TWO-MICROLOCAL REGULARITY OF QUASIMODES ON THE TORUS

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Abstract. We study the regularity of stationary and time-dependent solutions to strong perturbations of the free Schrödinger equation on two-dimensional flat tori. This is achieved by performing a second microlocalization related to the size of the perturbation and by analysing concentration and nonconcentration properties at this new scale. In particular, we show that sufficiently accurate quasimodes can only concentrate on the set of critical points of the average of the potential along closed geodesics.

1. Introduction

The high-frequency analysis of eigenfunctions of elliptic operators on a compact Riemannian manifold has been the subject of intensive study in the past fifty years. To this day, many questions remain open, even in the simplest cases. Here we focus on eigenfunctions of Schrödinger operators on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, the standard torus endowed with its canonical metric. Eigenfunctions of a Schrödinger operator on $\mathbb{T}^d$ are the solutions to the equation:

$$-\Delta u_\lambda(x) + V(x)u_\lambda(x) = \lambda^2 u_\lambda(x), \quad x \in \mathbb{T}^d, \qquad \|u_\lambda\|_{L^2(\mathbb{T}^d)} = 1, \tag{1}$$

where the potential V is real-valued and essentially bounded. In the free case $V = 0$, a straightforward computation shows that eigenfunctions of eigenvalue λ^2 are linear combinations of complex exponentials $e^{2i\pi k.x}$ with frequencies $k \in \mathbb{Z}^d$ lying on a circle of radius $\lambda/(2\pi) > 0$ centered at the origin. However, extracting from this exact representation formula an asymptotic description of eigenfuctions in the high frequency limit $\lambda \to +\infty$ is a hard problem, due to the fact that multiplicities of large eigenvalues can also be very big. Instead, one can try to describe particular features of high-frequency eigenfunctions, such as formation of (asymtotic) singularities.

A natural way to quantify these singularities is through the scale of L^p spaces. This has been a classical topic in harmonic analysis, that originates with the seminal result of Zygmund [30] showing that, for $d = 2$ and in the free case, there exists some universal constant C such that any solution u_λ of (1) verifies $\|u_\lambda\|_{L^4(\mathbb{T}^2)} \leq C$. Later on, Bourgain conjectured in [7] that, again for the free case and when $d \geq 3$, one must have $\|u_\lambda\|_{L^{\frac{2d}{d-2}}(\mathbb{T}^d)} \leq C_\delta \lambda^\delta$ for every $\delta > 0$. We refer the reader to [8, 10] for recent progress towards this conjecture.

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Note that the problem of showing the existence of an index $p > 2$ such that $\|u_\lambda\|_{L^p(\mathbb{T}^d)}$ is uniformly bounded remains open for $d \geq 3$.

There are alternative ways to describe the asymptotic structure of the solutions of (1). For instance, notice that a direct corollary of Zygmund's result is that, in the free case, any accumulation point of the sequence of probability measures,

$$\nu_\lambda(dx) = |u_\lambda(x)|^2 dx,$$

is a probability measure which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$ (it has in fact an L^2 density). This result was refined by Jakobson who showed that the density has to be a trigonometric polynomial whose frequencies enjoy certain geometric constraints [17]. It is natural to try to understand what happens when $d \geq 3$, where no analogue to Zygmund's result is known to hold, or when the Laplacian is perturbed by a lower order term, such as a potential. Note that the problem of identifying accumulation points of sequences of moduli squares of eigenfunctions has a long history and it is connected to fundamental questions in quantum mechanics.

In dimension $d \geq 3$ and for $V = 0$, Bourgain proved that any accumulation point has to be absolutely continuous even if we do not know *a priori* that the L^p norms of eigenfunctions are uniformly bounded for small $p > 2$, this result was reported in [17]. In the same reference, Jakobson obtained partial results on the structure of the densities of accumulation points. These results are based on harmonic analysis techniques and arguments on the geometry of lattice points. Absolute continuity of accumulation points also holds in the case of a non-zero potential $V \in L^\infty(\mathbb{T}^d)$, as was proved by Anantharaman and the first author [5]. The proof of that result is based on methods from semiclassical analysis for the time dependent Schrödinger equation that were introduced for the particular case $d = 2$ in [19]. In fact, the results in reference [5] apply to the more general problem:

$$\hat{P}_\epsilon(\hbar) u_\hbar = \frac{1}{2} u_\hbar + o(\hbar \epsilon_\hbar), \quad \|u_\hbar\|_{L^2(\mathbb{T}^d)} = 1, \tag{2}$$

where $\hbar \to 0^+$ is some semiclassical parameter, and where

$$\hat{P}_\epsilon(\hbar) := -\frac{\hbar^2 \Delta}{2} + \epsilon_\hbar^2 V, \tag{3}$$

with $0 \leq \epsilon_\hbar \leq \hbar$ for $\hbar$ small enough.[1] One of the main ingredients used in this approach are the two-microlocal techniques developed in [24, 23, 13, 14, 15] in a different context. The results in [5] were further extended to treat the case of more general completely integrable systems in [1]. This approach can also be used in order to analyse the Schrödinger equation on the planar disk [3, 4]. Note that studying the regularity of the solutions to (2) is also related to problems arising in control theory as was shown by Burq and Zworski [11]. We refer the reader to [2, 3, 5, 9, 11, 12, 20] for perspectives from the point of view of control theory.

A different but related approach consists in studying the wavefront set $WF_\hbar(u_\hbar)$ of solutions to (2). This was done in a series of works by Wunsch [27, 28] and Vasy–Wunsch [25]

[1] Note that, when $\hbar = \epsilon_\hbar = \lambda^{-1}$, equation (2) is essentially equation (1).

dealing with completely integrable systems in dimension $d = 2$. In these articles, the authors investigate the properties of the semiclassical wavefront set $WF_\hbar(u_\hbar)$ of solutions to (2) when $0 \leq \epsilon_\hbar \leq \hbar^{1+\delta}$ with $\delta > 0$. By proving some propagation of second microlocal wavefront sets, they showed that $WF_\hbar(u_\hbar)$ cannot be reduced to a single geodesic and has to fill a Lagrangian torus – see for instance [27, Th. B] or [28, Th. 3]. Note that, as in [1], the results of Vasy and Wunsch hold for general classes of nondegenerate completely integrable systems. Under the assumption that $\hbar^{1-\delta} \ll \epsilon_\hbar \ll 1$, Wunsch also exhibited examples of quasimodes of order $\mathcal{O}(\hbar^\infty)$ for the operator $\hat{P}_\epsilon(\hbar)$ which concentrate on closed geodesics. This result was reported in [1, Sect. 5.3], and it shows that $\epsilon_\hbar = \hbar$ is the critical size for which one can expect to have singular concentration phenomena for perturbations of the free semiclassical Schrödinger operator $-\frac{\hbar^2\Delta}{2}$. In particular, for stronger perturbation $\epsilon_\hbar \gg \hbar$, one cannot expect to have uniform bounds for L^p norms even for small range of p. A notable feature of Wunsch's construction is that the singularity is located on critical points of the potential V restricted to certain closed geodesics. In some sense, this type of singularities is similar to the ones that may occur in the case of Zoll manifolds [21, 22]. Motivated by this observation, we will combine the ideas from [5, 21] in order to derive some properties on the regularity of solutions to (2) when $\epsilon_\hbar \gg \hbar$. In particular, we will identify precisely the concentration phenomena that may occur and also show non-concentration properties by propagation of second microlocal data. Note that, when written in non-semiclassical terms, the regime we are interested in corresponds to the eigenvalue problem:

$$-\Delta u_\lambda(x) + f(\lambda)V(x)u_\lambda(x) = \lambda^2 u_\lambda(x), \quad x \in \mathbb{T}^d, \qquad \|u_\lambda\|_{L^2(\mathbb{T}^d)} = 1,$$

where $1 \ll f(\lambda) \ll \lambda^2$.

For the sake of simplicity, we will focus on the case of the rational torus $\mathbb{T}^2$ and assume that $V \in \mathcal{C}^\infty(\mathbb{T}^2;\mathbb{R})$. However, it is most likely that our analysis could be extended to more general completely integrable systems of dimension 2 following the approach of [1]. As the small perturbation regime[2] $0 \leq \epsilon_\hbar \leq \hbar$ was studied in great detail in all the above references, here we will focus on the strong perturbation regime and we shall assume all along the article that

$$\lim_{\hbar\to 0^+} \epsilon_\hbar = 0, \quad \text{and} \quad \lim_{\hbar\to 0^+} \hbar\epsilon_\hbar^{-1} = 0. \tag{4}$$

In order to state our results, we need some simple geometric preliminaries. Recall that the geodesics of $\mathbb{T}^2$ are either closed or dense curves. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 - \{0\}$ and $x \in \mathbb{T}^2$, the geodesic $s \mapsto x + s\xi$ is dense provided ξ_1 and ξ_2 are linearly independent over $\mathbb{Q}$, otherwise it is periodic. We denote by $\Omega_1 \subset \mathbb{R}^2 - \{0\}$ the set of ξ that generate a periodic geodesic and by Ω_2 its complementary in $\mathbb{R}^2 - \{0\}$. Consider the average of V along geodesics:

$$\mathcal{I}(V)(x,\xi) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T V(x + s\xi)ds.$$

[2]Note that, for the non-semiclassical version, it means that $f(\lambda) \leq 1$.

Clearly, $\mathcal{I}(V)$ is a zero-homogeneous function with respect to ξ. Moreover, a classical result by Kronecker implies that

$$\mathcal{I}(V)(x,\xi) = \begin{cases} \frac{1}{L_\xi}\int_0^{L_\xi} V\left(x + s\frac{\xi}{\|\xi\|}\right) ds & \text{if } \xi \in \Omega_1, \\ \int_{\mathbb{T}^2} V(y)dy & \text{if } \xi \in \Omega_2, \end{cases}$$

where L_ξ denotes the length of any geodesic with velocity ξ. In particular, despite $\mathcal{I}(V)$ is not continuous in general, one has $\mathcal{I}(V)(\cdot,\xi) \in \mathcal{C}^\infty(\mathbb{T}^2;\mathbb{R})$ for any $\xi \in \mathbb{R}^2 - \{0\}$, and $\|\mathcal{I}(V)\|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} \leq \|V\|_{L^\infty(\mathbb{T}^2)}$.

Then, we define the set of critical geodesics:

$$\mathcal{C}(V) := \left\{x_0 \in \mathbb{T}^2 : \ \exists\, \xi \in \Omega_1 \text{ s.t. } \partial_x\mathcal{I}(V)(x_0,\xi) = 0\right\}. \tag{5}$$

Note that $\mathcal{C}(V)$ is a union of closed geodesics of $\mathbb{T}^2$. For every closed geodesic γ of $\mathbb{T}^2$, we denote by δ_γ the normalized Lebesgue measure along this closed geodesic. Then, we define $\mathcal{N}(V)$ as the convex closure of the set of probability measures δ_γ where $\gamma \subset \mathcal{C}(V)$. With these conventions in mind, we can state our main result:

Theorem 1.1. *Suppose that $d = 2$ and that (4) holds. Let $(u_\hbar)_{\hbar\to 0^+}$ be a sequence satisfying (2). Then, for any accumulation point ν of the sequence of probability measures*

$$\nu_\hbar(dx) := |u_\hbar(x)|^2 dx,$$

and for any closed geodesic γ, one has

$$\nu(\gamma) \neq 0 \Longrightarrow \gamma \subset \mathcal{C}(V).$$

Moreover, ν can be decomposed as

$$\nu = f dx + \nu_{\text{sing}},$$

where $f \in L^1(\mathbb{T}^2)$ and where $\nu_{\text{sing}} \in \mathcal{N}(V)$.

Recall from the propagation properties of semiclassical measures [16, 29] that any ν as in Theorem 1.1 must a priori be a convex combination of the Lebesgue measure and of the measures δ_γ, where γ runs over the set of all closed geodesics. This Theorem shows that singular concentration along closed geodesics can only occur along certain closed orbits associated with critical points of the averages of V along closed geodesics. This result is sharp in the sense that Wunsch's construction in [1] shows that one can find quasimodes such that $\nu(\gamma) = 1$ for a given closed geodesic. Despite these unavoidable concentration phenomena, Theorem 1.1 also shows that the accumulation points enjoy certain regularity properties. This extra regularity will come out from our analysis by making a second microlocalization of size $\epsilon_\hbar$ along rational directions, and it will be induced by certain Lagrangian tori associated to our problem. Note that these two aspects are close to the situation of Zoll manifolds treated in [21, 22]. The main difference is that there exist infinitely many directions where the flow is periodic with periods tending to $+\infty$. We would like to treat these tori of periodic orbits as in this reference, and this can be achieved via rescaling the variables along these rational directions – see paragraph 3.4 for more details.

Finally, as we shall see it in Sections 2 and 3, our analysis holds in the more general context of the time dependent Schrödinger equation.

Organization of the article. Section 2 places our problem in the more general framework of the time-dependent Schrödinger equation associated with $\hat{P}_\epsilon(\hbar)$: Theorem 1.1 becomes a direct consequence of the more general Theorem 2.1 which deals with the evolution problem. The proof of this result is obtained by characterizing time-dependent semiclassical measures for solutions to the Schrödinger equation. Following a strategy similar to that in [5, 19], such a characterization can be obtained by using two-microlocal techniques. In Section 3, we introduce the 2-microlocal framework of our analysis that is needed to formulate our main results, Theorems 3.6 and 3.7. Section 4 presents several applications of these results. We first give the proof of Theorem 2.1, then we present a structure result for semiclassical measures of the evolution equation, Theorem 4.1, which we apply to compute the propagation of wave packet solutions (Proposition 4.3). This shows that Theorem 2.1 is sharp in some sense. The proofs of the 2-microlocal statements of Section 3 are given in Section 5. Finally, the article contains two appendices. Appendix A contains the proof of a geometric result which already appeared in [21] and which we adapt to the context of $\mathbb{T}^2$. In Appendix B, we collect a few tools from semiclassical analysis.

In the following (except in appendix B), we will always suppose that $d = 2$ and that (4) holds even if part of the results holds in greater generality.

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2. Semiclassical measures for the time-dependent Schrödinger equation

As was already mentionned, Theorem 1.1 is a consequence of our analysis of the time dependent semiclassical Schrödinger equation:

$$i\hbar\partial_t v_\hbar = \hat{P}_\epsilon(\hbar)v_\hbar, \quad v_\hbar|_{t=0} = u_\hbar \in L^2(\mathbb{T}^2), \quad \|u_\hbar\|_{L^2} = 1. \tag{6}$$

For the sake of simplicity, we shall focus on sequences of initial data oscillating at the frequency $\hbar^{-1}$. Thus, we will always assume that the following properties hold:

$$\limsup_{\hbar\to 0} \left\| \mathbf{1}_{[R,\infty)} \left(-\hbar^2\Delta\right) u_\hbar \right\|_{L^2(M)} \longrightarrow 0, \quad \text{as } R \longrightarrow \infty, \tag{7}$$

and

$$\limsup_{\hbar\to 0} \left\| \mathbf{1}_{[0,\delta]} \left(-\hbar^2\Delta\right) u_\hbar \right\|_{L^2(M)} \longrightarrow 0, \quad \text{as } \delta \longrightarrow 0^+. \tag{8}$$

Fix now a sequence of time scales $(\tau_\hbar)_{\hbar\to 0^+}$ such that

$$\lim_{\hbar\to 0^+} \tau_\hbar = +\infty.$$

We will deal with time-scaled solutions to the perturbed Schrödinger equation. More precisely, if $v_\hbar$ is a solution to (6), then we shall study the behavior of

$$t \longmapsto v_\hbar(\tau_\hbar t, \cdot).$$

As we will see below, the scale $\tau_\hbar = \epsilon_\hbar^{-1}$ is critical for this problem, and Theorem 1.1 follows from the analysis of the time-dependent equation in the regime $\tau_\hbar \gg \epsilon_\hbar^{-1}$.

2.1. **Time-dependent semiclassical measures.** For a given t in $\mathbb{R}$, we denote the Wigner distribution at time t by

$$\langle w_\hbar(t), a\rangle := \langle v_\hbar(t), \mathrm{Op}_\hbar^w(a) v_\hbar(t)\rangle \,, \tag{9}$$

where $\mathrm{Op}_\hbar^w(a)$ is a $\hbar$-pseudodifferential operator with principal symbol $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2)$ – see Appendix B. Above, $v_\hbar(t)$ denotes the solution at time t of (6) with initial conditions satisfying the oscillating assumptions (7) and (8). This quantity represents the distribution of the L^2-mass of the solution to (6) in the phase space $T^*\mathbb{T}^2$. According to [18], we can extract a subsequence $\hbar_n \to 0^+$ as $n \to +\infty$ such that, for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$ and for every θ in $L^1(\mathbb{R})$,

$$\lim_{\hbar_n \to 0^+} \int_{\mathbb{R}\times T^*\mathbb{T}^2} \theta(t)\langle w_{\hbar_n}(t\tau_{\hbar_n}), a\rangle dt = \int_{\mathbb{R}\times T^*\mathbb{T}^2} \theta(t) a(x,\xi)\mu(t, dx, d\xi) dt,$$

where, for a.e. t in $\mathbb{R}$, $\mu(t)$ is a finite positive Radon measure on $T^*\mathbb{T}^2$. Recall also that, for a.e. $t \in \mathbb{R}$, $\mu(t)$ is in fact a *probability measure* which does not put any mass on the zero section, thanks to the frequency assumption (8). In other words,

$$\mu(t)(\mathring{T}^*\mathbb{T}^2) = 1, \text{ for a.e. } t \in \mathbb{R}, \tag{10}$$

where

$$\mathring{T}^*\mathbb{T}^2 := \left\{(x,\xi) \in T^*\mathbb{T}^2 : \xi \neq 0\right\}.$$

Moreover, for a.e. t in $\mathbb{R}$, $\mu(t)$ is *invariant by the geodesic flow* φ^s on $T^*\mathbb{T}^2$.

For instance, $\mu(t)$ can be the normalized Lebesgue measure along a closed orbit of the geodesic flow. We will denote by $\mathcal{M}(\tau, \epsilon)$ the set of accumulation points of the sequences $(\mu_\hbar)$, where $\mu_\hbar(t,\cdot) := w_\hbar(t\tau_\hbar, \cdot)$, as the sequence of initial data $(u_\hbar)$ varies among normalized sequences satisfying (7) and (8). Similarly, one can define $\mathcal{N}(\tau,\epsilon)$ to be the set of accumulation points of the sequences $(n_\hbar)$ of time-dependent probability measures on $\mathbb{T}^2$, $n_\hbar(t, dx) := |v_\hbar(t\tau_\hbar, x)|^2 dx$, obtained letting the initial data vary among sequences satisfying (7), (8). Using (7), one can verify that

$$\mathcal{N}(\tau,\epsilon) = \left\{ \int_{\mathbb{R}^2} \mu(t, x, d\xi) : \; \mu \in \mathcal{M}(\tau,\epsilon)\right\}. \tag{11}$$

2.2. **Statement of the results.** In order to relate the time-dependent approach to the quasimode case, we can remark that, given a sequence of quasimodes $(u_\hbar)_{\hbar\to 0^+}$ satisfying (2), we can always find a sequence of time scales $(\tau_\hbar)$ such that

$$\lim_{\hbar\to 0} \tau_\hbar \epsilon_\hbar = +\infty,$$

and, for every $t \in \mathbb{R}$:

$$\lim_{\hbar\to 0} \|v_\hbar(\tau_\hbar t, \cdot) - e^{-i\tau_\hbar t/2\hbar} u_\hbar\|_{L^2(\mathbb{T}^2)} = 0,$$

where $v_\hbar$ denotes the solution to (6) with initial condition $u_\hbar$. This choice of $(\tau_\hbar)$ ensures that any accumulation point ν of the sequence of probability measures $(|u_\hbar|^2 dx)$ belongs to $\mathcal{N}(\tau, \epsilon)$ (even though it is constant in t), since it is also an accumulation point of $(|v_\hbar(\tau_\hbar t, \cdot)|^2 dx)$. In particular, Theorem 1.1 follows from the more general statement:

Theorem 2.1. *Suppose that*

$$\lim_\hbar \tau_\hbar \epsilon_\hbar = +\infty.$$

Let $t \longmapsto \nu(t)$ be an element of $\mathcal{N}(\tau, \epsilon)$. Then, for any closed geodesic γ not included inside $\mathcal{C}(V)$ and for a.e. t in $\mathbb{R}$, one has

$$\nu(t)(\gamma) = 0.$$

Moreover, $\nu(t)$ can be decomposed as

$$\nu(t) = f(t)dx + \nu_{\text{sing}}(t),$$

where, for a.e. t in $\mathbb{R}$, $f(t) \in L^1(\mathbb{T}^2)$ and $\nu_{\text{sing}}(t) \in \mathcal{N}(V)$.

The first step in the proof of this result is the partition of $\mathbb{R}^2 - \{0\}$ into φ^s-invariant subsets that was used in [19, 5]. Recall that $\Lambda \subset \mathbb{Z}^2$ is a *primitive lattice of rank one* provided that $\dim\langle\Lambda\rangle = 1$ and that $\langle\Lambda\rangle \cap \mathbb{Z}^2 = \Lambda$, where $\langle\Lambda\rangle$ is the linear subspace of $\mathbb{R}^2$ spanned by Λ. We introduce the invariant set of rational covectors

$$\Omega_1 = \bigsqcup_{\Lambda \text{ rank 1 primitive}} \Lambda^\perp - \{0\},$$

and its complement Ω_2 inside $\mathbb{R}^2 - \{0\}$ which is still invariant. Observe that this is consistent with the conventions of the introduction. Because of (10), we can decompose the measure as follows:

$$\mu(t) = \mu(t)\rceil_{\mathbb{T}^2 \times \Omega_2} + \sum_{\Lambda \text{ rank 1 primitive}} \mu(t)\rceil_{\mathbb{T}^2 \times \Lambda^\perp - \{0\}}. \tag{12}$$

As a consequence of the invariance by the geodesic flow, it can be verified that $\mu(t)\rceil_{\mathbb{T}^2 \times \Omega_2}$ is in fact independent of the x-variable. Hence, in order to prove Theorem 2.1, one only has to study the regularity of $\mu(t)\rceil_{\mathbb{T}^2 \times \Lambda^\perp - \{0\}}$ for every rank 1 primitive sublattice Λ. This will be achieved using two-microlocal tools adapted to this problem. The end of the proof of Theorem 2.1 is presented in Section 4.1. For time scales $\tau_\hbar = \mathcal{O}(\epsilon_\hbar^{-1})$, we obtain a more precise result, in the sense that each component of the time dependent semiclassical measure $\mu(t)$ according to the partition (12) can be completely determined from the initial data that were used to generate it. Again, the relation with the sequence of initial data is elucidated using the class of two-microlocal semiclassical measures that will be introduced in the next section. A precise statement is given in Theorem 4.1, Section 4.2.

Finally, in Section 4.3, we provide explicit computations of semiclassical measures associated to wave-packets (Proposition 4.3) that yield:

(1) if $\tau_\hbar \epsilon_\hbar \to 0$, then

$$\{\delta_\gamma \,:\, \gamma \text{ periodic geodesic of } \mathbb{T}^2\} \subset \mathcal{N}(\tau, \epsilon);$$

(2) if $\tau_\hbar = \epsilon_\hbar^{-1}$, then

$$\{\delta_\gamma \,:\, \gamma \in \mathcal{C}(V)\} \subset \mathcal{N}(\tau, \epsilon).$$

3. Invariance and propagation of 2-microlocal distributions

In this section, we present our main result on the 2-microlocal structure of solutions to the time-dependent Schrödinger equation along covectors in Ω_1. In particular, we show how solutions of (6) can concentrate along rational covectors.

Before stating the result, we need some additional notation. For every primitive rank 1 lattice Λ of $\mathbb{Z}^2$, we set $\mathfrak{e}_\Lambda$ to be an element in Λ such that $\mathbb{Z}\mathfrak{e}_\Lambda = \Lambda$, and $\mathfrak{e}_\Lambda^\perp$ to be the vector of same length which is directly orthogonal to $\mathfrak{e}_\Lambda$. We define

$$L_\Lambda := \|\mathfrak{e}_\Lambda\|.$$

We define two Hamiltonian maps associated to Λ as follows:

$$H_\Lambda(\xi) := \frac{1}{L_\Lambda}\langle \xi, \mathfrak{e}_\Lambda\rangle \text{ and } H_\Lambda^\perp(\xi) := \frac{1}{L_\Lambda}\langle \xi, \mathfrak{e}_\Lambda^\perp\rangle.$$

Note that $(H_\Lambda, H_\Lambda^\perp)$ defines a (nondegenerate) completely integrable system and that

$$\|\xi\|^2 = H_\Lambda(\xi)^2 + H_\Lambda^\perp(\xi)^2.$$

3.1. Two-microlocal distributions.

We aim at studying the concentration of solutions to (6) over $\mathbb{T}^2 \times \Lambda^\perp$ where $\Lambda \subset \mathbb{Z}^2$ is a primitive rank 1 sublattice and where $\Lambda^\perp$ denotes the set of covectors ξ such that $H_\Lambda(\xi) = 0$. For that purpose, we consider a two-microlocal scale $\alpha_\hbar \to 0^+$ satisfying $\hbar\alpha_\hbar^{-1} \to 0$ and we define the following two-microlocal Wigner distribution:

$$w_{\Lambda,\hbar}(t) : a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}}) \longmapsto \left\langle v_\hbar(t), \operatorname{Op}_\hbar^w\left(a\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right)\right) v_\hbar(t)\right\rangle.$$

Above, $\widehat{\mathbb{R}}$ is the compactified space $\mathbb{R} \cup \{\pm\infty\}$, $v_\hbar(t)$ is the solution of (6) at time t, and $\operatorname{Op}_\hbar^w(a)$ is a $\hbar$-pseudodifferential operator – see Appendix B.

Remark 3.1. Recall from (28) in Appendix B that the following useful relation holds:

$$\operatorname{Op}_\hbar^w\left(a\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right)\right) = \operatorname{Op}_{\hbar\alpha_\hbar^{-1}}^w\left(a\left(x, \alpha_\hbar\xi, H_\Lambda(\xi)\right)\right),$$

and that we have made the assumption that $\hbar\alpha_\hbar^{-1} \to 0$. Therefore, the operators involved in the definition of $w_{\Lambda,\hbar}$ are semiclassical pseudodifferential operators whose symbolic calculus enjoys a gain of $\hbar\alpha_\hbar^{-1}$.

Remark 3.2. The distributions $w_{\Lambda,\hbar}$ were introduced in [19, 5] for the critical case $\alpha_\hbar = \hbar$ under a slightly different form. There, the two microlocal variable η varies in the two-point compactification of $\langle\Lambda\rangle$. Of course, this is completely equivalent to our formulation for the two-dimensional torus, but turns out to be relevant when dealing with the higher dimensional case. As we will see, the fact that the two-microlocal scale is asymptotically bigger than $\hbar$ implies that the limiting objects are of a different nature than those obtained in [19, 5]. When $\hbar\alpha_\hbar^{-1} \to 0$, they are global variants on the torus of the two-scale semiclassical measures introduced in [14] – see also [2] for a related construction on the torus, in a context related to that of [5].

Recall that we introduced a time scale $\tau_\hbar \to \infty$. From now on, we shall fix the two-microlocal scale as follows:

$$\alpha_\hbar := \begin{cases} 1/\tau_\hbar & \text{if } \tau_\hbar\epsilon_\hbar^{-1} \to 0, \\ \epsilon_\hbar & \text{otherwise.} \end{cases} \tag{13}$$

As we shall explain it in paragraph 5.1, we can extract a subsequence $\hbar_n \to 0^+$ such that, for any $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$ and for any $\theta \in L^1(\mathbb{R})$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \theta(t)\langle w_{\Lambda,\hbar_n}(t\tau_{\hbar_n}), a\rangle dt = \int_{\mathbb{R}} \theta(t) \left(\int_{T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} a(x,\xi,\eta)\mu_\Lambda(t, dx, d\xi, d\eta) \right) dt,$$

where, for a.e. t in $\mathbb{R}$, $\mu_\Lambda(t)$ is an element of $\mathcal{B}'$ for some Banach space $\mathcal{B}$ that we will define in paragraph 5.1. We denote by $\mathcal{M}_\Lambda(\tau, \epsilon)$ the set of accumulation points obtained in this manner for initial data varying among subsequences verifying (7) and (8). The main new result of this article describes some invariance and propagation properties of these quantities depending on the relative size of $\tau_\hbar$ and $\epsilon_\hbar$.

For every primitive rank 1 sublattice, one has (see Remark 5.3),

$$\mathcal{M}(\tau,\epsilon) = \left\{ \int_{\widehat{\mathbb{R}}} \mu_\Lambda(t, x, \xi, d\eta) : \ \mu_\Lambda \in \mathcal{M}_\Lambda(\tau,\epsilon) \right\}. \tag{14}$$

3.2. **First properties.** Before proving our main results, we will verify a few preliminary results. First, one has

Proposition 3.3. *Let $\mu_\Lambda(t)$ be an element of $\mathcal{M}_\Lambda(\tau,\epsilon)$. Then, for a.e. t in $\mathbb{R}$, $\mu_\Lambda(t)$ is a positive finite Radon measure concentrated on $\mathring{T}^*\mathbb{T}^2 \times \widehat{\mathbb{R}}$.*

In what follows, we write

$$\tilde{\mu}_\Lambda(t) := \mu_\Lambda(t)\rceil_{\mathring{T}^*\mathbb{T}^2\times\mathbb{R}}, \quad \tilde{\mu}^\Lambda(t) := \mu_\Lambda(t)\rceil_{\mathring{T}^*\mathbb{T}^2\times\{\pm\infty\}}.$$

Hence, we can split the 2-microlocal measure as

$$\mu_\Lambda(t) = \tilde{\mu}_\Lambda(t) + \tilde{\mu}^\Lambda(t). \tag{15}$$

The measure $\tilde{\mu}_\Lambda(t)$ describes in some sense the way the solutions of (6) concentrate in an $\epsilon_\hbar$-neighborhood of the rational direction $\Lambda^\perp$. Let us now give some other simple properties of these functionals which are analoguous to the ones satisfied by time dependent semiclassical measures [18]. We shall also verify:

Proposition 3.4. *Let $\mu_\Lambda(t) \in \mathcal{M}_\Lambda(\tau,\epsilon)$. Then*

1. *$\tilde{\mu}_\Lambda(t)$ is a (finite) positive measure on $T^*\mathbb{T}^2 \times \mathbb{R}$ whose support is contained in $\mathbb{T}^2 \times (\Lambda^\perp - \{0\}) \times \mathbb{R}$;*
2. *for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$,*

$$\langle \tilde{\mu}_\Lambda(t), \xi.\partial_x a\rangle = \langle \tilde{\mu}^\Lambda(t), \xi.\partial_x a\rangle = 0.$$

Neither Proposition 3.3, nor part (1) of Proposition 3.4 uses that the functions used to generate $\mu_\Lambda(t)$ are solutions to (6). This fact is only used in the second part of Proposition 3.4. Note that all these properties follow from standard arguments which need to be slightly adapted in order to fit into the 2-microlocal set-up – see Section 5 for details.

3.3. **Main results.** Consider the Hamiltonian flow $\varphi_{H_\Lambda^\perp}$ associated with $H_\Lambda^\perp$. Note that, for a continuous function b on $T^*\mathbb{T}^2 \times \widehat{\mathbb{R}}$, we can define the average along this L_Λ-periodic flow as

$$\mathcal{I}_\Lambda(b)(x,\xi,\eta) := \frac{1}{L_\Lambda}\int_0^{L_\Lambda} b\left(\varphi^s_{H_\Lambda^\perp}(x,\xi),\eta\right) ds.$$

A direct computation gives

$$\mathcal{I}_\Lambda(b)(x,\xi,\eta) = \frac{1}{L_\Lambda}\int_0^{L_\Lambda} b\left(x + s\frac{\mathfrak{e}_\Lambda^\perp}{L_\Lambda},\xi,\eta\right) ds = \sum_{k\in\Lambda} \hat{b}_k(\xi,\eta)e^{2i\pi k.x},$$

provided b has the Fourier expansion $b(x,\xi,\eta) = \sum_{k\in\mathbb{Z}^2} \hat{b}_k(\xi,\eta)e^{2i\pi k.x}$. Moreover, if $\mathcal{I}(b)$ denotes the average of b along the geodesic flow

$$\varphi^s(x,\xi) = (x + s\xi, \xi)$$

on $T^*\mathbb{T}^2$, then the following holds:

$$\mathcal{I}(b)(x,\xi,\eta) = \mathcal{I}_\Lambda(b)(x,\xi,\eta), \quad \text{provided that } \xi \in \Lambda^\perp - \{0\}. \tag{16}$$

In the case where b only depends on x, as is the case with $b = V$, it is easy to check that $\mathcal{I}_\Lambda(V)$ does not depend on ξ and therefore we can identify it to an element in $\mathcal{C}^\infty(\mathbb{T}^2;\mathbb{R})$.

Remark 3.5. Part (2) of Proposition 3.4 implies that $\mu_\Lambda(t)$ is invariant under the geodesic flow φ^s. For b in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \mathbb{R})$, this observation combined with part (1) in Proposition 3.4 and identity (16) implies that, for a.e. t in $\mathbb{R}$,

$$\langle \mu_\Lambda(t), b\rangle = \langle \mu_\Lambda(t), \mathcal{I}_\Lambda(b)\rangle.$$

We shall use this property several times in our proof of Theorem 3.6 below.

We need to define an auxiliary Hamiltonian function on $\mathbb{T}^2 \times \Lambda^\perp \times \mathbb{R}$

$$p_\Lambda^V(x, \sigma\mathfrak{e}_\Lambda^\perp/L_\Lambda, \eta) := \frac{\eta^2}{2} + \mathcal{I}_\Lambda(V)(x). \tag{17}$$

Denote by $\varphi^t_{p_\Lambda^V}$ the flow of the vector field on $\mathbb{T}^2 \times \Lambda^\perp \times \mathbb{R}$:

$$\eta\frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x - \frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x\mathcal{I}_\Lambda(V)\partial_\eta.$$

This is the Hamiltonian vector field associated to p_Λ^V with respect to the symplectic form obtained by taking the push-forward of the canonical symplectic form on $T^*\mathbb{T}^2$ via the diffeomorphism

$$T^*\mathbb{T}^2 \ni (x,\xi) \longmapsto (x, H_\Lambda^\perp(x,\xi)\mathfrak{e}_\Lambda^\perp/L_\Lambda, H_\Lambda(x,\xi)) \in \mathbb{T}^2 \times \Lambda^\perp \times \mathbb{R}. \tag{18}$$

The flow $\varphi^t_{p_\Lambda^V}$ commutes with $\varphi^s_{H_\Lambda^\perp}$ when acting on $\mathbb{T}^2 \times \Lambda^\perp \times \mathbb{R}$.

We are now ready to state the main results of this article. The first one concerns the "compact" part of these two-microlocal distributions. Their possible behaviors are classified according to the limit of $\tau_\hbar\epsilon_\hbar$.

Theorem 3.6 (Invariance and propagation near Λ)**.** *Let Λ be a primitive rank 1 sublattice and let μ_Λ be an element of $\mathcal{M}_\Lambda(\tau,\epsilon)$ obtained as the limit of $(w_{\Lambda,\hbar}(t\tau_\hbar))$. Denote by μ_Λ^0 the limit of $(w_{\Lambda,\hbar}(0))$. The following results hold:*

(1) *If $\tau_\hbar\epsilon_\hbar\to 0$ as $\hbar\to 0^+$, then $t\mapsto\tilde{\mu}_\Lambda(t)$ is continuous, and one has, for every a in $\mathcal{C}_c^0(\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R})$,*
$$\tilde{\mu}_\Lambda(t)(a)=\tilde{\mu}_\Lambda^0(\mathcal{I}_\Lambda(a)\circ\varphi^t_{p_\Lambda^0}).$$

(2) *If $\tau_\hbar\epsilon_\hbar\to c>0$ as $\hbar\to 0^+$, then $t\mapsto\tilde{\mu}_\Lambda(t)$ is continuous, and one has, for every a in $\mathcal{C}_c^0(\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R})$,*
$$\tilde{\mu}_\Lambda(t)(a)=\tilde{\mu}_\Lambda^0(\mathcal{I}_\Lambda(a)\circ\varphi^{ct}_{p_\Lambda^V}).$$

(3) *If $\tau_\hbar\epsilon_\hbar\to+\infty$ as $\hbar\to 0^+$, then one has, for a.e. t in $\mathbb{R}$ and, for every a in $\mathcal{C}_c^0(\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R})$,*
$$\forall s\in\mathbb{R},\quad \tilde{\mu}_\Lambda(t)\,(a)=\tilde{\mu}_\Lambda(t)(a\circ\varphi^s_{p_\Lambda^V}).$$

Equivalently, this Theorem says that, besides invariance by the geodesic flow, the solutions of (6) satisfy some extra invariance properties in a shrinking neighborhood of the rational direction at least for times $\tau_\hbar\gg\epsilon_\hbar^{-1}$. For shorter times, the concentration in this shrinking neighborhood is completely determined by the initial data. The proof of this theorem is given in Section 5. Note that, when $\tau_\hbar\epsilon_\hbar\to 0$, the conclusion of part (1) holds even if $\epsilon_\hbar=\hbar$, this will be clear from the proof. Section 5.1 in reference [1] provides explicit computations of two-microlocal semiclassical measures in that regime.

It is interesting to compare part (2) of Theorem 3.6 with its counterpart in [5], where the regime $\epsilon_\hbar=\hbar$ is studied in detail in any dimension (not only in the two-dimensional case analysed here). First, the nature of the limiting object $\tilde{\mu}_\Lambda$ is rather different in that setting. It is no longer a positive measure, but rather a measure taking values in the set of Wigner transforms of positive Hermitian trace-class operators on the space $L^2(\mathbb{T}_\Lambda)$.[3] As a result, time-dependent semiclassical measures are absolutely continuous with respect to the Lebesgue measures in the x-variable. In that setting, the rôle of the flow $\varphi^s_{p_\Lambda^V}$ is played by the quantum flow $e^{-is(D_\Lambda^2+\mathcal{I}_\Lambda(V))}$ – see Corollary 25 in [5] for a precise statement.

The part at infinity satisfies an additional regularity property. Indeed, if we define
$$\mathcal{I}_0(a)(\xi,\eta):=\int_{\mathbb{T}^2}a(y,\xi,\eta)dy,$$
then the following holds:

[3]This space consists of those functions in $L^2(\mathbb{T}^2)$ that are invariant by translations in the direction $\Lambda^\perp$.

Theorem 3.7 (Regularity at infinity)**.** *Let Λ be a primitive rank 1 sublattice and let $\mu_\Lambda(t)$ be an element of $\mathcal{M}_\Lambda(\tau,\epsilon)$. Then, one has, for every a in $\mathcal{C}_c^\infty(\mathbb{T}^2\times\mathbb{R}^2\times\widehat{\mathbb{R}})$ and for a.e. t in $\mathbb{R}$,*

$$\langle \tilde{\mu}^\Lambda(t), \mathcal{I}_\Lambda(a) - \mathcal{I}_0(a)\rangle = 0.$$

In particular, the measure $\tilde{\mu}^\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\widehat{\mathbb{R}}}$ is constant in x.

In other words, the part at infinity has no (nonzero) Fourier coefficients in the Λ-direction. As for Theorem 3.6, this result depends highly on the choice of two-microlocal scale we have fixed from the beginning, and other scalings would have yield other properties. The first conclusion of this theorem is proved in Section 5. The last assertion follows from the invariance[4] of $\tilde{\mu}^\Lambda(t)$ under the geodesic flow, which implies that for every $a\in\mathcal{C}_c^0(T^*\mathbb{T}^2\times\widehat{\mathbb{R}})$:

$$\langle \tilde{\mu}^\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\widehat{\mathbb{R}}}, a\rangle = \langle \tilde{\mu}^\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\widehat{\mathbb{R}}}, \mathcal{I}_\Lambda(a)\rangle = \langle \tilde{\mu}^\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\widehat{\mathbb{R}}}, \mathcal{I}_0(a)\rangle.$$

Note also that the conclusion of Theorem 3.7 holds in the regime $\epsilon_\hbar=\hbar$ (in any dimension), see part (ii) of Theorem 12 in [5].

3.4. **Comparison with Zoll manifolds.** Theorem 3.6 shares also a lot of similarities with our main result on semiclassical measures for perturbations of Zoll Laplacians in [21, Sect. 2.2]. In that case, we were considering the semiclassical operator

$$-\frac{\hbar^2\Delta_g}{2}+\epsilon_\hbar^2 V,$$

where Δ_g is the Laplace Beltrami operator associated to a certain Zoll metric (say the standard metric on the canonical sphere). In the present article, we are analyzing the semiclassical measures associated to the same Schrödinger operator $\hat{P}_\epsilon(\hbar)$. Studying the "compact" part of elements inside $\mathcal{M}_\Lambda(\tau,\epsilon)$ is equivalent to understanding the solutions of (6) near submanifolds

$$\mathbb{T}^2\times\Lambda^\perp := \{(x,\xi)\in T^*\mathbb{T}^2 : H_\Lambda(\xi)=0\},$$

where the geodesic flow is periodic as in the Zoll case. In order to make the comparison clearer and to justify the rescaling of order $\epsilon_\hbar$, we can rewrite our operator in a form which is very close to what we did in the Zoll framework, i.e.

$$\hat{P}_\epsilon(\hbar)=\frac{1}{2}\operatorname{Op}_\hbar^w(H_\Lambda^\perp)^2+\epsilon_\hbar^2\operatorname{Op}_\hbar^w\left(\frac{1}{2}\left(\frac{H_\Lambda}{\epsilon_\hbar}\right)^2+V\right).$$

Thus, as in the Zoll case, we perturb in some sense a semiclassical operator $\operatorname{Op}_\hbar^w(H_\Lambda^\perp)^2$ asssociated to a "periodic" Hamiltonian flow and we obtain limit quantities which are invariant by the periodic flow and the Hamiltonian perturbation.

[4]Recall also that $\mu_\Lambda(t)$ is supported on $\mathring{T}^*\mathbb{T}^2\times\widehat{\mathbb{R}}$.

The main difference with the Zoll setting is that the perturbation depends on rescaled variables

$$\left(x, H_\Lambda^\perp(\xi), \frac{H_\Lambda(\xi)}{\epsilon_\hbar}\right) \in \mathbb{T}^2 \times \mathbb{R}^2 \simeq T^*\mathbb{T}^2.$$

For that reason, it is natural to test our Wigner distributions against symbols depending on these rescaled variables. Another notable difference with [21] is that, in the Zoll case, the critical time scale is of order $\epsilon_\hbar^{-2}$ while here, due to the use of rescaled variables, it is much shorter, i.e. of order $\epsilon_\hbar^{-1}$. Finally, in the Zoll case, a natural question was to discuss the case where the Radon transform of the perturbation identically vanishes [22]. Here, we emphasize that the $H_\Lambda^\perp$-average of the perturbation, namely $\frac{1}{2}\left(\frac{H_\Lambda}{\epsilon_\hbar}\right)^2 + \mathcal{I}_\Lambda(V)$ *cannot be equal to a constant* for this choice of 2-microlocal rescaling.

4. Applications of the 2-microlocal results

We present some applications of the results of the preceding section.

4.1. **Proof of Theorem 2.1.** Recall that only the structure of the terms $\mu(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}$ in the decomposition (12) needs to be clarified. Thanks to (14) and to Proposition 3.4, we deduce

$$\mu(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}} = \mu(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp} = \int_{\mathbb{R}} \tilde{\mu}_\Lambda(t,\cdot,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp} + \int_{\{\pm\infty\}} \tilde{\mu}^\Lambda(t,\cdot,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp}.$$

According to Theorem 3.7, the contribution from the part at infinity is independent of x. Hence, we are left with studying the regularity of the measures on $\mathbb{T}^2$:

$$\int_{\Lambda^\perp\times\mathbb{R}} \tilde{\mu}_\Lambda(t,\cdot,d\xi,d\eta).$$

The measure $\tilde{\mu}_\Lambda$ is invariant under the Hamiltonian flow $\varphi^t_{H_\Lambda^\perp}$ (see Remark 3.5) and, by part (3) of Theorem 3.6, it is also invariant under the Hamiltonian flow $\varphi^t_{p_\Lambda^V}$, which commutes with $\varphi^t_{H_\Lambda^\perp}$. Using Appendix A which describes the regularity of biinvariant measures, we can conclude the proof of Theorem 2.1. More specifically, part 1 follows from Proposition A.1 and part 2 from Corollary A.3.

4.2. **Semiclassical measures up the critical time scale $\tau_\hbar = \epsilon_\hbar^{-1}$.** At the time-scales up to the critical scale $\epsilon_\hbar^{-1}$, we can completely determine μ_t in terms of the initial data:

Theorem 4.1. *Let $\mu \in \mathcal{M}(\tau,\epsilon)$. Suppose that it is generated by some sequence of initial data $(u_\hbar)_{\hbar\to0^+}$. For every rank-one primitive lattice Λ, let $\tilde{\mu}_\Lambda^0$ be the restriction to $\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}$ of the two-microlocal measure associated with $(u_\hbar)_{\hbar\to0^+}$, and denote by μ^0 the semiclassical measure of $(u_\hbar)_{\hbar\to0^+}$.*

(1) *If $\tau_\hbar = \epsilon_\hbar^{-1}$, then, for every $a \in \mathcal{C}_c^0(\mathbb{T}^2 \times \mathbb{R}^2)$, the following holds:*

$$\int_{\mathbb{T}^2\times\mathbb{R}^2} a(x,\xi)\mu(t,dx,d\xi) = \int_{\mathbb{T}^2\times\mathbb{R}^2} \mathcal{I}_0(a)(\xi)\mu^0(dx,d\xi) + \sum_{\Lambda \text{ rank 1 primitive}} \int_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}} (\mathcal{I}_\Lambda(a) - \mathcal{I}_0(a))(\varphi^t_{p_\Lambda^V}(x,\xi,\eta))\tilde{\mu}^0_\Lambda(dx,d\xi,d\eta).$$

(2) *If $\tau_\hbar\epsilon_\hbar \to 0$, then the same result holds, provided we replace $\varphi^t_{p_\Lambda^V}$ by $\varphi^t_{p_\Lambda^0}$ in the formula above.*

The proof is as follows. Let $\mu \in \mathcal{M}(\tau,\epsilon)$, and decompose it as in (12). Using the lift property (14), we can further decompose μ as follows:

$$\mu(t) = \mu(t)\rceil_{\mathbb{T}^2\times\Omega_2} + \sum_{\Lambda \text{ rank 1 primitive}} \int_{\{\pm\infty\}} \tilde{\mu}^\Lambda(t,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp} + \sum_{\Lambda \text{ rank 1 primitive}} \int_{\mathbb{R}} \tilde{\mu}_\Lambda(t,\cdot,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp}.$$

Thanks to the invariance by the geodesic flow and to Theorem 3.7, we can conclude one more time that the first two terms on the right-hand side of the equality are independent of x. Thanks to the second part of Theorem 3.6, we can also write:

$$\tilde{\mu}_\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}} = (\varphi^t_{p_\Lambda^V})_*\left(\tilde{\mu}^0_\Lambda\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}}\right) \quad (\text{resp. } \tilde{\mu}_\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}} = (\varphi^t_{p_\Lambda^0})_*\left(\tilde{\mu}^0_\Lambda\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}}\right)),$$

when $\tau_\hbar = \epsilon_\hbar^{-1}$ (resp. $\tau_\hbar\epsilon_\hbar \to 0$). The result follows from the fact that the zero Fourier coefficient of $\mu(t)$ is itself equal to the zero Fourier coefficient of μ^0 thanks to the following adaptation of Proposition 29 from [5].

Lemma 4.2. *Suppose that*

$$\lim_{\hbar\to0^+} \tau_\hbar\epsilon_\hbar^2 = 0.$$

Let μ be an element in $\mathcal{M}(\tau,\epsilon)$ and let μ^0 be the semiclassical measure of the sequence of initial data used to generate μ. Then, one has, for a.e. t in $\mathbb{R}$, and for every $b \in \mathcal{C}_c(\mathbb{R}^2)$:

$$\int_{\mathbb{T}^2\times\mathbb{R}^2} b(\xi)\mu(t,dx,d\xi) = \int_{\mathbb{T}^2\times\mathbb{R}^2} b(\xi)\mu^0(dx,d\xi).$$

4.3. **Propagation of wave packets.** An application of Theorem 2.1 is the computation of semiclassical meaures for wave-packet type solutions to (6).

Let us first define wave-packet data on the torus. Take $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ supported in a small neighborhood of the origin such that $\|\rho\|_{L^2(\mathbb{R}^2)} = 1$. Let $(x_0,\xi_0) \in \mathring{T}^*\mathbb{T}^2$ and set

$$U_\hbar^{x_0,\xi_0}(x) := \frac{1}{\sigma_\hbar}\rho\left(\frac{x-x_0}{\sigma_\hbar}\right)e^{i\frac{\xi_0\cdot x}{\hbar}},$$

where $\sigma_\hbar \to 0^+$ and $\sigma_\hbar \gg \hbar$. Finally, write

$$u_\hbar^{x_0,\xi_0}(x) = \sum_{k\in\mathbb{Z}^2} U_\hbar^{x_0,\xi_0}(x+k). \tag{19}$$

If the support of ρ is small enough, then $\|u_\hbar^{x_0,\xi_0}\|_{L^2(\mathbb{T}^2)} = 1$. These initial data concentrate around x_0 and oscillate in the direction of ξ_0. Moreover, it is straightforward to check that $(u_\hbar^{x_0,\xi_0})$ satisfies (7) and (8). We next compute the time-dependent semiclassical measure of the sequence $(v_\hbar^{x_0,\xi_0})$ of solutions to (6) issued from the initial data $(u_\hbar^{x_0,\xi_0})$.

Proposition 4.3. *Suppose that the concentration scale* $(\sigma_\hbar)$ *satisfies* $\hbar(\epsilon_\hbar\sigma_\hbar)^{-1} \to 0$ *and that* $\xi_0 \in \Omega_1$. *Let* $\mu^{x_0,\xi_0} \in \mathcal{M}(\tau,\epsilon)$ *be generated by the initial data* $(u_\hbar^{x_0,\xi_0})$. *Let* $\gamma(x,\xi_0)$ *denote the geodesic in* $\mathbb{T}^2$ *issued from* (x,ξ_0) *and* $\delta_{\gamma(x,\xi_0)}$ *the uniform probability measure on that geodesic. The following hold:*

(1) *if* $\tau_\hbar\epsilon_\hbar \to 0$, *then*
$$\mu^{x_0,\xi_0}(t,dx,d\xi) = \delta_{\gamma(x_0,\xi_0)}(dx)\delta_{\xi_0}(d\xi);$$
(2) *if* $\tau_\hbar = \epsilon_\hbar^{-1}$, *then*
$$\mu^{x_0,\xi_0}(t,dx,d\xi) = \delta_{\gamma(x(t),\xi_0)}(dx)\delta_{\xi_0}(d\xi),$$
where $x(t)$ *is the projection on* $\mathbb{T}^2$ *of* $\varphi^t_{p^V_{\Lambda_{\xi_0}}}(x_0,\xi_0,0)$ *with* $\Lambda_{\xi_0} = \{\xi_0\}^\perp \cap \mathbb{Z}^2$. *If* x_0 *is a critical point of* $\mathcal{I}_{\Lambda_{\xi_0}}(V)$ *then* $x(t) = x_0$ *for all* $t \in \mathbb{R}$. *In that case,* μ^{x_0,ξ_0} *is also constant in time.*

Proof. Lemma 4.2 ensures that $\mu(t)$ is supported on $\mathbb{T}^2 \times \langle\xi_0\rangle$ for a.e. $t \in \mathbb{R}$. Therefore, in virtue of (14):
$$\mu(t) = \int_{\widehat{\mathbb{R}}} \mu_{\Lambda_{\xi_0}}(t,\cdot,d\eta)\rceil_{\mathbb{T}^2\times\langle\xi_0\rangle},$$
where $\mu_{\Lambda_{\xi_0}} \in \mathcal{M}_{\Lambda_{\xi_0}}(\tau,\epsilon)$ is generated by $(u_\hbar^{x_0,\xi_0})$. Let $\mu^0_{\Lambda_{\xi_0}}$ be an accumulation point of $(w_{\hbar,\Lambda_{\xi_0}}(0))$. Since $\hbar\sigma_\hbar^{-1} \ll \epsilon_\hbar \le \tau_\hbar^{-1}$, one can verify that, in every regime,
$$\mu^0_{\Lambda_{\xi_0}}(dx,d\xi,d\eta) = \delta_{x_0}(dx)\delta_{\xi_0}(d\xi)\delta_0(d\eta),$$
e.g. see the the proof of Proposition 5.2 in [1]. The result then follows from Theorem 2.1. □

5. Proof of the 2-microlocal statements

From this point on, we fix a primitive sublattice Λ of $\mathbb{Z}^2$ of rank 1 and we will proceed to the proofs of the results on 2-microlocal distributions. Namely, we will first recall how to extract converging subsequences from the sequences $(w_{\Lambda,\hbar}(t\tau_\hbar))_{\hbar\to0^+}$. Then, we will briefly recall how to adapt the proofs from [5] in order to prove Propositions 3.3 and 3.4. Finally, we will give the proofs of Theorems 3.6 and 3.7.

5.1. Extracting subsequences.

Recall that, following [19, 5, 1], we have introduced an auxiliary linear form whose invariance properties will be analyzed precisely. For every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, we have set
$$\langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle := \left\langle v_\hbar(t\tau_\hbar), \mathrm{Op}_\hbar^w\left(a\left(x,\xi,\frac{H_\Lambda(\xi)}{\alpha_\hbar}\right)\right)v_\hbar(t\tau_\hbar)\right\rangle,$$

where, recall, $\alpha_\hbar$ is given by (13). It will be useful to keep in mind Remark 3.1 throughout this section.

Remark 5.1. We emphasize that, for a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$, one has

$$\langle w_\hbar(t\tau_\hbar), a\rangle = \langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle.$$

Our first step is to explain how to extract converging subsequences following more or less standard procedures [16, 18, 5, 29]. For the sake of completeness, we briefly recall it. For that purpose, we denote by

$$\mathcal{B} := \mathcal{C}_0^D(\mathbb{T}^2 \times \mathbb{R}^2 \times \widehat{\mathbb{R}}),$$

the space of $\mathcal{C}^D$ functions on $\mathbb{T}^2 \times \mathbb{R}^2 \times \widehat{\mathbb{R}}$ all of whose derivatives tend to 0 at infinity. We choose $D > 0$ large enough so that Theorem B.2 holds for functions in $\mathcal{B}$.

We endow this space with its natural topology of Banach space. According to Theorem B.2, one knows that, for every a in $\mathcal{C}_c^\infty(\mathbb{R} \times T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, one has

$$|\langle w_{\Lambda,\hbar}(t\tau_\hbar), a(t)\rangle| \leq C \sum_{|\alpha|\leq D} \left(\hbar\alpha_\hbar^{-1}\right)^{\frac{|\alpha|}{2}} \|\partial^\alpha a(t)\|_\infty. \tag{20}$$

Thus, the map $t \mapsto w_{\Lambda,\hbar}(t\tau_\hbar)$ defines a bounded sequence in $L^1(\mathbb{R}, \mathcal{B})'$, and, after extracting a subsequence, one finds that there exists μ_Λ in $L^1(\mathbb{R}, \mathcal{B})'$ such that, for every a in $\mathcal{C}_c^\infty(\mathbb{R} \times T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, one has

$$\lim_{\hbar\to 0^+} \int_{\mathbb{R}\times T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} a(t,x,\xi,\eta) w_{\Lambda,\hbar}(t\tau_\hbar, dx, d\xi, d\eta)dt = \int_{\mathbb{R}\times T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} a(t,x,\xi,\eta)\mu_\Lambda(dt,dx,d\xi,d\eta).$$

Thanks to (20) and to the fact that $\hbar\alpha_\hbar^{-1} \to 0^+$, recall that, for every θ in $\mathcal{C}_c^\infty(\mathbb{R})$ and for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, one has

$$\left|\int_{\mathbb{R}\times T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} \theta(t) a(x,\xi,\eta)\mu_\Lambda(dt,dx,d\xi,d\eta)\right| \leq C\|\theta\|_{L^1(\mathbb{R})}\|a\|_{\mathcal{C}_0^0(T^*\mathbb{T}^2\times\widehat{\mathbb{R}})}.$$

Hence, μ_Λ is absolutely continuous with respect to the t variable, i.e. for every θ in $L^1(\mathbb{R})$ and every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, one has

$$\lim_{\hbar\to 0^+} \int_{\mathbb{R}} \theta(t)\langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle dt = \int_{\mathbb{R}} \theta(t)\langle \mu_\Lambda(t), a\rangle dt.$$

Moreover, for a.e. t in $\mathbb{R}$, $\mu_\Lambda(t)$ is a finite Radon measure on $T^*\mathbb{T}^2 \times \widehat{\mathbb{R}}$.

5.2. **Proof of Proposition 3.3.** We already know that the linear functionals μ_Λ are Radon measures. It remains to verify that they are positive. To see this, take $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$ such that $a \geq 0$. Using Gårding inequality (Th. 4.32 in [29]), we deduce that

$$\langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle \geq \mathcal{O}(\hbar\alpha_\hbar^{-1}) = o(1);$$

Remark 5.2. Note that the proof of the Gårding inequality in [29] is given in the case of $\mathbb{R}^d$. The extension to compact manifolds usually requires to deal with symbols that decay in ξ as we differentiate with respect to ξ. Yet, in the case of the torus, we can verify that this property remains true for an observable a all of whose derivatives are bounded (i.e. not necessarily decaying in ξ) as in $\mathbb{R}^d$. For that purpose, one can start from the Gårding inequality on $\mathbb{R}^d$ and apply the arguments of the proof of [29, Th. 5.5] which shows L^2-boundedness of pseudodifferential of order 0 on $\mathbb{T}^d$.

After integrating against a test function θ in $L^1(\mathbb{R})$ and passing to the limit $\hbar \to 0$, one finds that, for a.e. t in $\mathbb{R}$,

$$\langle \mu_\Lambda(t), a\rangle \geq 0.$$

This concludes the proof that μ_Λ is a positive, finite Radon measure on $T^*\mathbb{T}^2 \times \widehat{\mathbb{R}}$ and one sets $\tilde{\mu}_\Lambda(t) = \mu_\Lambda(t)\rceil_{T^*\mathbb{T}^2\times\mathbb{R}}$ and $\tilde{\mu}^\Lambda(t) = \mu_\Lambda(t)\rceil_{T^*\mathbb{T}^2\times\{\pm\infty\}}$. Thanks to the frequency assumption (8), one has, for a.e. t in $\mathbb{R}$,

$$\mu_\Lambda(t)(\{\xi = 0\}) = 0. \tag{21}$$

Remark 5.3. Remark 5.1 implies that, for a.e. t in $\mathbb{R}$, the time-dependent semiclassical measure $\mu(t)$ can be obtained by

$$\mu(t) = \int_{\widehat{\mathbb{R}}} \mu_\Lambda(t, \cdot, d\eta). \tag{22}$$

5.3. **Proof of Proposition 3.4.** Concerning the support of $\tilde{\mu}_\Lambda(t)$, we let a be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \mathbb{R})$ whose support does not intersect $\mathbb{T}^2 \times \Lambda^\perp \times \mathbb{R}$. Using Remark 3.1, one has

$$\operatorname{Op}_\hbar^w\left(a\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right)\right) = \operatorname{Op}_{\hbar\alpha_\hbar^{-1}}^w\left(a\left(x, \alpha_\hbar \xi, H_\Lambda(\xi)\right)\right).$$

Hence, this operator is equal to 0 when $\hbar$ is small enough (thanks to our assumption on the support of a). This concludes the proof of the first part of Proposition 3.4.

Let us now discuss invariance by the geodesic flow which is the only property that uses the particular form of $v_\hbar(t\tau_\hbar)$ so far. Again, we start with the "compact" part and we fix a to be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \mathbb{R})$. Using composition rules for pseudodifferential operators, we write

$$\begin{aligned}\frac{d}{dt}\langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle &= \tau_\hbar \langle w_{\Lambda,\hbar}(t\tau_\hbar), \xi.\partial_x a\rangle \\ &\quad + \frac{i\tau_\hbar \epsilon_\hbar^2}{\hbar}\langle v_\hbar(t\tau_\hbar), \left[V, \operatorname{Op}_{\hbar\alpha_\hbar^{-1}}^w\left(a(x, \alpha_\hbar\xi, H_\Lambda(\xi))\right)\right] v_\hbar(t\tau_\hbar)\rangle.\end{aligned}$$

Using Theorem B.3 (more specifically Remark B.4) one more time, we have that

$$\left[V, \operatorname{Op}_{\hbar\alpha_\hbar^{-1}}^w\left(a(x, \alpha_\hbar\xi, H_\Lambda(\xi))\right)\right] = -\frac{\hbar}{i\alpha_\hbar}\operatorname{Op}_\hbar^w\left(\frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x V \partial_\eta a\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right)\right) + \mathcal{O}(\hbar^3(\alpha_\hbar)^{-3}).$$

Combining these two identities to the fact $\hbar\alpha_\hbar^{-1} = o(1)$ and $\epsilon_\hbar\alpha_\hbar^{-1} = \mathcal{O}(1)$, we find that

$$\frac{d}{dt}\langle w_{\Lambda,\hbar}(t\tau_\hbar), a\rangle = \tau_\hbar\left(\left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \xi.\partial_x a - \frac{\epsilon_\hbar^2}{\alpha_\hbar}\frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x V \partial_\eta a\right\rangle + o(\hbar)\right).$$

Let now θ be an element in $\mathcal{C}^1_c(\mathbb{R})$. Integrating the previous equality against θ and integrating by parts, we find

$$\int_{\mathbb{R}} \theta(t) \left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \xi.\partial_x a - \frac{\epsilon_\hbar^2}{\alpha_\hbar} \frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x V \partial_\eta a \right\rangle dt = \mathcal{O}(\tau_\hbar^{-1}) + o(\hbar),$$

which implies the result for every a in $\mathcal{C}^\infty_c(T^*\mathbb{T}^2 \times \mathbb{R})$ when we let $\hbar$ goes to 0. Note that we used the Calderón-Vaillancourt Theorem B.2 to bound the $\epsilon_\hbar^2 \alpha_\hbar^{-1}$ term on the left hand side of this equality.

It now remains to treat the part at infinity. Let a be an element in $\mathcal{C}^\infty_c(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$. For every $R \geq 1$ and for every smooth cutoff function near 0, we set

$$a^R(x,\xi,\eta) := a(x,\xi,\eta) \left(1 - \chi\left(\frac{\eta}{R}\right)\right).$$

The same argument as before allows to prove that, for every θ in $\mathcal{C}^1(\mathbb{R})$, one has

$$\int_{\mathbb{R}} \theta(t) \left\langle w_{\Lambda,\hbar}(t\tau_\hbar), (\xi.\partial_x a)^R - \frac{\epsilon_\hbar^2}{\alpha_\hbar} \frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x V \partial_\eta a^R \right\rangle dt = o(1).$$

Thus, we can take the limit $\hbar \to 0$ and conclude the proof by letting R goes to $+\infty$.

5.4. **Invariance and propagation of 2-microlocal distributions.** We now turn to the proofs of our main statements, namely Theorems 3.6 and 3.7. Analogously to [5], we define the differential operators

$$D_\Lambda := \frac{1}{i} \frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\nabla \quad \text{and} \quad D_\Lambda^\perp := \frac{1}{i} \frac{\mathfrak{e}_\Lambda^\perp}{L_\Lambda}.\nabla$$

associated with the Hamiltonians H_Λ and $H_\Lambda^\perp$. One has

$$-\Delta = (D_\Lambda^\perp)^2 + D_\Lambda^2. \tag{23}$$

Recall also that, for every smooth compactly supported function b on $T^*\mathbb{T}^2$, the Egorov theorem is exact for these operators and it tells us that

$$\operatorname{Op}_\hbar^w(\mathcal{I}_\Lambda(b)) = \frac{1}{L_\Lambda} \int_0^{L_\Lambda} e^{isD_\Lambda^\perp} \operatorname{Op}_\hbar^w(b) e^{-isD_\Lambda^\perp} ds. \tag{24}$$

and that

$$[D_\Lambda^\perp, \operatorname{Op}_\hbar^w(\mathcal{I}_\Lambda(b))] = 0. \tag{25}$$

As mentioned before, this construction (that was originally presented in [5]) is reminiscent to the averaging argument of Weinstein [26] applied to certain one-dimensional tori that depend on Λ.

5.4.1. *Proof of Theorem 3.6.* Let a be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{R})$. We start our proof by computing the derivative of the 2-microlocal Wigner distribution. One has

$$\frac{d}{dt}\langle w_{\Lambda,\hbar}(t\tau_\hbar),\mathcal{I}_\Lambda(a)\rangle=\frac{i\tau_\hbar}{\hbar}\left\langle v_\hbar(t\tau_\hbar),\left[\frac{\hbar^2}{2}(D_\Lambda^\perp)^2+\frac{\hbar^2}{2}D_\Lambda^2+\epsilon_\hbar^2V,\mathrm{Op}_\hbar^w\left(a_{\Lambda,\hbar}\right)\right]v_\hbar(t\tau_\hbar)\right\rangle,$$

where

$$a_{\Lambda,\hbar}(x,\xi):=\mathcal{I}_\Lambda(a)\left(x,\xi,\frac{H_\Lambda(\xi)}{\alpha_\hbar}\right).$$

Using (25), we deduce that

$$\frac{d}{dt}\langle w_{\Lambda,\hbar}(t\tau_\hbar),\mathcal{I}_\Lambda(a)\rangle=\frac{i\tau_\hbar}{\hbar}\left\langle v_\hbar(t\tau_\hbar),\left[\frac{\hbar^2}{2}D_\Lambda^2+\epsilon_\hbar^2V,\mathrm{Op}_\hbar^w\left(a_{\Lambda,\hbar}\right)\right]v_\hbar(t\tau_\hbar)\right\rangle,$$

Thanks to the commutation properties of the Weyl quantization from Remark B.4, one has

$$\frac{d}{dt}\langle w_{\Lambda,\hbar}(t\tau_\hbar),\mathcal{I}_\Lambda(a)\rangle=\mathcal{O}(\tau_\hbar\epsilon_\hbar^2\hbar^2(\alpha_\hbar)^{-3})$$

(26)

$$+\alpha_\hbar\tau_\hbar\left\langle v_\hbar(t\tau_\hbar),\mathrm{Op}_\hbar^w\left(\frac{H_\Lambda(\xi)}{\alpha_\hbar}\frac{\mathfrak{e}_\Lambda.\partial_x\mathcal{I}_\Lambda(a)(x,\xi,H_\Lambda(\xi)/\alpha_\hbar)}{L_\Lambda}-\frac{\epsilon_\hbar^2}{\alpha_\hbar^2}\partial_\eta\mathcal{I}_\Lambda(a)\frac{\mathfrak{e}_\Lambda.\partial_xV}{L_\Lambda}\right)v_\hbar(t\tau_\hbar)\right\rangle.$$

Our assumption $\hbar\ll\epsilon_\hbar\ll\alpha_\hbar$ ensures that the remainder is in fact of order $o(\hbar\tau_\hbar)$.

We now distinguish three regimes.

First, we suppose that $\epsilon_\hbar\tau_\hbar\to 0$ as $\hbar\to 0^+$. In particular, $\alpha_\hbar=\tau_\hbar^{-1}\gg\epsilon_\hbar$. Thanks to the Calderón-Vaillancourt Theorem B.2, we can verify that last term in the right hand-side of equality (26) is in fact $o(1)$ uniformly for t in $\mathbb{R}$. Letting $\hbar\to 0$, one finds that, for a.e. t in $\mathbb{R}$,

$$\frac{d}{dt}\langle\mu_\Lambda(t),\mathcal{I}_\Lambda(a)\rangle=\left\langle\mu_\Lambda(t),\eta\frac{\mathfrak{e}_\Lambda}{L_\Lambda}.\partial_x\mathcal{I}_\Lambda(a)\right\rangle.$$

Combining Proposition 3.4 with (21), one has then $\langle\mu_\Lambda(t),a\rangle=\left\langle\mu_\Lambda^0,\mathcal{I}_\Lambda(a)\circ\varphi^t_{p_\Lambda^0}\right\rangle$ for a.e. t in $\mathbb{R}$, which proves point (1) of the Theorem.

Suppose now that $\tau_\hbar\epsilon_\hbar\to c>0$. Letting $\hbar\to 0$, the limit measure satisfies the following transport equation, for all $\theta\in\mathcal{C}_c^1(\mathbb{R})$:

$$-\int_{\mathbb{R}}\theta'(t)\langle\mu_\Lambda(t),\mathcal{I}_\Lambda(a)\rangle\,dt=c\int_{\mathbb{R}}\theta(t)\left\langle\mu_\Lambda(t),\eta\frac{\mathfrak{e}_\Lambda.\partial_x\mathcal{I}_\Lambda(a)}{L_\Lambda}-\partial_\eta\mathcal{I}_\Lambda(a)\frac{\mathfrak{e}_\Lambda.\partial_xV}{L_\Lambda}\right\rangle dt.$$

Using again Proposition 3.4 with (21), one deduces that

$$\partial_t\langle\mu_\Lambda(t),\mathcal{I}_\Lambda(a)\rangle=c\left\langle\mu_\Lambda(t),\eta\frac{\mathfrak{e}_\Lambda.\partial_x\mathcal{I}_\Lambda(a)}{L_\Lambda}-\partial_\eta\mathcal{I}_\Lambda(a)\frac{\mathfrak{e}_\Lambda.\partial_x\mathcal{I}_\Lambda(V)}{L_\Lambda}\right\rangle.$$

This proves point (2) of the Theorem.

Finally, we suppose that $\tau_\hbar\epsilon_\hbar\to+\infty$. Let θ be an element in $\mathcal{C}_c^1(\mathbb{R})$. We integrate one more time equality (26) against θ, and we make an integration by parts on the left-hand side of the equality. Then, we make use of the Calderón-Vaillancourt Theorem B.2 to bound the left-hand-side. After letting $\hbar$ goes to 0, one finds that, for every θ in $\mathcal{C}_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \theta(t) \left\langle \mu_\Lambda(t), \eta \frac{\mathfrak{e}_\Lambda.\partial_x \mathcal{I}_\Lambda(a)}{L_\Lambda} - \partial_\eta \mathcal{I}_\Lambda(a) \frac{\mathfrak{e}_\Lambda.\partial_x \mathcal{I}_\Lambda(V)}{L_\Lambda} \right\rangle dt = 0,$$

where we used one more time Proposition 3.4 with (21) in order to replace V by its Λ-average $\mathcal{I}_\Lambda(V)$. This implies point (3) of the Theorem.

5.4.2. *Proof of Theorem 3.7.* Let now a be an element in $\mathcal{C}_c^\infty(\mathbb{R}^2 \times \widehat{\mathbb{R}})$ and let k be an element in $\Lambda - \{0\}$. It suffices to show that:

$$\langle \tilde{\mu}^\Lambda(t), e^{-2i\pi k.x} a(\xi, \eta) \rangle = 0.$$

We fix $\chi_1(\eta) \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ which is equal to 1 for $\eta \geq 1$ and to 0 for $\eta \leq 1/2$. For every $R \geq 1$, we set

$$a_\pm^{R,k}(x, \xi, \eta) := e^{-2i\pi k.x} a(\xi, \eta) \chi_1\left(\pm \frac{\eta}{R}\right).$$

Remark 5.4. Let θ be an element in $\mathcal{C}_c^1(\mathbb{R})$. One has

$$\int_{\mathbb{R}} \theta(t) \frac{d}{dt} \left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \frac{1}{\eta} a_\pm^{R,k} \right\rangle dt = -\int_{\mathbb{R}} \theta'(t) \left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \frac{1}{\eta} a_\pm^{R,k} \right\rangle dt.$$

Thanks to the Calderón-Vaillancourt Theorem B.2, one knows that

$$\left\| \mathrm{Op}_\hbar^w \left(\chi\left(\frac{H_\Lambda(\xi)}{R\alpha_\hbar}\right) a\left(\xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) e^{-2i\pi k.x} \frac{\alpha_\hbar}{H_\Lambda(\xi)} \right) \right\|_{L^2 \to L^2} = \mathcal{O}(R^{-1}).$$

Thus, one has

$$\int_{\mathbb{R}} \theta(t) \frac{d}{dt} \left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \frac{1}{\eta} a_\pm^{R,k} \right\rangle dt = \mathcal{O}(R^{-1}).$$

In order to prove the proposition, we will now compute explicitely the derivative of $\left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \frac{1}{\eta} a_\pm^{R,k} \right\rangle$. For that purpose, we need to compute the following bracket:

$$\left[-\frac{\hbar^2 \Delta}{2} + \epsilon_\hbar^2 V, \mathrm{Op}_\hbar^w \left(a_\pm^{R,k}\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) \frac{\alpha_\hbar}{H_\Lambda(\xi)} \right) \right].$$

Using again (25), this commutator is in fact equal to

$$\left[\frac{\hbar^2 D_\Lambda^2}{2} + \epsilon_\hbar^2 V, \mathrm{Op}_\hbar^w \left(a_\pm^{R,k}\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) \frac{\alpha_\hbar}{H_\Lambda(\xi)} \right) \right].$$

We split this commutator in two parts. Thanks to remark B.4, one has

$$\left[\frac{\hbar^2 D_\Lambda^2}{2}, \mathrm{Op}_\hbar^w \left(a_\pm^{R,k}\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) \frac{\alpha_\hbar}{H_\Lambda(\xi)} \right) \right] = -2\pi\hbar\alpha_\hbar \mathrm{Op}_\hbar^w \left(\frac{\mathfrak{e}_\Lambda}{L_\Lambda}.k a_\pm^{R,k}\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) \right).$$

For the other part of the commutator, we use one more time the commutation rule for pseudodifferential operators and the Calderón Vaillancourt Theorem B.2. We find that

$$\left[V, \mathrm{Op}_\hbar^w \left(a_\pm^{R,k}\left(x, \xi, \frac{H_\Lambda(\xi)}{\alpha_\hbar}\right) \frac{\alpha_\hbar}{H_\Lambda(\xi)} \right) \right] = \mathcal{O}_{L^2 \to L^2}\left(\hbar \alpha_\hbar^{-1} R^{-1} + \hbar^3 \alpha_\hbar^{-3} \right).$$

As $\hbar\epsilon_\hbar^{-1} \to 0$ and $\epsilon_\hbar = \mathcal{O}(\alpha_\hbar)$, we finally get that

$$\frac{d}{dt}\left\langle w_{\Lambda,\hbar}(t\tau_\hbar), \frac{1}{\eta}a_\pm^{R,k}\right\rangle = -\frac{2\pi\tau_\hbar\alpha_\hbar\mathfrak{e}_\Lambda.k}{L_\Lambda}\left\langle w_{\Lambda,\hbar}(t\tau_\hbar), a_\pm^{R,k}\right\rangle + \mathcal{O}\left(\tau_\hbar\epsilon_\hbar R^{-1}\right) + o(\tau_\hbar\hbar).$$

Let now θ be an element in $\mathcal{C}_c^1(\mathbb{R})$. We integrate these expressions against θ. Using Remark 5.4 and making the assumption that $\limsup_{\hbar\to 0^+}\tau_\hbar\alpha_\hbar > 0$, we obtain

$$\forall k \in \Lambda - \{0\},\ \int_{\mathbb{R}}\theta(t)\left\langle w_{\Lambda,\hbar}(t\tau_\hbar), a_\pm^{R,k}\right\rangle dt = o(1) + \mathcal{O}(R^{-1}).$$

We now let $\hbar$ goes to 0, and we get that, for every $R > 0$,

$$\forall k \in \Lambda - \{0\},\ \int_{\mathbb{R}}\theta(t)\left\langle \mu_\Lambda(t), a_\pm^{R,k}\right\rangle dt = \mathcal{O}(R^{-1}).$$

To get the conclusion, we let R goes to $+\infty$.

Remark 5.5. From this Theorem, we deduce that, for every $a(x,\xi,\eta)$ in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\widehat{\mathbb{R}})$ and for a.e. t in $\mathbb{R}$,

$$\tilde{\mu}^\Lambda(t)(\mathcal{I}_\Lambda(a)) = \int_{T^*\mathbb{T}^2\times\{\pm\infty\}}\widehat{a}_0(\xi,\eta)\mu_\Lambda(t,d\xi,d\eta).$$

Appendix A. Regularity of bi-invariant measures

In this appendix, we fix Λ a primitive sublattice of $\mathbb{Z}^2$ of rank 1, and we aim at analyzing the regularity of the set of finite measures on $T^*\mathbb{T}^2$ which are invariant by the Hamiltonian flows[5] $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$. We will now recall the results from section 4 of [21] and explain how they can be adapted to the present framework. We refer the reader to this reference for the detailed proofs. We introduce the critical set in the direction of Λ:

$$\mathrm{Crit}_\Lambda(V) := \{(x,\xi)\in T^*\mathbb{T}^2 : H_\Lambda(\xi) = 0 \text{ and } \partial_x\mathcal{I}_\Lambda(V) = 0\}.$$

This is a closed subset of $T^*\mathbb{T}^2$ which is invariant by the Hamiltonian flows $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$, and we introduce its complement

$$\mathcal{R}(\Lambda) := T^*\mathbb{T}^2 - \mathrm{Crit}_\Lambda(V).$$

The map

$$\phi : \mathbb{R}^2\times\mathcal{R}(\Lambda) \ni (s,t,x,\xi) \longmapsto \varphi^s_{H_\Lambda^\perp}\circ\varphi^t_{p_\Lambda^V}(x,\xi) \in \mathcal{R}(\Lambda),$$

is a group action of $\mathbb{R}^2$ on $\mathcal{R}(\Lambda)$. Moreover, for any $(x_0,\xi_0)\in\mathcal{R}(\Lambda)$, the map

$$\phi_{x_0,\xi_0} : \mathbb{R}^2 \ni (s,t) \longmapsto \varphi^s_{H_\Lambda^\perp}\circ\varphi^t_{p_\Lambda^V}(x_0,\xi_0) \in \mathcal{R}(\Lambda),$$

is an immersion. Therefore, the stabilizer group G_{x_0,ξ_0} of (x_0,ξ_0) under ϕ is discrete. This proves that the orbits of the action ϕ are either diffeomorphic to the torus $\mathbb{T}^2$, to the cylinder $\mathbb{T}\times\mathbb{R}$ or to $\mathbb{R}^2$. On the other hand, the moment map,

$$\Phi : \mathcal{R}(\Lambda) \ni (x,\xi) \longmapsto (H_\Lambda^\perp(\xi), p_\Lambda^V(x,\xi)) \in \mathbb{R}^2,$$

[5]By making a slight abuse of notation, we shall identify $\varphi^t_{p_\Lambda}$, a flow *a priori* defined on $\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}$, to a flow on $T^*\mathbb{T}^2$ via the diffeomorphism (18). Recall that $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$ commute.

is a submersion, and, for every $(H,J)\in\Phi(\mathcal{R}(\Lambda))$ the level set

$$\mathcal{L}_{(H,J)}:=\Phi^{-1}(H,J),$$

is a smooth submanifold of $\mathcal{R}(\Lambda)$ of dimension two. To summarize, the couple $(H_\Lambda^\perp,p_\Lambda^V)$ forms a completely integrable system on $\mathcal{R}(\Lambda)$, and the map ϕ_{x_0,ξ_0} induces a diffeomorphism:

$$\forall(x_0,\xi_0)\in\mathcal{R}(\Lambda),\quad \phi_{x_0,\xi_0}:\mathbb{R}^2/G_{x_0,\xi_0}\longrightarrow\mathcal{L}^{x_0,\xi_0}_{(H_0,J_0)},\qquad \text{for } (H_0,J_0):=\Phi(x_0,\xi_0).$$

Here, $\mathcal{L}^{x_0,\xi_0}_{(H_0,J_0)}$ denotes the connected component of $\mathcal{L}_{(H_0,J_0)}$ that contains (x_0,ξ_0). Therefore, if $\mathcal{L}^{x_0,\xi_0}_{(H_0,J_0)}$ is compact then it is an embedded Lagrangian torus in $T^*\mathbb{T}^2$. In that case, we shall write $\mathbb{T}^2_{x_0,\xi_0}:=\mathbb{R}^2/G_{x_0,\xi_0}$. In the following, we denote by $\mathcal{R}_c(\Lambda)$ the set formed by those $(x,\xi)\in\mathcal{R}(\Lambda)$ such that $\mathcal{L}^{x,\xi}_{\Phi(x,\xi)}$ is compact. Mimicking the proof of proposition 4.2 in [21], one can show that the following holds:

Proposition A.1. *Let μ be a probability measure on $\mathcal{R}(\Lambda)$ that is invariant by $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$. Set $\overline{\mu}:=\Phi_*\mu$. Then, for every $a\in\mathcal{C}_c(\mathcal{R}(\Lambda))$, one has*

$$\int_{\mathcal{R}(\Lambda)}a(x,\xi)\mu(dx,d\xi)=\int_{\Phi(\mathcal{R}(\Lambda))}\int_{\mathcal{L}_{(H,J)}}a(x,\xi)\lambda_{H,J}(dx,d\xi)\overline{\mu}(dH,dJ),$$

where, for $(H,J)\in\Phi(\mathcal{R}(\Lambda))$, the measure $\lambda_{H,J}$ is a convex combination of the (normalized) Haar measures on the tori $\mathcal{L}^{x_0,\xi_0}_{(H,J)}$ for $(x_0,\xi_0)\in\mathcal{L}_{(H,J)}\cap\mathcal{R}_c(\Lambda)$. In particular, for every (x,ξ) in $\mathcal{R}(\Lambda)$, one has

$$\mu\left(\left\{\varphi^s_{H_\Lambda^\perp}(x,\xi):0\leq s\leq L_\Lambda\right\}\right)=0.$$

An explicit formula for the restriction of the measure $\lambda_{H,J}$ to a connected component $\mathcal{L}^{x,\xi}_{(H,J)}$ with $(x,\xi)\in\mathcal{R}_c(\Lambda)\cap\mathcal{L}_{(H,J)}$ is the following:

$$\int_{\mathcal{L}^{x_0,\xi_0}_{(H,J)}}a(x,\xi)\lambda_{H,J}(dx,d\xi)=c\int_{\mathbb{T}^2_{x_0,\xi_0}}a(\phi_{x_0,\xi_0}(s,t))dsdt, \tag{27}$$

for some constant $c\in[0,1]$.

We will now discuss the regularity of the projections of bi-invariant measures following the proof from paragraph 4.2 in [21]. We denote by $\Pi:T^*\mathbb{T}^2\to\mathbb{T}^2$ the canonical projection. The main result from section 4 in [21] was the following

Theorem A.2. *Let μ be a probability measure on $\mathcal{R}(\Lambda)$ that is invariant by $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$. Then, $\nu:=\Pi_*\mu$ is a probability measure on $\mathbb{T}^2$ that is absolutely continuous with respect to the Lebesgue measure.*

Denote by $\mathcal{N}(\Lambda)$ the convex closure of the set of measures $\delta_{\Pi\circ\Gamma}$ where $\Gamma\subset T^*\mathbb{T}^2$ ranges over the orbits of $\varphi_{H_\Lambda^\perp}$ that are contained in $\mathrm{Crit}_\Lambda(V)$. A direct consequence of the previous Theorem is the following:

Corollary A.3. *The projection $\nu := \Pi_* \mu$ of a probability measure μ on $T^*\mathbb{T}^2$ that is invariant by $\varphi^t_{H_\Lambda^\perp}$ and $\varphi^t_{p_\Lambda^V}$ can be decomposed as:*

$$\nu = f \operatorname{vol} + \alpha \nu_{\mathrm{sing}}$$

where $f \in L^1(\mathbb{T}^2)$, $\alpha \in [0,1]$ and $\nu_{\mathrm{sing}} \in \mathcal{N}(\Lambda)$.

Note that, for a "generic" choice of V, the set of points x satisfying $\partial_x \mathcal{I}_\Lambda(V) = 0$ consists of finitely many closed geodesics of $\mathbb{T}^2$. In particular, ν_{sing} is a finite combination of measures carried by closed geodesics.

Proof. As it is simple to explain in the current framework, we briefly explain how the proof of Theorem 4.6 in [21] can be adapted to prove Theorem A.2 – see also Lemma 2.1 in [6]. Recall that it is sufficient to fix some (x_0, ξ_0) in $\mathcal{R}_c(\Lambda)$ and to prove that the set of points where

$$\phi_{x_0,\xi_0} : (s,t) \in \mathbb{T}^2_{x_0,\xi} \mapsto \Pi \circ \varphi^s_{H_\Lambda^\perp} \circ \varphi^t_{p_\Lambda^V}(x_0, \xi_0) \in \mathbb{T}^2$$

is not a local diffeomorphism is made of finitely many disjoint $\mathcal{C}^1$ closed curves. Such curves are called caustics. This can be proved as follows. One can verify that the points where we do not have a local diffeomorphism are defined by the points (s,t) satisfying

$$H_\Lambda \left(\phi_{x_0,\xi_0}(s,t) \right) = 0.$$

Note that, for every s in $\mathbb{R}$,

$$H_\Lambda \left(\varphi^t_{p_\Lambda^V}(x_0, \xi_0) \right) = H_\Lambda \left(\phi_{x_0,\xi_0}(s,t) \right).$$

As (x_0, ξ_0) belongs to the $\varphi^t_{p_\Lambda^V}$-invariant set $\mathcal{R}(\Lambda)$, we know that

$$\partial_x \mathcal{I}_\Lambda(V) \left(\varphi^t_{p_\Lambda^V}(x_0, \xi_0) \right) \neq 0.$$

Thus, from the Hamilton-Jacobi equations, we deduce that there exists a small open neighborhood $(t-\eta, t+\eta)$ of t such that, for every $t' \in (t-\eta, t+\eta) - \{t\}$,

$$H_\Lambda \circ \varphi^{t'}_{p_\Lambda^V}(x_0, \xi_0) \neq 0.$$

In particular, there are ony finitely many values of t such that $H_\Lambda \circ \varphi^t_{p_\Lambda^V}(x_0, \xi_0) \neq 0$ and thus, there are only finitely many closed curves on $\mathbb{T}^2_{x_0,\xi_0}$ where the map ϕ_{x_0,ξ_0} is not a local diffeomorphism. □

Appendix B. Background on semiclassical analysis

In this appendix, we give a brief reminder on semiclassical analysis and we refer to [29] (mainly Chapters 1 to 5) for a more detailed exposition. Given $\hbar > 0$ and a in $\mathcal{S}(\mathbb{R}^{2d})$ (the Schwartz class), one can define the Weyl quantization of a as follows:

$$\forall u \in \mathcal{S}(\mathbb{R}^d),\ \operatorname{Op}^w_\hbar(a)u(x) := \frac{1}{(2\pi\hbar)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

This definition can be extended to any observable a with uniformly bounded derivatives, i.e. such that for every $\alpha \in \mathbb{N}^{2d}$, there exists $C_\alpha > 0$ such that $\sup_{x,\xi} |\partial^\alpha a(x,\xi)| \leq C_\alpha$. More generally, we will use the convention, for every $m \in \mathbb{R}$ and every $k \in \mathbb{Z}$,

$$S^{m,k} := \left\{ (a_\hbar(x,\xi))_{0<\hbar\leq 1} : \ \forall (\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d, \sup_{(x,\xi)\in\mathbb{R}^{2d};0<\hbar\leq 1} |\hbar^k \langle\xi\rangle^{-m} \partial_x^\alpha \partial_\xi^\beta a_\hbar(x,\xi)| < +\infty \right\},$$

where $\langle\xi\rangle := (1 + \|\xi\|^2)^{1/2}$. For such symbols, $\mathrm{Op}_\hbar^w(a)$ defines a continuous operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which acts by duality on $\mathcal{S}'(\mathbb{R}^d)$.

Remark B.1. We also note that we have the following relation that we use at different stages of our proof:

$$\forall \delta > 0, \ \forall a \in S^{m,k}, \mathrm{Op}_\hbar^w(a(x,\xi)) = \mathrm{Op}_{\hbar\delta^{-1}}^w(a(x,\delta\xi)). \tag{28}$$

Among the above symbols, we distinguish the family of $\mathbb{Z}^d$-periodic symbols that we denote by $S^{m,k}_{per}$. Note that any a in $\mathcal{C}^\infty(T^*\mathbb{T}^d)$ (with bounded derivatives) defines an element in $S^{0,0}_{per}$. Similarly to the proof of Th. 4.19 in [29], one can verify that, for any $a \in S^{m,k}_{per}$,

$$\mathrm{Op}_\hbar^w(a)(e_k) = \sum_{q\in\mathbb{Z}^d} e_q \hat{a}_{q-k}(\pi\hbar(q+k)),$$

where $e_k(x) := e^{2i\pi k.x}$, and $\hat{a}_p(\xi) := \int_{\mathbb{T}^d} a(x,\xi) e^{-2i\pi p.x} dx$. In particular, for any $a \in S^{m,k}_{per}$, the operator $\mathrm{Op}_\hbar^w(a)$ maps trigonometric polynomials into a smooth $\mathbb{Z}^d$-periodic function, and more generally any smooth $\mathbb{Z}^d$-periodic function into a smooth $\mathbb{Z}^d$-periodic function. Thus, for every a in $S^{m,k}_{per}$, $\mathrm{Op}_\hbar^w(a)$ acts by duality on the space of distributions $\mathcal{D}'(\mathbb{T}^d)$. An important feature of this quantization procedure is that it defines a bounded operator on $L^2(\mathbb{T}^d)$ [29, Ch. 5]:

Theorem B.2. *[Calderón-Vaillancourt] There exists a constant $C_d > 0$ and an integer $D > 0$ such that, for every a in $S^{0,0}_{per}$, one has, for every $0 < \hbar \leq 1$,*

$$\|\mathrm{Op}_\hbar^w(a)\|_{L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \leq C_d \sum_{|\alpha|\leq D} \hbar^{\frac{|\alpha|}{2}} \|\partial^\alpha a\|_\infty.$$

Another important feature of the Weyl quantization procedure is the composition formula:

Theorem B.3. *[Composition formula] Let $a \in S^{m_1,k_1}$ and $b \in S^{m_2,k_2}$. Then, one has, for any $0 < \hbar \leq 1$*

$$\mathrm{Op}_\hbar^w(a) \circ \mathrm{Op}_\hbar^w(b) = \mathrm{Op}_\hbar^w(a \sharp_\hbar b),$$

in the sense of operators from $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, where $a\sharp_\hbar b$ has uniformly bounded derivatives, and, for every $N \geq 0$

$$a\sharp_\hbar b \sim \sum_{k=0}^N \frac{1}{k!} \left(\frac{i\hbar}{2} D \right)^k (a,b) + \mathcal{O}(\hbar^{N+1}),$$

where $D(a,b)(x,\xi) = (\partial_x\partial_\nu - \partial_y\partial_\xi)(a(x,\xi)b(y,\nu))\rceil_{y=x,\nu=\xi}$.

We refer to chapter 4 of [29] for a detailed proof of this result. We observe that for $N = 0$, the coefficient is given by the symbol ab, and for $N = 1$, it is given by $\frac{\hbar}{2i}\{a, b\}$, where $\{.,.\}$ is the Poisson bracket. As before, we can restrict this result to the case of periodic symbols, and we can check that the composition formula remains valid for operators acting on $\mathcal{C}^\infty(\mathbb{T}^d)$.

Remark B.4. We note that the formula for the composed symbols is quite symmetric, and we have in fact the following useful property, for every $N \geq 0$,

$$a\sharp_\hbar b - b\sharp_\hbar a \sim \sum_{k=0}^{N} \frac{2}{(2k+1)!}\left(\frac{i\hbar}{2}D\right)^{2k+1}(a, b) + \mathcal{O}(\hbar^{2N+3}),$$

Finally, note that, if $b(\xi)$ is a polynomial in ξ of order ≤ 2, one has, the exact formula:

$$a\sharp_\hbar b - b\sharp_\hbar a = \frac{\hbar}{2i}\{a, b\}.$$

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