Université Lille 1, Sciences et Technologies
2013/2014 – Master degree in mathematical engineering
Refresher course in physics

Final exam.
October, 16^{th} 2013. Duration : **3 hours.**

Cell phones and calculators are forbidden. You may use your own course notes, but no books.

Each exercise should be done on a different sheet of paper.

1. Mechanical waves

Consider the membrane of a *circular drum* which is centered at O in the (xOy)-plane. The radius of the drum is R and its mass per unit of surface is μ.

We neglect the friction forces and the force due do to gravitation. The only force that we take into account is the tension force. We make the assumption that the tension force on a small piece of drum dl located in $M = (x, y)$ is of strength

$$dF = T(x, y)dl,$$

where $T(x, y)$ does not depend on the time t. Moreover, the direction of the force is orthogonal to dl. We want to describe the motion of the membrane on the z-direction, i.e. describe

$$z(x, y, t),$$

which depends only on the variables (x, y, t). We will denote by (r, θ) the polar coordinates in the (xOy)-plane.

Finally, for any integer $n \in \mathbb{Z}$, we introduce the Bessel equation

$$u^2h''(u) + uh'(u) + (u^2 - n^2)h(u) = 0.$$

We will admit that this equation has a solution J_n which is well defined in $u = 0$.

(1) Write the Newton's laws on the (x, y, z) directions for a small piece of drum $dS = dxdy$.

(2) Deduce that $T(x, y)$ is a constant that we will denote T_0.

(3) Deduce that
$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right),$$
and express c in terms of the physical quantities of the problem.

(4) What are the two limit conditions in $r = R$?

(5) We search for a solution of the form $z = \psi(r,\theta)\cos(\omega t)$. What property is satisfied by such a wave?

(6) Show that
$$r\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\omega^2 r^2}{c^2}\psi = 0.$$

(7) What property satisfies $\psi(r,\theta)$ as a function of θ. Deduce that $\psi(r,\theta)$ can be written
$$\psi(r,\theta) = A_0(r) + \sum_{n=1}^{+\infty} A_n(r)\cos(n\theta) + A_{-n}(r)\sin(n\theta).$$

(8) Express A_k in terms of the Bessel function J_k.

(9) What are the restrictions on the values of ω? For which physical reason these restrictions appear?

2. Quantum mechanics

Consider the classical Hamiltonian
$$H(q,p) = \frac{1}{2}p^2 - Fq,$$
where $F > 0$.

(i) Write, then solve, the corresponding Hamiltonian equations of motion.

We now consider the following one-dimensional Schrödinger equation ($y \in \mathbb{R}$)
$$i\hbar\partial_t\psi_t(y) = \frac{1}{2}(P^2\psi_t)(y) - F(Q\psi_t)(y), \qquad \psi_t(y) = \varphi_0(y),$$
where $F > 0$. Let $\hat{\psi}_t$ be the Fourier transform ψ_t.

(ii) Compute
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi_t, Q\psi_t \rangle, \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \psi_t, P\psi_t \rangle,$$
and then determine explicitly $\langle \psi_t, Q, \psi_t \rangle$.

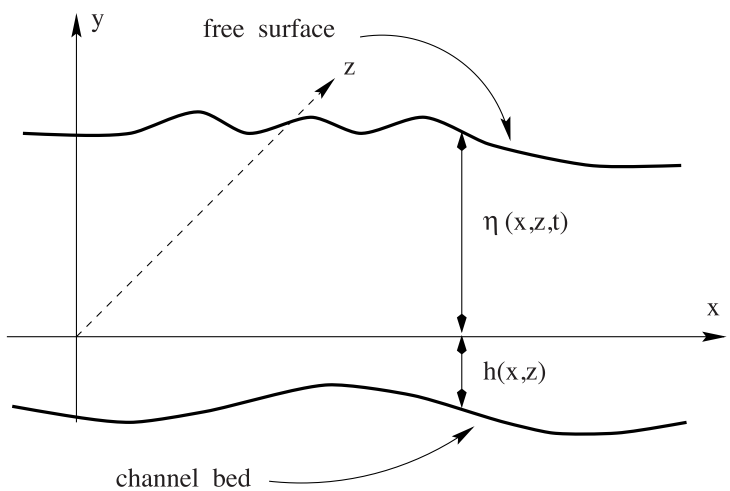
(iii) Show that
$$i\hbar\partial_t\hat{\psi}_t(p) = \frac{1}{2}p^2\hat{\psi}(p) - Fi\hbar\partial_p\hat{\psi}_t(p).$$

(iv) Introduce the function
$$f(p,t) := \hat{\psi}_t(p+Ft,t),$$
and show that $i\hbar\partial_t f(p,t) = \frac{1}{2}p^2 f(p,t)$.

(v) Give an expression for $\psi_t(y)$ in the form
$$\psi_t(y) = \int_{\mathbb{R}} \exp(\frac{i}{\hbar}\phi(t,p))\ \hat{\varphi}(p)\frac{\mathrm{d}p}{\sqrt{2\pi\hbar}}.$$
Identify the function $\phi(t,p)$ explicitly.

3. Fluid dynamics

Consider the flow of water in a channel and assume that the water is incompressible, non-viscous, non-heat conducting and subject to gravitational forces. The horizontal plane



is given by the coordinates x and z and the vertical direction is given by y. Denote the gravitational forces by $\rho\mathbf{g}$ where $\mathbf{g} = (0,-g,0)$ and g is the acceleration due to gravity. Two important boundaries of the three dimensional domain are the bottom of the channel, denoted by $y = -h(x,z)$ which is fixed in time, and the free surface under gravity $y = \eta(x,z,t)$, which depends on space and time.

(1) Give the law of conservation of mass and the momentum equation and simplify them in the case of incompressible and non-viscous flow.

(2) Assuming that the vertical component of the acceleration can be neglected, show that we obtain the following *hydrostatic pressure relation :*

$$p = \rho g(\eta - y) + p_{atm},$$

where $p = p(x, y, z, t)$ and p_{atm} is the atmospheric pressure at $y = \eta$, assumed horizontally uniform.

(3) Show that we can write

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_1}{\partial z} = -g \frac{\partial \eta}{\partial x},$$
$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_3 \frac{\partial u_3}{\partial z} = -g \frac{\partial \eta}{\partial z}.$$

(4) The Shallow Water Equations are an approximation to the free-surface problem when the fluid is incompressible. The boundary conditions are $\frac{D}{Dt}(\eta - y) = 0$ at $y = \eta$ and $\frac{D}{Dt}(h + y) = 0$ at $y = -h$.

(a) Integrate the continuity equation with respect to y from $y = -h$ to $y = \eta$.

(b) By expanding the boundary conditions, according to the definition of the Lagrangian derivative, and using the Leibniz integral rule (for differentiation under the integral sign), show that

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u_1 \, dy + \frac{\partial}{\partial z} \int_{-h}^{\eta} u_3 \, dy = 0.$$

(c) Observing that u_1 and u_3 are independent of y and h is independent of time, and using $\eta^* = \eta + h$, show that we obtain

$$\frac{\partial \eta^*}{\partial t} + \tilde{\nabla} \cdot (\eta^* \mathbf{u}) = 0,$$

where $\tilde{\nabla} \cdot \mathbf{u}$ denotes the horizontal divergence.

(d) Taking each equation in question (3), premultiplied by η^*, and added to equation (4.c), multiplied respectively by u_1 or u_3, show that the momentum equation becomes

$$\frac{\partial(\phi u_1)}{\partial t} + \frac{\partial(\phi u_1^2 + \frac{1}{2}\phi^2)}{\partial x} + \frac{\partial(\phi u_1 u_3)}{\partial z} = g\phi \frac{\partial h}{\partial x},$$
$$\frac{\partial(\phi u_3)}{\partial t} + \frac{\partial(\phi u_1 u_3)}{\partial x} + \frac{\partial(\phi u_3^2 + \frac{1}{2}\phi^2)}{\partial z} = g\phi \frac{\partial h}{\partial z},$$

where $\phi = g\eta^*$ is called the *geopotential.*

(e) Finally, we can write the two-dimensional Shallow Water Equations in the following compact conservation form with source terms

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial z} = \mathbf{S}(\mathbf{U}).$$

Give $\mathbf{U}, \mathbf{F}, \mathbf{G}$ and $\mathbf{S}$.