \wedge dx_k \otimes \mathbf{c}_j^\Lambda(x, y).$$

Moreover, one knows that the support of U_j^Λ is equal to $\overline{W^u(\Lambda)}$

5.1.2. *Closed orbits.* Suppose that Λ is a closed orbit of V of index k, i.e. such that $\dim W^s(\Lambda) = k$. The situation is now slightly more complicated. We denote by $\mathcal{P}_\Lambda$ the minimal period of the closed orbit. Thanks to the hypothesis that V is $\mathcal{C}^\infty$-diagonalizable, we can find a system of smooth coordinates $(x, y, \theta) \in \mathbb{R}^{k-1} \times \mathbb{R}^{n-k} \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$ near Λ such that the flow can be put under the following normal form

$$\varphi^t(x, y) = \left(P(\theta + t) e^{tA} P(\theta)^{-1}(x, y), \theta + t\right),$$

where

- A is of the form $\mathrm{diag}(A_s^\Lambda, A_u^\Lambda)$ with A_s^Λ (resp. A_u^Λ) belonging to $M_{k-1}(\mathbb{R})$ (resp. $M_{n-k}(\mathbb{R})$) and diagonalizable (in $\mathbb{C}$). Moreover, A_s^Λ (resp. A_u^Λ) has all its eigenvalues on the half plane $\mathrm{Re}(z) < 0$ (resp. $\mathrm{Re}(z) > 0$).
- $P(\theta)$ is $2\mathcal{P}_\Lambda$-periodic. Moreover, it satisfies $P(0) = \mathrm{Id}_{\mathbb{R}^{n-1}}$, $J_\Lambda := P(\mathcal{P}_\Lambda) = \mathrm{diag}(\pm 1)$ and $P(\theta + \mathcal{P}_\Lambda) = J_\Lambda P(\theta)$.

The manifold $W^s(\Lambda)$ (and thus $W^u(\Lambda)$) is orientable if and only if the determinant of J_Λ restricted to $\mathbb{R}^{k-1} \oplus \{0\}$ is equal to 1. We set $\varepsilon_\Lambda = 0$ whenever Λ is orientable and $\varepsilon_\Lambda = 1/2$ otherwise. It is related to the twisting index as follows $\Delta_\Lambda = e^{2i\pi\varepsilon_\Lambda}$. In this system of coordinates, Λ is given by the set $\{(0,0), \theta)\}$, the stable manifold by $\{P(\theta)(x, 0), \theta)\}$ and the unstable one by $\{P(\theta)(0, y), \theta)\}$. As for critical points, we can construct a moving frame $(\mathbf{c}_1^\Lambda, \dots, \mathbf{c}_N^\Lambda)$ of $\mathcal{E}$ such that there exists $\gamma_1^\Lambda, \dots, \gamma_N^\Lambda$ in $[0, 1)$ for which one has $\nabla \mathbf{c}_j^\Lambda = \frac{2i\pi\gamma_j^\Lambda}{\mathcal{P}_\Lambda} \mathbf{c}_j^\Lambda d\theta$ for every $0 \leq j \leq N$ – see e.g. [15, Sect. 4] for details. Here, we note that the $e^{2i\pi\gamma_j^\Lambda}$ are the eigenvalues of the monodromy matrix $M_{\mathcal{E}}(\Lambda)$ for the parallel transport around Λ.

Following one more time our previous work [15, Sect. 7], we can associate families of Pollicott-Ruelle resonant states of degree $k-1$ and k. They are now indexed by $(p, j) \in \mathbb{Z} \times \{1, \dots, N\}$ and we denote them by $U_{j,p}^\Lambda$ and $\tilde{U}_{j,p}^\Lambda$. Their local form near Λ is given by

$$U_{j,p}^\Lambda = e^{\frac{2i\pi(p+\varepsilon_\Lambda)\theta}{\mathcal{P}_\Lambda}} \left(P(\theta)^{-1}\right)^* \left(\delta_0(x_1, \dots, x_{k-1}) dx_1 \wedge \dots \wedge dx_{k-1}\right) \otimes \mathbf{c}_j^\Lambda(x, y, \theta),$$

and

$$\tilde{U}_{j,p}^{\Lambda}=e^{\frac{2i\pi(p+\varepsilon_\Lambda)\theta}{\mathcal{P}_\Lambda}}\left(P(\theta)^{-1}\right)^*(\delta_0(x_1,\ldots,x_{k-1})dx_1\wedge\ldots\wedge dx_{k-1})\wedge d\theta\otimes\mathbf{c}_j^{\Lambda}(x,y,\theta).$$

For a fixed (j,p), the resonant states $U_{j,p}^{\Lambda}$ and $\tilde{U}_{j,p}^{\Lambda}$ are associated with the resonance

$$z_0=-\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p+\varepsilon_\Lambda+\gamma_j^{\Lambda}\right),$$

and the support of $U_{j,p}^{\Lambda}$ is equal to $\overline{W^u(\Lambda)}$

5.1.3. *Resonant states on the imaginary axis.* Finally, the main result of [15, Sect. 7] states that the sets

$$\bigcup_{\Lambda\text{ critical point:}\dim W^s(\Lambda)=k}\left\{U_j^{\Lambda}:1\leq j\leq N\right\},$$

$$\bigcup_{\Lambda\text{ closed orbit:}\dim W^s(\Lambda)=k}\left\{\tilde{U}_{j,p}^{\Lambda}:1\leq j\leq N,\ p\in\mathbb{Z}\right\},$$

and

$$\bigcup_{\Lambda\text{ closed orbit:}\dim W^s(\Lambda)=k+1}\left\{U_{j,n}^{\Lambda}:1\leq j\leq N,\ p\in\mathbb{Z}\right\}$$

generate all the Pollicott-Ruelle resonances of degree k associated with a resonance lying on the imaginary axis. Moreover, all these states are linearly independent. We also recall from [15] that each of these states verifies $(\mathcal{L}_{V,\nabla}-z_0)U^{\Lambda}=0$ on $W^u(\Lambda)$ but that it only verifies a generalized eigenvalue equation $(\mathcal{L}_{V,\nabla}-z_0)^mU^{\Lambda}=0$ on M.

Remark 5.1. We note that there is no resonant state of degree k associated with $z_0=0$ if the Morse-Smale flow is nonsingular and if Δ_Λ is not an eigenvalue of $M_{\mathcal{E}}(\Lambda)$ for every closed orbit Λ.

5.2. **Proof of Theorem 2.4.** In order to prove Theorem 2.4, we start by fixing some critical point Λ of index k and $1\leq j\leq N$. Applying the contraction operator ι_V, we find that $\iota_V(U_j^{\Lambda})$ is equal to 0 near Λ. Thus, as $\iota_V(U_j^{\Lambda})$ verifies $\mathcal{L}_{V,\nabla}\circ\iota_V(U_j^{\Lambda})=0$ on $W^u(\Lambda)$ and as it is supported on $\overline{W^u(\Lambda)}$, we find that $\iota_V(U_j^{\Lambda})$ is supported on $\overline{W^u(\Lambda)}-W^u(\Lambda)$. As $(C^{\bullet}_{V,\nabla},\iota_V)$ is an homological complex, we know that $\iota_V(U_j^{\Lambda})$ will be a linear combination of elements inside $C^{k-1}_{V,\nabla}(0)$. According to the above discussion, such elements are supported either by the closure of unstable manifolds of dimension $n-k+1$ (in the case of a critical point) or by the closure of unstable manifolds of dimension $n-k+1$ or $n-k+2$ (in the case of closed orbits). Yet, according to Smale [52, Lemma 3.1] – see [14] for a brief reminder, such unstable submanifolds cannot be contained in the closure of $W^u(\Lambda)$. It means that the linear combination is equal to 0, i.e. $\iota_V(U_j^{\Lambda})=0$.

Fix now a closed orbit Λ such that[9] $\dim W^s(\Lambda)=k$ and (j,p) such that $U_{j,p}^{\Lambda}$ and $\tilde{U}_{j,p}^{\Lambda}$ are associated with the resonance 0. Let us first compute $\iota_V(U_{j,p}^{\Lambda})$. As before, we find, by using the local form of the resonant state and the eigenvalue equation on $W^u(\Lambda)$, that $\iota_V(U_{j,p}^{\Lambda})$

[9]Hence, $\dim W^u(\Lambda)=n-k+1$.

is supported on $\overline{W^u(\Lambda)} - W^u(\Lambda)$ which is thanks to Smale's Lemma [52, Lemma 3.1] the union of unstable manifolds whose dimension is $\leq n-k+1$. Recall also that if $W^u(\Lambda') \subset \overline{W^u(\Lambda)} - W^u(\Lambda)$ with $\dim W^u(\Lambda') = n-k+1$, then Λ' is a closed orbit. As we have an homological complex, $\iota_V(U_{j,p}^\Lambda)$ is an element in $C_{V,\nabla}^{k-2}(0)$. From the description of the resonant we just gave, we can verify as above that this subspace is generated by critical elements associated with unstable manifolds of dimension $\geq n-k+2$. This implies that $\iota_V(U_{j,p}^\Lambda) = 0$. It remains to compute $\iota_V(\tilde{U}_{j,p}^\Lambda)$ which is now an element in $C_{V,\nabla}^{k-1}(0)$. Contrary to the previous cases, $\iota_V(\tilde{U}_{j,p}^\Lambda)$ is supported by $\overline{W^u(\Lambda)}$ and it is equal to $U_{j,p}^\Lambda$ near Λ. Hence, from the eigenvalue equation, $\iota_V(\tilde{U}_{j,p}^\Lambda) - U_{j,p}^\Lambda$ is now supported by $\overline{W^u(\Lambda)} - W^u(\Lambda)$ which is a union of unstable manifold of dimension $\leq n-k+1$. Thus, using Smale's result one more time, we can conclude by similar arguments that

$$\iota_V(\tilde{U}_{j,p}^\Lambda) = U_{j,p}^\Lambda + \sum_{(\Lambda',j',p'):\Lambda'\neq\Lambda,\ \dim W^s(\Lambda')=k} \alpha_{\Lambda',j',p'} U_{j',p'}^{\Lambda'}.$$

In any degree, we can finally conclude that

$$\operatorname{Ker}\left(\iota_V^{(k)}\right) / \operatorname{Ran}\left(\iota_V^{(k+1)}\right) \simeq \operatorname{span}\left\{U_j^\Lambda:\ \dim\Lambda = k,\ 1 \leq j \leq N\right\},$$

where the kernel and the range are understood as for operators acting on $C_{V,\nabla}^\bullet(0)$.

5.3. **The case $z_0 \neq 0$.** As for the case of the coboundary operator, we can verify that the homology of the complex $(C_{V,\nabla}^\bullet(z_0), \iota_V)$ is trivial for any $z_0 \neq 0$. If we restrict to the case where $z_0 \in i\mathbb{R}^*$, we can in fact be slightly more precise. In fact, we should emphasize that, in the second part of the argument, we did not use the fact that z_0 is equal to 0 and the argument is in fact valid for any resonance z_0 lying on the imaginary axis $i\mathbb{R}^*$. More precisely, for every (Λ, j, p) such that

$$z_0 = -\frac{2i\pi(p + \varepsilon_\Lambda + \gamma_j^\Lambda)}{\mathcal{P}_\Lambda},$$

we will have $\iota_V(U_{j,p}^\Lambda) = 0$ and

$$\iota_V(\tilde{U}_{j,p}^\Lambda) = U_{j,p}^\Lambda + \sum_{(\Lambda',j',p'):\Lambda'\neq\Lambda,\ \dim W^s(\Lambda')=k} \alpha_{\Lambda',j',p'} U_{j',p'}^{\Lambda'}.$$

In particular, we have, for $z_0 \in i\mathbb{R}^*$ and for all $0 \leq k \leq n$,

$$\operatorname{Ker}\left(\iota_V^{(k)}\rceil_{C_{V,\nabla}^k(z_0)}\right) = \operatorname{span}\left\{U_{j,p}^\Lambda:\ \dim W^s(\Lambda) = k+1 \text{ and } z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma_j^\Lambda\right)\right\}.$$

6. Torsion of Pollicott-Ruelle resonant states

In this section, we will discuss the topological properties of the resonant states lying on the imaginary axis together with the proof of Theorem 2.7 which is related to the spectral zeta functions associated with the Pollicott-Ruelle resonances on the imaginary axis.

In the present section, we first present a new definition of torsion for the infinite dimensional chain complex of Pollicott–Ruelle resonances states corresponding by the spectrum on the imaginary axis which is directly related to Theorem 2.7. More specifically, we know from paragraph 4.2 that, for every $z_0 \in i\mathbb{R}^*$,

$$0 \xrightarrow{d^\nabla} C^0_{V,\nabla}(z_0) \xrightarrow{d^\nabla} C^1_{V,\nabla}(z_0) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} C^n_{V,\nabla}(z_0) \xrightarrow{d^\nabla} 0.$$

defines an acyclic cohomological complex. Moreover, we constructed in [15] a natural basis for this complex – see paragraph 5.1.2 for a brief reminder. Recall now that, as soon as we are given a cohomological complex with a preferred basis, we can compute its torsion which is just a certain determinant associated with d^∇ and computed in this basis. Hence, we will first compute the torsion of each individual complex $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ for a purely imaginary resonance z_0 in the basis of paragraph 5.1.2. Then, mimicking standard procedures used in spectral theory and QFT, we will define the **torsion of the infinite dimensional** complex $\bigoplus_{z_0\in i\mathbb{R}}(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ as the zeta regularized infinite product of the individual torsions in order to compute the torsion of the complex carried by the entire imaginary axis. Finally, we will prove Theorem 2.7 which is in fact related to the above regularized torsion.

From this point on, we will suppose that V is a Morse-Smale vector field which is $\mathcal{C}^\infty$-diagonalizable and that ∇ preserves an Hermitian structure on $\mathcal{E}$.

6.1. **Torsion of individual complexes** $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$**.** As was already pointed out, the finite dimensional complex $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ is acyclic as soon as $z_0 \neq 0$. Recall also from paragraph 5.1.2 that it has a preferred basis which is associated to the unstable manifolds of the closed orbits of the vector field V. Thus, given $z_0 \neq 0$ and its preferred basis of Pollicott-Ruelle resonant states, it is natural to compute the torsion of the complex $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$ [27, 37] and we shall start with this preliminary calculation when $z_0 \in i\mathbb{R}^*$.

Using the conventions of paragraph 5.1.2, we have, for every $0 \leq k \leq n$,

$$\bigoplus_{k=0}^n C^k_{V,\nabla}(z_0) = \text{span}\left\{ U^\Lambda_{j,p}, \tilde{U}^\Lambda_{j,p} : \ \Lambda \text{ closed orbit and } z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma^\Lambda_j\right)\right\},$$

which can be split as the direct sum

$$\bigoplus_{k=0}^n C^k_{V,\nabla}(z_0) = C^{\text{even}}_{V,\nabla}(z_0) \oplus C^{\text{odd}}_{V,\nabla}(z_0).$$

Recall from remark 4.3 that R is a chain contraction map for the complex $(C^\bullet_{V,\nabla}(z_0), d^\nabla)$. Then, according to [27, p. 30], the torsion of this finite dimensional complex (with respect to its preferred basis) is equal to

$$\tau\left(C^\bullet_{V,\nabla}(z_0), d^\nabla\right) = \left|\det\left(d^\nabla + R\right)_{C^{\text{even}} \to C^{\text{odd}}}\right|, \tag{20}$$

where the determinant is computed in the basis $(U^\Lambda_{j,p}, \tilde{U}^\Lambda_{j,p})_{j,p,\Lambda}$. In order to make this computation, we will to prove a preliminary result:

Lemma 6.1. *Let* (Λ, j, p) *be such that*

$$z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma^\Lambda_j\right).$$

Then, the supports of $\mathcal{L}_{V,\nabla}^{-1}(U_{j,p}^{\Lambda})$ *and* $\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j,p}^{\Lambda})$ *are equal to* $\overline{W^u(\Lambda)}$.

Proof. Recall from paragraph 5.1.2 that the supports of $U_{j,p}^{\Lambda}$ and $\tilde{U}_{j,p}^{\Lambda}$ are equal to $\overline{W^u(\Lambda)}$. According to Smale's Theorem A.1, $\overline{W^u(\Lambda)}$ is the union of unstable manifolds $W^u(\Lambda')$ with $\dim W^u(\Lambda') \leq \dim W^u(\Lambda)$. As $\mathcal{L}_{V,\nabla}^{(k)}$ preserves $C^k_{V,\nabla}(z_0)$ and as the support of $\mathcal{L}_{V,\nabla}U_{j,p}^{\Lambda}$ is contained in $\overline{W^u(\Lambda)}$, we can write

$$\mathcal{L}_{V,\nabla}U_{j,p}^{\Lambda} = \alpha_{j,p,\Lambda}U_{j,p}^{\Lambda} + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k} \alpha_{\Lambda',j',p'}U_{\Lambda',j',p'},$$

where $\leqq$ is the partial order relation of Smale's Theorem A.1 and where α_* are complex numbers. Similarly,

$$\begin{aligned}\mathcal{L}_{V,\nabla}(\tilde{U}_{j,p}^{\Lambda}) &= \tilde{\alpha}_{j,p,\Lambda}\tilde{U}_{j,p}^{\Lambda} + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k} \tilde{\alpha}_{\Lambda',j',p'}\tilde{U}_{j',p'}^{\Lambda'} \\ &\quad + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k+1} \tilde{\alpha}_{\Lambda',j',p'}U_{j',p'}^{\Lambda'},\end{aligned}$$

where α_* and $\tilde{\alpha}_*$ are complex numbers. From the explicit expressions of $U_{j,p}^{\Lambda}$ and $\tilde{U}_{j,p}^{\Lambda}$, we can already observe that $\alpha_{j,p,\Lambda} = \tilde{\alpha}_{j,p,\Lambda} = -z_0$. As $z_0 \neq 0$, we can also apply $\mathcal{L}_{V,\nabla}^{-1}$ to both sides of these equalities and we find that

$$U_{j,p}^{\Lambda} = -z_0\mathcal{L}_{V,\nabla}^{-1}(U_{j,p}^{\Lambda}) + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k} \alpha_{\Lambda',j',p'}\mathcal{L}_{V,\nabla}^{-1}(U_{j',p'}^{\Lambda'}),$$

and

$$\begin{aligned}\tilde{U}_{j,p}^{\Lambda} &= -z_0\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j,p}^{\Lambda}) + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k} \tilde{\alpha}_{\Lambda',j',p'}\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j',p'}^{\Lambda'}) \\ &\quad + \sum_{\Lambda' \leqq \Lambda, j',p' : \dim W^s(\Lambda')=k+1} \tilde{\alpha}_{\Lambda',j',p'}\mathcal{L}_{V,\nabla}^{-1}(U_{j',p'}^{\Lambda'}).\end{aligned}$$

Thus, determining the support of $\mathcal{L}_{V,\nabla}^{-1}(U_{j,p}^{\Lambda})$ and of $\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j,p}^{\Lambda})$ follows from the fact that we are able to compute the support of their analogues for $\Lambda' \leqq \Lambda$ with $\dim W^s(\Lambda') = k$ or $k+1$. Recall that the partial order relation $\leqq$ is defined in Theorem A.1 and that all the sums are implicitely over indices (Λ', j', p') such that

$$z_0 = -\frac{2i\pi}{\mathcal{P}_{\Lambda'}}\left(p' + \varepsilon_{\Lambda'} + \gamma_{j'}^{\Lambda'}\right). \tag{21}$$

If we consider the smallest such Λ' and if we are able to prove that the corresponding supports are equal to $\overline{W^u(\Lambda')}$, then the lemma will follow from an induction argument on $\dim W^u(\Lambda')$. Fix now $\Lambda' \leqq \Lambda$ which is minimal and (j', p') such that (21) is satisfied. As Λ' is minimal, we can decompose $\mathcal{L}_{V,\nabla}U_{j',p'}^{\Lambda'}$ and $\mathcal{L}_{V,\nabla}\tilde{U}_{j',p'}^{\Lambda'}$ but there is now no remainder term, i.e. one has

$$\mathcal{L}_{V,\nabla}U_{j',p'}^{\Lambda'} = -z_0U_{j',p'}^{\Lambda'} \text{ and } \mathcal{L}_{V,\nabla}\tilde{U}_{j',p'}^{\Lambda'} = -z_0\tilde{U}_{j',p'}^{\Lambda'}.$$

Applying $\mathcal{L}_{V,\nabla}^{-1}$ and using the fact $z_0 \neq 0$, we can conclude that the supports of $\mathcal{L}_{V,\nabla}^{-1} U_{j',p'}^{\Lambda'}$ and $\mathcal{L}_{V,\nabla}^{-1} \tilde{U}_{j',p'}^{\Lambda'}$ are equal to $\overline{W^u(\Lambda')}$. By induction, we can then recover that the supports of $\mathcal{L}_{V,\nabla}^{-1}(U_{j,p}^{\Lambda})$ and of $\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j,p}^{\Lambda})$ are equal to $\overline{W^u(\Lambda)}$. $\square$

We can now compute the torsion $(C_{V,\nabla}^{\bullet}(z_0), d^{\nabla})$ using formula (20). For that purpose, we fix Λ to be a closed orbit of the flow such that $\dim W^s(\Lambda) = k$. Suppose also that (j,p) verifies the equality

$$z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma_j^\Lambda\right).$$

On the one hand, if k is even, we would like to express $(d^\nabla + R)(\tilde{U}_{j,p}^{\Lambda})$ in the preferred basis of $C_{V,\nabla}^{\text{odd}}(z_0)$. On the other hand, if k is odd, we are interested in the expression of $(d^\nabla + R)(U_{j,p}^{\Lambda})$. In order to compute these determinants, it will be convenient to order these basis which are indexed by (Λ, j, p) according to Smale's partial order on the unstable manifolds. More precisely, we will order them in such a way that (Λ', j', p') is less than (Λ, j, p) whenever $\Lambda' \leqq \Lambda$.

6.1.1. *The case k even.* Let us start with the case $k \equiv 0 \mod 2$. Recall that near Λ, $\tilde{U}_{j,p}^{\Lambda}$ is of the form

$$\tilde{U}_{j,p}^{\Lambda} = e^{\frac{2i\pi(p+\varepsilon_\Lambda)\theta}{\mathcal{P}_\Lambda}}\left(P(\theta)^{-1}\right)^*\left(\delta_0(x_1,\ldots,x_{k-1})dx_1\wedge\ldots\wedge dx_{k-1}\right)\wedge d\theta\otimes \mathbf{c}_j^\Lambda(x,y,\theta).$$

First, from this expression, we can verify that $d^\nabla \tilde{U}_{j,p}^{\Lambda}$ is equal to 0 in a neighborhood of Λ. Moreover, as it satisfies $\mathcal{L}_{V,\nabla} d^\nabla \tilde{U}_{j,p}^{\Lambda}$ is equal to 0 on $W^u(\Lambda)$ and as $d^\nabla \tilde{U}_{j,p}^{\Lambda}$ is supported on $\overline{W^u(\Lambda)}$, we can deduce that $d^\nabla \tilde{U}_{j,p}^{\Lambda}$ is supported on $\overline{W^u(\Lambda)} - W^u(\Lambda)$. From Smale's Theorem [52] and from the fact that $(C_{V,\nabla}^{\bullet}(z_0), d^\nabla)$ is a cohomological complex, we can deduce that

$$d^\nabla \tilde{U}_{j,p}^{\Lambda} = \sum_{\Lambda'\leqq\Lambda, j',p':\dim W^s(\Lambda')=k+2} \alpha_{\Lambda',j',p'} U_{j',n'}^{\Lambda'} + \sum_{\Lambda'\leqq\Lambda, j',p':\dim W^s(\Lambda')=k+1} \alpha_{\Lambda',j',p'} \tilde{U}_{j',p'}^{\Lambda'}.$$

where we only sum over the (Λ', j', p') satisfying

$$z_0 = -\frac{2i\pi}{\mathcal{P}_{\Lambda'}}\left(p' + \varepsilon_{\Lambda'} + \gamma_{j'}^{\Lambda'}\right).$$

Now, we can compute $R\tilde{U}_{j,p}^{\Lambda}$. First, we write that $\mathcal{L}_{V,\nabla}(\tilde{U}_{j,p}^{\Lambda})$ belongs to $C_{V,\nabla}^k(z_0)$ and that its support is contained in $\overline{W^u(\Lambda)}$. Hence, one can write

$$\begin{aligned}\mathcal{L}_{V,\nabla}(\tilde{U}_{j,p}^{\Lambda}) &= \alpha_{j,n,\Lambda}\tilde{U}_{j,p}^{\Lambda} + \sum_{\Lambda'\leqq\Lambda, j',p':\dim W^s(\Lambda')=k} \alpha_{\Lambda',j',p'} \tilde{U}_{j',n'}^{\Lambda'} \\ &\quad + \sum_{\Lambda'\leqq\Lambda, j',p':\dim W^s(\Lambda')=k+1} \alpha_{\Lambda',j',p'} U_{j',p'}^{\Lambda'}.\end{aligned}$$

From the expression of $\tilde{U}_{j,p}^{\Lambda}$ near Λ, we know that $\alpha_{\Lambda,j,p}=-z_0$. Hence, we find

$$\begin{aligned}\mathcal{L}_{V,\nabla}^{-1}(\tilde{U}_{j,p}^{\Lambda}) &= -\frac{1}{z_0}\tilde{U}_{j,p}^{\Lambda}-\sum_{\Lambda'\leqq\Lambda,j',p':\dim W^s(\Lambda')=k}\alpha_{\Lambda',j',p'}\mathcal{L}_{V,\nabla}^{-1}\tilde{U}^{\Lambda'}j',p'\\ &-\sum_{\Lambda'\leqq\Lambda,j',p':\dim W^s(\Lambda')=k+1}\alpha_{\Lambda',j',p'}\mathcal{L}_{V,\nabla}^{-1}U_{j',p'}^{\Lambda'}.\end{aligned}$$

Recall now that $R=\iota_V\mathcal{L}_{V,\nabla}^{-1}$ when it acts on $C_{V,\nabla}^{\bullet}(z_0)$. Hence, one has

$$\begin{aligned}R(\tilde{U}_{j,p}^{\Lambda}) &= -\frac{1}{z_0}\iota_V(\tilde{U}_{j,p}^{\Lambda})-\sum_{\Lambda'\leqq\Lambda,j',p':\dim W^s(\Lambda')=k}\alpha_{\Lambda',j',p'}\iota_V\left(\mathcal{L}_{V,\nabla}^{-1}\tilde{U}_{j',p'}^{\Lambda'}\right)\\ &-\sum_{\Lambda'\leqq\Lambda,j',p':\dim W^s(\Lambda')=k+1}\alpha_{\Lambda',j',p'}\iota_V\left(\mathcal{L}_{V,\nabla}^{-1}U_{j',p'}^{\Lambda'}\right).\end{aligned}$$

Arguing as in the proof of Theorem 2.4 – see paragraph 5.3, we can verify that $\iota_V(\tilde{U}_{j,p}^{\Lambda})$ is equal to $U_{j,p}^{\Lambda}$ plus some remainder term which is carried on $\overline{W^u(\Lambda)}-W^u(\Lambda)$. Hence to summarize, one finds that

$$(d^{\nabla}+R)(\tilde{U}_{j,p}^{\Lambda})=-\frac{1}{z_0}U_{j,p}^{\Lambda}+T_{j,p}^{\Lambda}, \tag{22}$$

with $T_{j,p}^{\Lambda}$ belonging to $C_{V,\nabla}^{\mathrm{odd}}(z_0)$ and supported in $\overline{W^u(\Lambda)}-W^u(\Lambda)$.

6.1.2. *The case k odd.* Suppose now that $k\equiv 1 \bmod 2$. Near Λ, $U_{j,p}^{\Lambda}$ is of the form

$$U_{j,p}^{\Lambda}=e^{\frac{2i\pi(p+\varepsilon_\Lambda)\theta}{\mathcal{P}_\Lambda}}\left(P(\theta)^{-1}\right)^*\left(\delta_0(x_1,\ldots,x_{k-1})dx_1\wedge\ldots\wedge dx_{k-1}\right)\otimes\mathbf{c}_j^{\Lambda}(x,y,\theta).$$

From paragraph 5.3, we know that $\iota_V(U_{j,p}^{\Lambda})=0$. Hence, from the definition of R, one finds that $(d^{\nabla}+R)(U_{j,p}^{\Lambda})=d^{\nabla}(U_{j,p}^{\Lambda})$. Applying d^{∇} to $U_{j,p}^{\Lambda}$, we find that, in a neighborhood of Λ, it is equal to $z_0\tilde{U}_{j,p}^{\Lambda}$ (recall from paragraph 5.1.2 that $\nabla\mathbf{c}_j^{\Lambda}=\frac{2i\pi\gamma_j^{\Lambda}}{\mathcal{P}_\Lambda}\mathbf{c}_j^{\Lambda}$). The current $d^{\nabla}(U_{j,p}^{\Lambda})-z_0\tilde{U}_{j,p}^{\Lambda}$ belongs to $C_{V,\nabla}^{k}(z_0)$, it is supported inside $\overline{W^u(\Lambda)}$ and it identically vanishes near Λ. Hence, by propagation, it is supported inside $\overline{W^u(\Lambda)}-W^u(\Lambda)$. Equivalently, one has

$$(d^{\nabla}+R)(U_{j,p}^{\Lambda})=z_0\tilde{U}_{j,p}^{\Lambda}+T_{j,p}^{\Lambda}, \tag{23}$$

with $T_{j,p}^{\Lambda}$ belonging to $C_{V,\nabla}^{\mathrm{odd}}(z_0)$ and supported in $\overline{W^u(\Lambda)}-W^u(\Lambda)$.

6.1.3. *Conclusion.* Combining (20) with (22) and 23, one finally finds the following expression for the torsion:

$$\tau\left(C_{V,\nabla}^{\bullet}(z_0),d^{\nabla}\right)=\prod_{(\Lambda,j,p):(*)}\left|\frac{2\pi(n+\varepsilon_\Lambda+\gamma_j^{\Lambda})}{\mathcal{P}_\Lambda}\right|^{(-1)^{n+\dim\ W^u(\Lambda)}}, \tag{24}$$

where $(*)$ means that we take the product over the triples (Λ, j, p) satisfying

$$z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma_j^\Lambda\right).$$

In other words, the torsion $\tau\left(C^\bullet_{V,\nabla}(z_0), d^\nabla\right)$ is of the form $|z_0|^{(-1)^n m(z_0)}$ with

$$m(z_0) = \sum_{\Lambda \text{ closed orbit}} (-1)^{n+\dim\ W^u(\Lambda)} \left|\left\{(p,j) : z_0 = -\frac{2i\pi}{\mathcal{P}_\Lambda}\left(p + \varepsilon_\Lambda + \gamma_j^\Lambda\right)\right\}\right|.$$

In a more spectral manner, we have, using the results of paragraphs 5.3,

$$\ln\ \tau\left(C^\bullet_{V,\nabla}(z_0), d^\nabla\right) = \ln|z_0| \sum_{k=0}^n (-1)^{k+1} \dim\left(C^k_{V,\nabla}(z_0) \cap \mathrm{Ker}(\iota_V)\right). \tag{25}$$

Observe that, if we set

$$\begin{aligned} \zeta_{z_0}(s) \ &:= \sum_{(\Lambda,j,p):(*)} (-1)^{n+\dim\ W^u(\Lambda)} \left|\frac{2\pi(p+\varepsilon_\Lambda+\gamma_j^\Lambda)}{\mathcal{P}_\Lambda}\right|^{-s} \\ &= \frac{1}{|z_0|^s} \sum_{k=0}^n (-1)^{k+1} \dim\left(C^k_{V,\nabla}(z_0) \cap \mathrm{Ker}(\iota_V)\right), \end{aligned}$$

then one easily finds that

$$\tau\left(C^\bullet_{V,\nabla}(z_0), d^\nabla\right) = e^{-\zeta'_{z_0}(0)}, \tag{26}$$

which motivates the upcoming definitions.

6.2. **Another zeta function associated to the flow.** For every $0 \leq k \leq n$, we define the infinite dimensional vector space of Pollicott-Ruelle resonant states:

$$C^k_{V,\nabla}(i\mathbb{R}^*) = \bigoplus_{z_0 \in \mathcal{R}_k(V,\nabla) \cap i\mathbb{R}^*} C^k_{V,\nabla}(z_0).$$

This induces an infinite dimensional complex $(C^\bullet_{V,\nabla}(i\mathbb{R}^*), d^\nabla)$ and we define the following zeta function:

$$\zeta_{V,\nabla}(s) := \sum_{\Lambda,j,p} (-1)^{n+\dim\ W^u(\Lambda)} \left|\frac{\mathcal{P}_\Lambda}{2\pi(p+\varepsilon_\Lambda+\gamma_j^\Lambda)}\right|^s, \tag{27}$$

where the sum runs over the closed orbits Λ of the flow, $1 \leq j \leq N$ and $p \in \mathbb{Z}$ with the assumption that $p + \varepsilon_\Lambda + \gamma_j^\Lambda \neq 0$. In a more spectral manner, this can be written

$$\zeta_{V,\nabla}(s) = \sum_{k=0}^n (-1)^{k+1} \sum_{z_0 \in \mathcal{R}_k(V,\nabla) \cap i\mathbb{R}^*} \frac{1}{|z_0|^s} \dim\left(C^k_{V,\nabla}(z_0) \cap \mathrm{Ker}(\iota_V)\right).$$

We shall explain below by classical arguments from Hodge theory [37] that, up to the modulus, this zeta function is related to the spectral zeta function ζ_{RS} from corollary 2.9. Yet, we emphasize that *it appears here really from the computation of the torsion of an acyclic complex in a preferred basis* and not just as a reproduction of Ray-Singer's definition for the Laplace operator. In any case, this function is well defined for $\mathrm{Re}(s) > 1$ and we aim at describing its meromorphic extension to $\mathbb{C}$. In particular, motivated by (26), we would like to define the torsion of the cohomological complex $(C^\bullet_{V,\nabla}(i\mathbb{R}^*), d^\nabla)$ as $e^{-\zeta'_{V,\nabla}(0)}$ provided that it makes sense.

Let us now study the meromorphic continuation of the zeta functions we have just defined. For that purpose, we write

$$\zeta_{V,\nabla}(s) = \sum_{\Lambda, 1\leq j\leq N} (-1)^{n+\dim\, W^u(\Lambda)} \left(\frac{1}{2\pi}\right)^s \sum_{p\in\mathbb{Z}:\; p+\varepsilon_\Lambda+\gamma_j^\Lambda\neq 0} \left|\frac{\mathcal{P}_\Lambda}{(p+\varepsilon_\Lambda+\gamma_j^\Lambda)}\right|^s.$$

Hence, equivalently, it amounts to understand the meromorphic continuation of

$$\xi_{j,\Lambda}(s) := \sum_{p\in\mathbb{Z}:\; p+\varepsilon_\Lambda+\gamma_j^\Lambda\neq 0} \left|\frac{1}{(p+\varepsilon_\Lambda+\gamma_j^\Lambda)}\right|^s,$$

for every closed orbit and for every $1\leq j\leq N$. This can in fact easily be rewritten in terms of Hurwitz zeta functions and of Riemann zeta functions. Recall that $\gamma_j^\Lambda \in [0,1)$ and that $\varepsilon_\Lambda \in \{0, 1/2\}$ (the value depending on the orientability of $W^u(\Lambda)$). In particular, there exists an unique $q_j^\Lambda \in (0,1]$ such that $\varepsilon_\Lambda + \gamma_j^\Lambda$ is equal to q_j^Λ modulo 1. Introduce now the so-called Hurwitz zeta function [3, Ch. 12]:

$$\forall q \in (0,1],\ \zeta(s,q) := \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}.$$

Hence, for $q_j^\Lambda \neq 1$, one has

$$\xi_{j,\Lambda}(s) = \zeta(s, q_j^\Lambda) + \zeta(s, 1-q_j^\Lambda),$$

while, for $q_j^\Lambda = 1$,

$$\xi_{j,\Lambda}(s) = 2\zeta(s,1),$$

which is nothing else than twice the Riemann zeta function. From [3, Th. 12.4-5], we can conclude that, for every $1\leq j\leq N$ and for every closed orbit Λ, $\xi_{j,\Lambda}(s)$ has a meromorphic extension to $\mathbb{C}$ which is analytic except for a simple pole at $s=1$ (with residue 2). In particular, $\zeta_{V,\nabla}(s)$ extends meromorphically to $\mathbb{C}$ with an unique pole at $s=1$ which is simple and whose residue is given by

$$\frac{N}{\pi} \sum_{\Lambda\ \text{closed orbit}} (-1)^{n+\dim\, W^u(\Lambda)} \mathcal{P}_\Lambda.$$

6.3. **The regularized torsion of the infinite dimensional complex $(C^\bullet_{V,\nabla}(i\mathbb{R}^*), d^\nabla)$.** We define the **regularized torsion** of the infinite dimensional complex $(C^\bullet_{V,\nabla}(i\mathbb{R}^*), d^\nabla)$ as follows:

$$T\left(C^\bullet_{V,\nabla}(i\mathbb{R}^*), d^\nabla\right) := e^{-\zeta'_{V,\nabla}(0)},$$

which depends implicitely on the choice of the basis from paragraph 5.1.2. Let us compute the contribution coming from the derivative of the zeta function. We have

$$\zeta'_{V,\nabla}(s) = (-1)^n \sum_{\Lambda, 1\leq j\leq N} (-1)^{\dim\ W^u(\Lambda)} \left(\frac{1}{2\pi}\right)^s \left(\xi'_{j,\Lambda}(s) + \ln\left(\frac{\mathcal{P}_\Lambda}{2\pi}\right)\xi_{j,\Lambda}(s)\right).$$

In order to compute the derivative at $s = 0$, we shall come back to the expression in terms of Hurwitz zeta function. According to [3, Th. 12.13], we know that $\zeta(0, q) = \frac{1}{2} - q$. Hence, for $q_j^\Lambda = 1$, one has $\xi_{j,\Lambda}(0) = -1$ while, for $q_j^\Lambda \neq 1$, $\xi_{j,\Lambda}(0) = 0$. For the derivatives, one has $\xi'_{j,\Lambda}(0) = -\ln(2\pi)$ if $q_j^\Lambda = 1$ and $\xi'_{j,\Lambda}(0) = -\ln(2\sin(\pi q_j^\Lambda))$ otherwise [57, p. 271] [41, p. 195].

Recall that we used the notations $\Delta_\Lambda = e^{2i\varepsilon_\Lambda\pi}$ and $M_{\mathcal{E}}(\Lambda)$ for the monodromy matrix around Λ. Also, we have set $q_j^\Lambda = \varepsilon_\Lambda + \gamma_j^\Lambda$ mod 1 where $e^{2i\pi\gamma_j^\Lambda}$ are the eigenvalues of the monodromy matrix $M_{\mathcal{E}}(\Lambda)$. With these notations, we finally find

$$\zeta'_{V,\nabla}(0) = -\sum_{\Lambda, 1\leq j\leq N : q_j^\Lambda\neq 1} (-1)^{n+\dim\ W^u(\Lambda)} \ln\left(2\sin(\pi q_j^\Lambda)\right) - \sum_\Lambda (-1)^{n+\dim\ W^u(\Lambda)} m_\Lambda \ln \mathcal{P}_\Lambda,$$

and thus

$$e^{-\zeta'_{V,\nabla}(0)} = \prod_{\Lambda, 1\leq j\leq N : \Delta_\Lambda e^{2i\pi\gamma_j^\Lambda}\neq 1} \left|1 - \Delta_\Lambda e^{2i\pi\gamma_j^\Lambda}\right|^{(-1)^{n+\dim\ W^u(\Lambda)}} \prod_\Lambda \mathcal{P}_\Lambda^{(-1)^{n+\dim\ W^u(\Lambda)} m_\Lambda}.$$

Note that if Δ_Λ is not an eigenvalue of $M_{\mathcal{E}}(\Lambda)$ for every closed orbit[10], then we find

$$e^{-\zeta'_{V,\nabla}(0)} = \prod_\Lambda \left|\det\left(\mathrm{Id} - \Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)\right|^{(-1)^{n+\dim\ W^u(\Lambda)}},$$

which is equal to the Reidemester torsion (if V is also nonsingular) thanks to the works of Fried in [27, Sect. 3]. Recall that the left hand side can be interpreted either in terms of the nonzero Pollicott-Ruelle resonances on the imaginary axis or in terms of the torsion of the corresponding resonant states. To summarize, we have shown

Proposition 6.2. *Suppose that the assumptions of Theorem 2.7 are satisfied. Then, one has*

- $\zeta_{V,\nabla}(s)$ *has a meromorphic extension to* $\mathbb{C}$ *with a unique pole at* $s = 1$ *which is simple and whose residue equals*

$$\frac{N}{\pi} \sum_{\Lambda\ \textit{closed orbit}} (-1)^{n+\dim\ W^u(\Lambda)} \mathcal{P}_\Lambda.$$

- *one has*

$$e^{(-1)^n\zeta'_{V,\nabla}(0)} = \prod_{\Lambda, 1\leq j\leq N : \Delta_\Lambda e^{2i\pi\gamma_j^\Lambda}\neq 1} \left|1 - \Delta_\Lambda e^{2i\pi\gamma_j^\Lambda}\right|^{(-1)^{\dim\ W^u(\Lambda)+1}} \prod_\Lambda \mathcal{P}_\Lambda^{(-1)^{\dim\ W^u(\Lambda)+1} m_\Lambda},$$

[10] This is the assumption made in [27].

where $(e^{2i\pi\gamma_j^\Lambda})_{j=1,\dots,N}$ *are the eigenvalues of* $M_{\mathcal{E}}(\Lambda)$.

- *if* V *is nonsingular and* $m_\Lambda = 0$ *for every closed orbit, then* $e^{(-1)^n\zeta'_{V,\nabla}(0)}$ *is equal to the Reidemeister torsion of* $(\mathcal{E},\nabla)$.

6.4. **Twisted Fuller measures.** In this paragraph, we prove Theorem 2.7. First of all, we rewrite the Fuller measure in terms of the dimension of the unstable manifolds of the critical elements, i.e.

$$\begin{aligned}\mu_{V,\nabla}(t) &= -\frac{N}{t}\sum_{\Lambda \text{ fixed point}}(-1)^{\dim W^u(\Lambda)} \\ &\quad - \sum_{\Lambda \text{ closed orbit}}\sum_{m\geq 1}\frac{1}{m}(-1)^{\dim W^u(\Lambda)}\Delta_\Lambda^m \mathrm{Tr}\left(M_{\mathcal{E}}(\Lambda)^m\right)\delta(t-m\mathcal{P}_\Lambda).\end{aligned}$$

The Morse ineqality from Corollary 2.3 turns out to be an **equality** in the case $k=n$, thus we find that $\sum_{\Lambda \text{ fixed point}}(-1)^{\dim W^u(\Lambda)} = \sum_k(-1)^k b_k(M,\mathcal{E}) = \chi(M,\mathcal{E})$. Therefore the above formula can be rewritten in a more compact way as :

$$\mu_{V,\nabla}(t) = -\frac{\chi(M,\mathcal{E})}{t} - \sum_{\Lambda \text{ closed orbit}}\sum_{m\geq 1}\frac{1}{m}(-1)^{\dim W^u(\Lambda)}\Delta_\Lambda^m \mathrm{Tr}\left(M_{\mathcal{E}}(\Lambda)^m\right)\delta(t-m\mathcal{P}_\Lambda). \tag{28}$$

More explicitely, we can also write the sum on the r.h.s of the above identity in terms of the eigenvalues of the monodromy matrices:

$$t\mu_{V,\nabla}(t) = -N\sum_{\Lambda \text{ fixed point}}(-1)^{\dim W^u(\Lambda)} - \sum_{\Lambda,j}\mathcal{P}_\Lambda(-1)^{\dim W^u(\Lambda)}\sum_{m\geq 1}\left(e^{2i\pi\frac{\gamma_j^\Lambda+\varepsilon_\Lambda}{\mathcal{P}_\Lambda}}\right)^{m\mathcal{P}_\Lambda}\delta(t-m\mathcal{P}_\Lambda).$$

For $T>0$ and $\gamma\in\mathbb{R}$, recall that the Poisson formula implies that :

$$\sum_{m\in\mathbb{Z}}\delta(t-mT)e^{2i\pi\gamma mT} = \frac{1}{T}\sum_{l\in\mathbb{Z}}e^{\frac{2i\pi}{T}(l+\gamma T)}.$$

We can now apply the above formula which gives us :

$$t\mu_{V,\nabla}(t) = -N\sum_{\Lambda \text{ fixed point}}(-1)^{\dim W^u(\Lambda)} - \sum_{\Lambda,j}(-1)^{\dim W^u(\Lambda)}\sum_{l\in\mathbb{Z}}e^{2i\pi t\frac{l+\gamma_j^\Lambda+\varepsilon_\Lambda}{\mathcal{P}_\Lambda}},$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R}_+^*)$.

Finally, this quantity can be rewritten in a more spectral manner thanks to the results of paragraphs 5.2 and 5.3. More precisely,

$$\boxed{t\mu_{V,\nabla}(t) = \sum_{k=0}^n(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}\dim\left(C_{V,\nabla}^k(z_0)\cap\mathrm{Ker}(\iota_V)\right)e^{z_0 t},} \tag{29}$$

which concludes the proof of Theorem 2.7.

6.5. **Fried's torsion functions.** It now remains to prove Corollary 2.9 related to the torsion function. Recall that we introduced in subsection 2.6:

$$\zeta^{\flat}_{V,\nabla}(s,z)=\frac{1}{\Gamma(s)}\int_0^{+\infty}e^{-tz}t\mu_{V,\nabla}(t)t^{s-1}dt,$$

which, by definition of the twisted Fuller measure, can be rewritten thanks to (28) as

$$\zeta^{\flat}_{V,\nabla}(s,z)=-\frac{\chi(M,\mathcal{E})}{z^s}-\frac{1}{\Gamma(s)}\sum_{\Lambda \text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}\mathcal{P}_\Lambda^s\sum_{m\geq 1}\operatorname{Tr}\left(\left(e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)^m\right)m^{s-1}.$$

Recall that $\Gamma(s)^{-1}=s+o(s)$. Hence, if we differentiate $\zeta^{\flat}_{V,\nabla}$ with respect to s, then we find that, for $\operatorname{Re}(z)$ large enough,

$$\partial_s\zeta^{\flat}_{V,\nabla}(0,z)=\chi(M,\mathcal{E})\log z-\sum_{\Lambda \text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}\sum_{m\geq 1}\frac{\operatorname{Tr}\left(\left(e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)^m\right)}{m}$$

where one has

$$\sum_{m\geq 1}\frac{\operatorname{Tr}\left(\left(e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)^m\right)}{m}=-\log\det\left(\operatorname{Id}-e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)$$

as soon as $\operatorname{Re}(z)>0$ since $\|e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\|\leqslant e^{-\mathcal{P}_\Lambda Re(z)}<1$ by unitarity of $\Delta_\Lambda M_{\mathcal{E}}(\Lambda)$.

Equivalently, this can be rewritten as

$$\partial_s\zeta^{\flat}_{V,\nabla}(0,z)=\chi(M,\mathcal{E})\log z+\sum_{\Lambda \text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}\log\det\left(\operatorname{Id}-e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right).$$

Note that this expression is well defined for $\operatorname{Re}(z)>0$ and since the torsion function $Z_{V,\nabla}$ was defined as $Z_{V,\nabla}(z)=e^{-\partial_s\zeta^{\flat}_{V,\nabla}(0,z)}$, we deduce the following identity relating the torsion function and the weighted Ruelle zeta function :

$$Z_{V,\nabla}(z)=z^{-\chi(M,\mathcal{E})}\prod_{\Lambda \text{ closed orbit}}\det\left(\operatorname{Id}-e^{-\mathcal{P}_\Lambda z}\Delta_\Lambda M_{\mathcal{E}}(\Lambda)\right)^{-(-1)^{\dim W^u(\Lambda)}},$$

as was stated in paragraph 2.6.

In the case where V is a nonsingular Morse-Smale vector field, we recognize the torsion function introduced by Fried in [27, p. 51-53], also called twisted Ruelle zeta function. In any case, we can already verify from the exact expressions of the eigenvalues and their multiplicities (see paragraphs 5.1.1 and 5.1.2) that the poles and zeros of $Z_{V,\nabla}(z)$ are completely determined by the resonances on the imaginary axis.

Let us now come back to Corollary 2.9 and perform the Mellin transform on the right hand side of (29). We find that

$$\zeta^{\flat}_{V,\nabla}(s,z)=\frac{1}{\Gamma(s)}\sum_{k=0}^n(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)\int_0^{+\infty}e^{(z_0-z)t}t^{s-1}dt.$$

This is also equal to

$$\zeta^{\flat}_{V,\nabla}(s,z)=\sum_{k=0}^{n}(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)(z-z_0)^{-s}.$$

Remark 6.3. This expression can be *formally* differentiated at $s=0$ yielding :

$$\partial_s\zeta^{\flat}_{V,\nabla}(0,z)=-\sum_{k=0}^{n}(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)\log(z-z_0),$$

from which we deduce the **formal** expression relating Fried's torsion and some infinite product indexed by the resonances lying on the imaginary axis :

$$Z_{V,\nabla}(z)=\prod_{k=0}^{n}\prod_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}}(z-z_0)^{(-1)^{n-k+1}\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)}.$$

Using our expression for the kernel, we now find that

$$\begin{aligned}\zeta^{\flat}_{V,\nabla}(s,z) &= -\left(\chi(M,\mathcal{E})+\sum_{\Lambda\text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}m_\Lambda\right)z^{-s}\\ &+\sum_{k=0}^{n}(-1)^{n-k+1}\sum_{z_0\in\mathcal{R}_k(V,\nabla)\cap i\mathbb{R}^*}\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)(z-z_0)^{-s}.\end{aligned}$$

Now, in the spirit of Ray-Singer definition of analytic torsion, we can rewrite the second term in the right-hand side in a slightly different manner. As in the case of Hodge theory, we can observe that

$$\dim C^k_{V,\nabla}(z_0)=\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)+\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(d^{\nabla})\right)$$

and that

$$\dim\left(C^k_{V,\nabla}(z_0)\cap\operatorname{Ker}(\iota_V)\right)=\dim\left(C^{k+1}_{V,\nabla}(z_0)\cap\operatorname{Ker}(d^{\nabla})\right).$$

As in [37], this implies that

$$\zeta^{\flat}_{V,\nabla}(s,z)=-\left(\chi(M,\mathcal{E})+\sum_{\Lambda\text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}m_\Lambda\right)z^{-s}+\zeta_{RS}(s,z),$$

as expected.

Remark 6.4. Following Dyatlov and Zworski [18], we could also have defined the (twisted) dynamical zeta function as :

$$\tilde{\zeta}^{\flat}_{V,\nabla}(z):=\int_0^{+\infty}e^{-tz}t\mu_{V,\nabla}(t)dt,$$

which can be rewritten thanks to (28) as

$$\tilde{\zeta}^{\flat}_{V,\nabla}(z)=-\frac{\chi(M,\mathcal{E})}{z}-\sum_{\Lambda\text{ closed orbit}}(-1)^{\dim W^u(\Lambda)}\mathcal{P}_\Lambda\sum_{m\geq 1}\operatorname{Tr}\left(\Delta_\Lambda M_{\mathcal{E}}(\Lambda)e^{-z\mathcal{P}_\Lambda}\right)^m,$$

or equivalently

$$\tilde{\zeta}_{V,\nabla}^{\flat}(z) = -\frac{\chi(M,\mathcal{E})}{z} - \sum_{\Lambda \text{ closed orbit}} (-1)^{\dim W^u(\Lambda)} \operatorname{Tr}\left(\frac{\mathcal{P}_\Lambda \Delta_\Lambda M_{\mathcal{E}}(\Lambda) e^{-z\mathcal{P}_\Lambda}}{\operatorname{Id} - \Delta_\Lambda M_{\mathcal{E}}(\Lambda) e^{-z\mathcal{P}_\Lambda}}\right).$$

In that case, we would find that the poles of this zeta function are contained inside the intersection of the Pollicott-Ruelle resonances with the imaginary axis.

Appendix A. A brief reminder on Anosov and Morse-Smale flows

In this appendix, we briefly review some classical definitions and results from dynamical systems.

A.1. **Morse-Smale flows.** We say that $\Lambda \subset M$ is an elementary critical element if Λ is either a fixed point or a closed orbit of φ^t. Such an element is said to be hyperbolic if the fixed point or the closed orbit is hyperbolic – see [1] or the appendix of [14] for a brief reminder. Following [53, p. 798], φ^t is **a Morse-Smale flow** if the following properties hold:

(1) the non-wandering set $\mathrm{NW}(\varphi^t)$ is the union of finitely many elementary critical element $\Lambda_1, \dots, \Lambda_K$ which are hyperbolic,
(2) for every i, j and for every x in $W^u(\Lambda_j) \cap W^s(\Lambda_i)$, one has [11] $T_xM = T_xW^u(\Lambda_j) + T_xW^s(\Lambda_i)$.

We now briefly expose some important properties of Morse-Smale flows and we refer to [26, 43, 14] for a more detailed exposition on the dynamical properties of these flows. Under such assumptions, one can show that, for every x in M, there exists an unique couple (i, j) such that $x \in W^u(\Lambda_j) \cap W^s(\Lambda_i)$ (see e.g. Lemma 3.1 in [14]). In particular, *the unstable manifolds* $(W^u(\Lambda_j))_{j=1,\dots,K}$ *form a partition of* M, i.e.

$$M = \bigcup_{j=1}^{K} W^u(\Lambda_j), \quad \text{and} \quad \forall i \neq j,\ W^u(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset.$$

The same of course holds for stable manifolds. One of the main feature of such flows is the following result which is due to Smale [52, 53]:

Theorem A.1 (Smale)**.** *Suppose that φ^t is a Morse-Smale flow. Then, for every $1 \leq j \leq K$, the closure of $W^u(\Lambda_j)$ is the union of certain $W^u(\Lambda_{j'})$. Moreover, if we say that $W^u(\Lambda_{j'}) \leqq W^u(\Lambda_j)$ if $W^u(\Lambda_{j'})$ is contained in the closure of $W^u(\Lambda_j)$, then, $\leqq$ is a partial ordering. Finally if $W^u(\Lambda_{j'}) \leqq W^u(\Lambda_j)$, then* $\dim W^u(\Lambda_{j'}) \leq \dim W^u(\Lambda_j)$.

The partial order relation on the collection of subsets $W^u(\Lambda_j)_{j=1}^K$ defined above is called **Smale causality relation**. Following Smale, we define an oriented graph[12] D whose K vertices are given by $W^u(\Lambda_j)_{j=1}^K$. Two vertices $W^u(\Lambda_j), W^u(\Lambda_i)$ are connected by an oriented path starting at $W^u(\Lambda_j)$ and ending at $W^u(\Lambda_i)$ iff $W^u(\Lambda_j) \leqq W^u(\Lambda_i)$. Recall from

[11] See appendix of [14] for the precise definition of the stable/unstable manifolds $W^{s/u}(\Lambda)$.

[12] This diagram is the Hasse diagram associated to the poset $\left(W^u(\Lambda_j)_{j=1}^K, \leqq\right)$.

the works of Peixoto [44] that Morse-Smale flows form an open and dense of all smooth vector fields on surfaces while it is an open set in higher dimensions [42].

In the constructions from [14, 15], we needed to make extra assumptions on our flows, namely that they are linearizable near every critical element Λ_i. More precisely, we fix $1 \leq l \leq \infty$ and we say that the Morse-Smale flow is **$\mathcal{C}^l$-linearizable** if for every $1 \leq i \leq k$, the following hold:

- If Λ_i is a fixed point, there exists a $\mathcal{C}^l$ diffeomorphism $h : B_n(0, r) \to W$ (where W is a small open neighborhood of Λ_i and $B_n(0, r)$ is a small ball of radius r centered at 0 in $\mathbb{R}^n$) and a linear map A_i on $\mathbb{R}^n$ such that $V \circ h = dh \circ L$ where V is the vector field generating φ^t and where
$$L(x) = A_i x.\partial_x.$$
- If Λ_i is closed orbit of period $\mathcal{P}_{\Lambda_i}$ if there exists a $\mathcal{C}^l$ diffeomorphism $h : B_{n-1}(0, r) \times \mathbb{R}/(\mathcal{P}_{\Lambda_i}\mathbb{Z}) \to W$ (where W is a small open neighborhood of Λ_i and $r > 0$ is small) and a smooth map $A : \mathbb{R}/(\mathcal{P}_{\Lambda_i}\mathbb{Z}) \to M_{n-1}(\mathbb{R})$ such that $V \circ h = dh \circ L$ with
$$L_i(x, \theta) = A_i(\theta)x.\partial_x + \partial_\theta.$$

In other words, the flow can be put into a normal form in a certain chart of class $\mathcal{C}^l$. We shall say that **a Morse-Smale flow is $\mathcal{C}^l$-diagonalizable** if it is $\mathcal{C}^k$-linearizable and if, for every critical element Λ, either the linearized matrix $A \in GL_n(\mathbb{R})$ or the monodromy matrix M associated with $A(\theta)$ is diagonalizable in $\mathbb{C}$. Such properties are satisfied as soon as certain (generic) non resonance assumptions are made on the Lyapunov exponents thanks to the Sternberg-Chen Theorem [40, 56]. We refer to the appendix of [14] for a detailed description of these nonresonant assumptions.

A.2. **Anosov flows.** We say that a flow $\varphi^t : M \to M$ Λ is of **Anosov type** [2, 1] if there exist $C > 0$ and $\chi > 0$ and a family of spaces $E_u(\rho), E_s(\rho) \subset T_\rho\mathbb{M}$ (for every ρ in M) satisfying the following properties, for every ρ in M and for every $t \geq 0$,

(1) $T_\rho\mathbb{M} = \mathbb{R}V(\rho) \oplus E_u(\rho) \oplus E_s(\rho)$ with $V(\rho) = \frac{d}{d}(\varphi^t(\rho))\rceil_{t=0}$,
(2) $d_\rho\varphi^t E_{u/s}(\rho) = E_{u/s}(\varphi^t(\rho))$,
(3) for every v in $E_u(\rho)$, $\|d_\rho\varphi^{-t}v\| \leq Ce^{-\chi t}\|v\|$,
(4) for every v in $E_s(\rho)$, $\|d_\rho\varphi^t v\| \leq Ce^{-\chi t}\|v\|$.

Again, it is known from the works of Anosov that such flows form an open set inside smooth vector fields [2].

Appendix B. Review on anisotropic Sobolev spaces

B.1. **Escape function.** The key ingredient in Faure-Sjöstrand's analysis of transfer operators is the construction of a so-called **escape function**, or equivalently a Lyapunov function for the Hamiltonian flow on T^*M induced by
$$\forall (x, \xi) \in T^*M, \quad H_V(x, \xi) := \xi(V(x)).$$

Recall that the corresponding Hamiltonian flow is given by

$$\forall t \in \mathbb{R}, \quad \Phi_V^t(x,\xi) := \left(\varphi^t(x,\xi), \left(d\varphi^t(x)^T\right)^{-1}\xi\right),$$

where φ^t is the flow induced by the vector field V on M. We shall denote by $X_V(x,\xi)$ the Hamiltonian vector field induced by Φ_V^t. Using the terminology of [22], an **escape function** for the flow Φ_V^t on T^*M is a function of the form

$$G(x,\xi) := m(x,\xi)\log\sqrt{1+f(x,\xi)^2},$$

meeting the following requirements:

- m **is a symbol of order** 0. This means $m(x,\xi)$ belongs to $\mathcal{C}^\infty(T^*M)$ and, for all multi-indices $(\alpha,\beta)\in\mathbb{N}^{2n}$,
$$\partial_x^\alpha\partial_\xi^\beta m(x,\xi) = \mathcal{O}\left((1+\|\xi\|_x^2)^{-\frac{|\beta|}{2}}\right).$$
- f **has the correct homogeneity in** ξ. $f(x,\xi)$ belongs to $\mathcal{C}^\infty(T^*M)$, and, for $\|\xi\|_x\geq 1$, $f(x,\xi)>0$ is positively homegeneous of degree 1.
- **The symbol H_V is micro elliptic in some conical open set $\mathcal{N}_0$.** There exist a conical open set $\mathcal{N}_0$ and a constant $C>0$ such that, for every $(x,\xi)\in\mathcal{N}_0$ satisfying $\|\xi\|_x\geq C$, one has
$$|H_V(x,\xi)|\geq\frac{1}{C}(1+\|\xi\|_x).$$
- **Outside $\mathcal{N}_0$, G has uniform decrease along the Hamiltonian flow.** There exist $R\geq 1$ and $c>0$ such that, for every $(x,\xi)\in T^*M$ satisfying $\|\xi\|_x\geq R$,
$$X_V.G(x,\xi)\leq 0 \text{ and } (x,\xi)\notin\mathcal{N}_0 \Longrightarrow X_V.G(x,\xi)\leq -c.$$

In other words, the escape function is strictly decreasing except in the directions where the Hamiltonian H_V is elliptic. Observe that $\mathcal{N}_0$ and $\mathcal{N}_1$ can be even chosen to be empty – this was for instance the case for gradient flows [13]. Note that compared with Lemma 1.2 in [22], we do not require anything on the value of the **order function** m in certain directions of phase space. The reason for this is that the above axioms are the key ingredients to build a convenient spectral theory for the vector field V. The other requirements on m are additional informations on the structure Sobolev space which are specific to the Anosov or to the Morse-Smale framework. According to [22, 14], examples of vector fields possessing such an escape function are given by Anosov or Morse-Smale vector fields which are $\mathcal{C}^1$-linearizable. Motivated by the main results from [22], we say that a flow φ^t possessing such an escape function is **microlocally tame**. We shall make this assumption in the following and we keep in mind that the only known examples of such flows are the ones mentionned above.

B.2. **Anisotropic Sobolev spaces.** Let us now briefly recall the construction of Faure-Sjöstrand based on the existence of such a function [22]. Note that, in this reference, the authors only treated the case of the trivial bundle $M\times\mathbb{C}$. The extension of this microlocal approach to more general bundles was made by Dyatlov and Zworski in [18] by some slightly different point of view. Fix $s>0$ some large parameter (corresponding to the Sobolev regularity we shall require) and $0 \leq k \leq n$ (the degree of the forms). We also fix some Riemannian metric g on M and some hermitian structure $\langle ., \rangle_{\mathcal{E}}$ on $\mathcal{E}$, none of them being a priori related to V and ∇. We can fix the inner product $\langle , \rangle_{g^*}^{(k)}$ on $\Lambda^k(T^*M)$ which is induced by the metric g on M. Then, the the Hodge star operator is the unique isomorphism $\star_k : \Lambda^k(T^*M) \to \Lambda^{n-k}(T^*M)$ such that, for every ψ_1 in $\Omega^k(M) \simeq \Omega^k(M, \mathbb{C})$ and ψ_2 in $\Omega^{n-k}(M) \simeq \Omega^{n-k}(M, \mathbb{C})$,

$$\int_M \psi_1 \wedge \psi_2 = \int_M \langle \psi_1, \star_k^{-1} \psi_2 \rangle_{g^*(x)}^{(k)} \omega_g(x),$$

where ω_g is the volume form induced by the Riemannian metric on M. This induces a map $\star_k : \Omega^k(M, \mathcal{E}) \to \Omega^{n-k}(M, \mathcal{E})$ which acts trivially on the $\mathcal{E}$-coefficients. Using the Hermitian metric $g_{\mathcal{E}}$ on $\mathcal{E}$, we can also introduce the following pairing, for every $0 \leq k, l \leq n$,

$$\langle . \wedge . \rangle_{\mathcal{E}} : \Omega^k(M, \mathcal{E}) \otimes_{\mathcal{C}^\infty(M)} \Omega^l(M, \mathcal{E}) \to \Omega^{k+l}(M).$$

Combining these, we can define the positive definite Hodge inner product on $\Omega^k(M, \mathcal{E})$ as

$$(\psi_1, \psi_2) \in \Omega^k(M, \mathcal{E}) \times \Omega^k(M, \mathcal{E}) \mapsto \int_M \langle \psi_1 \wedge \star_k(\psi_2) \rangle_{\mathcal{E}}.$$

In particular, we can define $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$ as the completion of $\Omega^k(M, \mathcal{E})$ for this scalar product.

Set now

$$A_s^{(k)}(x, \xi) := \exp\left((sm(x, \xi) + n - k) \log \left(1 + f(x, \xi)^2 \right)^{\frac{1}{2}} \right), \tag{30}$$

where $G(x, \xi) = m(x, \xi) \log \left(1 + f(x, \xi)^2 \right)^{\frac{1}{2}}$ is the escape function. We can define

$$\mathbf{A}_s^{(k)}(x, \xi) := A_s^{(k)}(x, \xi) \mathbf{Id}_{\Lambda^k(T^*M) \otimes \mathcal{E}}$$

belonging to $\mathrm{Hom}(\Lambda^k(T^*M) \otimes \mathcal{E})$ and introduce an anisotropic Sobolev space of currents by setting

$$\mathcal{H}_k^{sm+n-k}(M, \mathcal{E}) = \mathrm{Op}(\mathbf{A}_s^{(k)})^{-1} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E}),$$

where $\mathrm{Op}(\mathbf{A}_s^{(k)})$ is a (essentially selfadjoint) pseudodifferential operator with principal symbol $\mathbf{A}_s^{(k)}$.

Remark B.1. Note that this requires to deal with symbols of variable order $m(x, \xi)$ whose symbolic calculus was described in Appendix A of [21]. This can be done as the symbol $m(x, \xi)$ belongs to the standard class of symbols $S^0(T^*M)$. We also refer to [18, App. C.1] for a brief reminder of pseudodifferential operators with values in vector bundles. In particular, adapting the proof of [21, Cor. 4] to the vector bundle valued framework, one can verify that $\mathbf{A}_s^{(k)}$ is an elliptic symbol, and thus $\mathrm{Op}(\mathbf{A}_s^{(k)})$ can be chosen to be invertible.

We observe from the composition rule for pseudodifferential operators that

$$\forall 0 \leq k \leq n-1, \quad d^\nabla : \mathcal{H}_k^{sm+n-k}(M, \mathcal{E}) \to \mathcal{H}_{k+1}^{sm+n-(k+1)}(M, \mathcal{E}). \tag{31}$$

Mimicking the proofs of [21], we can also deduce some properties of these spaces of currents. First of all, they are endowed with a Hilbert structure inherited from the L^2-structure on M. The space

$$\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})' = \mathrm{Op}(\mathbf{A}_s^{(k)})L^2(M,\Lambda^k(T^*M)\otimes\mathcal{E})$$

is the topological dual of $\mathcal{H}_k^m(M,\mathcal{E})$ which is in fact reflexive. The following injections holds and they are continuous:

$$\Omega^k(M,\mathcal{E}) \subset \mathcal{H}_k^m(M,\mathcal{E}) \subset \mathcal{D}^{\prime,k}(M,\mathcal{E}),$$

The hermitian metric on $\mathcal{E}$ allows to define a canonical isomorphism $\tau_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$ by setting, for every ψ in $\mathcal{E}$, $\tau_{\mathcal{E}}(\psi) = \langle \psi, . \rangle_{\mathcal{E}}$. Then, combined with the Hodge star map, it induces an isomorphism from $\mathcal{H}_k^m(M,\mathcal{E})'$ to $\mathcal{H}_{n-k}^{-m}(M,\mathcal{E}')$, whose Hilbert structure is given by the scalar product

$$(\psi_1,\psi_2) \in \mathcal{H}_{n-k}^{-m}(M,\mathcal{E}')^2 \mapsto \langle \star_k^{-1}\tau_{\mathcal{E}}^{-1}(\psi_1), \star_k^{-1}\tau_{\mathcal{E}}^{-1}(\psi_2)\rangle_{\mathcal{H}_k^m(M,\mathcal{E})'}.$$

Thus, the topological dual of $\mathcal{H}_k^m(M,\mathcal{E})$ can be identified with $\mathcal{H}_{n-k}^{-m}(M,\mathcal{E}')$, where, for every ψ_1 in $\Omega^k(M,\mathcal{E})$ and ψ_2 in $\Omega^{n-k}(M,\mathcal{E}')$, one has the following duality relation:

$$\begin{aligned} \langle \psi_2, \psi_1 \rangle_{\mathcal{H}_{n-k}^{-m}(M,\mathcal{E}')\times\mathcal{H}_k^m(M,\mathcal{E})} &= \int_M \psi_2 \wedge \psi_1 \\ &= \langle \mathrm{Op}(\mathbf{A}_m^{(k)})^{-1} \star_k^{-1} \tau_{\mathcal{E}}^{-1}(\overline{\psi_2}), \mathrm{Op}(\mathbf{A}_m^{(k)})\psi_1 \rangle_{L^2(M,\Lambda^k(T^*M)\otimes\mathcal{E})} \\ &= \langle \star_k^{-1}\tau_{\mathcal{E}}^{-1}(\psi_2), \psi_1\rangle_{\mathcal{H}_k^m(M,\mathcal{E})\times\mathcal{H}_k^m(M,\mathcal{E})'}. \end{aligned}$$

Appendix C. Proof of Theorem 4.1

In this last appendix, we briefly recall the proof of the classical de Rham theorem 4.1 adapted to the framework of anisotropic Sobolev spaces – see e.g. [19] which is based on the elliptic regularity of d^∇.

First of all, fix a Riemannian metric g on M and an hermitian structure $\langle ., . \rangle_{\mathcal{E}}$ on $\mathcal{E}$. Following [6, Th. 4.11], we can compute the formal adjoint for the Hilbert structure on $L^2(M,\Lambda^k(T^*M)\otimes\mathcal{E})$. More precisely, for ψ_1 in $\Omega^k(M,\mathcal{E})$ and for ψ_2 in $\Omega^{k+1}(M,\mathcal{E})$, one has

$$\langle d^\nabla\psi_1, \psi_2\rangle_{L^2} = \int_M d^\nabla\psi_1 \wedge \tau_{\mathcal{E}} \star_{k+1}(\psi_2) = (-1)^k \int_M \psi_1 \wedge d^{\nabla^\dagger}\tau_{\mathcal{E}} \star_{k+1}(\psi_2),$$

where the second equality follows from the Stokes Theorem. This tells us that[13] $(d^\nabla)^* = (-1)^k\star_k^{-1}\tau_{\mathcal{E}}^{-1}d^{\nabla^\dagger}\tau_{\mathcal{E}}\star_{k+1}$. We then form the corresponding Laplace Beltrami operator $\Delta_{g,\mathcal{E}} = d^\nabla(d^\nabla)^* + (d^\nabla)^*d^\nabla$, which is formally selfadjoint. According to [6, p.19], the principal symbol of this pseudodifferential operator is $-\|\xi\|_x^2\mathbf{Id}_{\Lambda^k(T^*M)\otimes\mathcal{E}}$. In particular, this defines an elliptic operator. We can now make use of elliptic regularity to prove De Rham Theorem.

Fix u in $\mathcal{D}^{\prime,k}(M,\mathcal{E})$ satisfying $d^\nabla u = 0$. According to [6, Part. I.3], there exists a pseudodifferential operator A_k of order -2 such that $u - \Delta_{g,\mathcal{E}}A_k u$ belongs to $\Omega^k(M,\mathcal{E})$. Using the fact that u is a cocycle, we find that $d^\nabla\Delta_{g,\mathcal{E}}A_k u = \Delta_{g,\mathcal{E}}d^\nabla A_k u$ belongs to

[13] Under the additional assumptions that ∇ preserves the Hermitian structure, we would have $\tau_{\mathcal{E}}^{-1}d^{\nabla^\dagger}\tau_{\mathcal{E}} = d^\nabla$.

$\Omega^{k+1}(M)$. Thus, by elliptic regularity, $d^\nabla A_k u$ belongs to $\Omega^{k+1}(M)$ and, by composing with $(d^\nabla)^*$, $(d^\nabla)^* d^\nabla A_k u$ belongs to $\Omega^k(M)$. Hence, we can find ω in $\Omega^k(M,\mathcal{E})$ such that $u-\omega = d^\nabla (d^\nabla)^* A_k u$ which proves point (2). Note that, if u belongs to $\mathcal{H}_k^{sm+n-k}(M,\mathcal{E})$, then we can verify from the composition rules of pseudodifferential operators that $(d^\nabla)^* A_k u$ belongs to $\mathcal{H}_{k-1}^{sm+n-k+1}(M,\mathcal{E})$ as pointed out in Remark 4.2.

For the proof of point (2), we proceed similarly except that we replace u by v which is not a priori a cocyle. Using the fact that $u = d^\nabla v$ is smooth, we can still conclude that there exists $\omega \in \Omega^{k-1}(M,\mathcal{E})$ such that $v - \omega = d^\nabla (d^\nabla)^* A_k v$. Applying d^∇, we get the expected conclusion.

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