ON THE LEBESGUE COMPONENT OF SEMICLASSICAL MEASURES FOR ABELIAN QUANTUM ACTIONS

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Abstract. For a large class of symplectic integer matrices, the action on the torus extends to a symplectic $\mathbb{Z}^r$-action with $r \geq 2$. We apply this to the study of semiclassical measures for joint eigenfunctions of the quantization of the symplectic matrices of the $\mathbb{Z}^r$-action. In the irreducible setting, we prove that the resulting probability measures are convex combinations of the Lebesgue measure with weight $\geq 1/2$ and a zero entropy measure. We also provide a general theorem in the reducible case showing that the Lebesgue components along isotropic and symplectic invariant subtori must have total weight $\geq 1/2$.

1. Introduction

The Quantum Ergodicity Theorem is a classical result in mathematical quantum chaos describing the equidistribution properties of stationary quantum states in the semiclassical limit [Šni74, Zel87, CdV85]. More precisely, given an orthonormal basis of Laplace eigenfunctions on a compact Riemannian manifold with *ergodic* geodesic flow, it states that most of the eigenfunctions become equidistributed in phase space in the large eigenvalue limit. In [RS94], Rudnick and Sarnak conjectured that, on negatively curved manifolds, *all* (and not only most) eigenfunctions must equidistribute. Over the last twenty years, this conjecture lead to many developments and we refer to [Ana22, Dya22] for recent reviews with many details and references on these results.

One way to get insights into this conjecture is to consider a basis of eigenfunctions having extra symmetries as in the seminal work [RS94]. Indeed, if there is an (or a family of) operator(s) commuting with the Laplacian, one can consider joint orthonormal basis of eigenfunctions and expect that the resulting basis has better equidistribution properties. In certain arithmetic cases, one can for instance show equidistribution of all eigenfunctions which are also eigenfunctions of the Hecke operators [BL03, Lin06, BL14, SV19].

Without the Hecke symmetry, Anantharaman and Silberman considered this problem on general compact locally symmetric spaces [AS13]. In that case, they proved entropic bounds for (accumulation points of) joint eigenfunctions of the entire algebra of translation-invariant differential operators. From this result, they deduced that, on compact quotients of $\mathrm{SL}(3, \mathbb{R})$, joint eigenfunctions have a Haar component of weight $\geq 1/4$ (thus exhibit some equidistribution) and they also extended this property to certain compact quotients of $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 4$ (with a weight in $(0, 1/2]$ depending on the situation). Motivated by the recent developments on the support properties of semiclassical measures for higher-dimensional quantum maps by Dyatlov-Jézéquel [DJ24] and Kim [KAO24], the goal of this article is to show how the results from [AS13] can be extended in the setting of unitary matrices quantizing symplectic linear maps of the torus $\mathbb{T}^{2d} := \mathbb{R}^{2d}/\mathbb{Z}^{2d}$.

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1.1. **Quantum maps.** Linear symplectic automorphisms $\mathrm{Sp}(2d,\mathbb{Z})$ of the torus $\mathbb{T}^{2d}$ provide a family of classical dynamical systems for which the above questions can be raised in a simple functional framework. The quantization of these classical systems in view of understanding questions from quantum chaos was introduced by Hannay and Berry in [HB80]. Namely, given any $\mathbf{N}\in\mathbb{N}$, one can define a natural Hilbert space[1] $\mathcal{H}_{\mathbf{N}}\simeq\ell^2((\mathbb{Z}/2\mathbf{N}\mathbb{Z})^d)$ respecting the periodic structure of the torus and on which the metaplectic representation $M_{\mathbf{N}}(A)$ of $A\in\mathrm{Sp}(2d,\mathbb{Z})$ acts unitarily. See §2 for a brief reminder or [DJ24, §2] for a detailed construction. Given a sequence $(\psi_k)_{k\geq 1}$ of normalized states in $\mathcal{H}_{\mathbf{N}_k}$ with $\mathbf{N}_k\to\infty$, one can define their Wigner distributions:

$$W_{\psi_k}: a\in\mathcal{C}^\infty(\mathbb{T}^{2d})\mapsto\left\langle \mathrm{Op}^w_{(4\pi\mathbf{N}_k)^{-1}}(a)\psi_k,\psi_k\right\rangle_{\mathcal{H}_{\mathbf{N}_k}},$$

where $\mathrm{Op}^w_h(a)$ is the Weyl quantization of a. Any accumulation point (as $k\to\infty$) of this sequence of distributions defines a probability measure on $\mathbb{T}^{2d}$ and the resulting measures are referred as the *semiclassical measures* of the sequence $(\psi_k)_{k\geq1}$. We denote this set by $\mathcal{P}((\psi_k)_{k\geq1})$. If we suppose in addition that the sequence is made of eigenvectors of $M_{\mathbf{N}_k}(A)$, then the limit measures are invariant under A. Again, we refer to §2 below or to [DJ24, §2] for more details.

The analogue of the Quantum Ergodicity Theorem holds true for this model [BDB96]. More precisely, if for every $\mathbf{N}\geq1$, we are given an orthonormal basis $(\psi_j^{\mathbf{N}})_{1\leq j\leq(2\mathbf{N})^d}$ of eigenfunctions of $M_{\mathbf{N}}(A)$, then most of the corresponding Wigner distributions converge to the Lebesgue measure on $\mathbb{T}^{2d}$ as soon as this measure is ergodic[2] for A.

1.2. **Main results.** We need to introduce a few conventions in order to state our main results. First, given $A\in\mathrm{Sp}(2d,\mathbb{Z})$, one can associate its characteristic polynomial and we will assume irreducibility over $\mathbb{Q}$ for our first result. We define for $A\in\mathrm{Sp}(2d,\mathbb{R})$ the integers

$$m(A)=\frac{1}{2}\#(\sigma(A)\cap\mathbb{R})\quad\text{and}\quad l(A)=\frac14\#(\sigma(A)\cap(\mathbb{C}\setminus(\mathbb{R}\cup\mathbb{S}^1)))$$

where $\sigma(A)=\{\lambda_1,\dots,\lambda_{2d}\}$ is the spectrum of A and where eigenvalues are counted with their multiplicity. Our first result reads as follows:

Theorem 1.1. *Let $d\geq2$ and $A\in\mathrm{Sp}(2d,\mathbb{Z})$ with irreducible characteristic polynomial over $\mathbb{Q}$ such that no ratio of eigenvalues $\frac{\lambda_i}{\lambda_j}$, $i\neq j$, is a root of unity. Furthermore assume $m(A)+l(A)\geq2$.*
Then, for every $\varepsilon>0$, one can find $B_\varepsilon\in\mathrm{Sp}(2d,\mathbb{Z})$ commuting with A such that

$$\forall\mathbf{N}\geq1,\quad M_{\mathbf{N}}(B_\varepsilon)M_{\mathbf{N}}(A)=M_{\mathbf{N}}(A)M_{\mathbf{N}}(B_\varepsilon).$$

and such that, for any sequence $(\psi_k)_{k\geq1}$ satisfying

$$\forall k\geq1,\quad M_{\mathbf{N}_k}(A)\psi_k=e^{i\beta_k(A)}\psi_k,\quad M_{\mathbf{N}_k}(B_\varepsilon)\psi_k=e^{i\beta_k(\varepsilon)}\psi_k,\quad\|\psi_k\|_{\mathcal{H}_{\mathbf{N}_k}}=1,$$

one has that, for every $\mu\in\mathcal{P}((\psi_k)_{k\geq1})$,

$$\mu=\alpha\,\mathrm{Leb}_{\mathbb{T}^{2d}}+(1-\alpha)\nu,$$

with

$$\alpha\geq\frac12-\varepsilon,$$

[1] Note that even values appear in view of the first remark of Section 2.1.
[2] This is equivalent to A not having roots of unity as eigenvalues.

and $h_{\mathrm{KS}}(\nu, B) = 0$ *for any* $B \in \langle A, B_\varepsilon \rangle \leq \mathrm{Sp}(2d, \mathbb{Z})$.

Here, $h_{\mathrm{KS}}(\nu, B)$ is the Kolmogorov-Sinai entropy of the measure ν with respect to B [Wal82, Ch. 4]. It is a nonnegative number which measures how much of the complexity of B is captured by ν. For instance, if ν is supported on a closed orbit, then the entropy vanishes while it is maximal for the Lebesgue measure.

The assumption of irreducibility and that no ratio of eigenvalues is a root of unity can be captured by the Galois group of the characteristic polynomial. In fact, this is equivalent to having χ_{A^k} irreducible for all $k \in \mathbb{N}$. It turns out that these are generic among symplectic matrices [KAO24] and we will explain how to construct explicitly matrices with this property and the additional assumption $m(A)+l(A) \geq 2$ (see §5). This assumption on the eigenvalues is made to ensure the existence of an abelian subgroup $\Lambda \leq \mathrm{Sp}(2d, \mathbb{Z})$ of rank $m(A) + l(A)$ containing A and the matrix B_ε will be picked in a convenient way inside this subgroup. In fact, we can choose Λ in such a way that

$$\forall B_1, B_2 \in \Lambda,\ \forall \mathbf{N} \geq 1, \quad M_{\mathbf{N}}(B_1)M_{\mathbf{N}}(B_2) = M_{\mathbf{N}}(B_2)M_{\mathbf{N}}(B_1),$$

and we could also consider joint eigenfunctions for *all* $M_{\mathbf{N}}(B)$, $B \in \Lambda$. In that case, our arguments will show that $\alpha \geq 1/2$ (see Theorem 4.14). In particular, if $m(A) + l(A) = 2$, we can pick $\varepsilon = 0$ in Theorem 1.1. For $m(A) + l(A) > 2$, if we pick an arbitrary matrix B in $\Lambda \setminus \langle A \rangle$ rather than B_ε, we will only get a lower bound $\alpha \geq c(A, B) > 0$ on the Lebesgue component with $c(A, B)$ depending on the Lyapunov exponents of A and B.

When ψ_k is only an eigenfunction of $M_{\mathbf{N}_k}(A)$, Kim recently proved that, under the same irreducibility condition and the assumption that $m(A) + l(A) \geq 1$, any limit measure $\mu \in \mathcal{P}((\psi_k)_{k\geq 1})$ has full support [KAO24]. See also [Sch24, DJ24] for earlier contributions of Schwartz when $d = 1$ and of Dyatlov and Jézéquel under more restrictive assumptions than in [KAO24] when $d \geq 1$. Theorem 1.1 shows that under one extra symmetry, there is a Lebesgue component with almost 1/2 weight. Kurlberg, Ostafe, Rudnick and Shparlinski also proved that, for a density 1 of integers $(\mathbf{N}_k)_{k\geq 1}$, one must have $\alpha = 1$ in Theorem 1.1 under some related assumptions on the matrix A [KORS24]. Here we do not make any assumption on the sequence of integers $\mathbf{N}$ at the expense of considering joint eigenfunctions and of having only $\alpha \geq 1/2 - o(1)$. Earlier works of the first author also show that any element $\in \mathcal{P}((\psi_k)_{k\geq 1})$ have positive entropy without any restriction on A or on the sequence of integers $(\mathbf{N}_k)_{k\geq 1}$ [Riv11]. This quantitative statement will in fact be one of the key ingredients of our proof. See also [FNDB03, BDB03, FN04, AN07, Bro10, Gut10] for earlier related results.

In §1.3, we will also discuss in more details results by Kelmer for joint eigenmodes of the Hecke operators [Kel10]. Roughly speaking, he picked joint eigenstates for all the generators of the Hecke group of A while we are using only two elements in this group. Let us just mention at this point that, under the assumption of Theorem 1.1, he proved that such joint arithmetic eigenmodes yield $\alpha = 1$ in the above statement. This property is referred as Arithmetic Quantum Unique Ergodicity for quantum maps. He also showed that, if A has an invariant isotropic subspace, then such a property fails even for the Hecke eigenmodes. For instance, when A is of the form $\mathrm{Diag}(\tilde{A}, (\tilde{A}^{-1})^T)$ with $\tilde{A} \in \mathrm{GL}(d, \mathbb{Z})$, then one can construct a sequence of eigenmodes whose limit measure is of the form $\mathrm{Leb}_{\mathbb{T}^d} \otimes \delta_0^{\mathbb{T}^d}$ [Gur06, App. B]. Motivated by this example, we can state a second application of the dynamical methods used in our work.

Theorem 1.2. *Let $d \geq 3$ and $\tilde{A} \in \mathrm{GL}(d, \mathbb{Z})$ with irreducible characteristic polynomial. Suppose also that $m(\tilde{A})+l(\tilde{A}) \geq 3$ and that no ratio of the eigenvalues of $A = \mathrm{Diag}(\tilde{A}, (\tilde{A}^{-1})^T) \in \mathrm{Sp}(2d, \mathbb{Z})$ is a root of unity.*
Then, for every $\varepsilon > 0$, one can find $\tilde{B}_\varepsilon \in \mathrm{GL}(d, \mathbb{Z})$ commuting with $\tilde{A}$ such that, letting $B_\varepsilon = \mathrm{Diag}(\tilde{B}_\varepsilon, (\tilde{B}_\varepsilon^{-1})^T)$, one has

$$\forall \mathbf{N} \geq 1, \quad M_{\mathbf{N}}(B_\varepsilon) M_{\mathbf{N}}(A) = M_{\mathbf{N}}(A) M_{\mathbf{N}}(B_\varepsilon).$$

and such that, for any sequence $(\psi_k)_{k\geq1}$ satisfying

$$\forall k \geq 1, \quad M_{\mathbf{N}_k}(A)\psi_k = e^{i\beta_k(A)}\psi_k, \quad M_{\mathbf{N}_k}(B_\varepsilon)\psi_k = e^{i\beta_k(\varepsilon)}\psi_k, \quad \|\psi_k\|_{\mathcal{H}_{\mathbf{N}_k}} = 1,$$

one has that, for every $\mu \in \mathcal{P}((\psi_k)_{k\geq1})$,

$$\mu = \alpha \, \mathrm{Leb}_{\mathbb{T}^{2d}} + \alpha_1 \nu_1 \otimes \mathrm{Leb}_{\mathbb{T}^d} + \alpha_2 \, \mathrm{Leb}_{\mathbb{T}^d} \otimes \nu_2 + (1 - \alpha - \alpha_1 - \alpha_2)\nu_0,$$

with

$$2\alpha + \alpha_1 + \alpha_2 \geq 1 - \varepsilon,$$

and $h_{\mathrm{KS}}(\nu_1, \tilde{B}_1) = h_{\mathrm{KS}}(\nu_2, \tilde{B}_2) = h_{\mathrm{KS}}(\nu_0, B) = 0$ for any $\tilde{B}_1 \in \langle \tilde{A}, \tilde{B}_\varepsilon \rangle \leq \mathrm{GL}(d, \mathbb{Z})$, $\tilde{B}_2 \in \langle \tilde{A}^T, \tilde{B}_\varepsilon^T \rangle \leq \mathrm{GL}(d, \mathbb{Z})$ and $B \in \langle A, B_\varepsilon \rangle \leq \mathrm{Sp}(2d, \mathbb{Z})$.

When A is a general element in $\mathrm{Sp}(2d, \mathbb{Z})$ such that no ratio of eigenvalues is a root of unity, our method still allows us to prove some similar regularity statement and we refer to Theorem 4.14 for a precise formulation of our general and main regularity result. Roughly speaking, it will state that any semiclassical measure can be decomposed as a sum of a zero entropy measure with weight $\leq 1/2$ and a measure whose projection along invariant isotropic and symplectic subtori is the Lebesgue measure. Under the present form, Theorem 1.2 already illustrates the kind of regularity property one can expect in a simplified setting. In particular, observe that if $\alpha = 0$ (i.e. the measure μ has no Lebesgue component), then the zero entropy part of the measure has weight at most ε. Again, if we consider joint eigenmodes for the full quantum action, we can pick $\varepsilon = 0$ (see Theorem 4.14).

1.3. **Comparison with Hecke operators for quantum maps.** Looking at joint eigenmodes associated with two commuting symplectic matrices is related to the notion of Hecke operators for quantum maps as it was introduced by Kurlberg and Rudnick in [KR00] for $d = 1$. This concept was extended to the case $d > 1$ by Kelmer [Kel10] and, using the conventions from this reference, we briefly review this construction for the sake of comparison. For simplicity, we suppose that A is irreducible and we refer to [Kel10] for the general separable case. The ring $\mathscr{D} := \mathbb{Z}[X]/(\chi_A)$, where χ_A is the characteristic polynomial of A, can be naturally embedded in $\mathrm{M}(2d, \mathbb{Z})$ by letting $\iota : p \mapsto p(A)$. Recall that, if $B \in \mathrm{GL}(2d, \mathbb{Z})$ commutes with A, then B is a rational polynomial in A (see e.g. Lemma 4.4). Given $p \in \mathscr{D}$, one can define p^* in such a way that $\iota(p^*) = p(A^{-1})$. Recall from [Kel10, Cor. 2.2] that $\iota(p) \in \mathrm{Sp}(2d, \mathbb{Z})$ if and only the "norm" $\mathcal{N}(p) := pp^*$ is equal to 1. In Theorem 1.1, the matrix B_ε is picked inside a fixed abelian subgroup Λ depending only on A and having rank $m(A) + l(A) \geq 2$. With the above conventions, one has $\langle A, B_\varepsilon \rangle \leq \Lambda \leq \mathrm{Ker}\, \mathcal{N}$.
According to [Kel10, §1.1.3], $M_{\mathbf{N}}(B)$ depends only B modulo $4\mathbf{N}$. Hence, one can introduce the map $\iota_{\mathbf{N}} : \mathscr{D}/4\mathbf{N}\mathscr{D} \to \mathrm{M}(2d, \mathbb{Z}/4\mathbf{N}\mathbb{Z})$ and the corresponding norm $\mathcal{N}_{\mathbf{N}} : \mathscr{D}/4\mathbf{N}\mathscr{D} \ni p \mapsto pp^* \in \mathscr{D}/4\mathbf{N}\mathscr{D}$. The Hecke group $C_A(\mathbf{N})$ is then as defined as a certain finite index subgroup of $\iota_{\mathbf{N}}\, (\mathrm{Ker}\, \mathcal{N}_{\mathbf{N}})$ with the index being bounded independently of $\mathbf{N}$. See [Kel10, §2] for more details. Kelmer then considered joint eigenfunctions for all the $M_{\mathbf{N}}(B)$ with B lying in the Hecke group. Thanks to [Kel10, Lemma 2.7], the number of elements in $C_A(\mathbf{N})$ is

bounded from below by $c_\epsilon \mathbf{N}^{d-\epsilon}$ and from above by $C_\epsilon \mathbf{N}^{d+\epsilon}$ and the lower bound plays an instrumental role in his proof of arithmetic quantum unique ergodicity. See for instance Propositions 3.6 and 3.7 in this reference.
In Theorem 1.1, the number of (implicitly) involved unitary matrices $M_{\mathbf{N}}(B)$ is given by $\sharp\iota_{\mathbf{N}}(\langle A, B_\varepsilon\rangle)$, and one has

$$\iota_{\mathbf{N}}(\langle A, B_\varepsilon\rangle) \leq \iota_{\mathbf{N}}(\Lambda) \leq \iota_{\mathbf{N}}(\text{Ker } \mathcal{N}) \leq \iota_{\mathbf{N}}(\text{Ker } \mathcal{N}_{\mathbf{N}}).$$

Hence, there are at most $\mathcal{O}(\mathbf{N}^{d+\epsilon})$ unitary matrices in that subgroup. However, the cardinal could be a priori much smaller in general and it is not clear if we could derive a good lower bound on this cardinal (that would allow to apply the arithmetic methods from [Kel10]) without further assumptions. Equivalently, we are picking only one (good) element B_ε mod $4\mathbf{N}$ in the Hecke group (on top of A mod $4\mathbf{N}$) and there is a priori no reason that these two elements generate[3] enough elements to apply the arithmetic averaging arguments from [Kel10, §3]. Despite that, Theorem 1.1 shows that one can already derive some equidistribution properties with only requiring to be a joint eigenmodes for 2 elements. Moreover, the dynamical argument we develop allows to deal with more general symplectic matrices as in Theorem 1.2 (see also Theorem 4.14). Finally, we refer to [RSOdA00] for numerics on the order of symplectic matrices modulo $\mathbf{N}$ when $d = 2$ and to [KORS24] for lower bounds on this order along typical sequences of integers.

1.4. **Organization of the article.** In §2, we briefly review the construction of semiclassical measures for symplectic linear automorphisms of the torus following [DJ24] and we recall what is known on the entropy of semiclassical measures in that setting. Then, in §3, we describe the centralizer of matrices in $\mathrm{Sp}(2d, \mathbb{Z})$. We gather these elements with rigidity results of Einsiedler and Lindenstrauss in §4 in order to derive the main theorem of this article, namely Theorem 4.14. Section 5 discusses the genericity of the assumption made on the matrix A in Theorem 1.1 and provides concrete examples of such matrices. Finally, Appendix A is a brief reminder on the metaplectic representation behind the construction of $M_{\mathbf{N}}(A)$.

2. Semiclassical measures for abelian actions

In this section, we briefly review the quantization procedure used to look at the quantum counterpart of a symplectic linear map acting on $\mathbb{T}^{2d}$. We closely follow the conventions from [DJ24] to which we refer for more details. We also recall the entropic results from [Riv11] that will be used in the subsequent sections.

Remark. Compared with [DJ24], the semiclassical parameter $\mathbf{N}$ lies here in $\mathbb{N}$ due to the fact that we deal with higher rank actions while [DJ24] allows for the more general case where $\mathbf{N} \in \frac{1}{2}\mathbb{N}$. Yet, as they picked the convention $\mathbf{N} \in \mathbb{N}$ (and not $\mathbf{N} \in \frac{1}{2}\mathbb{N}$) there is a factor 2 that differs from their convention in this brief exposition part.

2.1. **Quantum mechanics on $\mathbb{T}^{2d}$.** Let $\mathbf{N} \geq 1$ be a positive integer. One can set

$$\mathcal{H}_{\mathbf{N}} := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : U_w u = u \text{ for all } w = (q, p) \ \in \mathbb{Z}^{2d} \right\},$$

where

$$\forall u \in \mathcal{S}(\mathbb{R}^d), \quad U_{q,p} u(x) := e^{4i\pi \mathbf{N}\langle x, p\rangle} f(x - q).$$

[3]Recall for instance that, for $d = 1$, there exist subsequences of integers $(\mathbf{N}_k)_{k\geq 1}$ for which the number of elements generated by A is of order $\log \mathbf{N}_k$ [KR01].

One can verify that this defines a finite dimensional space of dimension $(2\mathbf{N})^d$ and this space is naturally endowed with an Hilbert structure. See [DJ24, Lemma 2.5] for more details on this construction.

Remark. We note that the general construction of these Hilbert spaces also involve a Floquet parameter $\theta \in \mathbb{T}^{2d}$ (which is fixed depending on A and $\mathbf{N}$) and that it can be carried out for $\mathbf{N} \in \frac{1}{2}\mathbb{N}$. Here, as we aim at dealing with quantum actions, we fix $\theta = 0$ and $\mathbf{N} \in \mathbb{N}$ so that the quantization condition [DJ24, Eq. (2.57)] on $(\theta, \mathbf{N})$ required to quantize a symplectic matrix are satisfied for any A.

Recall that a matrix B in $\mathrm{M}(2d, \mathbb{R})$ is said to be symplectic if $B^T J B = J$ where

$$J := \begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix},$$

where Id_d is the $d \times d$ identity matrix. The subgroup of symplectic matrices is denoted by $\mathrm{Sp}(2d, \mathbb{R})$. To any symplectic matrix B on $\mathbb{R}^{2d}$, one can associate its metaplectic representation $M_{\mathbf{N}}(B)$ of parameter $\mathbf{N} > 0$ which acts naturally on $\mathcal{S}'(\mathbb{R}^d)$ and which verifies, for every A_1 and A_2 in $\mathrm{Sp}(2d, \mathbb{R})$,

$$M_{\mathbf{N}}(A_1 A_2) = \pm M_{\mathbf{N}}(A_1) M_{\mathbf{N}}(A_2) \tag{2.1}$$

(see [Fol89, Ch. 4]). Note that this representation is projective, in the sense that it is defined up to a complex number of modulus 1. Yet, as explained in [Fol89, Th. 4.37], this number can be chosen so that (2.1) holds true. See also Appendix A for a brief reminder.
When $\mathbf{N}$ is a positive integer and when $A \in \mathrm{Sp}(2d, \mathbb{Z}) = \mathrm{Sp}(2d, \mathbb{R}) \cap \mathrm{GL}(2d, \mathbb{Z})$, one can look at the action of these matrices on the spaces $\mathcal{H}_{\mathbf{N}}$ defined above and verify that

$$M_{\mathbf{N}}(A) : \mathcal{H}_{\mathbf{N}} \to \mathcal{H}_{\mathbf{N}}.$$

See [DJ24, §2.2.4] for more details. Similarly, one has that

$$\forall a \in \mathcal{C}^\infty(\mathbb{T}^{2d}), \quad \mathrm{Op}^w_{(4\pi\mathbf{N})^{-1}}(a) : \mathcal{H}_{\mathbf{N}} \to \mathcal{H}_{\mathbf{N}},$$

where $\mathrm{Op}^w_h(a)$ is the Weyl quantization of the symbol a on $\mathbb{R}^{2d}$ [Zwo12, Ch. 4]. In order to emphasize the fact that one works with the restriction, we can set

$$\mathrm{Op}_{\mathbf{N}}(a) := \mathrm{Op}^w_{(4\pi\mathbf{N})^{-1}}(a)|_{\mathcal{H}_{\mathbf{N}}}.$$

A key property of this quantization procedure is the so-called Egorov property:

$$M_{\mathbf{N}}(A)^{-1} \mathrm{Op}_{\mathbf{N}}(a) M_{\mathbf{N}}(A) = \mathrm{Op}_{\mathbf{N}}(a \circ A). \tag{2.2}$$

2.2. **Semiclassical measures.** With these tools at hand and given $\psi \in \mathcal{H}_{\mathbf{N}}$ which is normalized, one can define the so-called Wigner distribution of ψ:

$$W_\psi : a \in \mathcal{C}^\infty(\mathbb{T}^{2d}) \mapsto \langle \mathrm{Op}_{\mathbf{N}}(a)\psi, \psi \rangle.$$

From the Calderón-Vaillancourt Theorem on $\mathbb{R}^d$ [Zwo12, Ch. 5] combined with [DJ24, Eq. (2.45)], one has

$$\| \mathrm{Op}_{\mathbf{N}}(a) \|_{\mathcal{L}(\mathcal{H}_{\mathbf{N}})} \leq C_d \sum_{|\alpha| \leq N_d} \mathbf{N}^{-\frac{|\alpha|}{2}} \|\partial^\alpha a\|_{\mathcal{C}^0}, \tag{2.3}$$

where $C_d, N_d > 0$ are constants depending only on the dimension. Hence, given a sequence

$$\psi_k \in \mathcal{H}_{\mathbf{N}_k}, \quad \|\psi_k\|_{\mathcal{H}_{\mathbf{N}_k}} = 1, \quad \lim_{k \to \infty} \mathbf{N}_k = \infty, \tag{2.4}$$

the sequence $(W_{\psi_k})_{k\geq 1}$ defines a bounded sequence in $\mathcal{D}'(\mathbb{T}^{2d})$ and we denote by $\mathcal{P}((\psi_k)_{k\to\infty})$ the corresponding set of accumulation points as $k\to\infty$. Thanks to the Gårding inequality [DJ24, Eq. (2.48)], any accumulation point is in fact a probability measure on $\mathbb{T}^{2d}$. If we suppose that, in addition to (2.4), the sequence ψ_k verifies

$$\mathcal{M}_{\mathbf{N}_k}(A)\psi_k = e^{i\beta_k}\psi_k, \text{ for some } \beta_k\in\mathbb{R}, \tag{2.5}$$

then any measure μ in $\mathcal{P}((\psi_k)_{k\to\infty})$ verifies $A_*\mu=\mu$ thanks to the Egorov property (2.2). The set of semiclassical measures for $A\in\mathrm{Sp}(2d,\mathbb{Z})$ is then defined as

$$\mathcal{P}_{\mathrm{sc}}(A) := \{\mu :\ \exists\ (\psi_k)_{k\geq 1} \text{ verifying (2.4) and (2.5) such that } \mu\in\mathcal{P}((\psi_k)_{k\to\infty})\}\,.$$

This defines a subset of the convex and compact[4] set $\mathcal{P}(A)$ made of A-invariant probability measure on $\mathbb{T}^{2d}$. The following holds true

Theorem 2.1 ([Riv11, Thm. 1.1])**.** *Let $A\in\mathrm{Sp}(2d,\mathbb{Z})$ and set*

$$\chi_+(A) \coloneqq \max\{\log|\lambda| :\ \lambda\in\sigma(A)\}\,.$$

Then, for every $\mu\in\mathcal{P}_{\mathrm{sc}}(A)$,

$$h_{\mathrm{KS}}(\mu,A) \geq \sum_{\lambda\in\sigma(A)} \max\left\{\log|\lambda| - \frac{\chi_+(A)}{2}, 0\right\},$$

where eigenvalues are counted with multiplicity and where $h_{\mathrm{KS}}(\mu,A)$ is the Kolmogorov-Sinai entropy of the measure μ with respect to A.

We also refer to [AS13] for analogues of this result in the context of compact locally symmetric spaces. See §2.3 for a reminder on the Kolmogorov-Sinai entropy.

Remark. Note that the theorem in [Riv11] is formulated for so-called quantizable matrices. This is to ensure that one can can also pick any $\mathbf{N}\in\frac{1}{2}\mathbb{N}$ (and thus a Floquet parameter $\theta\in\mathbb{T}^{2d}$ adapted to A). As we only deal with $\mathbf{N}\in\mathbb{N}$, we can always pick $\theta=0$ for any choice of symplectic matrix A.

Finally, the set of semiclassical measures for an abelian subgroup $\Lambda\leq\mathrm{Sp}(2d,\mathbb{Z})$ is defined as follows. The subgroup Λ is said to be *higher rank quantizable* if, for every $\mathbf{N}\in\mathbb{N}$ and for every $A,B\in\Lambda$,

$$M_{\mathbf{N}}(A)M_{\mathbf{N}}(B) = M_{\mathbf{N}}(B)M_{\mathbf{N}}(A).$$

Remark. If $\Lambda\leq\mathrm{Sp}(2d,\mathbb{Z})$ is abelian then it is finitely generated by [Seg83, Cor. 2.1]. Let $B_1,\dots,B_k$ be generators of Λ. Then the finite index subgroup generated by $B_1^2,\dots,B_k^2$ is higher rank quantizable thanks to (2.1). Moreover, if $A\in\Lambda$ then $\langle A,B_1^2,\dots,B_k^2\rangle$ is also a higher rank quantizable finite index subgroup of Λ containing A. In particular, if $\Lambda\leq\mathrm{Sp}(2d,\mathbb{Z})$ is abelian (and contains some element A) then there is a higher rank quantizable finite index subgroup of Λ (containing A).

Suppose now that, in addition to (2.4), the sequence $(\psi_k)_{k\geq 1}$ verifies

$$\forall A\in\Lambda,\quad M_{\mathbf{N}_k}(A)\psi_k = e^{i\beta_k(A)}\psi_k, \text{ for some } \beta_k(A)\in\mathbb{R}, \tag{2.6}$$

where $\Lambda\leq\mathrm{Sp}(2d,\mathbb{Z})$ is a higher rank quantizable subgroup. We can then define *the set of semiclassical measures for the abelian group* Λ as

$$\mathcal{P}_{\mathrm{sc}}(\Lambda) \coloneqq \{\mu :\ \exists\ (\psi_k)_{k\geq 1} \text{ verifying (2.4) and (2.6) such that } \mu\in\mathcal{P}((\psi_k)_{k\to\infty})\}\,.$$

[4] Recall that it is endowed with the weak-$\star$ topology induced by $\mathcal{C}^0(\mathbb{T}^{2d})$ on its dual.

From the Egorov theorem, one has $\mathcal{P}_{\text{sc}}(\Lambda) \subset \mathcal{P}_{\text{sc}}(A) \subset \mathcal{P}(A)$ for every $A \in \Lambda$. In particular, a measure $\mu \in \mathcal{P}_{\text{sc}}(\Lambda)$ is invariant under the action of Λ.

2.3. **A reminder on ergodic decomposition and Kolmogorov-Sinai entropy.** Let μ be an element in $\mathcal{P}(A)$. From the Birkoff ergodic theorem, one knows that there exists a subset Ω of $\mathbb{T}^{2d}$ such that $\mu(\Omega) = 1$ and such that, for every $f \in \mathcal{C}^0(\mathbb{T}^{2d}, \mathbb{C})$, one can find $f^* \in L^1(\mu)$ such that

$$\forall x \in \Omega, \quad \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} (f \circ A^k)(x) = f^*(x).$$

One can verify that the map

$$\mu_x : f \mapsto f^*(x), \quad f \in \mathcal{C}^0(\mathbb{T}^{2d}),$$

defines an element in $\mathcal{P}(A)$ which is ergodic for μ-almost every $x \in \mathbb{T}^{2d}$. This gives rise to the so-called *ergodic decomposition* of the measure μ [EW11, §6.1]:

$$\mu = \int_{\mathbb{T}^{2d}} \mu_x \, d\mu(x). \tag{2.7}$$

Fix now a partition $\mathcal{B} := (B_j)_{j=1,\dots K}$ of $\mathbb{T}^{2d}$ and denote by $\mathcal{B}^{(T)}$ the refined partition made of elements of the form

$$B_{\alpha_0} \cap A^{-1}(B_{\alpha_1}) \cap \ldots \cap A^{-T+1}(B_{\alpha_{T-1}}), \quad \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{T-1}) \in \{1, \ldots, K\}^T.$$

Given $T \geq 1$, one can associate to every point $x \in \mathbb{T}^{2d}$ a single element $B_T(x)$ in $\mathcal{B}^{(T)}$ such that $x \in B_T(x)$. The Shannon-McMillan-Breiman theorem ensures that, for μ-almost every $x \in \mathbb{T}^{2d}$, the limit $-\frac{1}{T} \ln \mu(B_T(x))$ exists and it is equal to the Kolmogorov-Sinai entropy of the measure μ_x with respect to the partition $\mathcal{B}$, i.e.

$$h_{\text{KS}}(\mu_x, A, \mathcal{B}) = \lim_{T \to \infty} -\frac{1}{T} \ln \mu(B_T(x))$$

(see [Par69, Ch. 3]). Moreover, the Kolmogorov-Sinai entropy of the measure μ (relative to $\mathcal{B}$) is given by

$$h_{\text{KS}}(\mu, A, \mathcal{B}) = \int_{\mathbb{T}^{2d}} h_{\text{KS}}(\mu_x, A, \mathcal{B}) d\mu(x).$$

Recall that the Kolmogorov-Sinai entropy $h_{\text{KS}}(\mu, A)$ of μ is then defined as the supremum over all the finite partitions $\mathcal{B}$. For other definitions of entropy, see [Wal82, Ch. 4]. One can show that

$$h_{\text{KS}}(\mu, A) = \int_{\mathbb{T}^{2d}} h_{\text{KS}}(\mu_x, A) d\mu(x).$$

Recall that these quantities are all nonnegative and that, for μ-almost every $x \in \mathbb{T}^{2d}$,

$$h_{\text{KS}}(\mu_x, A) \leq \sum_{\lambda \in \sigma(A)} \max \{\log |\lambda|, 0\}$$

where $\sigma(A)$ is the set of eigenvalues of A (counted with multiplicity) [Wal82, Th.8.15]. In particular, as a corollary of Theorem 2.1, one has

Corollary 2.2. *Let $A \in \mathrm{Sp}(2d, \mathbb{Z})$ and suppose that*

$$\chi_+(A) = \max\{\log|\lambda| : \lambda \in \sigma(A)\} > 0.$$

Then, for every $\mu \in \mathcal{P}_{\mathrm{sc}}(A)$,

$$\mu(\{x : h_{\mathrm{KS}}(\mu_x, A) > 0\}) \geq \frac{\sum_{\lambda\in\sigma(A)} \max\left\{\log|\lambda| - \frac{\chi_+(A)}{2}, 0\right\}}{\sum_{\lambda\in\sigma(A)} \max\{\log|\lambda|, 0\}},$$

where eigenvalues are counted with multiplicity.

3. Centralizers of symplectic matrices

In this section, we analyze the structure of the centralizer of a symplectic matrix with separable characteristic polynomial. The main result here is Theorem 3.12 describing the group structure of this centralizer. Before that, we also discuss two important cases that are used in the proof: the case of the linear group (Theorem 3.4) and the case where the characteristic polynomial is irreducible (Theorem 3.6).

Definition 3.1. A polynomial $f \in \mathbb{Q}[X]$ is called *separable* if it has no multiple roots in its splitting field (equivalently in $\mathbb{C}$).

Remark. If $f \in \mathbb{Q}[X]$ is irreducible, then f is separable. Indeed, if λ was a root of f of order ≥ 2 then $f(\lambda) = f'(\lambda) = 0$ which would contradict Bézout's identity.
More generally, if $f = p_1 \cdots p_k$ with $p_i \in \mathbb{Q}[X]$ irreducible, then f is separable if and only if all p_i are distinct. Clearly, if $p^2 \mid f$ for some non-constant irreducible $p \in \mathbb{Q}[X]$ then f has multiple root. Conversely, if f has a zero λ of order ≥ 2 then there is p_i such that $p_i(\lambda) = 0$. Since p_i is irreducible, p_i has no multiple zeros and hence $p_j(\lambda) = 0$ for some $j \neq i$. By Bézout's identity there are $r, s \in \mathbb{Q}[X]$ such that $rp_i + sp_j = 1$. This gives a contradiction when evaluating at λ.

3.1. Preliminary conventions.

We recall the following classical lemma in linear algebra:

Lemma 3.2. *Let $A \in \mathrm{GL}(m, \mathbb{Q})$.*

- *(i) If the characteristic polynomial χ_A of A is separable then the minimal polynomial m_A of A coincides with its characteristic polynomial χ_A.*
- *(ii) If $B \in \mathrm{M}(m, \mathbb{Q})$ commutes with A and $m_A = \chi_A$ then B is a rational polynomial in A. In particular, the centralizer of A in $\mathrm{GL}(m, \mathbb{Q})$ is abelian.*

Proof. Since $m_A \mid \chi_A \mid m_A^m$, m_A and χ_A have the same irreducible factors. By the assumption of separability each factor occurs exactly once so that both polynomials must be equal. For item (ii), we use the structure theorem for finitely generated modules over principal ideal domains. We find that, for the $\mathbb{Q}[X]$-module $\mathbb{Q}^m$ (where $X \cdot v = Av$), $\mathbb{Q}^m \simeq \bigoplus_i \mathbb{Q}[X]/(p_i^{m_i})$ for pairwise distinct irreducible polynomials $p_i \in \mathbb{Q}[X]$ with $\chi_A = m_A = \prod_i p_i^{m_i}$. The vector v with all components equal to $1 \in \mathbb{Q}[X]/(p_i^{m_i})$ is cyclic, i.e. $v, Av, \ldots, A^{m-1}v$ is a basis of $\mathbb{Q}^m$. Write then $Bv = \sum_{i=0}^{m-1} a_i A^i v$, $a_i \in \mathbb{Q}$, and it follows that $B = \sum_{i=0}^{m-1} a_i A^i$. □

For a semigroup G and $x \in G$ we denote by G_x the centralizer $G_x \coloneqq \{g \in G \mid gx = xg\}$. In order to make notation more natural, we now work over a general finite dimensional $\mathbb{Q}$-vector space V instead of $\mathbb{Q}^m$. The matrices with integer coefficients will be replaced by the following.

Definition 3.3. A *lattice* in a finite dimensional $\mathbb{Q}$-vector space V is a free abelian subgroup Γ of $(V,+)$ of rank $\dim V$, i.e. $\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{\dim V}$ for a basis (v_i) of V. We write $\mathrm{End}(\Gamma)$ for the endomorphisms B of V with $B(\Gamma) \subseteq \Gamma$ and $\mathrm{GL}(\Gamma) := \{B \in \mathrm{GL}(V) \mid B(\Gamma) = \Gamma\}$.

Note that $\mathrm{GL}(\Gamma)$ are the units in the ring $\mathrm{End}(\Gamma)$. Moreover, if we take a basis of Γ then the matrix of $B \in \mathrm{GL}(\Gamma)$ and of B^{-1} with respect to this basis has integer entries. In particular, $\mathrm{GL}(\mathbb{Z}^m) = \mathrm{GL}(m,\mathbb{Z})$. We also note that $\mathrm{GL}(\Gamma) = \{B \in \mathrm{End}(\Gamma) \mid \det B = \pm 1\}$. Indeed, if $B \in \mathrm{GL}(\Gamma)$ then we saw that suitable matrices representing B and B^{-1} have integer entries so that $\det B = \pm 1$. Conversely, if $B(\Gamma) \subset \Gamma$ and $\det B = \pm 1$ then $q = (1-(-1)^m \det B \chi_B)/X \in \mathbb{Z}[X]$ and $B^{-1} = q(B) \in \mathbb{Z}[B]$. Therefore, $B^{-1}(\Gamma) = q(B)(\Gamma) \subseteq \Gamma$ and $B \in \mathrm{GL}(\Gamma)$.

3.2. **The irreducible general linear case.** We begin with the following statement which is a consequence of Dirichlet's unit theorem.

Theorem 3.4. *Let $A \in \mathrm{GL}(\Gamma)$ with characteristic polynomial χ_A which is irreducible over $\mathbb{Q}$. Suppose that A has $r(A)$ real eigenvalues and $2c(A)$ eigenvalues in $\mathbb{C}\setminus\mathbb{R}$. Then $\mathrm{GL}(\Gamma)_A$ is an abelian group of rank $r(A)+c(A)-1$, i.e. $\mathrm{GL}(\Gamma)_A \simeq F \times \mathbb{Z}^{r(A)+c(A)-1}$ for a finite group F.*

Proof. We saw in Lemma 3.2 that $\mathrm{End}(V)_A = \mathbb{Q}[A]$. We also have $\mathbb{Q}[A] \simeq \mathbb{Q}[X]/(\chi_A)$ via $p(A) \leftrightarrow p + (\chi_A)$ and we also verified after Definition 3.3 that $\mathbb{Q}(A) = \mathbb{Q}[A]$. This defines an algebraic number field as χ_A is irreducible. Moreover, $\mathbb{Z}[A] \subseteq \mathrm{End}(\Gamma)_A \subseteq \mathbb{Q}[A] =: K$. Since $\mathbb{Z}[A]$ is a free abelian group of rank $\deg \chi_A = [K : \mathbb{Q}]$, $\mathrm{End}(\Gamma)_A =: \mathcal{O}$ is an order in K [Neu99, §I.12]. Its units $\mathcal{O}^\times$ are those elements of $\mathrm{End}(\Gamma)_A$ whose inverse are also in $\mathrm{End}(\Gamma)_A$: this means precisely $\mathcal{O}^\times = \mathrm{GL}(\Gamma)_A$. By Dirichlet's unit theorem (see e.g. [Neu99, Thm. 7.4, §I]) for the maximal order $\mathcal{O}_K$, one knows that $\mathcal{O}_K^\times$ is isomorphic to $F' \times \mathbb{Z}^{r(A)+c(A)-1}$ where F' is a finite cyclic group consisting of roots of unity. Then it follows from [Neu99, Th.12.12, §I] that $\mathcal{O}^\times \simeq F \times \mathbb{Z}^{r(A)+c(A)-1}$ where F is a finite group. □

Corollary 3.5. *Let $\varepsilon > 0$ and suppose that the assumptions of Theorem 3.4 are satisfied. Then there is $B \in \mathrm{GL}(\Gamma)_A$ such that*

$$\forall \lambda \in \sigma(B), \quad \frac{|\log|\lambda||}{\max\{|\log|\mu|| : \mu \in \sigma(B)\}} \in [0,\varepsilon] \cup [1-\varepsilon, 1],$$

with the convention $0/0 = 0$.

This corollary is in fact a consequence of the construction behind Dirichlet's unit theorem and we will explain it at the end of this paragraph. Besides proving this corollary, we also aim at deriving the analogues of these results in the symplectic setting. To that aim, the arguments used to show Dirichlet's unit theorem need to be discussed (and used) to take into account the symplectic structure. Hence, we briefly recall the main lines to prove Dirichlet's unit theorem in the case where the algebraic number field K is constructed from an element $A \in \mathrm{GL}(\Gamma)$ (with χ_A irreducible). We follow [Neu99, §I.7] and refer to it for more details.
Recalling that $K = \mathbb{Q}(A)$ and letting $W_\mathbb{R} = \mathbb{R}^{r(A)} \times \mathbb{C}^{c(A)}$, we define the (multiplicative) group morphism

$$j : K^\times \to W_\mathbb{R}^\times, \quad p(A) \mapsto (p(\lambda_1), \dots, p(\lambda_r), p(\mu_1), \dots, p(\mu_c)),$$

where $(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues of A and $(\mu_1, \overline{\mu}_1, \ldots, \mu_c, \overline{\mu}_c)$ are the ones in $\mathbb{C}\setminus\mathbb{R}$. Note that this defines an injective map. We also set the surjective morphism

$$\ell : \begin{cases} (W_{\mathbb{R}}^{\times}, \cdot) & \to \quad (\mathbb{R}^{r+c}, +), \\ (u_1, \ldots, u_r, z_1, \ldots, z_c) & \mapsto \quad \left(\log|u_1|, \ldots \log|u_r|, \log|z_1|^2, \ldots, \log|z_c|^2\right). \end{cases}$$

For $B \in K^{\times}$, we denote by $N_{K/\mathbb{Q}}(B) \in \mathbb{Q}^{\times}$ the determinant of the map $K \ni C \mapsto BC \in K$ viewed as a $\mathbb{Q}$-linear map. This is the field norm on $K/\mathbb{Q}$ and one has $N_{K/\mathbb{Q}} = N \circ j$ where

$$N(u_1, \ldots, u_r, z_1, \ldots, z_c) \coloneqq u_1 \ldots u_r |z_1|^2 \ldots |z_c|^2.$$

Recall now that Dirichlet's unit theorem is about the group structure of the multiplicative subgroup $\mathcal{O}^{\times} = \{B \in \mathcal{O} : N_{K/\mathbb{Q}}(B) \in \{\pm 1\}\}$. To study this question, one sets

$$\Lambda \coloneqq (\ell \circ j)(\mathcal{O}^{\times}) \subseteq H \coloneqq \left\{ X \in \mathbb{R}^{r+c} : \sum_{j=1}^{r+c} X_j = 0 \right\} \simeq \mathbb{R}^{r+c-1},$$

and one proves that this is a (full rank) lattice in that vector space [Neu99, Th. 7.3][5]. Letting $(v_1, \ldots, v_{r+c-1})$ be a $\mathbb{Z}$-basis for this lattice, its preimage $(g_1, \ldots, g_{r+c-1}) \in \mathcal{O}^{\times}$ by the map $\ell \circ j$ allows to define an abelian subgroup $G_0 = \langle g_1, \ldots, g_{r+c-1} \rangle$ which induces a surjective morphism onto $(\Lambda, +)$. Then, letting F' be the roots of unity lying in $\mathcal{O}$, one can prove that

$$1 \to F' \hookrightarrow \mathcal{O}^{\times} \xrightarrow{\ell \circ j} \Lambda \to 0$$

is an exact sequence [Neu99, Lemma 7.2] from which we deduce that

$$\mathcal{O}^{\times} = \left\{ f g^{n_1} \ldots g_{r+c-1}^{n_{r+c-1}} : \ f \in F', \ (n_1, \ldots, n_{r+c-1}) \in \mathbb{Z}^{r+c-1} \right\} \simeq F' \times \mathbb{Z}^{r+c-1}.$$

With these conventions at hand, we are ready to verify Corollary 3.5.

Proof of Corollary 3.5. For $B = p(A) \in \mathrm{GL}(\Gamma)_A$, the eigenvalues of B are exactly given by

$$(p(\lambda_1), \ldots, p(\lambda_r), p(\mu_1), p(\overline{\mu}_1), \ldots, p(\mu_c), p(\overline{\mu}_c)).$$

Hence, for every $\lambda \in \sigma(B)$, $\log|\lambda|$ (or $2\log|\lambda|$) is a coordinate of $\ell \circ j(B)$. Recall now that $\mathrm{GL}(\Gamma)_A = \mathrm{End}(\Gamma)_A^{\times}$ with $\mathrm{End}(\Gamma)_A$ being an order in K. Hence, there exists a compact set C in H so that $H = C + (\ell \circ j)(\mathrm{GL}(\Gamma)_A)$. For $M \in \mathbb{N}$, we now set $X_M = (M, -M, 0, \ldots 0)$ to be an element in H. Thanks to the above decomposition, X_M can be written as $x_M + z_M$ with $x_M \in C$ and $z_M \in \ell \circ j(\mathrm{GL}(\Gamma)_A)$. As C is a compact subset of H, it is contained inside $[-R_0, R_0]^{r+c}$ for some large enough $R_0 > 0$ (depending only on the lattice $\ell \circ j(\mathrm{GL}(\Gamma)_A)$). In particular,

$$z_M \in [M - R_0, M + R_0] \times [-M - R_0, -M + R_0] \times [-R_0, R_0]^{r+c-2}.$$

By construction, there exists $B_M \in \mathrm{GL}(\Gamma)_A$ such that $\ell \circ j(B_M) = z_M$ and B_M has the expected property if we pick M large enough (depending on ε). $\square$

[5] We note that in this reference they work with the maximal order $\mathcal{O}_K$ but everything works equally well with any other order $\mathcal{O}$.

3.3. **The irreducible symplectic case.** In this paragraph, we let (V,ω) be a finite dimensional symplectic vector space over $\mathbb{Q}$ and we let $A \in \mathrm{Sp}(\Gamma) := \mathrm{GL}(\Gamma) \cap \mathrm{Sp}(V,\omega)$ whose characteristic polynomial will be irreducible in the present §3.3. We aim at proving the following symplectic analogue of the result in the previous paragraph.

Theorem 3.6. *Let $A \in \mathrm{Sp}(\Gamma)$ with irreducible characteristic polynomial. Suppose that A have $2m(A)$ real eigenvalues and $4l(A)$ eigenvalues in $\mathbb{C} \setminus (\mathbb{R} \cup \mathbb{S}^1)$. Then $\mathrm{Sp}(\Gamma)_A$ is an abelian group of rank $m(A)+l(A)$, i.e. $\mathrm{Sp}(\Gamma)_A \simeq F \times \mathbb{Z}^{m(A)+l(A)}$ where F is a finite group.*

Remark. We note that, if $\chi_A \in \mathbb{Z}[X]$ is the characteristic polynomial of $A \in \mathrm{Sp}(\Gamma)$, then χ_A is *palindromic* of degree $2d = \dim V$ (i.e. $\chi_A(X^{-1})X^{2d} = \chi_A(X)$). Then $\chi_A' = 2dX^{2d-1}\chi_A(X^{-1}) - X^{2d-2}\chi_A'(X^{-1})$. In particular, $\chi_A'(\pm 1) = \pm 2d\chi_A(\pm 1) - \chi_A'(\pm 1)$. This implies that if $\chi_A(\pm 1) = 0$ then $\chi_A'(\pm 1) = 0$. Hence, ± 1 is not an eigenvalue of A if χ_A is separable.

Clearly, $\mathrm{Sp}(\Gamma)_A \leq \mathrm{GL}(\Gamma)_A$ and hence $\mathrm{Sp}(\Gamma)_A \simeq F' \times \mathbb{Z}^s$ with $F' = F \cap \mathrm{Sp}(\Gamma)$ and $s \leq r(A)+c(A)-1$. It remains to find s. In order to prove this theorem, we will let $\mathbb{Q}[A] =: K$. Recall that, thanks to our irreducibility assumption, ± 1 do not belong to $\sigma(A)$. Let us order the eigenvalues of A as follows: $\lambda_1, \lambda_1^{-1}, \dots \lambda_m, \lambda_m^{-1} \in \mathbb{R}$, $m = m(A)$, $\theta_1, \theta_1^{-1} = \overline{\theta_1}, \dots, \theta_k, \theta_k^{-1} \in \mathbb{S}^1$, $k = k(A)$ and $\mu_1, \overline{\mu_1}, \mu_1^{-1}, \overline{\mu_1}^{-1}, \dots, \mu_l, \overline{\mu_l}, \mu_l^{-1}, \overline{\mu_l}^{-1} \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{S}^1)$, $l = l(A)$. With the conventions of §3.2, one has $k + 2l = c$ and $2m = r$ as well as $W_\mathbb{R} = \mathbb{R}^{2m} \times \mathbb{C}^k \times \mathbb{C}^{2l}$ and we define j accordingly. From §3.2, $\ell \circ j(\mathrm{GL}(\Gamma)_A)$ is lattice of rank $r+c-1$ in $H \simeq \mathbb{R}^{r+c-1}$. We now define $W_\mathbb{R}' := \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{C}^l$ and

$$\mathcal{J}\colon \begin{cases} W_\mathbb{R} \to W_\mathbb{R}' \\ (u_1, \dots, u_{2m}, v_1, \dots, v_k, w_1 \dots, w_{2l}) \mapsto \begin{matrix}(u_1u_2, \dots, u_{2m-1}u_{2m}, \\ |v_1|^2, \dots, |v_k|^2, \\ w_1w_2, \dots, w_{2l-1}w_{2l}).\end{matrix} \end{cases}$$

One has the following characterization of symplectic matrices in K.

Lemma 3.7.

$$\mathrm{Sp}(\Gamma)_A = \{B \in \mathrm{GL}(\Gamma)_A \mid \mathcal{J}(j(B)) = (1, \dots, 1)\}$$

Proof. For $B \in \mathrm{End}(V)$, denote by B^* the unique endomorphism of V such that $\omega(Bv, w) = \omega(v, B^*w)$ for all $v, w \in V$.
If $B = p(A) \in \mathrm{End}(V)_A = K$ then $B^* = p(A^*) = p(A^{-1}) = p^\omega(A)$ where[6] $p^\omega = p(\frac{1-\chi}{X}) \in \mathbb{Q}[X]$. Therefore, $1 = B^*B$ if and only if $\chi_A \mid p^\omega p - 1$. This is equivalent to the fact that $p(s^{-1})p(s) = 1$ for every eigenvalue s of A. These are precisely the coordinates of $\mathcal{J} \circ j(B)$. By observing that $p(\theta_i^{-1}) = p(\overline{\theta_i}) = \overline{p(\theta_i)}$ we infer $\mathrm{Sp}(V,\omega)_A = \ker \mathcal{J} \circ j$. We finish the proof by intersecting with $\mathrm{GL}(\Gamma)_A$. □

Proof of Theorem 3.6. Let

$$G := \{w \in W_\mathbb{R}^\times \mid \mathcal{J}(w) = (1, \dots, 1)\} \leq (W_\mathbb{R}^\times, \cdot).$$

One has that $\ell \circ j(G)$ is a subgroup of $(\mathbb{R}^{2m+k+2l}, +)$. By construction, one has that

$$\ell(G) \subseteq \{(x, -x) \mid x \in \mathbb{R}\}^m \times \{0\}^k \times \{(y, -y) \mid y \in \mathbb{R}\}^l.$$

By putting $u_{2j+1} = e^x, u_{2j+2} = e^{-x}, v_j = 1, w_{2j+1} = e^{y/2}, w_{2j+2} = e^{-y/2}$, we see that equality holds. Hence, $\ell(G) \simeq \mathbb{R}^{m+l}$ as additive groups. Set now $U := G \cap j(\mathcal{O}^\times) =$

[6] In [Kel10] p^ω is denoted by p^*. In contrast to this notation we use $p^* = X^{\deg p}p(X^{-1})$.

$j(\mathrm{Sp}(\Gamma)_A)$, $\tilde{G} := N^{-1}(\{\pm 1\}) \leq W_{\mathbb{R}}^{\times}$ and $\tilde{U} := \tilde{G} \cap j(\mathcal{O}^{\times})$. By construction, G is a subgroup of $\tilde{G}$ and $U = \tilde{U} \cap G$. Hence, G/U embeds into $\tilde{G}/\tilde{U}$.
It follows from the discussion on Dirichlet's unit theorem in §3.2 that $\ell(\tilde{G})/\ell(\tilde{U})$ is compact, that $\ell(\tilde{U})$ is discrete and that $\ker \ell|_{\tilde{U}}$ is finite. Hence, $\ell(G)/\ell(U)$ is compact and, as $\ell(U)$ is discrete, we find that $\ell(U)$ is a full rank lattice in $\ell(G) \simeq \mathbb{R}^{m+l}$. Therefore, $\mathrm{Sp}(\Gamma)_A \simeq U \simeq \ker \ell|_U \times \mathbb{Z}^{m+l}$. □

As for $\mathrm{GL}(\Gamma)_A$, one has the following property as a consequence of the above construction.

Corollary 3.8 (of proof of Theorem 3.6)**.** *Let $\varepsilon > 0$. Then there is $B \in \mathrm{Sp}(\Gamma)_A$ such that*

$$\frac{\max\{\log|\lambda|, 0\}}{\max\{\log|\mu| : \mu \in \sigma(B)\}} \in [0, \varepsilon] \cup [1-\varepsilon, 1]$$

for all $\lambda \in \sigma(B)$ (with the convention $0/0 = 0$).

Proof. For this we observe that if $B = p(A) \in \mathrm{Sp}(\Gamma)_A$, then the eigenvalues of B are

$$p(\lambda_i), p(\lambda_i^{-1}), p(\theta_i), \overline{p(\theta_i)}, p(\mu_i), \overline{p(\mu_i)}, p(\mu_i^{-1}), \overline{p(\mu_i^{-1})}.$$

Therefore, one has again that $\log|\lambda|$ or $2\log|\lambda|$ (with $\lambda \in \sigma(B)$) correspond to the coordinates of $\ell(j(B))$. We saw above that $\ell(U)$ is a lattice of full rank in $\ell(G)$. Hence there is a compact set C in $\ell(G)$ such that $C + \ell(U) = \ell(G)$. We infer that for each $M \in \mathbb{N}$ there is $j(B_M) = u_M \in U$ and $x_M \in C$ such that $\ell(j(B_M)) + x_M = (M, -M, 0, \dots, 0) \in \ell(G)$ in the case $m \geq 1$. We have

$$\ell(j(B_M)) \in [M-R, M+R] \times [-M-R, -M+R] \times [-R, R]^{2m+k+2l-2}.$$

The claim follows by picking M large enough. The case $m = 0$ works similarly by picking the last coordinates. If $m = l = 0$ then all eigenvalues are of modulus 1 so we have $0/0$. □

3.4. **The general symplectic case.** We will now discuss the case of a general symplectic matrix in $\mathrm{Sp}(2d, \mathbb{Z})$ with separable characteristic polynomial. To that aim, we first collect a few statements taken from [Kel10, §2.2]. Given a polynomial $p \in \mathbb{Z}[X]$, we set $p^*(X) = X^{\deg p} p(X^{-1}) \in \mathbb{Z}[X]$. We can write

$$\chi_A = \prod_{i=1}^{r} p_i \prod_{j=1}^{s} \rho_j \rho_j^*,$$

with $p_i = p_i^*, \rho_j \in \mathbb{Z}[X]$ irreducible, pairwise distinct and $\rho_j \neq \rho_j^*$. Then, according to [Kel10, Prop. 2.4 and Rk. 2.4], one has

$$\mathbb{Q}^{2d} = \bigoplus_{i=1}^{r} \ker p_i(A) \oplus \bigoplus_{j=1}^{s} \left(\ker \rho_j(A) \oplus \ker \rho_j^*(A)\right), \tag{3.1}$$

where

- for every i, j, $\ker p_i(A)$ and $\ker \rho_j(A) \oplus \ker \rho_j^*(A)$ are orthogonal with respect to the symplectic form,
- for $i \neq i'$, $\ker p_i(A)$ and $\ker p_{i'}(A)$ are orthogonal with respect to the symplectic form,
- for $j \neq j'$, $\ker \rho_j(A) \oplus \ker \rho_j^*(A)$ and $\ker \rho_{j'}(A) \oplus \ker \rho_{j'}^*(A)$ are orthogonal with respect to the symplectic form,
- for every $1 \leq j \leq s$, $\ker \rho_j(A)$ and $\ker \rho_j^*(A)$ are isotropic spaces,

- for every i, j, $\ker p_i(A)$ and $\ker \rho_j(A) \oplus \ker \rho_j^*(A)$ are symplectic subspaces,
- for every i, j, $\ker p_i(A)$, $\ker \rho_j(A)$ and $\ker \rho_j^*(A)$ are irreducible subspaces for the action of A.

Remark. If we denote by $V_0^{\perp_\omega}$ the symplectic orthogonal of a linear subspace V_0, recall that V_0 is Lagrangian when $V_0^{\perp_\omega} = V_0$. When $V_0 \subseteq V_0^{\perp_\omega}$ (resp. $V_0^{\perp_\omega} \subseteq V_0$), we say that V_0 is isotropic (resp. coisotropic). When $V_0^{\perp_\omega} \cap V_0 = \{0\}$, the subspace is symplectic for $\omega|_{V_0}$.

In the following, we shall use the following convention, for every $1 \leq i \leq r$ and for every $1 \leq j \leq s$,

$$V_i := \ker p_i(A),\ W_j := \ker \rho_j(A) \oplus \ker \rho_j^*(A),\ \overline{W}_j := \ker \rho_j(A),\ \text{and}\ \overline{W}_j^* := \ker \rho_j^*(A). \tag{3.2}$$

By taking sums of the subspaces appearing in (3.2), we have a description of all the A-invariant subspaces of $\mathbb{Q}^{2d}$. We introduce the following sublattices:

$$\Delta_i := V_i \cap \mathbb{Z}^{2d},\ \Gamma_j := W_j \cap \mathbb{Z}^{2d},\ \overline{\Gamma}_j := \overline{W}_j \cap \mathbb{Z}^{2d} \text{ and } \overline{\Gamma}_j^* := \overline{W}_j^* \cap \mathbb{Z}^{2d}.$$

Their ranks are given by the dimension of V_i, W_j, $\overline{W}_j$ and $\overline{W}_j^*$ respectively. Indeed, if we start with a basis $v_1, \dots, v_n$ of one of the subspaces V_i, W_j, $\overline{W}_j$ or $\overline{W}_j^*$ then there is an integer N such that, for any i, Nv_i is in the sublattice. This shows that Δ_i, Γ_j, $\overline{\Gamma}_j$, and $\overline{\Gamma}_j^*$ are lattices in the respective subspaces. In particular, $\Gamma := \bigoplus_i \Delta_i \oplus \bigoplus_j \Gamma_j$ has finite index inside $\mathbb{Z}^{2d}$. One has

Lemma 3.9. *Let G be a subgroup of $\mathrm{GL}(2d, \mathbb{R})$. Then $G \cap \mathrm{GL}(2d, \mathbb{Z})$ has finite index in $G \cap \mathrm{GL}(\Gamma)$.*

Proof. Since the property of having finite index is stable under intersection with G, we can without loss of generality assume that $G = \mathrm{GL}(2d, \mathbb{R})$. There is $N \in \mathbb{N}$ such that $N\mathbb{Z}^{2d} \subseteq \Gamma$. Let $g \in \mathrm{GL}(\Gamma)$ then $g^{\pm 1}(\Gamma) \subseteq \Gamma$. Therefore, g acts on the finite space $\Gamma/N\Gamma$ so that there is $M \in \mathbb{N}$ such that $g^{\pm M}$ is the identity on $\Gamma/N\Gamma$. Hence, $g^{\pm M} = I + NX_\pm$ with $X_\pm(\Gamma) \subseteq \Gamma$. It follows that

$$(g^{\pm M} - I)(\mathbb{Z}^{2d}) \subseteq X_\pm(\Gamma) \subseteq \Gamma \subseteq \mathbb{Z}^{2d}.$$

This implies $g^M \in \mathrm{GL}(\mathbb{Z}^{2d})$. Since $\mathrm{GL}(\Gamma)$ is finitely generated (as it is isomorphic to $\mathrm{GL}(2d, \mathbb{Z})$ by choosing a basis), the lemma follows. □

In particular, this lemma shows that $\mathrm{Sp}(2d, \mathbb{Z})_A$ has finite index in

$$\mathrm{Sp}(\Gamma)_A \simeq \prod_{i=1}^{r} \mathrm{Sp}(\Delta_i)_{A|_{V_i}} \times \prod_{j=1}^{s} \mathrm{Sp}(\Gamma_j)_{A|_{W_j}}. \tag{3.3}$$

The group structure of $\mathrm{Sp}(\Delta_i)_{A|_{V_i}}$ was already described in Theorem 3.6 as the characteristic polynomial of $A|_{V_i}$ is $p_i = p_i^*$ which is irreducible. Hence, it remains to describe the group structure of $\mathrm{Sp}(\Gamma_j)_{A|_{W_j}}$. To that aim, observe first that the same argument as in the proof of Lemma 3.9 shows that $\mathrm{Sp}(\Gamma_j)_{A|_{W_j}}$ has finite index in $\mathrm{Sp}(\overline{\Gamma}_j \oplus \overline{\Gamma}_j^*)_{A|_{W_j}}$. Then it remains to use the following two lemmas:

Lemma 3.10. *Let $\tilde{A} \in \mathrm{Sp}(W, \omega)$ with characteristic polynomial $\chi_{\tilde{A}} = \rho\rho^*$ where $\rho \in \mathbb{Q}[X]$ irreducible and $\rho \neq \rho^*$. Then, there is an $\tilde{A}$-invariant Lagrangian subspace $\overline{W} \leq W$ such that $W \simeq \overline{W} \times \overline{W}^*$ (with $\overline{W}^*$ the dual space to $\overline{W}$) and $\omega((v, \lambda), (w, \mu)) = \lambda(w) - \mu(v)$. Moreover,* $\mathrm{Sp}(W, \omega)_{\tilde{A}} \simeq \{\mathrm{Diag}(B, (B^{-1})^T) \mid B \in \mathrm{GL}(\overline{W})_{\tilde{A}|_{\overline{W}}}\} \simeq \mathrm{GL}(\overline{W})_{\tilde{A}|_{\overline{W}}}$.

Remark. Under the assumption of this lemma, $\tilde{A}$ has no eigenvalues of modulus 1. Indeed, if $\rho(\theta) = 0$ for $|\theta| = 1$ then $\rho(\overline{\theta}) = 0 = \rho^*(\theta^{-1})$ which would contradict the separability as $\overline{\theta} = \theta^{-1}$.

Proof. We write $1 = r\rho + s\rho^*$. Let $v, v' \in \overline{W} =: \ker p(\tilde{A})$. Then $v = s\rho^*(\tilde{A})v$ and $\omega(v, v') = \omega(sp^*(\tilde{A}), v') = \omega(v, s^*p(\tilde{A})v) = 0$. Hence $\overline{W} \subseteq \overline{W}^{\perp\omega}$. Since $\dim \overline{W} = \dim \ker p^*(\tilde{A}) = \frac{1}{2}\dim W$ we have $\overline{W} = \overline{W}^{\perp\omega}$. Letting $\overline{W}' = \ker \rho^*(\tilde{A})$ we have $\overline{W}'^{\perp\omega} = \overline{W}'$ and $W = \overline{W} \oplus \overline{W}'$. It follows that $w' \mapsto \omega(w', \cdot)$ defines an isomorphism $J\colon \overline{W}' \to \overline{W}^*$. For $w \in \overline{W}$ and $\lambda \in \overline{W}^*$, we have $\omega(J^{-1}(\lambda), w) = \lambda(w)$. Hence, (W, ω) is $\overline{W} \times \overline{W}'$ with the standard symplectic form. Then clearly $B \in \mathrm{Sp}(W, \omega)_{\tilde{A}}$ if and only if B is of the form $\mathrm{Diag}(\overline{B}, (\overline{B}^{-1})^T)$ for $\overline{B} \in \mathrm{GL}(\overline{W})_{\tilde{A}|_{\overline{W}}}$. □

Lemma 3.11. *In the situation of Lemma 3.10 if $\overline{\Gamma}$ is a lattice in $\overline{W}$ and $\overline{\Gamma}^*$ is a lattice in $\overline{W}^*$ and $\tilde{A} \in \mathrm{Sp}(\overline{\Gamma} \otimes \overline{\Gamma}^*)$ then $\mathrm{Sp}(\overline{\Gamma} \oplus \overline{\Gamma}^*)_{\tilde{A}}$ is a finite index subgroup of $\mathrm{GL}(\overline{\Gamma})_{\tilde{A}|_{\overline{W}}}$ under the isomorphism $\mathrm{Sp}(W, \omega)_{\tilde{A}} \simeq \mathrm{GL}(\overline{W})_{\tilde{A}|_{\overline{W}}}$.*

Proof. As the isomorphism $\mathcal{R}\colon \mathrm{Sp}(W, \omega)_{\tilde{A}} \simeq \mathrm{GL}(\overline{W})_{\tilde{A}|_{\overline{W}}}$ is given by restriction to $\overline{W}$, $\mathrm{Sp}(\overline{\Gamma} \oplus \overline{\Gamma}^*)_{\tilde{A}}$ is mapped into $\mathrm{GL}(\overline{\Gamma})_{\tilde{A}|_{\overline{W}}}$. Let us define the so-called colattice $\mathrm{co}(\overline{\Gamma}) := \{\lambda \in \overline{W}^* \mid \lambda(\overline{\Gamma}) \subseteq \mathbb{Z}\}$. Then for $B = \mathrm{Diag}(\overline{B}, \overline{B}^{-T}) \in \mathrm{Sp}(W, \omega)_{\tilde{A}}$ with $\overline{B} \in \mathrm{GL}(\overline{\Gamma})$ we have $(\overline{B}^{-1})^T(\lambda)(\gamma) = \lambda(\overline{B}^{-1}\gamma) \in \mathbb{Z}$ for $\lambda \in \mathrm{co}(\overline{\Gamma})$ and $\gamma \in \overline{\Gamma}$. Hence, $B \in \mathrm{Sp}(\overline{\Gamma} \oplus \mathrm{co}(\overline{\Gamma}))$. It follows $\mathcal{R}(\mathrm{Sp}(\overline{\Gamma} \oplus \overline{\Gamma}^*)_{\tilde{A}}) \subseteq \mathrm{GL}(\overline{\Gamma})_{\tilde{A}|_{\overline{W}}} \subseteq \mathcal{R}(\mathrm{Sp}(\overline{\Gamma} \oplus \mathrm{co}(\overline{\Gamma}))_{\tilde{A}})$. By applying again Lemma 3.9 we find that $\mathrm{Sp}(\overline{\Gamma} \oplus \overline{\Gamma}^*)_{\tilde{A}}$ has finite index in $\mathrm{Sp}(\overline{\Gamma} \oplus \mathrm{co}(\overline{\Gamma}))_{\tilde{A}}$. This implies that $\mathcal{R}(\mathrm{Sp}(\overline{\Gamma} \oplus \overline{\Gamma}^*)_{\tilde{A}})$ has finite index in $\mathrm{GL}(\overline{\Gamma})_{\tilde{A}|_{\overline{W}}}$. □

As a consequence of this lemma and of the decomposition (3.3), we can deduce that $\mathrm{Sp}(2d, \mathbb{Z})_A$ has finite index in a subgroup isomorphic to

$$\prod_{i=1}^{r} \mathrm{Sp}(\Delta_i)_{A|_{V_i}} \times \prod_{j=1}^{s} \mathrm{GL}(\overline{\Gamma}_j)_{A|_{\overline{W}_j}}.$$

Combined with Theorems 3.4 and 3.6 and the fact that $A|_{\overline{W}_j}$ has no eigenvalue of modulus 1, we infer the following structure theorem:

Theorem 3.12. *Let $A \in \mathrm{Sp}(2d, \mathbb{Z})$ with separable characteristic polynomial and suppose that A has $2m(A)$ real eigenvalues and $4l(A)$ eigenvalues in $\mathbb{C} \setminus (\mathbb{R} \cup \mathbb{S}^1)$. Then, one has*

$$\mathrm{Sp}(2d, \mathbb{Z})_A \simeq F \times \mathbb{Z}^{m(A)+l(A)-I(A)}$$

where F is a finite abelian group (consisting of matrices B with $B^{|F|} = 1$) and $2I(A)$ is the number of irreducible isotropic invariant subspaces of A in $\mathbb{Q}^{2d}$.

4. Reducibility and rigidity of actions on tori

In this section, we combine the constructions from the previous sections with a rigidity result of Einsiedler and Lindenstrauss [EL03, EL22] in view of describing the regularity of semiclassical measures. Along the way, we also review some material from rigidity of $\mathbb{Z}^r$-actions on tori and describe criteria on the matrix A we started with where these results apply. This allows us to prove Theorems 1.1 and 1.2 from the introduction and to state and prove our main Theorem (Theorem 4.14) on semiclassical measures for joint eigenmodes.

4.1. **Preliminary conventions.** Let $m \in \mathbb{N}$. We consider the action of $\mathrm{GL}(m,\mathbb{Z})$ on $\mathbb{T}^m$. The following notion will be used in Theorem 4.2 below.

Definition 4.1 ([EL03]). (i) The action of a subgroup $\Lambda \subseteq \mathrm{GL}(m,\mathbb{Z})$ on $\mathbb{T}^m$ is called *irreducible* if every infinite Λ-invariant subgroup of $\mathbb{T}^m$ is dense.
(ii) The action is called *totally irreducible* if the action restricted to every finite index subgroup is irreducible.
(iii) The action is called *virtually cyclic* if there exists $g_0 \in \Lambda$ and $\Lambda' \leq \Lambda$ with finite index such that, for every $g \in \Lambda'$, one can find $k \in \mathbb{Z}$ such that $g = g_0^k$.

Observe that, if Λ is abelian and if the rank of Λ is ≥ 2 then there is an injective group morphism $\rho : \mathbb{Z}^2 \to \Lambda$, and the action of Λ is not virtually cyclic. Indeed, otherwise, there would exist $g_0 \in \Lambda$ and $\Lambda' \leq \Lambda$ of finite index such that, for all $g' \in \Lambda'$, one has $g' = g_0^k$ for some k. Finite index would then ensure the existence of $p \in \mathbb{Z}$ such that $g^p \in \Lambda'$ and $h^p \in \Lambda'$ (where $g = \rho(1,0)$ and $h = \rho(0,1)$ are given by the group morphism). One would have then $g^p = g_0^k$ and $h^p = g_0^l$ for some $k, l \in \mathbb{Z}$. Hence, $\rho(pl, -pk) = g^{pl}h^{-pk} = \mathrm{Id}_m$ which would contradict the injectivity of ρ.

4.2. **The irreducible case.** The following theorem by Einsiedler and Lindenstrauss is the key ingredient in our classification of semiclassical measures in the irreducible case.

Theorem 4.2 ([EL03, EL22]). *Let $\Lambda \leq \mathrm{GL}(m,\mathbb{Z})$ be a totally irreducible abelian subgroup*[7] *of rank ≥ 2. Let μ be an ergodic measure on $\mathbb{T}^m$ for the action of Λ. Then either μ is the Lebesgue measure or $h_{\mathrm{KS}}(\mu, B) = 0$ for all $B \in \Lambda$.*

Remark. We remark that the original version is more general since it allows solenoids instead of $\mathbb{T}^m$, non-faithful actions of $\mathbb{Z}^r$, as well as non-irreducible actions (see below). In these references, the authors made the "not virtually cyclic" assumption. Here, this is automatically satisfied as we supposed that the abelian subgroup Λ has rank ≥ 2 and we consider the natural action of $\mathrm{GL}(m,\mathbb{Z})$ on $\mathbb{T}^m$ which is faithful.

When applied to semiclassical measures, this theorem combined with Corollary 2.2 directly yields:

Corollary 4.3. *Let $\Lambda \leq \mathrm{Sp}(2d,\mathbb{Z})$ be an abelian subgroup which is quantizable and totally irreducible with rank ≥ 2. Then, for any $\mu \in \mathcal{P}_{\mathrm{sc}}(\Lambda)$, one has*

$$\mu = \alpha \,\mathrm{Leb}_{\mathbb{T}^{2d}} + (1-\alpha)\nu,$$

where $h_{\mathrm{KS}}(\nu,\gamma) = 0$ for every $\gamma \in \Lambda$ and where

$$\alpha \geq \max_{\gamma \in \Lambda : \chi_+(\gamma) > 0} \left\{ \frac{\sum_{\lambda \in \sigma(A)} \max\left\{\log|\lambda| - \frac{\chi_+(\gamma)}{2}, 0\right\}}{\sum_{\lambda \in \sigma(\gamma)} \max\left\{\log|\lambda|, 0\right\}} \right\},$$

with eigenvalues counted with multiplicity.

Proof. We write the ergodic decomposition of μ with respect to the action of Λ, i.e. $\mu = \int_{\mathcal{E}} e d\tau(e)$ where e runs over the set $\mathcal{E}$ of Λ-ergodic measures. From Theorem 4.2, e is either the Lebesgue measure or has zero entropy for every γ in Λ. So that we can decompose $\mu = \alpha \,\mathrm{Leb}_{\mathbb{T}^{2d}} + (1-\alpha)\nu$ where ν has zero entropy for every $\gamma \in \Lambda$. Applying Corollary 2.2 concludes the proof. ∎

[7]We note that Λ is finitely generated by [Seg83, Cor. 2.1].

We are now left with finding conditions ensuring that a quantizable action is totally irreducible with rank ≥ 2. To that aim, we state the following lemma.

Lemma 4.4. *(i) Let $A \in \mathrm{GL}(m, \mathbb{Z})$. The closed connected A-invariant subgroups of $\mathbb{T}^m$ are in one to one correspondence with the A-invariant subspaces of $\mathbb{Q}^m$. The correspondence is as follows: given a closed connected A-invariant subgroup T of $\mathbb{T}^m$, its tangent space is $V(T) \otimes \mathbb{R}$ where $V(T)$ is the corresponding A-invariant subspace of $\mathbb{Q}^m$.*

(ii) For $A \in \mathrm{GL}(m, \mathbb{Z})$ the only A-invariant subspaces of $\mathbb{Q}^m$ are $\{0\}$ and $\mathbb{Q}^m$ if and only if A has irreducible characteristic polynomial over $\mathbb{Q}$.

(iii) If $A \in \mathrm{GL}(m, \mathbb{Z})$ has separable characteristic polynomial and $\Lambda \leq \mathrm{GL}(m, \mathbb{Z})$ is an abelian subgroup containing A, then Λ is irreducible if and only if A has irreducible characteristic polynomial.

We note that in (iii) the possible Λ are subgroups of $\mathrm{GL}(m, \mathbb{Z})_A$. More precisely, if $\Lambda \leq \mathrm{GL}(m, \mathbb{Z})$ is abelian and $A \in \Lambda$, then $\Lambda \subseteq \mathrm{GL}(m, \mathbb{Z})_A$. Hence, (iii) can be applied to any abelian $\Lambda \leq \mathrm{GL}(m, \mathbb{Z})$ that contains some matrix with separable characteristic polynomial.

Remark. The case of invariant subsets instead of subgroups of $\mathbb{T}^m$ is characterized in [Ber83]. In this case one needs two more conditions for the absence infinite invariant closed subsets. In particular, one has to make assumptions on the eigenvalues and one needs higher rank.

Proof. The arguments we give are similar to [DJ24, Lemma 4.3] (see also [KAO24, App.]). For (i) let $T \leq \mathbb{T}^m$ be a connected closed A-invariant subgroup of $\mathbb{T}^m$. Then T is a Lie subgroup with Lie algebra $\mathfrak{t} \leq \mathbb{R}^m$ which is A-invariant. The exponential map of $\mathbb{T}^m$ is just the quotient map π. The exponential map for T is its restriction and is surjective since T is connected and abelian. Hence, $T = \pi(\mathfrak{t}) = (\mathfrak{t} + \mathbb{Z}^m)/\mathbb{Z}^m \simeq \mathfrak{t}/(\mathfrak{t} \cap \mathbb{Z}^m)$. Since T is compact, $\mathfrak{t} \cap \mathbb{Z}^m$ is a cocompact lattice in $\mathfrak{t}$. As T (and thus $\mathfrak{t}$) is A-invariant, we obtain that $V := \mathfrak{t} \cap \mathbb{Q}^m$ is an A-invariant subspace of $\mathbb{Q}^m$.
Conversely, if $V \leq \mathbb{Q}^m$ is an A-invariant subspace then $\pi(V \otimes \mathbb{R})$ is a connected A-invariant subgroup of $\mathbb{T}^m$. It is also closed as V is contained in $\mathbb{Q}^m$. Indeed, by Lemma 4.5 below we choose a complement W of V in $\mathbb{Q}^m$ such that $\mathbb{Z}^m = (V \cap \mathbb{Z}^m) + (W \cap \mathbb{Z}^m)$. If now $\pi(x_q) \to \pi(x)$ with $(x_q)_{q \geq 1} \in V \otimes \mathbb{R}$ and $x \in \mathbb{R}^m$ then there are $(z_q)_{g \geq 1} \in \mathbb{Z}^m$ such that $x_q + z_q \to x$. But $x = v + w \in (V \otimes \mathbb{R}) \oplus (W \otimes \mathbb{R})$ and $z_q = v_q + w_q \in (V \cap \mathbb{Z}^m) + (W \cap \mathbb{Z}^m)$. Therefore, $w_q \to w \in \mathbb{Z}^m$ and $\pi(x) = \pi(v) \in \pi(V \otimes \mathbb{R})$. The two constructions are clearly inverse to each other.
We now turn to the proof of the next items and we regard $\mathbb{Q}^m$ as $\mathbb{Q}[X]$-module where $X \cdot v = Av$. Then, by the structure theorem for finitely generated modules over principal ideal domains and the Chinese remainder theorem, one has $\mathbb{Q}^m = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m_i} \mathbb{Q}[X]/(p_i^{\nu_{i,j}})$ where p_i are irreducible and pairwise distinct and $\nu_{i,1} \leq \nu_{i,2} \leq \cdots \leq \nu_{i,m_i}$. One has

$$\chi_A = \prod_{i=1}^{l} p_i^{\sum_{j=1}^{m_i} \nu_{i,j}}.$$

The A-invariant subspaces of $\mathbb{Q}^m$ are precisely $\mathbb{Q}[X]$-submodules. By Bézout's identity, one finds

$$V_i = \bigoplus_{j=1}^{m_i} \mathbb{Q}[X]/\left(p_i^{\nu_{i,j}}\right) = \left\{v \in \mathbb{Q}^m \mid p_i^{\nu_{i,m_i}}(A)v = 0\right\}.$$

Moreover, using again Bézout's identity, each $\mathbb{Q}[X]$-submodule is a direct sum of submodules of V_i. For an irreducible p, the submodules of $\mathbb{Q}[X]/(p^\nu)$ are precisely the ones generated by $1, p, p^2, \dots, p^\nu$. Hence in order to have no submodules at all we must have $l = 1$, $m_i = 1$, and $v_{i,1} = 1$, i.e. $\mathbb{Q}^m = \mathbb{Q}[X]/(p)$ with p irreducible. This is equivalent to saying that χ_A is irreducible over $\mathbb{Q}$ and (ii) is proved. For (iii) we observe that by Lemma 3.2 the A-invariant subspaces and the Λ-invariant subspaces coincide. The claim follows by (ii). □

Lemma 4.5. *Let $V \leq \mathbb{Q}^m$ be a subspace. Then there exists a complement W of V in $\mathbb{Q}^m$ such that $\mathbb{Z}^m = (V \cap \mathbb{Z}^m) + (W \cap \mathbb{Z}^m)$.*

Proof. The sublattice $V \cap \mathbb{Z}^m$ is a subgroup of the free abelian group $\mathbb{Z}^m$. Hence there is a basis $v_1, \dots, v_m$ of $\mathbb{Z}^m$, $k \in \mathbb{N}$, and $d_1, \dots, d_k \in \mathbb{Z}$ such that $d_1 v_1, \dots, d_k v_k$ is a basis of $V \cap \mathbb{Z}^m$ (see e.g. [Art91, Thm. 4.11]) and thus $v_1, \dots, v_k$ is a basis of $V \cap \mathbb{Z}^m$). Indeed, for $i = 1, \dots k$, $d_i v_i \in V$ implies $v_i \in V \cap \mathbb{Z}^m$. It follows that there are $a_{i,j} \in \mathbb{Z}$ such that $v_i = \sum_{j=1}^k a_{i,j} d_j v_j$. This implies $a_{i,i} d_i = 1$ and therefore $d_i = \pm 1$.
We also observe that $V = \langle v_1, \dots, v_k \rangle_{\mathbb{Q}}$ as for every $v \in V$ there is $N \in \mathbb{Z}$ with $Nv \in \mathbb{Z}^m$. The subspace $W := \langle v_{k+1}, \dots, v_m \rangle_{\mathbb{Q}}$ is then a complement of V and we find that $\mathbb{Z}^m = (V \cap \mathbb{Z}^m) + (W \cap \mathbb{Z}^m)$. □

In order to deal with finite index subgroups we formulate the following lemma.

Lemma 4.6. *Let $A \in \mathrm{GL}(m, \mathbb{Q})$ with separable characteristic polynomial χ_A. Then χ_{A^k} is separable for all $k \in \mathbb{N}$ if and only if no quotient of eigenvalues is a root of unity, If λ/μ for two eigenvalues λ, μ of A is a root of unity, then $(\lambda/\mu)^N = 1$ for some $N \in \mathbb{N}$ with $\varphi(N) \leq m^2$ where φ is Euler's totient function.*

Here and after, eigenvalues are considered with their multiplicity. In particular, if no quotient of eigenvalues is a root of unity, all eigenvalues of A are distinct (which is exactly asking the characteristic polynomial to be separable).

Remark. If χ_A is irreducible and no ratio of eigenvalues of A is a root of unity, then χ_{A^k} is irreducible for all $k \in \mathbb{N}$. Using Lemma 3.2, one has that A is of the form $P(A^k)$ with $P \in \mathbb{Q}[X]$. So each A^k-invariant subspace is A-invariant and we can use Lemma 4.4.

Proof. Let $\chi_A = \prod(X - \lambda_i)$ with $\lambda_i \in \mathbb{C}$. Then $\chi_{A^k} = \prod(X - \lambda_i^k) \in \mathbb{Q}[X]$. By assumption $\lambda_i \neq \lambda_j$ for $i \neq j$ and we have to show that $\lambda_i^k \neq \lambda_j^k$ for $i \neq j$, i.e. $(\lambda_i/\lambda_j)^k \neq 1$. This is the assumption of the lemma.
If λ, μ are eigenvalues and λ/μ is a primitive N-th root of unity then

$$\varphi(N) = [\mathbb{Q}[\lambda/\mu] \colon \mathbb{Q}] \leq [\mathbb{Q}[\lambda, \mu] \colon \mathbb{Q}] \leq [\mathbb{Q}[\lambda] \colon \mathbb{Q}] \cdot [\mathbb{Q}[\mu] \colon \mathbb{Q}] = m^2.$$ □

Remark.
- If $\sigma(A) \subseteq \mathbb{R}_+$ then $\lambda/\mu \in \mathbb{R}_+$ for all eigenvalues λ and μ. Hence this ratio cannot be a non-trivial root of unity.
- $\varphi(N) \to \infty$ for $N \to \infty$. Therefore, we only have to check finitely many powers to apply the above lemma. More precisely, $\varphi(N) \geq \frac{N \log 2}{\log(2N)}$ for $N \geq 2$. We refer to [Rib88, Ch. 4.I.C] and [RS62, Thm. 15] for this and more explicit lower bounds.

As a corollary of Lemma 4.4, we directly obtain the following statement.

Corollary 4.7. *Let $A \in \mathrm{GL}(m, \mathbb{Z})$ such that no ratio of eigenvalues is a root of unity. An abelian subgroup $\Lambda \leq \mathrm{GL}(m, \mathbb{Z})$ containing A is totally irreducible if and only if χ_A is irreducible.*

Proof. If Λ is totally irreducible, then Λ must be irreducible so that by Lemma 4.4 χ_A is irreducible.

Conversely, let χ_A be irreducible and, using Lemma 4.4, let V be subspace of $\mathbb{Q}^m$ invariant under some finite index subgroup Λ' of Λ. There is some $k \in \mathbb{N}$ such that $A^k \in \Lambda'$. By Lemma 4.6 χ_{A^k} is separable and thus χ_{A^k} is the minimal polynomial of A^k. It follows from Lemma 3.2 that A is a rational polynomial in A^k. Therefore, V is A-invariant. By Lemma 4.4 V is $\{0\}$ or $\mathbb{Q}^m$. □

We are now ready to prove Theorem 1.1 from the introduction.

Proof of Theorem 1.1. We assumed that χ_A is irreducible and no ratio of eigenvalues is root of unity. Hence, by Corollary 4.7 any abelian subgroup of $\mathrm{Sp}(2d,\mathbb{Z})$ containing A is totally irreducible. We also assumed that $m(A) + l(A) \geq 2$. By Theorem 3.6 and Corollary 3.8 there is, for any $\varepsilon > 0$, $B_\varepsilon \in \mathrm{Sp}(2d,\mathbb{Z})_A$ such that

$$\forall \lambda \in \sigma(B_\varepsilon), \quad \frac{\max\{\log|\lambda|, 0\}}{\chi_+(B_\varepsilon)} \in [0,\varepsilon] \cup [1-\varepsilon, 1]$$

and $\langle A, B_\varepsilon\rangle$ has rank 2. We now apply Corollary 3.8 to a quantizable finite index subgroup of $\langle A, B_\varepsilon\rangle$ to obtain Theorem 1.1. □

4.3. **The general case.** In case the irreducibility of the characteristic polynomial does not hold, we saw that the abelian subgroup is not irreducible. Yet, Einsiedler and Lindenstrauss showed that one can still describe Λ-ergodic measures in that case. To state this result in a concise form, we introduce the following definition.

Definition 4.8. Let $\Lambda \leq \mathrm{GL}(m,\mathbb{Z})$ be an abelian subgroup. Let $T \leq \mathbb{T}^m$ be a closed and connected Λ-invariant subgroup[8]. We say that a Λ-invariant probability measure μ is an (T,Λ)-*admissible measure* if it is also T-invariant[9] and if the induced measure on $\mathbb{T}^m/T$ has zero Kolmogorov-Sinai entropy for every $B \in \Lambda$.

Note that, for $T = \{0\}$, (T,Λ)-admissible measures are just Λ-invariant measures with zero entropy, whereas for $T = \mathbb{T}^m$ the only (T,Λ)-admissible measure is the Lebesgue measure on $\mathbb{T}^m$. With this convention at hand, we can formulate the rigidity theorem as follows:

Theorem 4.9 ([EL03, EL22]). *Let Λ be an abelian subgroup of $\mathrm{GL}(m,\mathbb{Z})$ (of rank $r \geq 2$) that has no virtually cyclic factors. Let μ be a Λ-ergodic measure on $\mathbb{T}^m$. Then, there exist $\Lambda' \leq \Lambda$ of finite index and Λ'-invariant closed connected subgroups $T_1, \dots, T_M \leq \mathbb{T}^m$ such that*

$$\mu = \frac{1}{M}\left(\mu_1 + \ldots + \mu_M\right),$$

where each μ_j is a Λ'-ergodic measure which is (T_j,Λ')-admissible and where

$$\forall \gamma \in \Lambda, \quad \gamma_*\mu_j = \mu_i \text{ and } \gamma(T_j) = T_i \text{ for some } i.$$

Given $\Lambda \leq \mathrm{GL}(m,\mathbb{Z})$ and $T \leq \mathbb{T}^m$ a connected closed Λ-invariant subgroup distinct from $\mathbb{T}^m$, recall that a factor of Λ is the induced action of Λ on $\mathbb{T}^m/T$.

Remark. The formulation in [EL22] does not assume connectedness for the $(T_j)_{1\leq j\leq M}$. Yet, we can assume all the T_j to be connected in this theorem. Indeed, if not all the T_j were connected, we could proceed as follows. Since $\gamma(T_j) = T_i$, we must have $\gamma(T_j^\circ) \subseteq T_i^\circ$ for all $\gamma \in \Lambda$ where we denote by T° the connected component of the neutral element. It follows that $\gamma(T_j^\circ) = T_i^\circ$. Moreover, μ_j is T_j°-invariant, and T_j° is a Λ'-invariant closed and

[8]Recall from Lemma 4.4 that this corresponds to a Λ-invariant subspace of $\mathbb{Q}^m$.

[9]Hence, the induced measure on T is the Haar measure.

connected subgroup. Observe now that T_j° has finite index in T_j so that $\mathbb{T}^m/T_j^\circ \to \mathbb{T}^m/T_j$ is a finite cover. Therefore, $h_{\mathrm{KS}}(\mu_{\mathbb{T}^m/T_j^\circ}, \gamma) = h_{\mathrm{KS}}(\mu_{\mathbb{T}^m/T_j}, \gamma) = 0$ for all $\gamma \in \Lambda'$. In other words, μ_j is (T_j°, Λ')-admissible.

When combined with Theorem 2.1, this theorem provides conditions on semiclassical measures for quantizable actions with no virtually cyclic factors. Recall indeed from the proof of Lemma 4.4 that, for a closed and connected Λ'-invariant subgroup $T \leq \mathbb{T}^m$, one can find a Λ'-invariant subspace $V(T)$ of $\mathbb{Q}^m$ such that the tangent space to T is given by $V(T) \otimes \mathbb{R}$. One has also [Wal82, Th. 8.15]

$$\forall \gamma \in \Lambda', \quad h_{\mathrm{KS}}(\mathrm{m}_T, \gamma) = \sum_{\lambda \in \sigma(\gamma|_{V(T)})} \max\{\log|\lambda|, 0\}, \tag{4.1}$$

where m_T is the Haar measure on T. Thanks to the facts that $h_{\mathrm{KS}}(\mu_j, \gamma) = h_{\mathrm{KS}}(\mathrm{m}_{T_j}, \gamma)$ and that the Kolmogorov-Sinai entropy is an affine function, one finds that the entropy of the measure μ in the above Theorem is given by

$$\forall \gamma \in \Lambda', \quad h_{\mathrm{KS}}(\mu, \gamma) = \frac{1}{M} \sum_{j=1}^{M} \sum_{\lambda \in \mathrm{Sp}(\gamma|_{V(T_j)})} \max\{\log|\lambda|, 0\}.$$

Hence, as Λ' has finite index, one has that, for every $\gamma \in \Lambda$, one has $\gamma^{[\Lambda:\Lambda']} \in \Lambda'$. The previous equality translates into

$$\forall \gamma \in \Lambda, \quad h_{\mathrm{KS}}(\mu, \gamma) = \frac{1}{M} \sum_{j=1}^{M} \sum_{\lambda \in \sigma(\gamma^{[\Lambda:\Lambda']}|_{V(T_j)})} \max\{\log|\lambda|, 0\}.$$

Together with Theorem 2.1, this provides constraints on the allowed semiclassical measures as any Λ-invariant measure can be decomposed as a convex sum of Λ-ergodic measures:

$$\mu = \int_{\mathcal{E}} \mathrm{e} \, d\tau(\mathrm{e}),$$

where $\mathcal{E}$ is the set of Λ-ergodic measures. The fact that entropy is affine implies that

$$\forall \gamma \in \Lambda, \quad h_{\mathrm{KS}}(\mu, \gamma) = \int_{\mathcal{E}} h_{\mathrm{KS}}(\mathrm{e}, \gamma) d\tau(\mathrm{e}) = \int_{\mathcal{E}} \frac{1}{M_{\mathrm{e}}} \sum_{j=1}^{M_{\mathrm{e}}} \sum_{\lambda \in \sigma(\gamma^{[\Lambda:\Lambda_{\mathrm{e}}]}|_{V(T_j)})} \max\{\log|\lambda|, 0\} \, d\tau(\mathrm{e}),$$

which can be compared with the lower bound in Theorem 2.1. Observe that these constraints are not so easy to exploit in general due to the fact that it involves a finite index subgroup Λ_{e} that depends on the ergodic component $\mathrm{e} \in \mathcal{E}$.
Motivated by the previous discussion, we can make the following definition that will provide simple settings to apply this theorem. We will give in §4.4 below two simple (and nontrivial) examples with this property.

Definition 4.10. We call a subgroup $\Lambda \leq \mathrm{GL}(m, \mathbb{Z})$ *tame* if there are at most finitely many closed connected Λ-invariant subgroups of $\mathbb{T}^m$ and each closed connected subgroup of $\mathbb{T}^m$ invariant by a finite index subgroup of Λ is already invariant by Λ.

Remark. When $A \in \mathrm{Sp}(2d, \mathbb{Z})$, recall from the decomposition (3.1) that, as soon as χ_A is separable, $\mathbb{T}^{2d}$ has finitely many closed connected A-invariant subgroups thanks to Lemma 4.4.

Theorem 4.9 now reads as follows for tame subgroups.

Corollary 4.11. *Let $\Lambda \leq \mathrm{GL}(m,\mathbb{Z})$ be a tame abelian subgroup with closed connected invariant subgroups $T_1,\dots,T_\ell \leq \mathbb{T}^m$ and with no virtually cyclic factors.*
Then, for any Λ-ergodic measure μ, one can find $1 \leq l \leq \ell$ such that μ is (T_l,Λ)-admissible.

Proof. Using Theorem 4.9 there is a finite index subgroup $\Lambda' \leq \Lambda$ and Λ'-invariant closed connected subgroups $\tilde{T}_1,\dots,\tilde{T}_M \leq \mathbb{T}^m$ such that $\mu = \frac{1}{M}(\mu_1 + \cdots \mu_M)$, where, for each $j = 1,\dots,M$, μ_j is Λ'-ergodic, $(\tilde{T}_j,\Lambda')$-admissible and for any $\gamma \in \Lambda$ there is i such that $\gamma_*\mu_j = \mu_i$ as well as $\gamma(\tilde{T}_j) = \tilde{T}_i$. Each $\tilde{T}_i$ is Λ-invariant as it is Λ'-invariant and Λ is tame. Hence, each $\tilde{T}_i$ is some T_l. Let $\mu_{T_l} \coloneqq \frac{1}{n_l}\sum_{i:\,\tilde{T}_i = T_l}\mu_i$ with $n_l \coloneqq \#\{i \mid \tilde{T}_i = T_l\}$. Then μ_{T_l} is a Λ-invariant probability measure as $\gamma \in \Lambda$ permutes the summands of μ_{T_l}. Moreover, μ_{T_l} is (T_l,Λ')-admissible as a sum of (T_l,Λ')-admissible measures. Since Λ' has finite index in Λ and T_l is Λ-invariant, μ_{T_l} is (T_l,Λ)-admissible. Moreover, $\mu = \sum_l \frac{n_l}{M}\mu_{T_l}$. Since μ is Λ-ergodic and μ_{T_l} is Λ-invariant, we infer that $\mu = \mu_{T_l}$ for some l. □

In the setting of Corollary 4.11, any Λ-invariant measure μ can be decomposed as

$$\mu = \sum_{l=1}^{\ell} \alpha_l \mu_l,$$

where $\sum_{l=1}^{\ell}\alpha_l = 1$ and where each μ_l is an (T_l,Λ)-admissible measure. The entropy can then be written as

$$h_{\mathrm{KS}}(\mu,\gamma) = \sum_{l=1}^{\ell}\alpha_l \sum_{\lambda\in\sigma(\gamma|_{V(T_l)})} \max\{\log|\lambda|,0\}, \quad \gamma\in\Lambda, \tag{4.2}$$

which can be more easily compared with the lower bound in Theorem 2.1. Recall that one of the T_l is reduced to $\{0\}$ in view of allowing zero entropy measures (like the one carried by the neutral element of $\mathbb{T}^m$).
Regarding semiclassical measures, we obtain the following analogue of Corollary 4.3 as a direct consequence of Theorem 2.1 and (4.2).

Corollary 4.12. *Let $\Lambda \leq \mathrm{Sp}(2d,\mathbb{Z})$ be an abelian subgroup which is quantizable and tame and which has no virtually cyclic factors. Let $T_1,\dots,T_\ell$ be the closed connected subgroups of $\mathbb{T}^{2d}$. Then, for any $\mu \in \mathcal{P}_{\mathrm{sc}}(\Lambda)$, one has*

$$\mu = \sum_{l=1}^{\ell}\alpha_l\mu_l \quad \textit{with} \quad \sum_{l=1}^{\ell}\alpha_l = 1,$$

where each μ_l is (T_l,Λ)-admissible and where, for any $\gamma\in\Lambda$,

$$\sum_{l=1}^{\ell}\alpha_l\left(\sum_{\lambda\in\sigma(\gamma|_{V(T_l)})}\max\{\log|\lambda|,0\}\right) \geq \sum_{\lambda\in\sigma(\gamma)}\max\left\{\log|\lambda| - \frac{\chi_+(\gamma)}{2},0\right\}.$$

Observe that, if without loss of generality $T_\ell = 1$ then $\alpha_1 + \cdots + \alpha_{\ell-1} > 0$. The following lemma ensuring tameness can be viewed as the analogue of Corollary 4.7 for the reducible case.

Lemma 4.13. *Let $A \in \mathrm{GL}(m,\mathbb{Z})$ such that no ratio of eigenvalues is a root of unity. Then, any abelian subgroup $\Lambda \leq \mathrm{GL}(m,\mathbb{Z})$ containing some power of A is tame. More precisely, the closed connected Λ-invariant subgroups correspond to direct sums of $\ker p_i(A)$ where $p_i \in \mathbb{Z}[X]$ are the irreducible factors of χ_A.*

Proof. As in the proof of Corollary 4.7, each subspace invariant under a finite index subgroup of Λ is already A^N-invariant for some power N. But, as χ_{A^N} is separable, by Lemma 4.6, such a subspace is A-invariant by Lemma 3.2. Hence, we only have to determine the A-invariant subspaces in $\mathbb{Q}^m$. We use again the structure theorem for the $\mathbb{Q}[X]$-module $\mathbb{Q}^m$ to see that $\mathbb{Q}^m \simeq \bigoplus_i \mathbb{Q}[X]/(p_i)$ where $p_i \in \mathbb{Z}[X]$ are the irreducible factors of χ_A. The invariant submodules are direct sums of the $\mathbb{Q}[X]/(p_i) \simeq \ker p_i(A)$. Hence there are only finitely many. □

We are now ready to state our main theorem in the reducible case.

Theorem 4.14. *Let $A \in \mathrm{Sp}(2d, \mathbb{Z})$ such that no ratio of eigenvalues is a root of unity. Let $\chi_A = \prod_{i=1}^r p_i \prod_{j=1}^s \rho_j \rho_j^*$, V_i, W_j, $\overline{W}_j$ as in (3.2). Assume that, for every $1 \le i \le r$ and $1 \le j \le s$,*

$$m(A|_{V_i}) + l(A|_{V_i}) \ge 2 \text{ and } m(A|_{W_j}) + l(A|_{W_j}) \ge 3. \tag{4.3}$$

Then, for any $\mu \in \mathcal{P}_{\mathrm{sc}}(\Lambda)$, with $\Lambda \le \mathrm{Sp}(2d, \mathbb{Z})_A$ quantizable and of finite index, one has

$$\mu = \sum_{I \subseteq \{V_i, \overline{W}_j, \overline{W}_j^*\}} \alpha_I \mu_I \quad \text{with} \quad \sum_{I \subseteq \{V_i, \overline{W}_j, \overline{W}_j^*\}} \alpha_I = 1,$$

where each μ_I is $(T_I, \mathrm{Sp}(2d, \mathbb{Z})_A)$-admissible where $T_I \coloneqq \bigoplus_{U \in I} U \otimes \mathbb{R}/(\mathbb{Z}^{2d} \cap \bigoplus_{U \in I} U \otimes \mathbb{R})$ and where,

$$\sum_{V_i \in I} \alpha_I \ge 1/2 \quad \text{and} \quad \sum_{\overline{W}_j, \overline{W}_j^* \in I} \alpha_I + \frac{1}{2} \left(\sum_{\overline{W}_j \in I, \overline{W}_j^* \notin I} \alpha_I \right) + \frac{1}{2} \left(\sum_{\overline{W}_j \notin I, \overline{W}_j^* \in I} \alpha_I \right) \ge \frac{1}{2}$$

for any choice of V_i and $\overline{W}_j$.

Note also that, for a given j, the assumption $m(A|_{W_j}) + l(A|_{W_j}) \ge 3$ is satisfied as soon as $\dim \overline{W}_j \ge 5$ (recall that separability implies that ρ_j does not cancel on $\mathbb{S}^1$).

Proof. Using Corollary 4.12 and Lemma 4.13 we only have to justify assumption of having no virtually cyclic factors and the resulting bounds on α. For both we can without loss of generality assume that $\Lambda = \mathrm{Sp}(2d, \mathbb{Z})_A$.
For the prior, we consider the induced action $\rho_{\mathbb{T}^{2d}/T}$ of $\mathrm{Sp}(2d, \mathbb{Z})_A$ on the factor $\mathbb{T}^{2d}/T$. Let $V \le \mathbb{Q}^{2d}$ the A-invariant subspace corresponding to T and W an A-invariant complement of V. Such a complement exists as V is the direct sum of some $V_i, \overline{W}_j, \overline{W}_j^*$ and W is chosen as the sum of the remaining ones. Let $\mathrm{cl}(W)$ denote the smallest A-invariant symplectic subspace of $\mathbb{Q}^{2d}$ containing W, i.e. $\mathrm{cl}(W)$ is obtained from W by adding all $\overline{W}_j^*$ if $\overline{W}_j \subseteq W$ and $\overline{W}_j$ if $\overline{W}_j^* \subseteq W$. We then consider the action $\rho_{\mathbb{T}^{2d}/T}$ restricted to the subgroup $\mathrm{Sp}(\mathrm{cl}(W) \cap \mathbb{Z}^{2d})_{A|_{\mathrm{cl}(W)}}$ of $\mathrm{Sp}(2d, \mathbb{Z})_A$. If $\rho_{\mathbb{T}^{2d}/T}$ restricted to this subgroup is injective, then we can combine the eigenvalue assumption (4.3) together with Theorem 3.12. This implies that $\rho_{\mathbb{T}^{2d}/T}(\mathrm{Sp}(\mathrm{cl}(W) \cap \mathbb{Z}^{2d})_{A|_{\mathrm{cl}(W)}}$ has rank ≥ 2 and is therefore not virtually cyclic.
In order to prove injectivity, we start by observing $\mathbb{T}^{2d}/T = \mathbb{R}^{2d}/(V \otimes \mathbb{R} + \mathbb{Z}^{2d})$. By Lemma 4.5 we can choose a complement W' in $\mathbb{Q}^{2d}$ of V such that $\mathbb{Z}^{2d} = (V \cap \mathbb{Z}^{2d}) \oplus (W' \cap \mathbb{Z}^{2d})$. Then $V \otimes \mathbb{R} + \mathbb{Z}^{2d} = V \otimes \mathbb{R} + W' \cap \mathbb{Z}^{2d}$ and $V \otimes \mathbb{R} \cap W' \cap \mathbb{Z}^{2d} = \{0\}$. Assume now that one has $B \in \mathrm{Sp}(\mathrm{cl}(W) \cap \mathbb{Z}^{2d})_{A|_{\mathrm{cl}(W)}}$ with $\rho_{\mathbb{T}^{2d}/T}(B) = \mathrm{Id}_{\mathbb{T}^{2d}/T}$. Then $h\colon x \in \mathbb{R}^{2d} \mapsto Bx - x \in \mathbb{R}^{2n}$ is linear and it has values in $V \otimes \mathbb{R} + (W' \cap \mathbb{Z}^{2d})$. Therefore, if we denote by pr the projection

$\mathbb{R}^{2d} = (V \otimes \mathbb{R}) \oplus (W' \otimes \mathbb{R}) \to W' \otimes \mathbb{R}$, then $\mathrm{pr} \circ h \colon \mathbb{R}^{2d} \to W' \otimes \mathbb{R}$ is linear and has values in $W' \cap \mathbb{Z}^{2d}$ and we infer that it must vanish. Consequently, $h(\mathbb{R}^{2d}) \subseteq V \otimes \mathbb{R}$ and also $h(\mathbb{Q}^{2d}) \subseteq V$. This implies $B|_W = \mathrm{Id}$ and since B is symplectic we also have $B|_{\mathrm{cl}(W)} = 1$ by Lemma 3.10. This proves injectivity and the above discussion shows that the action has no virtually cyclic factors.

For the bounds on α, recall that $\mathrm{Sp}(2d,\mathbb{Z})_A$ has finite index in

$$\prod_{i=1}^{r} \mathrm{Sp}(\Delta_i)_{A|_{V_i}} \times \prod_{j=1}^{s} \mathrm{GL}(\overline{\Gamma}_j)_{A|_{\overline{W}_j}}, \tag{4.4}$$

with the conventions of §3.4. In particular, recall that here $\prod_{j=1}^{s} \mathrm{Sp}(\overline{\Gamma}_j \oplus \overline{\Gamma}_j^*)_{A|_{W_j}}$ is identified with a subgroup of $\prod_{j=1}^{s} \mathrm{GL}(\overline{\Gamma}_j)_{A|_{\overline{W}_j}}$ though the map $\overline{B} \mapsto (\overline{B}, (\overline{B}^{-1})^T)$.

We first deal with the case of an invariant symplectic subspace V_i. Let $0 < \varepsilon < 1/2$. By Corollary 3.8, there is $B_i \in \mathrm{Sp}(\Delta_i)_{A|_{V_i}}$, such that, for every $\lambda \in \sigma(B_i)$,

$$\frac{\max\{\log|\lambda|, 0\}}{\chi_+(B_i)} \in [0,\varepsilon] \cup [1-\varepsilon, 1].$$

For all the factors in (4.4) different from V_i, we pick the matrix to be the identity. This yields a matrix B on the product space and, thanks to finite index, we can find some $N \geq 1$ such that B^N belongs to Λ. Applying the bound on α from Corollary 4.12 (for $\gamma = B^N$) turns into

$$\left(\sum_{V_i \in I} \alpha_I\right)\left(\sum_{\lambda \in \sigma(B_i)} \max\{\log|\lambda|, 0\}\right) \geq \sum_{\lambda \in \sigma(B_i)} \max\left\{\log|\lambda| - \frac{\chi_+(B_i)}{2}, 0\right\}.$$

It follows that $\sum_{V_i \in I} \alpha_I \geq \frac{1}{2} - C\varepsilon$, where C depends only on $\dim V_i$. As this is valid for any $\varepsilon > 0$, we get the expected lower bound.

For the other bound, consider $\overline{W}_j \leq W_j$. Applying Corollary 3.5 instead of Corollary 3.8, we find some $B \in \Lambda$ acting trivially on all other V_i and $W_{j'}$ and such that

$$\forall \lambda \in \sigma(B|_{\overline{W}_j}), \quad \frac{|\log|\lambda||}{\max\{\chi_+(B|_{\overline{W}_j}), \chi_+(B|_{\overline{W}_j^*})\}} \in [0,\varepsilon] \cup [1-\varepsilon, 1]. \tag{4.5}$$

We also observe that $\sigma(B|_{\overline{W}_j^*}) = \{\lambda^{-1} : \lambda \in \sigma(B|_{\overline{W}_j})\}$ and thus

$$\frac{1}{2} \sum_{\lambda \in \sigma(B|_{W_j})} \max\{\log|\lambda|, 0\} = \sum_{\lambda \in \sigma(B|_{\overline{W}_j})} \max\{\log|\lambda|, 0\} = \sum_{\lambda \in \sigma(B|_{\overline{W}_j^*})} \max\{\log|\lambda|, 0\}.$$

We also remark that $\chi_+(B|_{W_j}) = \max\{\chi_+(B|_{\overline{W}_j}), \chi_+(B|_{\overline{W}_j^*})\}$. The lower bound from Corollary 4.12 yields

$$\left(\sum_{\overline{W}_j, \overline{W}_j^* \in I} \alpha_I + \frac{1}{2} \sum_{\overline{W}_j \in I, \overline{W}_j^* \notin I} \alpha_I + \frac{1}{2} \sum_{\overline{W}_j \notin I, \overline{W}_j^* \in I} \alpha_I\right) \sum_{\lambda \in \sigma(B|_{W_j})} \max\{\log|\lambda|, 0\}$$
$$\geq \sum_{\lambda \in \sigma(B|_{W_j})} \max\left\{\log|\lambda| - \frac{\chi_+(B|_{W_j})}{2}, 0\right\}.$$

Combined with (4.5), this yields a lower bound of size $\frac{1}{2} - C\varepsilon$ where $C > 0$ depends only on the dimension of W_j. As this valid for any $\varepsilon > 0$, we obtain the expected lower bound. □

Remark. As in the irreducible case (Theorem 1.1), we could have picked in each symplectic factor one matrix B_ε and considered the subgroup generated by them to obtain a version similar to Theorem 1.1. See for instance Theorem 1.2 for such a formulation in the case where $r = 0$ and $s = 1$.

4.4. **Examples in the reducible case.** Let us give two examples to illustrate the use of Theorem 4.14.

4.4.1. *Example with Lagrangian invariant subspaces.* Let $A \in \mathrm{Sp}(2d, \mathbb{Z})$ with characteristic polynomial $\chi_A = pp^*$, $p \in \mathbb{Z}[X]$ irreducible and $p \neq p^*$. By Lemma 3.10 and without loss of generality, $A = \mathrm{Diag}(A', A'^{-T})$ with $\chi_{A'} = p$. The assumption $p \neq p^*$ means that A' is not itself symplectic with respect to some symplectic form on $\mathbb{Q}^d$. Let us also assume that $\ker p(A) \cap \mathbb{Z}^{2d} + \ker p^*(A) \cap \mathbb{Z}^{2d} = \mathbb{Z}^{2d}$ so that $A' \in \mathrm{GL}(d, \mathbb{Z})$. In general this holds up to finite index. The assumption that no ratio of eigenvalues of A is a root of unity transforms to no ratio and no product of eigenvalues of A' is a root of unity. The invariant subtori are $0, \mathbb{T}^n \times 0, 0 \times \mathbb{T}^n, \mathbb{T}^{2n}$. If we assume $m(A) + l(A) \geq 3$, i.e. $\#(\mathrm{Sp}(A') \cap \mathbb{R}) + \frac{1}{2}\#(\mathrm{Sp}(A') \cap (\mathbb{C} \setminus \mathbb{R})) \geq 3$, we obtain that for every $\mu \in \mathcal{P}_{\mathrm{sc}}(\Lambda)$, where $\Lambda \leq \mathrm{Sp}(2d, \mathbb{Z})_A$ is quantizable and has finite index, we have

$$\mu = \alpha \,\mathrm{Leb}_{\mathbb{T}^{2d}} + \alpha_2 \,\mathrm{Leb}_{\mathbb{T}^d} \otimes \nu_2 + \alpha_1 \nu_1 \otimes \mathrm{Leb}_{\mathbb{T}^d} + \alpha_0 \nu_0$$

with $\alpha + \alpha_2 + \alpha_1 + \alpha_0 = 1$ and, for any $B' \in \mathrm{GL}(d, \mathbb{Z})_{A'}$, $h_{\mathrm{KS}}(\nu_2, B^T) = h_{\mathrm{KS}}(\nu_1, B) = h_{\mathrm{KS}}(\nu_0, \mathrm{Diag}(B', B'^{-T})) = 0$. The bound on α rephrases to

$$\alpha + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 \geq \frac{1}{2}. \tag{4.6}$$

4.4.2. *Example with symplectic invariant subspaces.* In this second example, we consider the product situation. We consider the symplectic form ω on $\mathbb{R}^{2d_1+2d_2}$ given by the symplectic product structure. For $i = 1, 2$, we let $A_i \in \mathrm{Sp}(2d_i, \mathbb{Z})$ with irreducible distinct characteristic polynomial. Then $A := \mathrm{Diag}(A_1, A_2) \in \mathrm{Sp}(2d_1+2d_2, \mathbb{Z})$ has separable characteristic polynomial $\chi_{A_1}\chi_{A_2}$. The closed connected A-invariant subgroups are $0, 0 \times \mathbb{T}^{2d_2}, \mathbb{T}^{2d_1} \times 0, \mathbb{T}^{2d_1+2d_2}$. Again we assume that no ratio of eigenvalues of A is a root of unity. In addition to that we assume $m(A_i) + l(A_i) \geq 2$ for both $i = 1, 2$. Then we obtain that, for every $\mu \in \mathcal{P}_{\mathrm{sc}}(\Lambda)$, where $\Lambda \leq \mathrm{Sp}(2d_1 + 2d_2, \mathbb{Z})_A$ is quantizable and has finite index, we have

$$\mu = \alpha \,\mathrm{Leb}_{\mathbb{T}^{2d_1+2d_2}} + \alpha_2 \,\mathrm{Leb}_{\mathbb{T}^{d_1}} \otimes \nu_2 + \alpha_1 \nu_1 \otimes \mathrm{Leb}_{\mathbb{T}^{d_2}} + \alpha_0 \nu_0$$

with $\alpha + \alpha_2 + \alpha_1 + \alpha_0 = 1$ and, for any $B_i \in \mathrm{Sp}(2d_i, \mathbb{Z})_{A_i}$, $h_{\mathrm{KS}}(\nu_2, B_2) = h_{\mathrm{KS}}(\nu_1, B_1) = h_{\mathrm{KS}}(\nu_0, \mathrm{Diag}(B_1, B_2)) = 0$. The bound on α rephrases to

$$\alpha + \alpha_i \geq \frac{1}{2} \quad \text{for} \quad i = 1, 2.$$

5. The Galois condition in $\mathrm{Sp}(2d, \mathbb{Z})$ and some examples

In this section, we recall a criterion for the irreducibility of the characteristic polynomials in $\mathrm{Sp}(2d, \mathbb{Z})$ due to Anderson and Oliver [KAO24, Appendix B] which will also imply that no ratio of eigenvalues is a root of unity.

5.1. **A criterion for irreducibility and separability.** If χ is the characteristic polynomial of some element in $\mathrm{Sp}(2d,\mathbb{Z})$ then χ is palindromic or reciprocal, i.e. the coefficients of $\chi = \sum_{i=0}^{2d} a_i X^i$ satisfy $a_i = a_{2d-i}$ for all i. As a consequence, the roots are of the form $\lambda_1,\dots,\lambda_d,\lambda_1^{-1},\dots,\lambda_d^{-1}$. Every field automorphism $\sigma \in \mathrm{Gal}(\chi)$ must send λ_i^{-1} to $\sigma(\lambda_i)^{-1}$. This shows that the Galois group preserves the set of unordered pairs $\{\lambda_1,\lambda_1^{-1}\},\dots,\{\lambda_d,\lambda_d^{-1}\}$. The wreath product $S_2 \wr S_d$ is defined as the subgroup of S_{2d} preserving this set of unordered pairs so that $\mathrm{Gal}(\chi) \le S_2 \wr S_d$. Hence, the largest possible Galois group is $S_2 \wr S_d$ meaning $|\mathrm{Gal}(\chi)| \le |S_2 \wr S_d| = 2^d d!$. We say that $A \in \mathrm{Sp}(2d,\mathbb{Z})$ or χ satisfies the *Galois condition* if

$$|\mathrm{Gal}(\chi)| = 2^d d!. \tag{G}$$

Remark. If (G) holds for a palindromic polynomial χ with coefficients in $\mathbb{Z}$, then χ is irreducible over $\mathbb{Q}$ (thus over $\mathbb{Z}$ if $a_0 = 1$). Indeed, let $\lambda_1^{\pm 1},\dots,\lambda_d^{\pm 1}$ be the roots of χ. Then $[\mathbb{Q}(\lambda_i)\colon \mathbb{Q}] \le 2d$ since χ is a polynomial over $\mathbb{Q}$ such that $\chi(\lambda_i) = 0$. But then $\frac{\chi(X)}{(X-\lambda_i)(X-\lambda_i^{-1})} \in \mathbb{Q}(\lambda_i)[X]$ is of degree $2(d-1)$ and annihilates λ_j, $i \neq j$. Therefore, $[\mathbb{Q}(\lambda_i,\lambda_j)\colon \mathbb{Q}(\lambda_i)] \le 2(d-1)$. Inductively,

$$[\mathbb{Q}(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(r)})\colon \mathbb{Q}(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(r-1)})] \le 2(d-r+1),$$

for $\sigma \in S_d$. In particular, since for $Z \coloneqq \mathbb{Q}(\lambda_1,\dots,\lambda_d)$,

$$[Z\colon \mathbb{Q}] = [\mathbb{Q}(\lambda_1,\dots,\lambda_d)\colon \mathbb{Q}(\lambda_1,\dots,\lambda_{d-1})]\cdots[\mathbb{Q}(\lambda_1,\lambda_2)\colon \mathbb{Q}(\lambda_1)]\cdot[\mathbb{Q}(\lambda_1)\colon \mathbb{Q}] \le 2^d d!.$$

Hence under the assumptions of the lemma, all estimates are actually equalities. In particular $[\mathbb{Q}[\lambda_i]\colon \mathbb{Q}] = 2d$ for every $1 \le i \le d$ so that χ is irreducible. Yet, it is worth noticing that χ may be irreducible without (G) being satisfied. For instance, $\chi = X^4 + X^3 + X^2 + X + 1$ is irreducible with Galois group equal to $(\mathbb{Z}/5\mathbb{Z})^\times$.

The following lemma guarantees the applicability of Theorem 1.1 if additionally $m(A) + l(A) \ge 2$ holds.

Lemma 5.1 ([KAO24, Lemma B.2])**.** *Let $d \ge 2$. If $A \in \mathrm{Sp}(2d,\mathbb{Z})$ satisfies (G), then χ_A is irreducible and no ratio of eigenvalues of A is a root of unity.*

Proof. The latter statement is equivalent to χ_{A^k} being separable for all $k \in \mathbb{N}$ by Lemma 4.6. In [KAO24, Lemma B.2] it is shown that if A satisfies (G) then χ_{A^k} is not only separable but even irreducible for all $k \in \mathbb{N}$. □

Remark. We note that (G) does not imply $l(A) + m(A) \ge 2$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

has characteristic polynomial $x^4 - 2x^3 + x^2 - 2x + 1$ which has two real roots and two roots on the unit circle and has Galois group $S_2 \wr S_2$ [DJ24, App. A]. We also note that, inside the set of all palindromic polynomials, this is quite common. More precisely, if $f(x) = a_0 + \cdots + a_{2n}x^{2n}$ with $a_k = a_{2n-k} \in \mathbb{R}$ and $|a_k| \ge |a_n| \cos\left(\frac{\pi}{[\frac{n}{n-k}]+2}\right)$ for some $k = 0,\dots,n-1$, then f has a root on the unit circle [KM04].

In view of Example 4.4.1 we also formulate the following lemma.

Lemma 5.2. *If $A' \in \mathrm{GL}(d,\mathbb{Z})$, $d \geq 3$, satisfies $\mathrm{Gal}(\chi_{A'}) = S_d$ then $A := \mathrm{Diag}(A', A'^{-T}) \in \mathrm{Sp}(2d,\mathbb{Z})$ has separable characteristic polynomial $\chi_{A'}\chi^*_{A'}$ with both factors being irreducible and no ratio of eigenvalues of A is a root of unity. If $d = 2$ and in addition $\chi_{A'}$ is neither X^2+1, X^2+X+1, nor X^2-X+1 then the same conclusion holds.*

Proof. The arguments are similar to the one of [KAO24, Lemma B.2]. Let λ_i be the roots of $\chi_{A'}$ and $Z := \mathbb{Q}[\lambda_1,\dots,\lambda_d]$ be the splitting field of $\chi_{A'}$. One has

$$d! = [Z\colon \mathbb{Q}] = \prod_{i=1}^{d} [\mathbb{Q}[\lambda_1,\dots,\lambda_i]\colon \mathbb{Q}[\lambda_1,\dots,\lambda_{i-1}] \leq \prod_{i=1}^{d} d-i+1 = d!$$

by the same argument as for (G). In particular, $\mathbb{Q}[\lambda_i]$ has degree d over $\mathbb{Q}$. This implies that $\chi_{A'}$ is irreducible. It follows easily that $\chi^*_{A'} = \chi_{A'^{-1}} = \chi_{A'^{-T}}$ is irreducible as well.
The lemma will be proved if $\lambda_i \neq \lambda_j^{-1}$ for any i,j (separability property) and $\lambda_i^N \neq \lambda_j^N$ for $i \neq j$ for any N (not root of unity property). If $\lambda_i = \lambda_i^{-1}$ then $\lambda_i^2 = 1$ and therefore $\chi_{A'} \mid X^2-1$ by irreducibility. Then $Z = \mathbb{Q}$ contradicting $[Z:\mathbb{Q}] = |S_d| = d!$.
Let us therefore assume that $\lambda_i^N \in \mathbb{Q}[\lambda_j]$ for some $j \neq i$. Then $\lambda_i^N \in \mathbb{Q}[\lambda_i] \cap \mathbb{Q}[\lambda_j]$. By the Galois correspondence $\mathbb{Q}[\lambda_i] = Z^{G_i} = \{x \in Z \mid \sigma(x) = x \,\forall \sigma \in G_i\}$ with $G_i = \{\sigma \in S_d \mid \sigma(\lambda_i) = \lambda_i\}$. The field $\mathbb{Q}[\lambda_i] \cap \mathbb{Q}[\lambda_j]$ is the fixed field of the subgroup generated by G_i and G_j. Since $G_i \simeq S_{d-1}$ and G_j contains some element not fixing λ_i we have $\mathbb{Q}[\lambda_i] \cap \mathbb{Q}[\lambda_j] = Z^{S_d} = \mathbb{Q}$. Hence, $\lambda_i^N \in \mathbb{Q}$. Moreover, λ_i^N is a root of $\chi_{A^N} \in \mathbb{Z}[X]$ which is monic forcing $\lambda_i^N \in \mathbb{Z}$. Indeed, if $\lambda_i^N = p/q$ (with p and q coprime), then q divides p^{2d} and thus $q = \pm 1$. The same holds true for λ_i^{-N} so that $\lambda_i^N = \pm 1$ and $\lambda_i^{2N} = 1$. As before, as λ_i is a root of unity, $\chi_{A'}$ is a cyclotomic polynomial and $\mathrm{Gal}(\chi_{A'})$ is abelian. This is a contradiction if $d \geq 3$. The cyclotomic polynomials of degree 2 are the three listed ones. □

5.2. **Finding examples satisfying (G) and the eigenvalue condition.** We now describe a method of generating $A \in \mathrm{Sp}(2d,\mathbb{Z})$ with the required properties for Theorem 1.1. By Lemma 5.1, (G) and $m(A) + l(A) \geq 2$ is sufficient. We start with the following result:

Theorem 5.3 ([KAO24, Thm. B.1])**.** *For any $d \geq 2$ the matrices in $\mathrm{Sp}(2d,\mathbb{Z})$ with (G) have density one in $\mathrm{Sp}(2d,\mathbb{Z})$ (when ordered by some norm on $\mathrm{M}(2d,\mathbb{R})$).*

Below we will present a method to produce a matrix in $\mathrm{Sp}(2d,\mathbb{Z})$ with (G) and $m(A)+l(A) \geq 2$ out of a matrix of $\mathrm{Sp}(2d,\mathbb{Z})$ satisfying (only) (G). Since both assumptions only depend on the characteristic polynomial, the following result reduces the problem to finding suitable polynomials thanks to the next result.

Theorem 5.4 ([Kir69, Riv08, Thm. A.1])**.** *Let $f \in \mathbb{Z}[X]$ be a monic palindromic polynomial. Then there is $A \in \mathrm{Sp}(2d,\mathbb{Z})$ with characteristic polynomial f.*

The following two statements can be used to ensure (G). The first one is a classical method to determine the Galois group.

Lemma 5.5 ([Isa93, Thm. 28.23])**.** *Let $f \in \mathbb{Z}[X]$ be monic irreducible over $\mathbb{Q}$. For $d_1,\dots,d_l \in \mathbb{N}$ the following two statements are equivalent:*

(i) *$\mathrm{Gal}(f)$ contains a permutation (of the zeros of f) which is a disjoint product of cycles of length $d_1,\dots,d_l$.*
(ii) *There is a prime p such that $\overline{f} = \overline{f_1}\cdots\overline{f}_l$, for some irreducible $\overline{f}_i \in \mathbb{F}_p[X]$, $\overline{f}$ is separable, and $\deg \overline{f}_i = d_i$, where $\overline{f}$ denotes the reduction of f modulo p.*

Lemma 5.6 ([DDS98, Lemma 2]). *If $f \in \mathbb{Z}[X]$ is monic and palindromic, and* $\mathrm{Gal}(f)$ *contains a 2-cycle, a 4-cycle, a $(2d-2)$-cycle, and a $2d$-cycle then f satisfies* (G)*.*

With these lemmas at hand, we assume $d \geq 4$ so that $2d \neq 4 \neq 2d-2$ (the adjustments for $d = 2, 3$ are obvious). If f is the characteristic polynomial of some matrix satisfying (G) (which have density one by Theorem 5.3), then $\mathrm{Gal}(f)$ contains a 2-cycle, a 4-cycle, a $(2d-2)$-cycle and a $2d$-cycle and we can take the primes p_2, p_4, p_{2d-2} and p_{2d} given by Lemma 5.5 and corresponding respectively to these cycles in $\mathrm{Gal}(f)$. We observe that any polynomial $\tilde{f}$ that agrees with f modulo the primes p_2, p_4, p_{2d-2}, and p_{2d} also satisfies (G) by Lemma 5.5 and Lemma 5.6. By the Chinese remainder theorem such $\tilde{f}$ is unique mod $p_2p_4p_{2d-2}p_{2d}$. We will choose a suitable $\tilde{f}$ to ensure the eigenvalue condition.
To obtain a polynomial satisfying (G) and the eigenvalue condition, we now set $f_k := f + kp_2p_4p_{2d-2}p_{2d}X^d$, $k \in \mathbb{Z}$, which is still monic and palindromic. As explained above, f_k satisfies (G). We now define $g \in \mathbb{Z}[X]$ by $X^d g(X + X^{-1}) = f$. Obviously, $g_k := g + kp_2p_4p_{2d-2}p_{2d}$ satisfies $X^d g_k(X + X^{-1}) = f_k$. Pairs of roots $\{\alpha, \alpha^{-1}\}$ of f_k correspond to roots of g_k via $\{\alpha, \alpha^{-1}\} \mapsto \alpha + \alpha^{-1}$. Moreover, pairs of roots of f_k on $\mathbb{S}^1$ correspond to roots of g_k in the interval $[-2, 2]$ (see [KM04, Lemma 1]). Since $g([-2, 2])$ is a bounded set, $g_k = g + kp_2p_4p_{2d-2}p_{2d}$ has no roots in $[-2, 2]$ for $|k| \gg 1$. Hence f_k has no roots in $\mathbb{S}^1$ for $|k| \gg 1$. If $d = 2$ then g_k is a quadratic polynomial which has two real roots outside $[-2, 2]$ for $k \ll -1$. Hence f_k has 4 real roots for $k \ll -1$.
We conclude that $A \in \mathrm{Sp}(2d, \mathbb{Z})$ with characteristic polynomial f_k for $|k| \gg 1$ (resp. $k \ll -1$ if $d = 2$) satisfies (G) and $m(A) + 2l(A) = d$ (resp. $m(A) = 2$ if $d = 2$). In both cases $m(A)+l(A) \geq 2$. The existence of A with such a characteristic polynomial is finally provided by Theorem 5.4.

Remark. In the above construction, instead of starting with a characteristic polynomial satisfying (G) we could also choose the reductions of f modulo the primes directly. More precisely, let $\mathcal{P}_{2d}^{\mathrm{rec}}(\mathbb{F}_p, (i))$ be the set of monic palindromic polynomials of degree $2d$ over $\mathbb{F}_p$ consisting of polynomials factoring as an irreducible polynomial of degree i times a product of $2d - i$ distinct linear polynomials (see [KAO24, Appendix B]). By [KAO24, Lemma B.4] for any integer $1 \leq k \leq n$ we have

$$\#\mathcal{P}_{2d}^{\mathrm{rec}}(\mathbb{F}_p; (2k)) = \frac{1}{2^{d-k+1} \cdot k \cdot (d-k)!} p^d + \mathcal{O}(p^{d-1})$$

so that $\mathcal{P}_{2d}^{rec}(\mathbb{F}_p; (2k))$ is non-empty for almost all primes p.
Let us now pick distinct primes $p_2, p_4, p_{2d-2}, p_{2d}$ and $q_i \in \mathcal{P}_{2d}^{\mathrm{rec}}(\mathbb{F}_{p_i}; (i))$ for $i \in \{2, 4, 2d-2, 2d\}$. Let $f \in \mathbb{Z}[X]$ be monic and palindromic such that the reduction modulo the primes p_i gives q_i. We are now in the same situation as if f was the characteristic polynomial of a matrix in $\mathrm{Sp}(2d, \mathbb{Z})$ satisfying (G).

Example 5.7. The matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 3 \\ 0 & -1 & 3 & 4 \end{pmatrix}$$

has characteristic polynomial $\chi_A = X^4 - X^3 - 19X^2 - X + 1$ and eigenvalues $4.9059\ldots$ and $-3.850\ldots$ as well as their inverses. In particular, $m(A) = 2$. The polynomial $g = X^2 - X - 21$ satisfies $g(X + 1/X)X^2 = \chi_A$ and has two real roots $\frac{1 \pm \sqrt{85}}{2}$ lying outside $[-2, 2]$.

χ_A is irreducible $\mod 2$ and splits into $(X-3)(X-5)(X^2+1) \mod 7$. Hence $|\mathrm{Gal}(\chi_A)| = 8$ by Lemma 5.5 and 5.6, i.e. A satisfies (G). By Theorem 3.6 $\mathrm{Sp}(4,\mathbb{Z})_A = \{\pm I\} \times \mathbb{Z}^2$. Let

$$B = \begin{pmatrix} 1 & -27 & 45 & 114 \\ -27 & -62 & 114 & 311 \\ -45 & -114 & 208 & 564 \\ -114 & -311 & 564 & 1524 \end{pmatrix} = 9A^3 + 29A^2 - 21A + 3I$$

Clearly, $AB = BA$ and one checks $B \in \mathrm{Sp}(4,\mathbb{Z})$ and B has characteristic polynomial $\chi_B = X^4 - 1671X^3 + 17191X^2 - 1671X + 1$. We observe $\chi_A \equiv \chi_B \equiv X^4 + X^3 + X^2 + X + 1 \mod 2$ but $\chi_A \not\equiv \chi_B \equiv X^4 + 2X^3 + 6X^2 + 2X + 1 \equiv (X-3)(X-5)(X^2+3X+1) \mod 7$ so they define the same cycle type $\mod 2$ and $\mod 7$. Diagonalizing A with a matrix $S \in \mathrm{Sp}(4,\mathbb{R})$ gives

$$S^{-1}AS = \mathrm{Diag}(-3.8500\ldots, 4.9059\ldots, (-3.8500\ldots)^{-1}, (4.9059)^{-1})$$

and

$$S^{-1}BS = \mathrm{Diag}(0.0975\ldots, 1660.6486\ldots, (0.0975\ldots)^{-1}, (1660.6486\ldots)^{-1})$$

From the moduli of the first two entries it now follows easily that $A^kB^l = I$, $k,l \in \mathbb{Z}$, can only hold for $k = l = 0$, i.e. $\Lambda \coloneqq \{A^kB^l \mid k,l \in \mathbb{Z}\}$ is free abelian of rank 2. We conclude that Λ has finite index in $\mathrm{Sp}(4,\mathbb{Z})_A$. Hence, we can apply Theorem 1.1 to find $B_\varepsilon \in \Lambda$.

Appendix A. A brief reminder on the metaplectic group

In this section we review the metaplectic transformations and we refer to [dG11] for more details. See also [Fol89, Ch. 4].

The metaplectic group $\mathrm{Mp}(2d,\mathbb{R})$ is the unique double covering group of $\mathrm{Sp}(2d,\mathbb{R})$. More precisely, $\mathrm{Sp}(2d,\mathbb{R})$ has a Cartan decomposition $\mathrm{Sp}(2d,\mathbb{R}) \simeq K \times \mathfrak{p}$ where $K \simeq U(d)$ and $\mathfrak{p}$ is a vector space [Kna02, Thm. 6.31, p. 575]. This shows that the fundamental group $\pi_1(\mathrm{Sp}(2d,\mathbb{R})) = \pi_1(K)$ as $\mathfrak{p}$ is a vector space. In our case, $K \simeq U(d)$ has fundamental group $\mathbb{Z}$. Let for the moment $\widetilde{G}$ be the universal cover of $\mathrm{Sp}(2d,\mathbb{R})$ with covering map $\widetilde{q}$. Then $\mathrm{Sp}(2d,\mathbb{R}) \simeq \widetilde{G}/\ker\widetilde{q}$ and $\ker\widetilde{q} \simeq \mathbb{Z}$ is contained in the center of $\widetilde{G}$. All other coverings of $\mathrm{Sp}(2d,\mathbb{R})$ factor through this universal covering. This means they are given by subgroups $n\mathbb{Z}$, $n \in \mathbb{N}_0$, of $\mathbb{Z}$. In particular, we have $\widetilde{G} \to \widetilde{G}/2\mathbb{Z} \to \widetilde{G}/\mathbb{Z} = \mathrm{Sp}(2d,\mathbb{R})$ and the middle group $\widetilde{G}/2\mathbb{Z}$ is the metaplectic group $\mathrm{Mp}(2d,\mathbb{R})$. It is the unique connected double cover of $\mathrm{Sp}(2d,\mathbb{R})$. Let q denote the covering map. By construction, the map q is $2:1$, i.e. there is Z in the center of $\mathrm{Mp}(2d,\mathbb{R})$ such that $\ker q = \{1, Z\}$. It is worth noting that $\mathrm{Mp}(2d,\mathbb{R})$ has no finite-dimensional faithful representation, i.e. it cannot be described as a group of matrices. However, there is a faithful representation π_h on $L^2(\mathbb{R}^d)$ which is unique with the property

$$\forall (w,t) \in H_d, \quad \pi_h(g)T_{(w,t)}\pi_h(g)^{-1} = T_{(q(g)w,t)} \quad g \in \mathrm{Mp}(2d,\mathbb{R}), \tag{A.1}$$

where H_d is the Heisenberg group and the representation T of the Heisenberg group H_d is the unique unitary irreducible representation such that $T_{(0,t)} = e^{\frac{i}{h}t}$ (i.e. has central character $e^{\frac{i}{h}}$) by the Stone-von Neumann Theorem.

Remark. Recall from [DJ24, §2] that T is the representation used to defined the Weyl quantization.

The construction of $\pi_h(g)$ works as follows. For $g \in \mathrm{Sp}(2d,\mathbb{R})$, the map $(w,t) \mapsto T_{(gw,t)}$ defines another unitary irreducible representation of H_d with the same central character. Hence they are unitarily equivalent and there exits an isometry M_g of $L^2(\mathbb{R}^d)$ such that $M_g \circ T_{(w,d)} \circ M_g^{-1} = T_{(gw,d)}$. By Schur's Lemma M_g is unique up scalar. One can check that $g \mapsto [M_g]$ defines a projective representation of $\mathrm{Sp}(2d,\mathbb{R})$, i.e. a group homomorphism $\mathrm{Sp}(2d,\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^d))/(\mathbb{S}^1 \cdot \mathrm{Id})$. This representation does not come from an ordinary representation of $\mathrm{Sp}(2d,\mathbb{R})$ but it can be lifted to a representation of $\mathrm{Mp}(2d,\mathbb{R})$, where 'lifted' means precisely Equation (A.1). One then gets $\pi_h(Z) = -1$, since $Z^2 = 1 \in \mathrm{Mp}(2d,\mathbb{R})$ and $\pi_h(Z)$ must be scalar by Schur's Lemma and $\neq 1$ since otherwise it would be a representation of $\mathrm{Sp}(2d,\mathbb{R})$.

This allows to define the metaplectic representation $\widetilde{M}_h(A)$ of A as $\pi_h(\widetilde{A})$ for any $h > 0$, where we picked $\widetilde{A}$ to be a metaplectic lift of A (which is unique up to $\mathrm{Ker}(q) = \{1, Z\}$). This defines an operator on $L^2(\mathbb{R}^d)$ that can can restricted to the Schwartz class. Hence, by duality, it can be extended to tempered distributions and thus to the spaces $\mathcal{H}_{\mathbf{N}}$ from §2. This corresponds to the operators $M_{\mathbf{N}}(A)$ considered in this article.

Hence, if we are given two symplectic matrices A and B, we can choose two metaplectic lifts $\tilde{A}$ and $\tilde{B}$ in $\mathrm{Mp}(2d,\mathbb{R})$ which are unique up to $\mathrm{Ker}(q) = \{1, Z\}$. One has then

$$\widetilde{M}_h(A)\widetilde{M}_h(B) = \pi_h(\widetilde{A})\pi_h(\widetilde{B}) = \pi_h(\widetilde{A}\widetilde{B}) = \pm\widetilde{M}_h(AB).$$

In particular, if A and B commute, one finds that $\widetilde{M}_h(A)\widetilde{M}_h(B) = \widetilde{M}_h(B)\widetilde{M}_h(A)$ or $\widetilde{M}_h(A)\widetilde{M}_h(B)^2 = \widetilde{M}_h(B)^2\widetilde{M}_h(A)$. In that last case, this yields the commuting relation $\widetilde{M}_h(A)\widetilde{M}_h(B^2) = \widetilde{M}_h(B^2)\widetilde{M}_h(A)$.

References

[AN07] N. Anantharaman and S. Nonnenmacher. Entropy of semiclassical measures of the Walsh-quantized Baker's map. *Ann. Henri Poincaré*, 8(1):37–74, 2007. 3

[Ana22] N. Anantharaman. *Quantum ergodicity and delocalization of Schrödinger eigenfunctions*. Zur. Lect. Adv. Math. Berlin: European Mathematical Society (EMS), 2022. 1

[Art91] Michael Artin. *Algebra*. Englewood Cliffs, NJ: Prentice-Hall, 1991. 18

[AS13] N. Anantharaman and L. Silberman. A Haar component for quantum limits on locally symmetric spaces. *Isr. J. Math.*, 195:393–447, 2013. 1, 7

[BDB96] A. Bouzouina and S. De Bièvre. Equipartition of the eigenfunctions of quantized ergodic maps on the torus. *Commun. Math. Phys.*, 178(1):83–105, 1996. 2

[BDB03] F. Bonechi and S. De Bièvre. Controlling strong scarring for quantized ergodic toral automorphisms. *Duke Math. J.*, 117(3):571–587, 2003. 3

[Ber83] D. Berend. Multi-invariant sets on tori. *Trans. Am. Math. Soc.*, 280:509–532, 1983. 17

[BL03] J. Bourgain and E. Lindenstrauss. Entropy of quantum limits. *Comm. Math. Phys.*, 233(1):153–171, 2003. 1

[BL14] S. Brooks and E. Lindenstrauss. Joint quasimodes, positive entropy, and quantum unique ergodicity. *Invent. Math.*, 198(1):219–259, 2014. 1

[Bro10] S. Brooks. On the entropy of quantum limits for 2-dimensional cat maps. *Commun. Math. Phys.*, 293(1):231–255, 2010. 3

[CdV85] Y. Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.*, 102(3):497–502, 1985. 1

[DDS98] S. Davis, W. Duke, and X. Sun. Probabilistic Galois theory of reciprocal polynomials. *Expo. Math.*, 16(3):263–270, 1998. 27

[dG11] M. A. de Gosson. *Symplectic methods in harmonic analysis and in mathematical physics*, volume 7 of *Pseudo-Differ. Oper., Theory Appl.* Basel: Birkhäuser, 2011. 28

[DJ24] S. Dyatlov and M. Jézéquel. Semiclassical measures for higher-dimensional quantum cat maps. *Ann. Henri Poincaré*, 25(2):1545–1605, 2024. 1, 2, 3, 5, 6, 7, 17, 25, 28

[Dya22] S. Dyatlov. Around quantum ergodicity. *Ann. Math. Qué.*, 46(1):11–26, 2022. 1

[EL03] M. Einsiedler and E. Lindenstrauss. Rigidity properties of $\mathbb{Z}^d$-actions on tori and solenoids. *Electron. Res. Announc. Am. Math. Soc.*, 9:99–110, 2003. 15, 16, 19

[EL22] M. Einsiedler and E. Lindenstrauss. Rigidity properties for commuting automorphisms on tori and solenoids. *Ergodic Theory Dyn. Syst.*, 42(2):691–736, 2022. 15, 16, 19

[EW11] M. Einsiedler and T. Ward. *Ergodic theory. With a view towards number theory*, volume 259 of *Grad. Texts Math.* London: Springer, 2011. 8

[FN04] F. Faure and S. Nonnenmacher. On the maximal scarring for quantum cat map eigenstates. *Commun. Math. Phys.*, 245(1):201–214, 2004. 3

[FNDB03] F. Faure, S. Nonnenmacher, and S. De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Commun. Math. Phys.*, 239(3):449–492, 2003. 3

[Fol89] G. B. Folland. *Harmonic analysis in phase space*, volume 122 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 1989. 6, 28

[Gur06] S. Gurevich. Weil Representation, Deligne Sheaf, and Proof of the Kurlberg-Rudnick Conjecture. Preprint, arXiv:math-ph/0601031 (PhD Thesis, Tel Aviv), 2006. 3

[Gut10] B. Gutkin. Entropic bounds on semiclassical measures for quantized one-dimensional maps. *Commun. Math. Phys.*, 294(2):303–342, 2010. 3

[HB80] J. H. Hannay and M.V. Berry. Quantization of linear maps on a torus-fresnel diffraction by a periodic grating. *Physica D: Nonlinear Phenomena*, 1(3):267–290, 1980. 2

[Isa93] I.M. Isaacs. *Algebra: a graduate course*. Pacific Grove, CA: Brooks/Cole Publishing Company, 1993. 26

[KAO24] E. Kim, T.C. Anderson, and R.J. Lemke Oliver. Characterizing the support of semiclassical measures for higher-dimensional cat maps. Preprint, arXiv:2410.13449 [math.AP] (2024), 2024. 1, 3, 17, 24, 25, 26, 27

[Kel10] D. Kelmer. Arithmetic quantum unique ergodicity for symplectic linear maps of the multidimensional torus. *Ann. Math. (2)*, 171(2):815–879, 2010. 3, 4, 5, 12, 13

[Kir69] D. Kirby. Integer matrices of finite order. *Rend. Mat., VI. Ser.*, 2:403–408, 1969. 26

[KM04] J. Konvalina and V. Matache. Palindrome-polynomials with roots on the unit circle. *C. R. Math. Acad. Sci., Soc. R. Can.*, 26(2):39–44, 2004. 25, 27

[Kna02] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Prog. Math.* Boston, MA: Birkhäuser, 2nd ed. edition, 2002. 28

[KORS24] P. Kurlberg, A. Ostafe, Z. Rudnick, and I.E. Shparlinski. On quantum ergodicity for higher dimensional cat maps. Preprint, arXiv:2411.05997 [math.DS] (2024), 2024. 3, 5

[KR00] P. Kurlberg and Z. Rudnick. Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. J.*, 103(1):47–77, 2000. 4

[KR01] P. Kurlberg and Z. Rudnick. On quantum ergodicity for linear maps of the torus. *Commun. Math. Phys.*, 222(1):201–227, 2001. 5

[Lin06] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006. 1

[Neu99] J. Neukirch. *Algebraic number theory. Transl. from the German by Norbert Schappacher*, volume 322 of *Grundlehren Math. Wiss.* Berlin: Springer, 1999. 10, 11

[Par69] W. Parry. *Entropy and generators in ergodic theory.* Math. Lect. Note Ser. The Benjamin/Cummings Publishing Company, Reading, MA, 1969. 8

[Rib88] P. Ribenboim. *The book of prime number records*. Springer, New York, 1988. 18

[Riv08] I. Rivin. Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. *Duke Math. J.*, 142(2):353–379, 2008. 26

[Riv11] G. Rivière. Entropy of semiclassical measures for symplectic linear maps of the multidimensional torus. *Int. Math. Res. Not. IMRN*, (11):2396–2443, 2011. 3, 5, 7

[RS62] J. Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Ill. J. Math.*, 6:64–94, 1962. 18

[RS94] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994. 1

[RSOdA00] A. M. F. Rivas, M. Saraceno, and A. M. Ozorio de Almeida. Quantization of multidimensional cat maps. *Nonlinearity*, 13(2):341–376, 2000. 5

[Sch24] N. Schwartz. The full delocalization of eigenstates for the quantized cat map. *Pure Appl. Anal.*, 6(4):1017–1053, 2024. 3

[Seg83] Daniel Segal. *Polycyclic groups*, volume 82 of *Camb. Tracts Math.* Cambridge University Press, Cambridge, 1983. 7, 16

[Šni74] A. I. Šnirel'man. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk*, 29(6(180)):181–182, 1974. 1

[SV19] L. Silberman and A. Venkatesh. Entropy bounds and quantum unique ergodicity for Hecke eigenfunctions on division algebras. In *Probabilistic methods in geometry, topology and spectral theory.*, pages 171–197. Providence, RI: American Mathematical Society (AMS); Montreal: Centre de Recherches Mathématiques (CRM), 2019. 1

[Wal82] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982. 3, 8, 20

[Zel87] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987. 1

[Zwo12] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2012. 6

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