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# Equidistribution of toral eigenfunctions along hypersurfaces

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**Abstract.** We prove a new quantum variance estimate for toral eigenfunctions. As an application, we show that, given any orthonormal basis of toral eigenfunctions and any smooth embedded hypersurface with non-vanishing principal curvatures, there exists a density one subsequence of eigenfunctions that equidistribute along the hypersurface. This is an analogue of the *Quantum Ergodic Restriction* theorems in the case of the flat torus, which in particular verifies the Bourgain-Rudnick’s conjecture on  $L^2$ -restriction estimates for a density one subsequence of eigenfunctions in any dimension. Using our quantum variance estimates, we also obtain equidistribution of eigenfunctions against measures whose supports have Fourier dimension larger than  $d - 2$ . In the end, we also describe a few quantitative results specific to dimension 2.

## 1. Introduction

Let  $(M, g)$  be a smooth ( $C^\infty$ ), compact, oriented, Riemannian manifold without boundary and of dimension  $d \geq 2$ . Consider the following eigenvalue problem

$$(1.1) \quad -\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1.$$

Let now  $\Sigma$  be a smooth, compact, oriented and embedded submanifold of  $M$  of codimension<sup>1</sup> 1. Then, it is a natural question to estimate the  $L^2$ -norm of the restrictions of eigenfunction to  $\Sigma$  with respect to the induced hypersurface measure  $d\sigma$ . It was proved by Burq, Gérard and Tzvetkov [6] that, for any solution  $\psi_\lambda$  of (1.1), one has:

$$(1.2) \quad \|\psi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma \lambda^{\frac{1}{4}},$$

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<sup>1</sup>We will refer to such submanifolds as hypersurfaces.

where  $C_\Sigma > 0$  depends only on  $(M, g)$  and  $\Sigma$ . They also proved that this bound is sharp in general. In addition, they proved that for  $d = 2$  and if  $\Sigma$  has non vanishing geodesic curvature, the bound can be improved to

$$\|\psi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma \lambda^{\frac{1}{6}},$$

which is again sharp in general. Though these estimates are sharp, it must be noticed that sequences for which the upper bound is sharp are in some sense sparse. Indeed, by the Hörmander-Weyl's law [20], given any orthonormal basis (ONB)  $(\psi_j)_{j \in \mathbb{N}}$  of Laplace eigenfunctions, one has uniformly for  $x \in M$ ,

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} |\psi_j(x)|^2 = \frac{1}{\text{Vol}_g(M)} + \mathcal{O}(\lambda^{-1}),$$

where  $N(\lambda) := |\{j : \lambda_j \leq \lambda\}|$  is the spectral counting function. After integrating this asymptotics against the hypersurface measure  $\sigma$ , this implies that, for any given function  $R(\lambda) \rightarrow \infty$  and any ONB of eigenfunctions, there is a full density<sup>2</sup> subsequence  $(\psi_j)_{j \in S}$  of Laplace eigenfunctions such that

$$\forall j \in S, \quad \|\psi_{\lambda_j}\|_{L^2(\Sigma)} \leq R(\lambda_j).$$

Another way to look for improvements is to make some restriction on the geometry of the manifold. For instance, the authors of [6] also noted that, in the case of the 2-dimensional flat torus  $\mathbb{T}^2$ , their bound (1.2) can be drastically improved as the  $L^\infty$ -norms are controlled by a term of order  $\lambda^\delta$  for every  $\delta > 0$  without any extraction argument. Improving this bound in the case of the flat torus was recently pursued by Bourgain and Rudnick [5] who obtained uniform lower and upper bounds on  $\|\psi_\lambda\|_{L^2(\Sigma)}$  for  $d = 2$  or  $3$  when  $\Sigma$  is a real analytic hypersurface with nonvanishing curvatures – see below for a more precise statement. Bourgain and Rudnick also conjectured that these quantities should be uniformly bounded in any dimension whenever the hypersurface is real analytic [5, Conj. 1.9]. As an application of our analysis, we verify this conjecture *for a density 1 subsequence of eigenfunctions provided some nonvanishing curvature properties are satisfied by  $\Sigma$  and only smoothness* – see Corollary 2.2 below. More precisely, we show that for any given function  $R(\lambda) \rightarrow \infty$  and any ONB of eigenfunctions, there is a full density subsequence of eigenfunctions  $(\psi_j)_{j \in S}$ , along which one has

$$\int_\Sigma |\psi_j|^2 d\sigma = \text{Vol}(\Sigma) + \mathcal{O}\left(\lambda_j^{-\frac{1}{2}} R(\lambda_j) \log \lambda_j\right),$$

where  $\text{Vol}(\Sigma)$  is the hypersurface volume of  $\Sigma$ . Rather than geometric considerations, we emphasize that the real reason for the improvements obtained here or in [5] compared with (1.2) is due to the arithmetic structure of the rational torus and thus to number theory properties.

Related to these restriction estimates are the Quantum Ergodicity Restriction (QER) theorems of Toth–Zelditch [33, 34] and Dyatlov–Zworski [9]. Recall that

<sup>2</sup>Recall that  $S \subset \mathbb{N}$  has full density (or density 1) if  $|\{j \in S : j \leq N\}|/N \rightarrow 1$  as  $N \rightarrow +\infty$ .

the Quantum Ergodicity (QE) theorem states that almost all the eigenfunctions of a given orthonormal basis of Laplace eigenfunctions become equidistributed in the unit cotangent bundle if the Liouville measure is ergodic [32, 36, 7]. It is natural to ask if this equidistribution remains true along hypersurfaces and this question was recently answered by the above mentioned works on QER theorems. In particular, it implies that, for such subsequences, the  $L^2$ -restriction estimates can be improved as  $\|\psi_\lambda\|_{L^2(\Sigma)}^2$  converges to the volume of  $\Sigma$ . The QE theorem can be extended to other dynamical frameworks, and the analogue of this result was obtained by Marklof and Rudnick in the case of rational polygons including the case of the flat torus  $\mathbb{T}^d := \mathbb{R}^d / (2\pi\mathbb{Z})^d$  [27]. The main difference in this case is that most eigenfunctions become equidistributed in configuration space but not necessarily in phase space. Quantitative versions of this result were recently derived by techniques of different nature. Our proof in [18] was close to the original proof of the QE theorem while the proof of Lester and Rudnick in [24] made use of tools of arithmetic nature. The main purpose of the present article is to combine the methods from these two articles and obtain improvements on some of the results from these references and also to deduce analogues of the QER theorem in the case of the flat torus. In particular, we will prove that almost all toral eigenfunctions equidistribute along hypersurfaces with nonvanishing principal curvatures.

Finally, we observed in [17] that quantitative equidistribution properties of Laplace eigenfunctions can be used to get informations on the growth of their  $L^p$  norms in the large eigenvalue limit. Here, as a by-product of our analysis (see paragraph 3.5), we will recover the so-called Cooke-Zygmund's inequality [8, 39], i.e. for every solution  $\psi_\lambda$  to (1.1) on the rational torus  $\mathbb{T}^2$ , one has

$$\|\psi_\lambda\|_{L^4(\mathbb{T}^2)} \leq 3^{\frac{1}{4}}.$$

Yet, in higher dimensions, our argument would only give an upper bound of order  $\lambda_j^{\frac{d-2}{4}}$  which remains far<sup>3</sup> from [2, 3].

## 2. Statement of the main results

Let us fix some notations. Throughout the paper, we set  $d \geq 2$  and we denote by  $\mathbb{T}^d := \mathbb{R}^d / (2\pi\mathbb{Z})^d$  the rational torus with the standard metric, by  $dx$  the *normalized volume measure* induced by the standard metric, and by  $(\psi_j)_{j \geq 1}$  an orthonormal basis of Laplace eigenfunctions, i.e.

$$-\Delta\psi_j = \lambda_j^2\psi_j,$$

with  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \lambda_n \dots$ . Recall that any eigenfunction is in fact of the form

$$\psi_j = \sum_{n \in \mathbb{Z}^d: \|n\| = \lambda_j} \hat{c}_n(j) e^{in \cdot x}, \quad \sum_{n \in \mathbb{Z}^d: \|n\| = \lambda_j} |\hat{c}_n(j)|^2 = 1.$$

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<sup>3</sup>Recall that, for  $d \geq 3$ , Bourgain conjectured that the  $L^{\frac{2d}{d-2}}$ -norm should be of order  $\mathcal{O}(\lambda^\epsilon)$  for every  $\epsilon > 0$  [2].

Hence, eigenvalues are of the form  $\|n\|$  with  $n \in \mathbb{Z}^d$  and they can be indexed (counted with multiplicities) by lattice points of  $\mathbb{Z}^d$ , i.e.

$$\{\lambda_j^2 : j \geq 1\} = \{\|n\|^2 : n \in \mathbb{Z}^d\}.$$

We shall denote by  $N(\lambda)$  the spectral counting function over *long intervals*,

$$N(\lambda) := |\{j \geq 1 : \lambda_j \leq \lambda\}| = |\{n \in \mathbb{Z}^d : \|n\| \leq \lambda\}|.$$

## 2.1. Equidistribution along hypersurfaces

The nicest consequence of our main result (Th. 2.5 below) is the following:

**Theorem 2.1.** *Let  $\Sigma \subset \mathbb{T}^d$  be a smooth compact embedded and oriented submanifold of dimension  $d - 1$  which has no boundary. Suppose that, for every  $x$  in  $\Sigma$ , all the principal curvatures of  $\Sigma$  at  $x$  are different from 0. Then, for any  $a$  in  $C^\infty(\mathbb{T}^d)$ , there exists a constant  $C_a > 0$  such that, for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , one has*

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\Sigma} a |\psi_j|^2 d\sigma - \int_{\Sigma} a d\sigma \right|^2 \leq C_a \frac{(\log \lambda)^2}{\lambda},$$

where  $\sigma$  is the induced hypersurface measure on  $\Sigma$ .

In other words, this theorem states that, on average, eigenfunctions are equidistributed along  $\Sigma$  with a precise rate of convergence. In fact the proof will show that the error is of order  $\frac{\log \lambda}{\lambda}$  for  $d \geq 3$  – see Remark 3.4. As we shall see below, this theorem is a direct consequence of our Theorem 2.5 and of Littman’s Theorem [25] – see also [19, 16] for earlier results in the case of convex bodies. Recall that this latter Theorem states that, under the geometric assumption of Theorem 2.1, there exists some constant  $C > 0$  such that, for every  $n \in \mathbb{Z}^d \setminus \{0\}$ ,

$$(2.1) \quad \left| \int_{\Sigma} e^{-in \cdot x} d\sigma(x) \right|^2 \leq C \|n\|^{1-d}.$$

An extraction argument [36, 7] allows to show that:

**Corollary 2.2** (Equidistribution along hypersurfaces). *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , there exists a density 1 subset  $S$  of  $\mathbb{N}$  such that, for every  $a \in C^0(\mathbb{T}^d)$ ,*

$$(2.2) \quad \lim_{j \rightarrow +\infty, j \in S} \int_{\Sigma} a |\psi_j|^2 d\sigma = \int_{\Sigma} a d\sigma,$$

where  $\sigma$  is the induced hypersurface measure on  $\Sigma$ . Moreover, for any function  $R(\lambda) \rightarrow \infty$ , there is a density 1 subset  $S$  along which, in addition to (2.2), one also has

$$\left| \int_{\Sigma} |\psi_j|^2 d\sigma - \text{Vol}(\Sigma) \right| \leq \frac{R(\lambda_j) \log \lambda_j}{\lambda_j^{\frac{1}{2}}},$$

where  $\text{Vol}(\Sigma)$  is the hypersurface volume of  $\Sigma$ .

This result is the analogue in the case of the flat torus of the Quantum Ergodicity Restriction theorems from [33, 34, 9]. The results from these references are stronger in the sense that equidistribution also holds in phase space but we emphasize they do not provide any rate of convergence of the variance. It is plausible that, combined with the arguments from [38, 31], one would get a logarithmic decay rate when  $(M, g)$  is negatively curved. Recall also that these QER statements are valid under the assumptions that the geodesic flow is ergodic for the Liouville measure (which is not the case here) and that the hypersurface verifies a certain asymmetry condition on the geodesics passing through  $\Sigma$ . While these conditions are of dynamical nature, we emphasize that ours are purely geometrical even if the number theoretic nature of the problem plays of course a crucial role in our proofs.

A direct consequence of this corollary is that, along a density one subsequence, the  $L^2$ -restriction estimates of [6] can be improved. Recall, that for a smooth closed embedded and oriented curve  $\Sigma \subset \mathbb{T}^2$  with nonvanishing curvature, Bourgain and Rudnick proved [5] that there exist  $0 < c \leq C$  such that, for every  $\psi_\lambda$  satisfying  $\Delta\psi_\lambda = -\lambda^2\psi_\lambda$ , one has

$$(2.3) \quad c\|\psi_\lambda\|_{L^2(\mathbb{T}^2)} \leq \|\psi_\lambda\|_{L^2(\Sigma)} \leq C\|\psi_\lambda\|_{L^2(\mathbb{T}^2)}.$$

They also proved (in a more involved proof) that this remains true in dimension 3 for *real analytic*<sup>4</sup> hypersurfaces with nonzero curvature. In the higher dimensional case, it seems that the question remains open. It is in fact conjectured in [5] that, under appropriate assumptions on  $\Sigma$ , these two bounds should hold in any dimension. In particular, the above corollary is consistent with that conjecture as it shows it is true in any dimension for a *density 1* subsequence of eigenfunctions provided that the principal curvatures of  $\Sigma$  do not vanish. Along this typical sequence, the result is even better than expected as it shows that  $\|\psi_j\|_{L^2(\Sigma)}$  converges in any dimension. Applying Theorems 0.2 and 0.3 [35] would give upper bounds on the size of the nodal sets restricted to  $\Sigma$  but this would not be better than the bounds from [5] which are valid without extracting subsequences.

## 2.2. Rate of decay of the quantum variance

Let us now come back to the more classical framework of the quantum ergodicity theorems where we consider equidistribution on  $\mathbb{T}^d$  and not along hypersurfaces. In this case, we can consider averages over shorter spectral intervals  $[\lambda - 1, \lambda]$ . If we denote by  $N(\lambda - 1, \lambda)$  the number of indices  $j$  such that  $\lambda_j \in [\lambda - 1, \lambda]$ , then we proved in Theorem 1.2 of [18] that, for any smooth function  $a$ ,

$$(2.4) \quad \frac{1}{N(\lambda - 1, \lambda)} \sum_{j: \lambda - 1 \leq \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a|\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq C_a \lambda^{-\frac{2}{3}},$$

for some constant  $C_a > 0$  depending on a certain number of derivatives of  $a$ . Our proof was modeled on the classical proof of the quantum ergodicity theorem [32,

<sup>4</sup>In the case  $d = 2$ , the short proof given in p.880–881 [5] does not require analyticity.

36, 7]. In [24], this result was improved by Lester and Rudnick who considered the moments of order 1. They obtained the following<sup>5</sup>:

$$(2.5) \quad \frac{1}{N(\lambda-1, \lambda)} \sum_{j: \lambda-1 \leq \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right| \leq \frac{C_a}{\lambda},$$

for some constant  $C_a > 0$  depending again on a certain number of derivatives of  $a$ . Observe now that

$$\left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq 2 \|a\|_{C^0} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|,$$

which yields the following improvement of (2.4):

$$(2.6) \quad \frac{1}{N(\lambda-1, \lambda)} \sum_{j: \lambda-1 \leq \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq \frac{C_a}{\lambda}.$$

We cannot expect to have a better rate of decay than  $\lambda^{-1}$  (at least in dimension 2). Indeed, the Weyl's law [10] tells us that, in dimension  $d = 2$ ,

$$N(\lambda-1, \lambda) \sim \lambda.$$

If we had a better rate, then it would mean that every sequence of eigenfunctions equidistribute on  $\mathbb{T}^2$  as  $\lambda \rightarrow +\infty$  and this is not the case – see for instance Section 3 of [24]. However, we will show that the combination of the semiclassical methods from [18] with the arithmetic methods from [24] allows to obtain improvements on the involved constant  $C_a$  for the moments of order 2:

**Theorem 2.3.** *There exists a universal constant  $C_d > 0$  such that, for any  $a$  in  $L^2(\mathbb{T}^d)$ , for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , and for any  $\lambda > 0$ , one has*

$$\frac{1}{N(\lambda-1, \lambda)} \sum_{j: \lambda-1 \leq \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq \frac{C_d}{\lambda} \|a\|_{L^2(\mathbb{T}^d)}^2.$$

The proof of this Theorem will be given in Paragraph 3.2. In dimension 2, this type of upper bound on the variance is consistent with the ones appearing in the physics literature [13, 11] – see also [26] in the case of an arithmetic surface, [38, 31] for general negatively curved surfaces, and [23] for quantum cat-maps. We observe that Theorem 2.3 does not seem to imply (2.5).

### 2.3. Main result on the quantum variance

Coming back to the case of long spectral intervals, we have the following crucial estimate on the quantum variance:

<sup>5</sup>The proof of Proposition 2.4 in [24] is given for long spectral intervals but it can be adapted to shorter intervals by using (3.3) below.

**Theorem 2.4.** *There exists a universal constant  $C_d > 0$  such that, for any  $a$  in  $L^2(\mathbb{T}^d)$ , for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , and for any  $\lambda > 0$ , one has*

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq \frac{C_d}{\lambda} \sum_{n: 1 \leq \|n\| \leq 2\lambda} \frac{|\hat{a}_n|^2}{\|\mathfrak{p}(n)\|},$$

where  $\mathfrak{p}(n)$  is the element of minimal norm in  $\mathbb{Z}^d \cap \mathbb{R}_+^* n$  and

$$\hat{a}_n := \int_{\mathbb{T}^d} a(x) e^{-inx} dx.$$

This Theorem will be proved in Paragraph 3.2. In [24, Prop. 2.4], Lester and Rudnick obtained an analogous statement on moments of order 1:

$$(2.7) \quad \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right| \leq \frac{C_d}{\lambda} \sum_{n: 1 \leq \|n\| \leq 2\lambda} \frac{|\hat{a}_n|}{\|\mathfrak{p}(n)\|}.$$

Yet, as we shall briefly explain it below, this bound on moments of order 1 is not sufficient to prove Theorem 2.1 except in dimension 2. Here, we will crucially use the fact that our upper bound involves factors of type  $|\hat{a}_n|^2$ .

Coming back to Theorem 2.1, we also note that it is natural to study a generalized framework when the hypersurface measure is replaced by a general measure  $\mu$  carried possibly on a complicated set<sup>6</sup>. In fact, as shown in Paragraph 3.3, our Theorem 2.4 implies:

**Theorem 2.5.** *Let  $\mu$  be a probability measure on  $\mathbb{T}^d$  such that there exists  $\alpha \geq 0$  and  $C > 0$  satisfying*

$$\forall n \in \mathbb{Z}^d \setminus \{0\}, \quad |\hat{\mu}(n)|^2 \leq C \|n\|^{-\alpha}.$$

*Then, for any  $a$  in  $C^\infty(\mathbb{T}^d)$ , there exists a constant  $C_a > 0$  (depending on a finite number<sup>7</sup> of derivatives of  $a$ ) such that, for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , one has*

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 d\mu - \int_{\mathbb{T}^d} a d\mu \right|^2 \leq C_a (\log \lambda)^2 \lambda^{\max(d-2-\alpha, -1)}.$$

Observe that Theorem 2.1 becomes a direct corollary of this Theorem and of Littman's upper bound (2.1). Our assumption on the measure  $\mu$  is related to the notion of Fourier dimension and Salem sets in harmonic analysis – see p. 168 in [28]. From Remark 3.4 below, the proof will in fact yield the better estimate  $\lambda^{-1}$  for  $\alpha > d - 1$  which is consistent with Theorem 2.3. We also note that if,

<sup>6</sup>The recent work of Eswarathasan and Pramanik [12] extends the results of [6] when the hypersurface measure is replaced by some measure  $\mu$  carried by a random fractal set.

<sup>7</sup>This number is independent of  $a$ .

instead of moments of order 2, we had considered moments of order 1, then the upper bound (2.7) from [24] would lead us to estimate

$$\sum_{1 \leq \|n\| \leq \lambda} \frac{1}{\|\mathbf{p}(n)\| \|n\|^{\frac{\alpha}{2}}},$$

which would lead to stronger constraints on the value of  $\alpha$ . Arguing as in Paragraph 3.3, this would in fact require to impose  $\alpha > 2(d-2)$  which is a worst constraint than requiring  $\alpha > d-2$  as soon as  $d \geq 3$ . For instance, if we keep in mind that our main application follows from Littman's upper bound (2.1), we need to allow  $\alpha = d-1$ , and this would not be authorized by the assumption  $\alpha > 2(d-2)$  in dimensions  $d \geq 3$ .

Finally, we record the following Corollary:

**Corollary 2.6.** *Let  $\mu$  be a probability measure on  $\mathbb{T}^d$  such that there exists  $\alpha > d-2$  and  $C > 0$  satisfying*

$$\forall n \in \mathbb{Z}^d \setminus \{0\}, \quad |\hat{\mu}(n)|^2 \leq C \|n\|^{-\alpha}.$$

*Then, for any orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , there exists a density 1 subset  $S$  of  $\mathbb{N}$  such that, for every  $a \in C^0(\mathbb{T}^d)$*

$$\lim_{j \rightarrow +\infty, j \in S} \int_{\mathbb{T}^d} a |\psi_j|^2 d\mu = \int_{\mathbb{T}^d} a d\mu.$$

### 3. Proof of the main Theorems

In this section, we give the proof of the variance estimates from the introduction. We will first prove a general variance estimate valid on any spectral interval and reducing the problem to a number theory question (Paragraph 3.1). Then, we will deduce the proofs of Theorems 2.3, 2.4 and 2.5 from Proposition 3.2 and from number theory results. Finally, in the last paragraphs, we will derive a few other consequences of our analysis (Cooke-Zygmund's inequality and variance on individual space) and recall a few facts on the related question of eigenfunctions averages.

The main point compared with the ‘‘semiclassical’’ approach of [18] is that we try to optimize our argument by implementing some of the lattice points properties proved in [30] and used in [24]. Let  $a$  be a smooth function on  $\mathbb{T}^d$ . We write its Fourier decomposition:

$$a = \sum_{p \in \mathbb{Z}^d} \hat{a}_p \mathbf{e}_p,$$

where  $\mathbf{e}_p(x) = e^{ip \cdot x}$ . Suppose for the sake of simplicity that  $a$  is real valued.



### 3.1. Estimating the variance

We introduce the following sum:

$$S_2(a, \lambda) := \sum_{j: \lambda_j = \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2.$$

If we set<sup>8</sup>  $a_\lambda = \sum_{n: \|n\| \leq 2\lambda} \hat{a}_n \mathbf{e}_n$ , then we can replace  $a$  by its truncated Fourier series and verify that

$$S_2(a, \lambda) = \sum_{j: \lambda_j = \lambda} \left| \int_{\mathbb{T}^d} a_\lambda |\psi_j|^2 dx - \int_{\mathbb{T}^d} a_\lambda dx \right|^2.$$

We will now proceed as in the proof of [18] and first use the fact that

$$\forall t \in \mathbb{R}, e^{it\Delta} \psi_j = e^{-it\lambda_j^2} \psi_j.$$

This implies that, for every  $T > 0$ , one has

$$\int_{\mathbb{T}^d} a_\lambda |\psi_j|^2 dx = \left\langle \psi_j, \left( \frac{1}{T} \int_0^T e^{-it\Delta} a_\lambda e^{it\Delta} dt \right) \psi_j \right\rangle.$$

In order to alleviate notations, let us introduce the self-adjoint operator:

$$A(T, \lambda) := \frac{1}{T} \int_0^T e^{-it\Delta} a_\lambda e^{it\Delta} dt - \int_{\mathbb{T}^d} a_\lambda dx.$$

With these conventions, one has

$$S_2(a, \lambda) = \sum_{j: \lambda_j = \lambda} |\langle \psi_j, A(T, \lambda) \psi_j \rangle|^2.$$

Apply now the Cauchy-Schwarz inequality which yields

$$S_2(a, \lambda) \leq \sum_{j: \lambda_j = \lambda} \|A(T, \lambda) \psi_j\|^2.$$

Using the fact that the trace of the operator  $A(T, \lambda)^2$  is independent of the choice of orthonormal basis, this inequality can be rewritten as

$$S_2(a, \lambda) \leq \sum_{k: \|k\| = \lambda} \|A(T, \lambda) \mathbf{e}_k\|^2.$$

We now expand

$$\begin{aligned} A(T, \lambda) \mathbf{e}_k &= \frac{1}{T} \int_0^T e^{-it\Delta} \left( \sum_{1 \leq \|n\| \leq 2\lambda} \hat{a}_n \mathbf{e}_n \right) e^{it\Delta} \mathbf{e}_k dt \\ &= \sum_{1 \leq \|n\| \leq 2\lambda} \hat{a}_n \left( \frac{1}{T} \int_0^T e^{it(\|k+n\|^2 - \|k\|^2)} dt \right) \mathbf{e}_{k+n}. \end{aligned}$$

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<sup>8</sup>Observe that this is still real-valued.

Up to this point, this is exactly the proof from [18], but from this point on we proceed differently. First, we observe that, up to now every estimate and every identity is valid for any  $T > 0$  and any  $\lambda > 0$ . By letting  $T$  tend to infinity first, we deduce that

$$S_2(a, \lambda) \leq \sum_{k: \|k\|=\lambda} \left\| \sum_{\substack{1 \leq \|n\| \leq 2\lambda \\ \|k\|^2 = \|k+n\|^2}} \hat{a}_n \mathbf{e}_{k+n} \right\|^2,$$

By application of Plancherel's formula, the upper bound reduces to

$$S_2(a, \lambda) \leq \sum_{k: \|k\|=\lambda} \sum_{\substack{1 \leq \|n\| \leq 2\lambda \\ \|k\|^2 = \|k+n\|^2}} |\hat{a}_n|^2,$$

which, after changing the order of summation, yields

$$(3.1) \quad S_2(a, \lambda) \leq \sum_{1 \leq \|n\| \leq 2\lambda} |\hat{a}_n|^2 |\{k : \|k\| = \|k+n\| = \lambda\}|.$$

*Remark 3.1.* At this stage of the proof, we would like to emphasize an important point of our argument. In the standard proof of the Quantum Ergodicity Theorem [32, 36, 7] and also in the case of its analogue on  $\mathbb{T}^d$  [27], one always start by letting  $\lambda$  tends to  $+\infty$  and then  $T$  to  $+\infty$ . Here, because of the specific structure of the torus, we can first take the large time limit and then the large eigenvalue limit. Usually, one would try to take  $T$  depending on  $\lambda$  [38, 31, 17, 18] in order to optimize the arguments but to still keep track of the underlying classical dynamics. Here, by letting  $T$  tend to  $+\infty$  with  $\lambda$  fixed, we reduce the question to a number theory question which does not seem to have any dynamical interpretation. This observation is responsible for the improvements we get compared with the proof from [18].

Our estimate above for  $S_2(a, \lambda)$  is valid for any eigenvalue  $\lambda$ . Therefore, if we sum over a range of eigenvalues  $[c(\lambda), \lambda]$  (with  $0 \leq c(\lambda) \leq \lambda$ ) and if we set

$$V_2(a, c(\lambda), \lambda) := \frac{1}{N(c(\lambda), \lambda)} \sum_{j: c(\lambda) \leq \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2,$$

with  $N(c(\lambda), \lambda)$  the number of indices  $j$  such that  $\lambda_j \in [c(\lambda), \lambda]$ , then we get

**Proposition 3.2.** *Let  $0 \leq c(\lambda) \leq \lambda$ . For every  $a$  in  $L^2(\mathbb{T}^d)$  and for every orthonormal basis  $(\psi_j)_{j \geq 1}$  of eigenfunctions of  $\Delta$ , one has*

$$V_2(a, c(\lambda), \lambda) \leq \sum_{1 \leq \|n\| \leq 2\lambda} |\hat{a}_n|^2 \frac{|\{k \in \mathbb{Z}^d : \|k\| = \|k+n\| \in [c(\lambda), \lambda]\}|}{|\{k \in \mathbb{Z}^d : \|k\| \in [c(\lambda), \lambda]\}|}.$$

### 3.2. Proofs of Theorems 2.4 and 2.3

Thanks to Proposition 3.2, we have reduced our question to a problem of counting lattice points which is close to the one appearing in [24] – see also [30]. Precisely, according to Lemma 2.3 in [24], one has

**Lemma 3.3.** *There exists a constant  $c_d > 0$  such that, for every  $n$  in  $\mathbb{Z}^d \setminus \{0\}$  and for every  $\lambda > 1$ ,*

$$(3.2) \quad |\{k : 0 \leq \|k\| \leq \lambda \text{ and } \|k\|^2 = \|n+k\|^2\}| \leq c_d \frac{\lambda^{d-1}}{\|\mathbf{p}(n)\|},$$

where  $\mathbf{p}(n)$  is the point in  $\mathbb{Z}^d \cap \mathbb{R}_+^* n$  having minimal norm.

Together with Proposition 3.2 and the Weyl's law, this upper bound implies Theorem 2.4. Recall that this Lemma comes from the fact that we have to count the number of lattice points which are orthogonal to  $p$  because  $\|k\|^2 = \|n+k\|^2$  is equivalent to the fact that  $\langle n, n-2k \rangle = 0$  – see [24] for details. Note that, if we consider shorter spectral intervals, the argument from Lemma 2.3 in [24] still allows to conclude that, for every  $n$  in  $\mathbb{Z}^d \setminus \{0\}$ ,

$$(3.3) \quad |\{k : \lambda - 1 \leq \|k\| \leq \lambda \text{ and } \|k\|^2 = \|n+k\|^2\}| \leq c'_d \lambda^{d-2},$$

for some constant  $c'_d > 0$  depending only on  $d$ . Combined with Proposition 3.2 and Weyl's law on the torus, this implies Theorem 2.3 from the introduction.

### 3.3. Proof of Theorem 2.5

Fix now a probability measure  $\mu$  satisfying the properties of Theorem 2.5. It can be viewed as a distribution on  $\mathbb{T}^d$ . Hence, one can find a sequence of smooth functions  $(\mu_m)_{m \geq 1}$  such that  $\mu_m \rightarrow \mu$  in  $\mathcal{D}'(\mathbb{T}^d)$ . We set  $a$  to be a smooth function on  $\mathbb{T}^d$  and we apply Theorem 2.4 to the test function  $a\mu_m$ :

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a\mu_m |\psi_j|^2 dx - \int_{\mathbb{T}^d} a\mu_m dx \right|^2 \leq \frac{C_d}{\lambda} \sum_{n: 1 \leq \|n\| \leq 2\lambda} \frac{|\int_{\mathbb{T}^d} a(x)\mu_m(x)e^{-inx} dx|^2}{\|\mathbf{p}(n)\|}.$$

This inequality is valid for any  $m \geq 1$  and any  $\lambda \geq 1$ . By letting  $m \rightarrow +\infty$ , we find

$$(3.4) \quad \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 d\mu - \int_{\mathbb{T}^d} a d\mu \right|^2 \leq \frac{C_d}{\lambda} \sum_{n: 1 \leq \|n\| \leq 2\lambda} \frac{|\int_{\mathbb{T}^d} a(x)e^{-inx} d\mu(x)|^2}{\|\mathbf{p}(n)\|}.$$

Recall now that, for every  $n$  in  $\mathbb{Z}^d$ ,

$$|\widehat{\mu}(n)|^2 \leq C(1 + \|n\|)^{-\alpha}.$$

Hence, one has, for every  $n$  in  $\mathbb{Z}^d \setminus \{0\}$ ,

$$\left| \int_{\mathbb{T}^d} a(x)e^{-inx} d\mu(x) \right| \leq \sum_{p \in \mathbb{Z}^d} |\widehat{a}_p| |\widehat{\mu}(p-n)| \leq \sqrt{C} \sum_{p \in \mathbb{Z}^d} |\widehat{a}_p| (1 + \|p-n\|)^{-\frac{\alpha}{2}},$$

from which one can deduce, by writing  $\|n\| \leq \|p-n\| + \|p\|$ ,

$$\left| \int_{\mathbb{T}^d} a(x)e^{-inx} d\mu(x) \right| \leq \frac{\sqrt{C}}{(1 + \|n\|)^{\frac{\alpha}{2}}} \sum_{p \in \mathbb{Z}^d} |\widehat{a}_p| (1 + \|p\|)^{\frac{\alpha}{2}} = \mathcal{O}_a(\|n\|^{-\frac{\alpha}{2}}),$$

where the constant in the remainder depends on a certain number of derivatives of  $a$ . From this, one can infer

$$(3.5) \quad \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 d\mu - \int_{\mathbb{T}^d} a d\mu \right|^2 = \mathcal{O}_a(\lambda^{-1}) \sum_{n: 1 \leq \|n\| \leq 2\lambda} \frac{1}{\|n\|^\alpha \|\mathbf{p}(n)\|}.$$

We then argue as in [24]:

$$\sum_{1 \leq \|n\| \leq \lambda} \frac{1}{\|\mathbf{p}(n)\| \|n\|^\alpha} = \sum_{1 \leq m \leq \lambda} \frac{1}{m^\alpha} \sum_{1 \leq \|\mathbf{p}\| \leq \lambda/m} \frac{1}{\|\mathbf{p}\|^{\alpha+1}}.$$

We distinguish several cases:

- $\alpha > d - 1$ : the sum  $\sum_{1 \leq \|\mathbf{p}\| \leq \lambda/m} \frac{1}{\|\mathbf{p}\|^{\alpha+1}}$  is bounded.
- $\alpha = d - 1$ : the sum  $\sum_{1 \leq \|\mathbf{p}\| \leq \lambda/m} \frac{1}{\|\mathbf{p}\|^{\alpha+1}}$  is of order  $\mathcal{O}(\log(\lambda/m))$ .
- $\alpha < d - 1$ : the sum  $\sum_{1 \leq \|\mathbf{p}\| \leq \lambda/m} \frac{1}{\|\mathbf{p}\|^{\alpha+1}}$  is of order  $(\lambda/m)^{d-1-\alpha}$ .

In any case, this gives the upper bound

$$\sum_{1 \leq \|n\| \leq \lambda} \frac{1}{\|\hat{n}\| \|n\|^\alpha} \lesssim (\log \lambda)^2 \lambda^{\max(d-1-\alpha, 0)},$$

where  $\lesssim$  means that the upper bound involves a constant which is independent of  $\lambda$ .

*Remark 3.4.* For  $d \geq 3$  and  $\alpha < d - 1$ , the  $(\log \lambda)^2$  factor can be removed. For  $d \geq 3$  and  $\alpha = d - 1$ , we have a term of order  $\log \lambda$  instead of  $(\log \lambda)^2$ . Finally, for  $\alpha > d - 1$ , the  $(\log \lambda)^2$  factor can be removed again.

Combined with (3.5), we finally get

$$\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 d\mu - \int_{\mathbb{T}^d} a d\mu \right|^2 = \mathcal{O}_a \left( (\log \lambda)^2 \lambda^{\max(d-2-\alpha, -1)} \right),$$

which concludes the proof of Theorem 2.5.

### 3.4. Average of eigenfunctions over hypersurfaces

In this paragraph, we make a few well-known remarks on the related question of the average of toral eigenfunctions over an hypersurface  $\Sigma$  endowed with the induced Riemannian measure  $\sigma$ . More precisely, given a solution  $\psi_\lambda$  of (1.1), we want to estimate

$$\int_{\Sigma} \psi_\lambda(x) d\sigma(x).$$

Using Zelditch's Kuznecov trace formula [37, Eq. (3.4)], one knows that for a density one subsequence of eigenfunctions of a given orthonormal basis, this quantity decays as  $\lambda^{\frac{1-d}{2}} R(\lambda)$  for any choice of  $R(\lambda) \rightarrow +\infty$ . In addition, the trace formula

of [37] also shows that for all eigenfunctions this quantity is uniformly bounded. In the case of the rational torus  $\mathbb{T}^d$ , we would like to recall how one can easily say a little bit more provided some assumptions are made on the curvature of  $\Sigma$ . To see this, we write

$$\psi_\lambda = \sum_{k: \|k\|=\lambda} \hat{\psi}_\lambda(k) \mathbf{e}_k,$$

and we find using the Cauchy-Schwarz inequality

$$\left| \int_\Sigma \psi_\lambda(x) d\sigma(x) \right| = \left| \sum_{k: \|k\|=\lambda} \hat{\psi}_\lambda(k) \hat{\sigma}(-k) \right| \leq \left( \sum_{k: \|k\|=\lambda} |\hat{\sigma}(k)|^2 \right)^{\frac{1}{2}}.$$

Suppose now that, for every  $x$  in  $\Sigma$ , all the principal curvatures at  $x$  do not vanish, so that we can apply Littman's Theorem again. This yields

$$\left| \int_\Sigma \psi_\lambda(x) d\sigma(x) \right| \leq \left( \sum_{k: \|k\|=\lambda} \frac{1}{\|k\|^{d-1}} \right)^{\frac{1}{2}} \leq \frac{1}{\lambda^{\frac{d-1}{2}}} |\{k : \|k\| = \lambda\}|^{\frac{1}{2}}.$$

Hence, everything boils down to an estimate on the number  $r_d(\lambda)$  of lattice points on a circle of radius  $\lambda$  centered at 0 which is a standard problem in number theory. We briefly recall classical results from the literature. First of all, for  $d \geq 5$ , it follows from [15, p. 155] that  $r_d(\lambda)$  is of order  $\lambda^{d-2}$ . If  $d = 4$ , we can deduce from Theorem 13.12 in [1] and from Theorem 4 in [15, p.30] that  $r_4(\lambda)$  is bounded by  $\lambda^{2+\delta}$  for every  $\delta > 0$ . When  $d = 3$ , we have that  $r_3(\lambda)$  is bounded by  $\lambda^{1+\delta}$  for any positive  $\delta$  as a consequence of Theorem 13.12 in [1] and of Equation (4.10) in [15, p.55]. For  $d = 2$ , it follows from Theorem 13.12 [1] and from Theorem 3 in [15, p. 15] that  $r_2(\lambda)$  is of order  $\lambda^\delta$  for every  $\delta > 0$ . Gathering these bounds, we get:

**Proposition 3.5.** *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for any solution  $\psi_\lambda$  of (1.1), one has, for every  $\delta > 0$ ,*

$$\int_\Sigma \psi_\lambda(x) d\sigma(x) = \mathcal{O}(\lambda^{-\frac{1}{2}+\delta}),$$

with  $\delta$  that can be taken to be 0 when  $d \geq 5$ .

We emphasize that the assumption on  $\Sigma$  is sharp in the sense that, if  $\Sigma := \mathbb{T}^{d-1} \subset \mathbb{T}^d$ , then it is easy to construct a subsequence of eigenfunctions verifying  $\int_\Sigma \psi_\lambda(x) d\sigma(x) = 1$  (take  $\psi_k(x) = \exp(ikx_d)$ ).

### 3.5. Recovering Cooke-Zygmund's estimate

In dimension 2, Proposition 3.2 (with  $c(\lambda) = \lambda$ ) yields that for each  $\psi_j$ ,

$$\left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq \sum_{1 \leq \|n\| \leq 2\lambda} |\hat{a}_n|^2 |\{k : \|k\| = \|k+n\| = \lambda\}|.$$

But since the number of lattice points lying on two circles of radius  $\lambda$  centered at 0 and  $n \neq 0$  is at most 2, this implies that

$$\left| \int_{\mathbb{T}^2} a |\psi_j|^2 dx - \int_{\mathbb{T}^2} a dx \right|^2 \leq 2 \int_{\mathbb{T}^2} \left| a(x) - \int_{\mathbb{T}^2} a(y) dy \right|^2 dx.$$

By plugging  $a = |\psi_j|^2$  into this we obtain

$$\left( \int_{\mathbb{T}^2} |\psi_j|^4 dx - 1 \right)^2 \leq 2 \int_{\mathbb{T}^2} (|\psi_j|^2 - 1)^2 dx,$$

which implies

$$\forall j \geq 1, \quad \|\psi_j\|_{L^4} \leq 3^{\frac{1}{4}}.$$

Hence, as pointed in the introduction, our argument recovers Cooke-Zygmund's classical result on uniform boundedness of  $L^4$  norms [8, 39].

### 3.6. Quantum variance on individual eigenspaces

In this section, we continue to record some remarks when the variance is considered on an eigenspace i.e.  $c(\lambda) = \lambda$ . By combining Proposition 3.2 with (3.3), we get (3.6)

$$\frac{1}{|\{j : \lambda_j = \lambda\}|} \sum_{j: \lambda_j = \lambda} \left| \int_{\mathbb{T}^d} a |\psi_j|^2 dx - \int_{\mathbb{T}^d} a dx \right|^2 \leq \|a\|_{L^2(\mathbb{T}^d)}^2 \min \left\{ 1, c'_d \frac{\lambda^{d-2}}{r_d(\lambda)} \right\},$$

where  $r_d(\lambda)$  is the number of lattice points lying on the circle of radius  $\lambda$  centered at 0 – see paragraph 3.4 for a brief reminder on its asymptotic properties. In particular, the upper bound does not necessarily go to 0 as  $\lambda \rightarrow +\infty$ . For instance, in dimension  $d \geq 5$ , one has  $r_d(\lambda) \sim \lambda^{d-2}$ . Yet, in lower dimension, it is well known that the value of  $r_d(\lambda)$  may fluctuate depending on the value of  $\lambda$ . Let us briefly recall some results in this direction.

In dimension 2, if we denote by  $\sigma(\Delta)$  the integers which are a sum of two squares (equivalently the set of Laplace eigenvalues), then, according to Corollary 3.6 in [22], one can find, for every  $\delta > 0$ , a density 1 subset  $S_\delta$  of  $\sigma(\Delta)$  such that, for every  $\lambda^2 \in S_\delta$ ,

$$r_2(\lambda) \geq (\log \lambda)^{\frac{\log 2}{2} - \delta}.$$

In particular, for every  $\lambda^2 \in S_\delta$ ,

$$\frac{1}{|\{j : \lambda_j = \lambda\}|} \sum_{j: \lambda_j = \lambda} \left| \int_{\mathbb{T}^2} a |\psi_j|^2 dx - \int_{\mathbb{T}^2} a dx \right|^2 \leq \frac{c'_2 \|a\|_{L^2(\mathbb{T}^2)}^2}{(\log \lambda)^{\frac{\log 2}{2} - \delta}}.$$

More generally, the variance goes to 0 along any subsequence of eigenvalues such that  $r_2(\lambda) \rightarrow +\infty$ . On the other hand, for every  $q \geq 0$ , one has  $r_2(3^q) = 4$  [15, p. 15]. Hence,  $r_2(\lambda)$  remains bounded along certain subsequences tending to  $+\infty$ .

The same discussion occurs in dimensions  $d = 3, 4$ . Indeed, one has  $r_3(2^q) = 6$  and  $r_4(2^q \sqrt{2}) = 24$  for every  $q \geq 0$  [15, p.30–38] while, along good subsequences of

eigenvalues,  $\frac{\lambda^{d-2}}{r_d(\lambda)}$  may go to 0 as  $\lambda$  tends to  $+\infty$ . For instance, in dimension 4, we can set, for every  $n \geq 1$ ,  $\lambda(n)^2 = \prod_{p \leq n: p \in \mathcal{P}} p$  where  $\mathcal{P}$  is the set of prime numbers. It follows from [1, p.38] and [15, p.30] that  $r_4(\lambda(n)) = \prod_{p \leq n: p \in \mathcal{P}} (1+p)$ . Hence, one has

$$\log r_4(\lambda(n)) = \sum_{p \leq n: p \in \mathcal{P}} \log p + \sum_{p \leq n: p \in \mathcal{P}} \log \left(1 + \frac{1}{p}\right) = \sum_{p \leq n: p \in \mathcal{P}} \log p + \sum_{p \leq n: p \in \mathcal{P}} \frac{1}{p} + \mathcal{O}(1).$$

Then, from Mertens asymptotics [1, p. 90], one finds

$$r_4(\lambda(n)) = \lambda(n)^2 \exp \left( \sum_{p \leq n: p \in \mathcal{P}} \frac{1}{p} + \mathcal{O}(1) \right) = e^{\mathcal{O}(1)} \lambda(n)^2 \log n.$$

In particular, the upper bound in (3.6) tends to 0 for this sequence of eigenvalues.

**Sharpness in dimension 2.** When  $d = 2$ , we have shown that

$$\frac{1}{|\{j : \lambda_j = \lambda\}|} \sum_{j: \lambda_j = \lambda} \left| \int_{\mathbb{T}^2} a |\psi_j|^2 dx - \int_{\mathbb{T}^2} a dx \right|^2 \leq \frac{2 \|a\|_{L^2}^2}{r_2(\lambda)},$$

and we emphasized that the upper bound may not go to 0. Here, we exhibit a sequence of eigenvalues  $\lambda(q)^2 \rightarrow +\infty$  such that the variance does not tend to 0 as  $q$  tends to  $+\infty$ . For this purpose, recall that Iwaniec proved the existence of a sequence of integers such that  $n_q \rightarrow +\infty$  and such that  $\lambda(q)^2 := n_q^2 + 1$  is the product of at most two primes [21]. Then, we know from p. 15 in [15] that, for every  $q$  large enough,  $8 \leq r_2(\lambda(q)) \leq 16$ , hence if the variance sum goes to zero then each individual term must go to zero. However, if we select

$$\varphi_q(x) := \frac{1}{\sqrt{2}} \left( e^{i(n_q x_1 + x_2)} + e^{i(n_q x_1 - x_2)} \right) = \sqrt{2} e^{i n_q x_1} \cos(x_2),$$

then for every  $a$  in  $\mathcal{C}^\infty(\mathbb{T}^2)$  and for every  $q \geq 1$ , one has

$$\int_{\mathbb{T}^2} a(x) |\varphi_q(x)|^2 dx = \int_{\mathbb{T}^2} 2a(x) \cos^2(x_2) dx,$$

which obviously does not converge to  $\int_{\mathbb{T}^2} a(x) dx$ .

#### 4. The case $d = 2$

In this last section, we focus on the two-dimensional case. More precisely, we draw a couple of simple consequences related to our results from an arithmetic Lemma due to Bourgain and Rudnick [4].

From Landau Theorem [15, p. 22], the number of integers in  $\{1, \dots, N\}$  that can be written as the sum of two squares is of order  $N/\sqrt{\log N}$  as  $N \rightarrow +\infty$ . Among these numbers, we know that most of them have the property of having well separated solutions (see Lemma 5 in [4]):

**Lemma 4.1.** *Let  $\delta > 0$ . Then, for all but  $\mathcal{O}(N^{1-\frac{\delta}{3}})$  integers  $E \leq N$ , one has*

$$\min_{k \neq \ell \in \mathbb{Z}^2: \|k\|^2 = \|\ell\|^2 = E} |k - \ell| > E^{\frac{1-\delta}{2}}.$$

In other words, for almost all eigenvalues  $\lambda^2 \leq \Lambda^2$ , the associated lattice points are well separated. In the following, we denote by  $\sigma(\Delta)$  the set of integers which are a sum of two squares, equivalently the set of Laplace eigenvalues. A nice corollary of this Lemma is the following:

**Corollary 4.2.** *Let  $d = 2$ . Then, there exists a density 1 subset  $S$  of  $\sigma(\Delta)$  such that, for every sequence of  $L^2$  normalized solutions  $\psi_\lambda$  of*

$$\Delta \psi_\lambda = -\lambda^2 \psi_\lambda,$$

one has,

$$\forall a \in \mathcal{C}^0(\mathbb{T}^2), \quad \lim_{\lambda \rightarrow +\infty, \lambda^2 \in S} \int_{\mathbb{T}^2} a(x) |\psi_\lambda(x)|^2 dx = \int_{\mathbb{T}^2} a(x) dx.$$

In other words, for a generic sequence of eigenvalues, *all* the Laplace eigenfunctions are equidistributed on the configuration space as  $\lambda \rightarrow +\infty$ .

*Proof.* By a density argument, it is sufficient to prove this for  $a = \mathbf{e}_p$  for every  $p$  in  $\mathbb{Z}^2$ . We write

$$\psi_\lambda = \sum_{k: \|k\|=\lambda} \hat{c}_k \mathbf{e}_k, \quad \sum_{k: \|k\|=\lambda} |\hat{c}_k|^2 = 1.$$

Then, one has

$$\int_{\mathbb{T}^2} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 dx = \sum_{k, \ell: \|k\|=\|\ell\|=\lambda} \hat{c}_k \overline{\hat{c}_\ell} \int_{\mathbb{T}^2} \mathbf{e}_{p+k-\ell}(x) dx = \sum_{k, \ell: \|k\|=\|\ell\|=\lambda, \ell-k=p} \hat{c}_k \overline{\hat{c}_\ell}.$$

From Lemma 4.1, we know that, for a fixed  $p \neq 0$ , this sum is empty for  $\lambda$  large enough (depending on  $p$ ). On the other hand, for  $p = 0$ , Lemma 4.1 and the normalization imply that the sum is equal to 1. Hence, for  $\lambda$  large enough (in a way that depends on  $p$  in  $\mathbb{Z}^2$ ), one has

$$\int_{\mathbb{T}^2} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 dx = \int_{\mathbb{T}^2} \mathbf{e}_p(x) dx,$$

which concludes the proof of the corollary.  $\square$

*Remark 4.3.* In a recent article, Granville and Wigman [14] obtained improvements on the Bourgain-Rudnick lemma and used it to get some results on *small scale equidistribution* property of toral eigenfunctions in dimension 2. See [18, 24, 29] for earlier work on this topic.

Once we have noticed this simple corollary of Bourgain-Rudnick's Lemma, the next question to ask is whether this remains true for equidistribution along hyper-surfaces. This is in fact the case thanks to Littman's Theorem and to the upper bound from (2.3):



**Proposition 4.4.** *Let  $\Sigma \subset \mathbb{T}^2$  be a smooth closed embedded and oriented curve with nonvanishing curvature. Then, there exists a density 1 subset  $S$  of  $\sigma(\Delta)$  such that, for every sequence of  $L^2$  normalized solutions  $\psi_\lambda$  of*

$$\Delta\psi_\lambda = -\lambda^2\psi_\lambda,$$

one has, as  $\lambda \rightarrow +\infty$ ,

$$\forall a \in C^0(\mathbb{T}^2), \quad \lim_{\lambda \rightarrow +\infty, \lambda^2 \in S} \int_{\Sigma} a(x) |\psi_\lambda(x)|^2 d\sigma(x) = \int_{\Sigma} a(x) d\sigma(x).$$

*Proof.* Using Bourgain-Rudnick's upper bound (2.3) which is uniform in  $\lambda$ , we can make use of a density argument and it is again sufficient to prove this result when  $a = \mathbf{e}_p$  for every fixed  $p$  in  $\mathbb{Z}^2$ . Indeed, given  $a$  and  $b$  in  $C^0(\mathbb{T}^2)$ , one has

$$\begin{aligned} \left| \int_{\Sigma} a(x) |\psi_\lambda(x)|^2 d\sigma(x) - \int_{\Sigma} b(x) |\psi_\lambda(x)|^2 d\sigma(x) \right| &\leq \|a - b\|_{C^0} \int_{\Sigma} |\psi_\lambda(x)|^2 d\sigma(x) \\ &\leq C \|a - b\|_{C^0}, \end{aligned}$$

where the second inequality follows from (2.3). In order to prove the Proposition for  $a = \mathbf{e}_p$ , we write

$$\psi_\lambda = \sum_{k: \|k\|=\lambda} \hat{c}_k \mathbf{e}_k, \quad \sum_{k: \|k\|=\lambda} |\hat{c}_k|^2 = 1.$$

We find that

$$\int_{\Sigma} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 d\sigma(x) = \sum_{k, \ell: \|k\|=\|\ell\|=\lambda} \hat{c}_k \bar{\hat{c}}_\ell \int_{\Sigma} \mathbf{e}_{p+k-\ell}(x) d\sigma(x).$$

Using the normalization and applying Littman's Theorem, one finds that

$$\int_{\Sigma} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 d\sigma(x) = \int_{\Sigma} \mathbf{e}_p d\sigma + \mathcal{O}(1) \sum_{k \neq \ell: \|k\|=\|\ell\|=\lambda} (1 + \|p + k - \ell\|)^{-\frac{1}{2}} |\hat{c}_k| |\hat{c}_\ell|.$$

From Lemma 4.1, one knows that, along a density 1 subset,  $\|k - \ell\| \geq \lambda^{1-\delta}$  for  $k \neq \ell$  lying on the same circle of radius  $\lambda$ . Hence, along this density one subset, one finds

$$\int_{\Sigma} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 d\sigma(x) = \int_{\Sigma} \mathbf{e}_p d\sigma + \mathcal{O}_p \left( \lambda^{-\frac{1-\delta}{2}} \right) \sum_{k \neq \ell: \|k\|=\|\ell\|=\lambda} |\hat{c}_k| |\hat{c}_\ell|.$$

For every  $\delta > 0$ , recall from paragraph 3.4 that the number of lattice points on the circle  $\lambda\mathbb{S}^1$  is of order  $\mathcal{O}(\lambda^{\delta/2})$ . Hence, thanks to the Cauchy-Schwarz inequality, the contribution of the nondiagonal terms will be of order  $\mathcal{O}(\lambda^{\delta/2})$ . This implies that

$$\int_{\Sigma} \mathbf{e}_p(x) |\psi_\lambda(x)|^2 d\sigma(x) = \int_{\Sigma} \mathbf{e}_p(x) d\sigma(x) + \mathcal{O}_p(\lambda^{-\frac{1}{2}+\delta}).$$

□

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