ENTROPY OF SEMICLASSICAL MEASURES FOR SYMPLECTIC LINEAR MAPS OF $\mathbb{T}^{2d}$

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ABSTRACT. For a linear symplectic map A of the $2d$-torus $\mathbb{T}^{2d}$, one can associate a sequence of unitary matrices $(M_N(A))_{N\in\mathbb{N}}$ acting on Hilbert spaces of dimension N^d. The eigenstates of this sequence of matrices are said to be the stationary states of the quantum system. In this article, we give quantitative properties on the distribution on $\mathbb{T}^{2d}$ of these stationary states in the so-called semiclassical limit ($\hbar = (2\pi N)^{-1} \to 0$). To do this, we use semiclassical measures which are A-invariant probability measures associated to these sequences of states and we give a lower bound for the Kolmogorov-Sinai entropy of these measures. Our main result is that, for any *quantizable* matrix A in $Sp(2d,\mathbb{Z})$ and any semiclassical measure μ associated to it, the Kolmogogorov-Sinai entropy of μ with respect to A is bounded from below by $\sum_{\beta\in\mathrm{sp}(A)} \max\left(\log|\beta| - \frac{\lambda_{\max}}{2}, 0\right)$, where the sum is taken over the spectrum of A (counted with multiplicities) and $\lambda_{\max}$ is the supremum of $\{\log|\beta| : \beta \in \mathrm{sp}(A)\}$. In particular, our result implies that if A has an eigenvalue outside the unit circle, then a semiclassical measure cannot be carried only by periodic orbits of A.

1. Introduction

The semiclassical principle asserts that one can reconstruct objects from classical mechanics by looking at the semiclassical limit of quantum objects. In the more specific setting of quantum chaos, one can try to understand the range of validity of this principle and more precisely the influence of the chaotic properties of the dynamical system (Anosov property, ergodicity, etc.) on the semiclassical behavior of stationary states.

The first result in this direction is due to Shnirelman [28], Zelditch [30] and Colin de Verdière [10]. It states that, given an ergodic geodesic flow on a riemannian manifold M, *almost all* sequences of eigenfunctions of Δ are equidistributed on S^*M in the high energy limit. This phenomenon is known as quantum ergodicity and has many extensions. Rudnick and Sarnak formulated the so-called "Quantum Unique Ergodicity Conjecture" which states that for manifolds of negative curvature, all the eigenfunctions of Δ are equidistributed in the high energy limit [27]. This conjecture remains widely open in the general case.

In order to study quantum chaos, an approach is to study toy models, i.e. simple symplectic dynamical systems that are highly chaotic and that admits a "good" quantization procedure [20]. The main advantages of such systems is that the classical dynamics is in general simpler than the one of "realistic" models and the quantum states belong to finite dimensional spaces. These properties make these models easier to manipulate than the more "realistic" ones (like geodesic flows). Moreover, these toy models share enough properties with the "realistic" ones so that it is often easier to understand the properties of the simple models and then try to adapt the method in the more complicated situation.

Among the several toy models is the family of symplectic linear automorphisms on the $2d$-torus $\mathbb{T}^{2d}$. We say that a matrix A in $Sp(2d,\mathbb{Z})$ is *quantizable* if it does not have 1 as an eigenvalue or if it belongs to the subset

$$Sp_\theta(2d,\mathbb{Z}) := \left\{ \begin{pmatrix} E & F \\ G & H \end{pmatrix} \in Sp(2d,\mathbb{Z}) : EF^t \equiv GH^t \equiv 0 \mod 2 \right\}.$$

Under one of these assumptions, the map A can be quantized following the method from [6]. In this case, the phase space $\mathbb{T}^{2d}$ is compact and in particular, one can associate a finite dimensional Hilbert space for every integer N: each space will be denoted $\mathcal{H}_N$ (see section 2) and will be of dimension N^d. The semiclassical parameter is denoted $\hbar$ and it satisfies $2\pi\hbar N = 1$ (where N is an integer). So looking at the semiclassical limit $\hbar \to 0$ is equivalent to making the parameter N tends to infinity. The set of classical observables will be the set $\mathcal{C}^\infty(\mathbb{T}^{2d})$ of smooth functions on the torus. There exists a positive quantization procedure $\mathrm{Op}_N^{AW}(.)$ that associates to each observable a a linear operator $\mathrm{Op}_N^{AW}(a)$ on $\mathcal{H}_N$. This procedure is called the anti-Wick quantization and is constructed from a family of coherent states [7]. Moreover, there is a quantum propagator $M_N(A)$ corresponding to A which acts on $\mathcal{H}_N$. This propagator satisfies the Egorov property:

$$M_N(A)^{-1}\mathrm{Op}_N^{AW}(a)M_N(A) = \mathrm{Op}_N^{AW}(a \circ A) + O_a(N^{-1}). \tag{1}$$

For any eigenvector φ^N of $M_N(A)$ in the Hilbert space $\mathcal{H}_N$, one can define the following measure on the torus:

$$\tilde{\mu}_{\varphi^N}(a) := \langle \varphi^N | \mathrm{Op}_N^{AW}(a) | \varphi^N \rangle_{\mathcal{H}_N}. \tag{2}$$

This quantity gives a description of the stationary state in terms of the position and the momentum, i.e. of the stationary state and its N-Fourier transform. Thanks to Egorov theorem, we have that any weak limit of the corresponding $(\tilde{\mu}_{\varphi^N})$ in the semiclassical limit (i.e. as N tends to infinity) is an A-invariant measure on the torus. We call a semiclassical measure an accumulation point of a sequence $(\tilde{\mu}_{\varphi^N})_N$, where $(\varphi^N)_N$ is a sequence of stationary states in $\mathcal{H}_N$ (see section 2). In this setting, Bouzouina and de Bièvre proved an analogue of Shnirelman's theorem [7]. Precisely, if A is *ergodic*[1], they prove that for any sequence of orthonormal basis $(\varphi_j^N)_{1\leq j\leq N^d, N\in\mathbb{N}}$ of $\mathcal{H}_N$ made of eigenvectors of $M_N(A)$, there exists a sequence of $J_N \subset \{1, \cdots, N^d\}$ satisfying

$$\lim_{N\to+\infty} \frac{|J_N|}{N^d} = 1 \text{ and } \forall a \in \mathcal{C}^\infty(\mathbb{T}^{2d}),\ \forall \{j(N) \in J_N : N \in \mathbb{N}\},\ \lim_{N\to+\infty} \tilde{\mu}_{\varphi^N_{j(N)}}(a) = \mathrm{Leb}(a).$$

It means that in this sense, *almost all* the eigenvectors converge weakly to the Lebesgue measure on the torus. The analogue of Quantum Unique Ergodicity in this setting would be to prove that for any orthonormal basis of eigenvectors, one can take $J_N = \{1, \cdots, N^d\}$. An important property of these models is that there exist symplectic matrices and associated sequences of orthonormal basis for which one can take $J_N = \{1, \cdots, N^d\}$ [18], [17] but there are also sequences for which one can not take $J_N = \{1, \cdots, N^d\}$ [13], [17]. In the case $d = 1$, it was proved by de Bièvre, Faure and Nonnenmacher that $\frac{1}{2}(\delta_0 + \mathrm{Leb})$ is a semiclassical measure [13]. In higher dimensions and under arithmetic assumptions on A, Kelmer constructed semiclassical measures supported on submanifolds of $\mathbb{T}^{2d}$ [17].

Even if we know that the set of semiclassical measures is not reduced to the Lebesgue measure for quantized maps of the torus, one can ask about the properties of these semiclassical measures. For instance, it was shown in [6] and [14] that if we split the semiclassical measure into its pure point, Lebesgue and singular continuous components, $\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{Leb}} + \mu_{\mathrm{sc}}$, then $\mu_{\mathrm{pp}}(\mathbb{T}^2) \leq \mu_{\mathrm{Leb}}(\mathbb{T}^2)$ and in particular $\mu_{\mathrm{pp}}(\mathbb{T}^2) \leq 1/2$.

1.1. Statement of the main theorem.

In [1], Anantharaman proved that for a compact riemannian manifold M with Anosov geodesic flow, the Kolmogorov-Sinai entropy of any semiclassical measure associated to a sequence of eigenfunctions of Δ is positive (see section 5 or [29] (chapter 4) for a definition of the entropy). Her result proves in particular that eigenfunctions of the Laplacian cannot concentrate only on a closed geodesic, in the large eigenvalue limit. Translated in the context of our toy models, her result says that, for any symplectic and *hyperbolic* matrix A, the Kolmogorov-Sinai entropy of a semiclassical measure is positive. In particular, a semiclassical measure cannot be supported only on closed orbits of A. In subsequent works with Koch and Nonnenmacher [3], [2], they gave quantitative lower bounds on the Kolmogorov-Sinai entropy of semiclassical measures. Translated in the model of quantized maps of the $2d$-torus, their result can be written, for any semiclassical measure associated to an *hyperbolic* matrix A:

$$h_{KS}(\mu, A) \geq \sum_{i=1}^{2d} \max(\log|\beta_i|, 0) - \frac{d}{2}\lambda_{\max}, \tag{3}$$

[1] It means that no eigenvalue of A is a root of unity.

where $\{\beta_i : 1 \leq i \leq 2d\}$ is the spectrum of A (counted with multiplicities) and $\lambda_{\max}$ is the maximum of the $\log|\beta_i|$ [23]. As A is hyperbolic, it has exactly d eigenvalues (counted with multiplicities) of modulus larger than 1. We underline that their result was more general as it also deals with varying Lyapunov exponents. One can remark that if $\lambda_{\max}$ is very large compared with some of the $\log|\beta_i|$, the previous lower bound can be negative (and so the result empty). Regarding their result (and the different counterexamples), they conjectured some optimal lower bound on the entropy of semiclassical measures [1], [3], [2]. Translated in our context, their conjecture states that, for any semiclassical measure μ associated to an *hyperbolic* matrix A,

$$h_{KS}(\mu, A) \geq \sum_{i=1}^{2d} \max\left(\frac{\log|\beta_i|}{2}, 0\right). \tag{4}$$

Again, their conjecture was more general as they expected it to hold for situations where there are varying Lyapunov exponents. In this article, we will show:

Theorem 1.1. *Let A be a* quantizable *matrix in $Sp(2d,\mathbb{Z})$, i.e. such that 1 is not an eigenvalue of A or such that A belongs to $Sp_\theta(2d,\mathbb{Z})$. Let μ be a semiclassical measure on $\mathbb{T}^{2d}$ associated to A. One has*

$$h_{KS}(\mu, A) \geq \sum_{i=1}^{2d} \max\left(\log|\beta_i| - \frac{\lambda_{\max}}{2}, 0\right), \tag{5}$$

where $\{\beta_i : 1 \leq i \leq 2d\}$ is the spectrum of A and $\lambda_{\max}$ is the maximum of the $\log|\beta_i|$.

A first remark about this result is that it holds for weakly chaotic systems that can have only one instability, i.e. one positive Lyapunov exponent (the result is trivial if A has no positive Lyapunov exponent). In particular, we do not assume A to be hyperbolic and our result implies that if A has an eigenvalue outside the unit circle, then a semiclassical measure of $(\mathbb{T}^{2d}, A)$ cannot only be carried by periodic orbits of A. For instance, such a property is satisfied by an ergodic symplectic matrix [22].

We also underline that we do not obtain exactly the lower bound (4) expected by Anantharaman, Koch and Nonnenmacher. Compared with their result, the lower bound of our theorem improves their bound (3) and it always defines a nonnegative quantity. However, regarding the semiclassical measures constructed in [17], the lower bound of our theorem should be suboptimal. In fact, recall that the simplest example of exceptional semiclassical measure as they were constructed in [17] happens in the case where $A := \begin{pmatrix} B^t & 0 \\ 0 & B^{-1} \end{pmatrix}$, with $B \in SL(d,\mathbb{Z})$. For this matrix, one can show that there exists a subsequence of stationary states converging to the Lebesgue measure on the submanifold $X_0 := \{(0,\xi) \in \mathbb{T}^{2d} : \xi \in \mathbb{T}^d\}$. We also recall that the Kolmogorov-Sinai entropy satisfies the Ruelle inequality, i.e. for any A-invariant measure μ,

$$h_{KS}(\mu, A) \leq \sum_{i=1}^{2d} \max\left(\log|\beta_i|, 0\right),$$

with equality if $\mu = \text{Leb}$ [29] (chapter 8). In particular, Kelmer's counterexample has entropy equal to $\sum_{i=1}^{2d} \max\left(\frac{\log|\beta_i|}{2}, 0\right)$.

In the case of varying Lyapunov exponents, the conjecture of Anantharaman, Koch and Nonnenmacher has been shown to be true when one has only a one dimensional unstable direction [16], [25]. At this point, it is not clear to us how to combine both methods in order to obtain an explicit nonnegative lower bound on the entropy of semiclassical measures in a general setting. We also underline that, in the case of hyperbolic automorphisms of $\mathbb{T}^2$, stronger results on the entropy of semiclassical measures were obtained by Brooks [9]. Finally, in the case of locally symmetric spaces of rank ≥ 2, a similar lower bound as the one of theorem 1.1 was obtained by Anantharaman and Silberman [4].

1.2. **Strategy of the proof.** Compared with the original result of Anantharaman, the proof of inequality (3) in [2] was simplified by the use of an entropic uncertainty principle due to Maassen and Uffink [21]. This principle is a consequence of the Riesz-Thorin interpolation theorem and it can be stated as follows (in [3], see theorem 2.1 combined with remark 2.2):

Theorem 1.2 (Maassen-Uffink)**.** *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two Hilbert spaces. Let U be an unitary operator on $\tilde{\mathcal{H}}$. Suppose $(\pi_i)_{i=1}^D$ is a family of operators from $\tilde{\mathcal{H}}$ to $\mathcal{H}$ that satisfies the following property of partition of identity:*

$$\sum_{i=1}^D \pi_i^\dagger \pi_i = Id_{\tilde{\mathcal{H}}}.$$

Then, for any unit vector ψ, we have

$$(6)\qquad -\sum_{i=1}^D \|\pi_i\psi\|_{\mathcal{H}}^2 \log\|\pi_i\psi\|_{\mathcal{H}}^2 - \sum_{i=1}^D \|\pi_i U\psi\|_{\mathcal{H}}^2 \log\|\pi_i U\psi\|_{\mathcal{H}}^2 \geq -2\log\sup_{i,j}\|\pi_i U\pi_j^\dagger\|_{\mathcal{L}(\mathcal{H})}.$$

In [2], the method was to use this principle for eigenfunctions of the Laplacian on M and a well-chosen partition of $\mathrm{Id}_{L^2(M)}$ so that the quantity in the left side of (6) can be interpreted as the usual entropy from information theory [29]. One of the main difficulty (that already appeared in [1]) was then to give a sharp estimate on the quantity $\|\pi_i U\pi_j^\dagger\|_{L^2(M)\to L^2(M)}$ for this choice. In [26], we show that for a good choice of partition, the quantity $\|\pi_i U\pi_j^\dagger\|_{\mathcal{H}_N\to\mathcal{H}_N}$ was easier to bound than the corresponding one in [2] and in [23]. More precisely, the bound could be directly derived from estimates on the propagation of coherent states under the quantum propagator as in [5] and [13]. Our strategy will be to generalize our method for $d = 1$ to higher dimensions. To do this, we will introduce a new quantization procedure adapted to the classical dynamics for which good choice of partitions can also be made.

1.3. **Organization of the article.** In section 2, we recall how the dynamical system $(\mathbb{T}^{2d}, A)$ can be quantized. In section 3, we collect some facts about the reduction of symplectic matrices. Then, in section 4, we construct a quantization procedure adapted to the classical dynamics induced by A. In section 5, we apply the entropic uncertainty principle to derive theorem 1.1. Finally, in section 6, we prove a crucial estimate on our quantization procedure that we used in section 5. This estimate is similar to the ones obtained for the propagation of coherent states in [5], [13]. The appendices are devoted to the proof of crucial and technical lemmas that we admitted at different steps of the article.

2. Quantum mechanics on the $2d$-torus

In this section, we recall some basic facts about quantization of linear symplectic toral automophisms. Our goal is to describe a procedure which will associate [20]

- a family of finite dimensional Hilbert spaces $\mathcal{H}_N$ indexed by the integers to the compact phase space $\mathbb{T}^{2d}$;
- an operator $\mathrm{Op}_N(a)$ acting on $\mathcal{H}_N$ for every observable a in $\mathcal{C}^\infty(\mathbb{T}^{2d})$;
- a unitary matrix $M_N(A)$ acting on $\mathcal{H}_N$ to the symplectic matrix A (and related to A by an Egorov property).

To do this, we will follow the approach and notations used by de Bièvre & al. in previous articles [6], [7]. We refer the reader to them for further details and references. We denote $\mathbb{T}^{2d} := \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ the $2d$-torus.

2.1. **Quantization of the phase space.** In physical words, $\mathbb{R}^d$ (or $\mathbb{T}^d$) is called the configuration space and $\mathbb{R}^{2d}$ (or $\mathbb{T}^{2d}$) is the phase space associated to it. In this article, $\rho = (x, \xi)$ will denote a point of the phase space, i.e. points of $\mathbb{R}^{2d}$ or $\mathbb{T}^{2d}$. The usual scalar product on $\mathbb{R}^{2d}$ is denoted $\langle a, b\rangle$ and σ is the usual symplectic form on $\mathbb{R}^{2d}$, i.e. $\sigma(\rho, \rho') = \langle \rho, J\rho'\rangle$ where $J := \begin{pmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^d} \\ \mathrm{Id}_{\mathbb{R}^d} & 0 \end{pmatrix}$.

For $\psi \in \mathcal{S}'(\mathbb{R}^d)$, we can define $(\mathbf{q}_j\psi)(x) := x_j\psi(x)$ and $(\mathbf{p}_j\psi)(x) := \frac{\hbar}{\imath}\frac{\partial\psi}{\partial x_j}(x)$. This allows to define translation operators acting on a tempered distribution as:

$$U_\hbar(x,\xi) := e^{\frac{\imath}{\hbar}\sigma((x,\xi),(\mathbf{q},\mathbf{p}))}.$$

The action of this operator on $L^2(\mathbb{R}^d)$ is given, for any $\psi \in L^2(\mathbb{R}^d)$, by

$$\left[U_\hbar(x,\xi)\psi\right](y) = e^{\frac{\imath}{\hbar}\left\langle y-\frac{x}{2},\xi\right\rangle}\psi(y-x).$$

We underline that it is unitary on $L^2(\mathbb{R}^d, dx)$ and standard facts about this representation of the Heisenberg group can be found in the book by Folland [15]. In particular, it can be shown that:

$$U_\hbar(\rho)U_\hbar(\rho') = e^{\frac{\imath}{2\hbar}\sigma\left(\rho,\rho'\right)}U_\hbar(\rho+\rho'). \tag{7}$$

In order to define the quantum states associated to the phase space $\mathbb{T}^{2d}$, we first underline that the phase space is $\mathbb{R}^{2d}$ where we have identified two points which are $\mathbb{Z}^{2d}$-equivalent. So, we can require that the Hilbert space we will construct will be made of quantum states in $\mathcal{S}'(\mathbb{R}^d)$ which satisfy an invariance under the translation operators $U_\hbar(q,p)$ for $(q,p) \in \mathbb{Z}^{2d}$. It means that the quantum states should have the same periodicity as the phase space. To do this, we fix some $\kappa = (\kappa_1, \kappa_2)$ in $[0, 2\pi[^{2d}$. We will require that, for all $(q,p) \in \mathbb{Z}^{2d}$, a quantum state ψ should check the following condition:

$$U_\hbar(q,p)\psi = e^{\frac{\imath}{2\hbar}\langle q,p\rangle}e^{-\imath\langle\kappa_1,q\rangle+\imath\langle\kappa_2,p\rangle}\psi.$$

It can be remarked that κ different from 0 is allowed as, for $\alpha \in \mathbb{R}$, ψ and $e^{\imath\alpha}\psi$ represent the same quantum state. The states in $\mathcal{S}'(\mathbb{R}^d)$ that satisfy the previous conditions are said to be the quantum states on the $2d$-torus and their set is denoted $\mathcal{H}_N(\kappa)$ where N is related to the semiclassical parameter by

$$2\pi\hbar N = 1. \tag{8}$$

It defines tempered distributions of period 1 (modulo phase factors) whose $\hbar$-Fourier transform is also 1-periodic. These distributions are said to be "quasiperiodic". The following lemma can be shown [7]:

Lemma 2.1. *Let κ be an element in $[0, 2\pi[^{2d}$. Let $\hbar$ and N be positive and such that $2\pi\hbar N = 1$. The subspace $\mathcal{H}_N(\kappa)$ is not reduced to 0 iff $N \in \mathbb{N}^*$. In this case, $\dim \mathcal{H}_N(\kappa) = N^d$. Moreover, for all $r \in \mathbb{Z}^{2d}$, $U_\hbar(\frac{r}{N})\mathcal{H}_N(\kappa) = \mathcal{H}_N(\kappa)$ and there is a unique Hilbert structure such that $U_\hbar(\frac{r}{N})$ is unitary for each $r \in \mathbb{Z}^{2d}$.*

From this point, we will fix N to be an integer in order to have a nontrivial Hilbert space. The Hilbert structure on $\mathcal{H}_N(\kappa)$ is not very explicit [7]. However, one can make it more clear using the following map which defines a surjection of $\mathcal{S}(\mathbb{R}^d)$ (Schwartz functions) onto $\mathcal{H}_N(\kappa)$:

$$S_N(\kappa) := \sum_{(n,m)\in\mathbb{Z}^{2d}} (-1)^{N\langle n,m\rangle}e^{\imath(\langle\kappa_1,n\rangle-\langle\kappa_2,m\rangle)}U_\hbar(n,m). \tag{9}$$

This surjection associates to each state in $\mathcal{S}(\mathbb{R}^d)$ a state which is periodic in position and momentum. Using it, we can define $|\phi,\kappa,N\rangle := S_N(\kappa)|\phi\rangle$ and $|\phi',\kappa,N\rangle := S_N(\kappa)|\phi'\rangle$ for $|\phi\rangle$ and $|\phi'\rangle$ in $\mathcal{S}(\mathbb{R}^d)$. Then, the following link between scalar products on $L^2(\mathbb{R}^d)$ and $\mathcal{H}_N(\kappa)$ holds:

$$\langle N,\kappa,\phi|\phi',\kappa,N\rangle_{\mathcal{H}_N(\kappa)} = \sum_{n,m\in\mathbb{Z}^{2d}} (-1)^{N\langle n,m\rangle}e^{\imath(\langle\kappa_1,n\rangle-\langle\kappa_2,m\rangle)}\langle\phi|U_\hbar(n,m)\phi'\rangle_{L^2(\mathbb{R})}. \tag{10}$$

Finally, the following decomposition into irreducible subrepresentations of the discrete Weyl-Heisenberg group $\{(\frac{r}{N},\phi) : r \in \mathbb{Z}^{2d}, \phi \in \mathbb{R}\}$ can be written [7]:

$$L^2(\mathbb{R}^d) \cong \int_{[0,2\pi[^{2d}} \mathcal{H}_N(\kappa)d\kappa \text{ and } U_\hbar\left(\frac{r}{N}\right) = \int_{[0,2\pi[^{2d}} U_{N,\kappa}\left(\frac{r}{N}\right)d\kappa,$$

where the integral is a direct integral.

2.2. **Weyl quantization.** In the case of $\mathbb{R}^{2d}$, classical observables are functions of $\rho=(x,\xi)$ that belong to a certain class of symbols. We will use the following class of symbols:

$$S^k_\nu(1):=\{(a_\hbar)_{\hbar>0}\in\mathcal{C}^\infty(\mathbb{R}^{2d}):\ \text{for all multiindices }\alpha,\ \|\partial^\alpha_\rho a_\hbar\|_\infty\leq\hbar^{-k-\nu|\alpha|}C_{\alpha,a}\},$$

where $\nu\leq\frac{1}{2}$. We underline that smooth $\mathbb{Z}^{2d}$-periodic functions belongs to these class of symbols and in this case, the semi-norms involved in the definition of $S^0_\nu(1)$ control the growth of the derivatives in $\hbar$. An usual way to quantize these observables is to use the Weyl quantization [11], [12]. Let us recall the standard definition of this operator for an observable a:

$$[\mathrm{Op}^w_\hbar(a)u](x):=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}}e^{\frac{\imath}{\hbar}\langle x-y,\xi\rangle}a\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi.$$

We recall that the product of two such pseudodifferential operators is given by

$$\mathrm{Op}^w_\hbar(a)\circ\mathrm{Op}^w_\hbar(b)=\mathrm{Op}^w_\hbar(a\sharp b),$$

where $a\sharp b$ is the Moyal product of the two observables [11], [12]. At several points of the article we will use the following expression of the Moyal product for a and b in $\mathcal{S}(\mathbb{R}^{2d})$ (that can depend on $\hbar$):

$$a\sharp b(\rho):=\frac{1}{(\pi\hbar)^{2d}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}e^{-\frac{2\imath}{\hbar}\sigma(w_1,w_2)}a(\rho+w_1)b(\rho+w_2)dw_1dw_2.$$

We also recall that, from the Calderón-Vailancourt theorem (theorem 7.11 in [11] or theorem 4.22 in [12]), we know that there exists an integer D and a constant C (depending only on d) such that

$$\forall\,0\leq\nu\leq\frac{1}{2},\ \forall a\in S^0_\nu(1),\ \|\mathrm{Op}^w_\hbar(a)\|_{L^2(\mathbb{R}^d)\rightarrow L^2(\mathbb{R}^d)}\leq C\sum_{|\alpha|\leq D}\hbar^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_\infty. \tag{11}$$

We underline that the case $\nu=\frac{1}{2}$ is authorized for this last result [11]. In the case of the $2d$-torus, classical observables are $\mathcal{C}^\infty$ functions on $\mathbb{T}^{2d}$ which are $\hbar$ independant (they can be seen as a subset of $S^0(1)$). It can be shown that for $a\in\mathcal{C}^\infty(\mathbb{T}^{2d})$,

$$\mathrm{Op}^w_\hbar(a)=\sum_{r\in\mathbb{Z}^{2d}}a_rU_\hbar(2\pi\hbar r),$$

where a_r is the r coefficient of the Fourier serie of a, i.e. $a(\rho)=\sum_{a\in\mathbb{Z}^{2d}}a_re^{-2\imath\pi\langle Jr,\rho\rangle}$. Using the fact that $U_\hbar(r)\mathrm{Op}^w_\hbar(a)U_\hbar(r)^*=\mathrm{Op}^w_\hbar(a)$ (thanks to (7)), it follows that $\mathrm{Op}^w_\hbar(a)\mathcal{H}_N(\kappa)\subset\mathcal{H}_N(\kappa)$. In view of this remark, we shall denote $\mathrm{Op}^w_{N,\kappa}(a)$ the restriction of $\mathrm{Op}^w_\hbar(a)$ to $\mathcal{H}_N(\kappa)$. We underline that this operator still makes sense if a is an element of $\mathcal{C}^\infty(\mathbb{T}^{2d})$ that depends nicely on $\hbar$ (in the sense of the definition of $S^0_\nu(1)$). Finally, the following decomposition holds:

$$\mathrm{Op}^w_\hbar(a)=\int_{[0,2\pi[^{2d}}\mathrm{Op}^w_{N,\kappa}(a)d\kappa,$$

where the integral is a direct integral. Recall from [24] (theorem XIII.83) that such a decomposition implies:

$$\sup_{\kappa\in[0,2\pi[^{2d}}\|\mathrm{Op}^w_{N,\kappa}(a)\|_{\mathcal{L}(\mathcal{H}_N(\kappa))}=\|\mathrm{Op}^w_\hbar(a)\|_{L^2(\mathbb{R}^d)}. \tag{12}$$

2.3. **Quantization of toral automorphisms.** Let A be a matrix in $Sp(2d,\mathbb{Z})$. We would like now to quantize the dynamics associated to A on the phase space, i.e. define a quantum progator associated to A. This can be done using the metaplectic "representation" of $Sp(2d,\mathbb{R})$ [15] that defines, for each matrix A and for each $\hbar>0$, the unique (up to a phase) operator which satisfies:

$$\forall\rho\in\mathbb{R}^{2d},\ M_\hbar(A)U_\hbar(\rho)M_\hbar(A)^{-1}=U_\hbar(A\rho).$$

$M_\hbar(A)$ is called the quantum propagator associated to A. It is a unitary operator on $L^2(\mathbb{R}^d)$ (and by duality, it acts also on $\mathcal{S}'(\mathbb{R}^d)$). It can be shown [6]:

Lemma 2.2. *Let A be an element in $Sp(2d,\mathbb{Z})$ such that 1 is not an eigenvalue of A. For each $N\in\mathbb{N}^*$, there exists at least one $\kappa_A(N)\in[0,2\pi[^{2d}$ such that*

$$M_\hbar(A)\mathcal{H}_N(\kappa_A(N))=\mathcal{H}_N(\kappa_A(N)).$$

$M_{N,\kappa_A(N)}(A)$ denotes then the restriction of $M_\hbar(A)$ to $\mathcal{H}_N(\kappa_A(N))$. It is an unitary operator for the Hilbert structure on $\mathcal{H}_N(\kappa_A(N))$.

Remark. Even if it was only stated for ergodic matrices, the proof of this lemma was given in [6] (lemma 2.2). The hypothesis that 1 is not an eigenvalue is crucial in the proof of [6]. Yet, one can check in this same proof that the property of the lemma still holds if we take $\kappa_A(N)=0$ for a matrix A in

$$Sp_\theta(2d,\mathbb{Z}):=\left\{\begin{pmatrix} E & F\\ G & H\end{pmatrix}\in Sp(2d,\mathbb{Z}) : EF^t\equiv GH^t\equiv 0 \mod 2\right\}.$$

We say that an element A in $Sp(2d,\mathbb{Z})$ is *quantizable* if 1 is not an eigenvalue of A or if A is in $Sp_\theta(2d,\mathbb{Z})$. We underline that the matrices in $Sp_\theta(2d,\mathbb{Z})$ were the ones considered by Kelmer in [17].

Notations. From this point of the article, we fix A to be a quantizable matrix and for every integer N, we fix some $\kappa_A(N)$ which satisfies the property of the previous lemma. For simplicity of notations, we will omit to mention κ (except if there is an ambiguity) and we will denote $\mathrm{Op}_N(a)$, S_N, $\mathcal{H}_N$, $M_N(A)$, $U_N(\rho)$ instead of $\mathrm{Op}_{N,\kappa}(a)$, $S_N(\kappa)$, $\mathcal{H}_N(\kappa)$, $M_{N,\kappa}(A)$, $U_{N,\kappa}(\rho)$. When using the subscript N, we will refer to objects living on the torus and when using $\hbar$, we will refer to objects living on $\mathbb{R}^{2d}$.

From all this, the following "exact" Egorov property can be shown for each $a\in\mathcal{C}^\infty(\mathbb{T}^{2d})$ [15], [7]:

$$M_N(A)^{-1}\mathrm{Op}_N^w(a)M_N(A)=\mathrm{Op}_N^w(a\circ A).$$

Remark. We underline that we do not need any assumption on A (except that is is symplectic) to define $M_\hbar(A)$ on $L^2(\mathbb{R}^d)$. In particular, for every Q in $Sp(2d,\mathbb{R})$, we have that

$$\forall\, a\in S^0_\nu(1),\ M_\hbar(Q)^{-1}\mathrm{Op}_\hbar^w(a)M_\hbar(Q)=\mathrm{Op}_\hbar^w(a\circ Q).$$

2.4. **Anti-Wick quantization.** The Weyl quantization has the nice properties that it satisfies an exact Egorov property and that for a symbol a, $\mathrm{Op}_\kappa^w(a)^*=\mathrm{Op}_\kappa^w(\overline{a})$. However, it does not satisfy the property that if a is nonnegative then $\mathrm{Op}_\kappa^w(a)$ is also nonnegative. As our goal is to construct measures using a quantization procedure, we would like for simplicity to consider a positive quantization. This can be achieved by considering the anti-Wick quantization. To describe this quantization, we define the coherent state at point 0 on $\mathbb{R}^d$:

$$|0,\hbar\rangle(x):=\left(\frac{1}{\pi\hbar}\right)^{\frac{d}{4}}e^{-\frac{\|x\|^2}{2\hbar}}.$$

We define the translated coherent state at point $\rho\in\mathbb{R}^{2d}$ as $|\rho,\hbar\rangle:=U_\hbar(\rho)|0,\hbar\rangle$. Using these coherent states, we can define a quantization procedure for a bounded symbol a in $\mathcal{C}^0(\mathbb{R}^{2d})$:

$$\mathrm{Op}_\hbar^{AW}(a):=\int_{\mathbb{R}^{2d}}a(\rho)|\rho,\hbar\rangle\langle\hbar,\rho|\frac{d\rho}{(2\pi\hbar)^d}.$$

We underline that $|\rho,\hbar\rangle\langle\hbar,\rho|$ is a rank-one operator defined for $|\psi\rangle$ in $L^2(\mathbb{R}^d)$ by $|\rho,\hbar\rangle\langle\hbar,\rho|\psi\rangle:=\langle\hbar,\rho|\psi\rangle_{L^2}|\rho,\hbar\rangle$. It is obvious that this quantization is positive. It can be verified also that it satisfies the property of resolution of identity:

$$\mathrm{Op}_\hbar^{AW}(1)=\mathrm{Id}_{L^2(\mathbb{R}^d)}=\int_{\mathbb{R}^{2d}}|\rho,\hbar\rangle\langle\hbar,\rho|\frac{d\rho}{(2\pi\hbar)^d}.$$

This quantization is related to the Weyl quantization. To see this, we can define the gaussian observable $\tilde{G}_\hbar(x,\xi):=\frac{1}{(\pi\hbar)^d}e^{-\frac{\|x\|^2+\|\xi\|^2}{\hbar}}$. For a smooth bounded observable a, the relation between the two procedures of quantization is $\mathrm{Op}_\hbar^{AW}(a)=\mathrm{Op}_\hbar^w(a\star\tilde{G}_\hbar)$. Using Calderón-Vaillancourt theorem and the previous property, one can verify that $\|\mathrm{Op}_\hbar^{AW}(a)-\mathrm{Op}_\hbar^w(a)\|_{L^2(\mathbb{R}^d)}=O_a(\hbar)$. Now one can construct a positive quantization on the torus mimicking this positive quantization on $\mathbb{R}^{2d}$. To do this, we project the coherent states on the Hilbert space $\mathcal{H}_N$:

$$|\rho, N\rangle := S_N|\rho, \hbar\rangle,$$

where S_N is the surjection defined by (9). We define the anti-Wick quantization of an observable a in $\mathcal{C}^\infty(\mathbb{T}^{2d})$ as follows:

$$\mathrm{Op}_N^{AW}(a) := \int_{\mathbb{T}^{2d}} a(\rho)|\rho, N\rangle\langle N, \rho|\frac{d\rho}{(2\pi\hbar)^d}.$$

In this case, $|\rho, N\rangle\langle N, \rho|$ is a rank-one matrix defined for $|\psi\rangle$ in $\mathcal{H}_N$ by $|\rho, N\rangle\langle N, \rho|\psi\rangle := \langle N, \rho|\psi\rangle_{\mathcal{H}_N}|\rho, N\rangle$. This quantization procedure satisfies that for a symbol a, $\mathrm{Op}_N^{AW}(a)^* = \mathrm{Op}_N^{AW}(\overline{a})$ and that the quantization is nonnegative. As in the case of the Weyl quantization, it is related to the quantization on $\mathbb{R}^{2d}$ by the integral representation [7]:

$$\mathrm{Op}_\hbar^{AW}(a) = \int_{[0,2\pi[^{2d}} \mathrm{Op}_{N,\kappa}^{AW}(a)d\kappa.$$

It also satisfies a resolution of identity property [7]:

$$\mathrm{Op}_N^{AW}(1) = \mathrm{Id}_{\mathcal{H}_N} = \int_{\mathbb{T}^{2d}} |\rho, N\rangle\langle N, \rho|\frac{d\rho}{(2\pi\hbar)^d}.$$

It follows from the reoslution of identity and from the definition of $\mathrm{Op}_N^{AW}(a)$ that $\|\mathrm{Op}_N^{AW}(a)\|_{\mathcal{L}(\mathcal{H}_N)} \leq \|a\|_\infty$.

2.5. **Semiclassical measures.** All these definitions allow to introduce the notion of semiclassical measures for the quantized cat-maps [7]:

Definition 2.3. Let A be a *quantizable* matrix in $Sp(2d, \mathbb{Z})$. We call semiclassical measure of $(\mathbb{T}^{2d}, A)$ any accumulation point of a sequence of measures of the form

$$\forall a \in \mathcal{C}^\infty(\mathbb{T}^{2d}, \mathbb{C}),\ \tilde{\mu}^{N_k}(a) := \langle\psi^{N_k}|\mathrm{Op}_{N_k}^{AW}(a)\psi^{N_k}\rangle_{\mathcal{H}_{N_k}} = \int_{\mathbb{T}^{2d}} a(\rho)N_k\left|\left\langle\psi^{N_k}|\rho, N_k\right\rangle_{\mathcal{H}_{N_k}}\right|^2 d\rho,$$

where $(\psi^{N_k})_{N_k}$ is a sequence made of eigenvectors of $M_{N_k}(A)$ in $\mathcal{H}_{N_k}$ with $N_k \to +\infty$ as $k \to +\infty$.

Remark. The set of semiclassical measure defines a nonempty set of probability measures on the torus $\mathbb{T}^{2d}$. We underline that we have previously fixed $\kappa_A(N)$ (section 2.3) for every integer N. In particular, the set of semiclassical measures depends implicitely on our choice of quantization procedure as it depends on the sequence $\kappa_A(N)$ we have fixed. We could have allowed κ to vary in our definition and we would have obtained a bigger set of accumulation points. The entropic properties of this bigger set would be the same and for simplicity of the notations, we prefer to keep the sequence $\kappa_A(N)$ fixed for the article.

An important property is that a semiclassical measure is A-invariant thanks to the following Egorov property:

Proposition 2.4 (Egorov property). *Let A be a* quantizable *matrix in $Sp(2d, \mathbb{Z})$. For every a in $\mathcal{C}^\infty(\mathbb{T}^{2d})$, one has*

$$\forall t \in \mathbb{Z},\ M_N(A)^{-t}Op_N^{AW}(a)M_N(A)^t = Op_N^{AW}(a \circ A^t) + O_{a,t}(N^{-1}),$$

where the constant involved in the remainder depends on a and t.

To conclude the presentation of our system, we underline that if μ is a semiclassical measure associated to a sequence (ψ_{N_k}), then μ is also an accumulation point of the sequence of linear form

$$a \mapsto \langle\psi_{N_k}|\mathrm{Op}_{N_k}(a)\psi_{N_k}\rangle_{\mathcal{H}_{N_k}},$$

where Op_N is any quantization that satisfies, in the semiclassical limit,

$$\forall a \in \mathcal{C}^\infty(\mathbb{T}^{2d}),\ \left\|\mathrm{Op}_N(a) - \mathrm{Op}_N^{AW}(a)\right\|_{\mathcal{L}(\mathcal{H}_N)} = o_a(1). \tag{13}$$

3. Symplectic linear algebra and Lyapunov exponents

In this section, we collect some facts about symplectic matrices that we will use crucially in our proof. We refer the reader to chapter 1 of [19] for more details. We fix a *quantizable* matrix A in $Sp(2d,\mathbb{Z})$, i.e. such that 1 is not an eigenvalue of A or such that $A\in Sp_\theta(2d,\mathbb{Z})$. As theorem 1.1 is trivial in the case where the spectrum is included in $\{z\in\mathbb{C}:|z|=1\}$, we also make the assumption that A has an eigenvalue of modulus larger than 1. We will denote

$$\lambda_{\max}=\sup\{\log|\beta|:\beta\text{ is in the spectrum of }A\}.$$

Remark. According to Kronecker's theorem (theorem 2.5 in [22]), we know that if A is an *ergodic* matrix in $SL(2d,\mathbb{Z})$ (i.e. no eigenvalue of A is a root of unity), then $\lambda_{\max}>0$.

One can decompose $\mathbb{R}^{2d}$ into A-invariant subspaces called the stable, neutral and unstable spaces, i.e.

$$\mathbb{R}^{2d}:=E^-\oplus E^0\oplus E^+.$$

These subspaces satisfy various properties that we will use. The spectrum of the restriction of A on the neutral space E^0 is included in $\{z\in\mathbb{C}:|z|=1\}$. The dimension of E^0 is even and we will denote it $2d_0$. The restriction of A on the stable (resp. unstable) space E^- (resp. E^+) has a spectrum included in $\{z\in\mathbb{C}:|z|<1\}$ (resp. $\{z\in\mathbb{C}:|z|>1\}$). These two subspaces have the same dimension equal to $d-d_0$ (which is by assumption positive). Moreover, there exist r in $\mathbb{N}$ and $0<\lambda_1^+<\cdots<\lambda_r^+$ such that E^+ (resp. E^-) can be decomposed into A-invariant subspaces as follows:

$$E^+=E_1^+\oplus\cdots\oplus E_r^+\text{ and }E^-=E_1^-\oplus\cdots\oplus E_r^-,$$

where the spectrum of the restriction of A to E_i^+ (resp. E_i^-) is included in $\{z\in\mathbb{C}:|z|=e^{\lambda_i^+}\}$ (resp. $\{z\in\mathbb{C}:|z|=e^{-\lambda_i^+}\}$). Moreover, one can verify that the subspaces E_i^+ and E_i^- have the same dimension that we will denote d_i. The coefficients λ_i^+ are called the positive Lyapunov exponents of A. We underline that $\lambda_r^+=\lambda_{\max}$. With these notations, theorem 1.1 can be rewritten that for any semiclassical measure μ associated to A, one has

$$h_{KS}(\mu,A)\geq\sum_{i=1}^r d_i\max\left(\lambda_i^+-\frac{\lambda_{\max}}{2},0\right). \tag{14}$$

For the sake of simplicity, we will denote

$$\Lambda_+:=\sum_{i=1}^r d_i\lambda_i^+ \tag{15}$$

and

$$\Lambda_0:=\sum_{i=1}^r d_i\max\left(\lambda_i^+-\frac{\lambda_{\max}}{2},0\right). \tag{16}$$

Our decomposition is exactly the Oseledets decomposition associated to the dynamical system $(\mathbb{T}^{2d},A,\mu)$ [29]. For our proof, we will need something stronger in order to apply tools of semiclassical analysis. Precisely, we will need a symplectic decomposition of $\mathbb{R}^{2d}$ into these subspaces. According to [19] (section 1.4 to 1.7), this decomposition is possible and we now recall the results from [19] that we will need. To do this, we introduce the $\diamond$-product of two matrices. Consider two real matrices M_1 in $M(2d',\mathbb{R})$ and M_2 in $M(2d'',\mathbb{R})$ of the block form

$$M_1:=\begin{pmatrix}A_1&B_1\\C_1&D_1\end{pmatrix}\text{ and }M_2:=\begin{pmatrix}A_2&B_2\\C_2&D_2\end{pmatrix},$$

where A_1, B_1, C_1 and D_1 are in $M(d',\mathbb{R})$ and A_2, B_2, C_2 and D_2 are in $M(d'',\mathbb{R})$. The $\diamond$-sum of M_1 and M_2 is defined as the following $2(d'+d'')$ matrix:

$$M_1\diamond M_2:=\begin{pmatrix}A_1&0&B_1&0\\0&A_2&0&B_2\\C_1&0&D_1&0\\0&C_2&0&D_2\end{pmatrix}.$$

We can use this $\diamond$-sum to rewrite our symplectic matrix A in an adapted symplectic basis [19] (section 1.7-theorem 3). Precisely, for every $1 \leq i \leq r$, one can construct an adapted D_i in $Gl(d_i, \mathbb{R})$ such that the spectrum of D_i is included in $\{z \in \mathbb{C} : |z| = e^{\lambda_i^+}\}$ and denote $A_i := \text{diag}(D_i, D_i^{*-1})$ (setion 1.7 in [19]). There exists also an adapted A_0 in $Sp(2d_0, \mathbb{R})$ such that the spectrum[2] of A_0 is included in $\{z \in \mathbb{C} : |z| = 1\}$ (section 1.5 and 1.6 in [19]). Using these matrices, it can be shown that there exists a symplectic matrix Q in $Sp(2d, \mathbb{R})$ such that

$$A = Q\,(A_0 \diamond A_1 \diamond \cdots A_r)\,Q^{-1}. \tag{17}$$

This tells us that we have a symplectic reduction adapted to the Oseledets decomposition. The results in [19] are more precise and we have only stated what we will need for our proof of theorem 1.1.

4. Positive quantization adapted to the dynamics

We have defined the set of semiclassical measures starting from the anti-Wick quantization. In this section, we will construct a new (positive) quantization procedure that is adapted to the classical dynamics and that is equivalent to the Weyl quantization (and so to the anti-Wick one) in the sense of equation (13). To do this, we will mimick the construction of the anti-Wick quantization. In this case, we have seen that it corresponds to the Weyl quantization applied to the observable $a \star \tilde{G}_\hbar$. It means that we have made the convolution of the observable a with a Gaussian observable which is localized in a ball of radius $\sqrt{\hbar}$.

Our strategy is to make a slightly different choice of function $G_\hbar$ which will be localized on an ellipsoid with lengths on each direction that depend on the Lyapunov exponent. For instance, if the Lyapunov exponent associated to the variable (x_1, ξ_1) is larger than the one associated to the variable (x_2, ξ_2), the ellipsoid will be larger in the second direction. We should also take care of not violating the uncertainty principle and as a consequence the radius of the ellipsoid will always be bounded from below by $\sqrt{\hbar}$.

In this section, we make this argument precise in the case of $\mathbb{R}^{2d}$ and then periodize the new quantization to get a quantization on the torus.

4.1. **An adapted convolution observable.** To construct our new quantization on $\mathbb{R}^{2d}$, we introduce a Gaussian observable $G(x, \xi) := \exp\left(-\pi\|(x,\xi)\|^2\right)$, where $\|.\|$ is the euclidian norm on $\mathbb{R}^{2d}$. In the case of the anti-Wick quantization, we took the convolution of any bounded observable a with $\tilde{G}_\hbar = (\pi\hbar)^{-d} G \circ \left((\pi\hbar)^{-\frac{1}{2}}\text{Id}\right)$ to construct our quantization. Regarding the Oseledets decomposition of A (see (17)), we would like to make a more strategical choice than $(\pi\hbar)^{-\frac{1}{2}}\text{Id}$ for our choice of matrix. To do this, we use the notations of section 3 and for $\hbar > 0$, we introduce a matrix $B(\hbar)$ of the following form:

$$B(\hbar) := Q \begin{pmatrix} D(\hbar) & 0 \\ 0 & D(\hbar) \end{pmatrix} Q^{-1},$$

where $D(\hbar)$ is an element in $GL(d, \mathbb{R})$ of the form

$$D(\hbar) := \left(\hbar^{-\frac{\epsilon_0}{2\lambda_{\max}}}\text{Id}_{\mathbb{R}^{d_0}}, \hbar^{-\frac{\lambda_1^+}{2\lambda_{\max}}}\text{Id}_{\mathbb{R}^{d_1}}, \cdots, \hbar^{-\frac{\lambda_r^+}{2\lambda_{\max}}}\text{Id}_{\mathbb{R}^{d_r}}\right).$$

In the previous definition, ϵ_0 is some small fixed positive number that we keep fixed until the end of the proof and that we suppose to be very small compared with λ_1^+ (to avoid complications). To simplify the expressions, we introduce the notation $\gamma_j^+ := \frac{\lambda_j^+}{2\lambda_{\max}}$ and $\gamma_0^+ := \frac{\epsilon_0}{2\lambda_{\max}}$. In particular, we have that the supremum $\|B(\hbar)^{-1}\|_\infty$ of the modulus of the coefficient of $B(\hbar)$ is a $\mathcal{O}(\hbar^\gamma)$ where $\gamma = \frac{\epsilon_0}{2\lambda_{\max}}$. Finally, we can define an "adapted" $\hbar$-Gaussian observable

$$G_\hbar := 2^{\frac{d}{2}} |\det B(\hbar)|^{\frac{1}{2}} G \circ B(\hbar).$$

We underline that $2^{d/2}$ is only a normalization constant.

[2]We underline that d_0 will be equal to 0 if all the Lyapunov exponents of A are nonzero.

4.2. **Positive quantization on $\mathbb{R}^{2d}$.** We have constructed a convolution which is adapted to the dynamics. We would also like to keep the nice 'u^*u'-structure of the anti-Wick quantization. So, for a bounded observable a in $\mathcal{C}^\infty(\mathbb{R}^{2d})$, we define

$$\mathrm{Op}_\hbar^+(a) := \mathrm{Op}_\hbar^w\left(a \star (G_\hbar \sharp G_\hbar)\right),$$

where $a \star b$ is the convolution product of two observables and $a\sharp b$ is the Moyal product of two observables (i.e. the symbol of $\mathrm{Op}_\hbar^w(a) \circ \mathrm{Op}_\hbar^w(b)$ [11]). We verify that

$$\mathrm{Op}_\hbar^+(a) = \int_{\mathbb{R}^{2d}} a(\rho_0)\mathrm{Op}_\hbar^w\left((G_\hbar \sharp G_\hbar)(\bullet - \rho_0)\right) d\rho_0 = \int_{\mathbb{R}^{2d}} a(\rho_0)\mathrm{Op}_\hbar^w\left(G_\hbar^{\rho_0}\right)^* \circ \mathrm{Op}_\hbar^w\left(G_\hbar^{\rho_0}\right) d\rho_0,$$

where $G_\hbar^{\rho_0}(\rho) := G_\hbar(\rho - \rho_0)$. So if $a \geq 0$, this defines a nonnegative operator. The following lemma says that $\mathrm{Op}_\hbar^+$ is a nice quantization procedure:

Lemma 4.1. *Let a be an observable in $S^0(1)$. We have*

$$\left\| Op_\hbar^w(a) - Op_\hbar^+(a)\right\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O_a(\hbar^\gamma),$$

for some fixed positive γ (depending only on A and on ϵ_0).

We postpone the proof of this lemma to appendix A. The strategy is the same as when one proves the equivalence of the anti-Wick quantization and the Weyl one. Precisely, we can prove that there exists an explicit kernel $K_\hbar(\rho_0)$ such that

$$a \star (G_\hbar \sharp G_\hbar)(\rho) = \int_{\mathbb{R}^{2d}} a(\rho + B(\hbar)^{-1}\rho_0) K_\hbar(\rho_0) d\rho_0,$$

where $\int_{\mathbb{R}^{2d}} K_\hbar(\rho_0)d\rho_0 = 1$.

4.3. **Periodization of observables.** We have just defined a new quantization procedure on $\mathbb{R}^{2d}$ which is related to the classical dynamics associated to the matrix A. To study our problem, we need to restrict this quantization procedure to $\mathcal{H}_N$. To do this, we define

$$\mathrm{Op}_N^+(a) := \mathrm{Op}_N^w\left(a \star (G_\hbar \sharp G_\hbar)\right). \tag{18}$$

This definition makes sense for a smooth and $\mathbb{Z}^{2d}$-periodic observable a (that can also depend on $\hbar$). Thanks to lemma 4.1 and to the decomposition of $L^2(\mathbb{R}^d)$ along the spaces $\mathcal{H}_N(\kappa)$, we know that $\|\mathrm{Op}_N^+(a) - \mathrm{Op}_N^w(a)\|_{\mathcal{L}(\mathcal{H}_N)} = O_a(\hbar^\gamma)$ (see also [24] (theorem XIII.83)). The explicit form of this procedure is given by

$$\mathrm{Op}_N^+(a) = \sum_{r\in\mathbb{Z}^{2d}} \left(\int_{\mathbb{T}^{2d}} e^{2\imath\pi\langle\rho, Jr\rangle}\left(\int_{\mathbb{R}^{2d}} a(\rho_0)\,(G_\hbar\sharp G_\hbar)(\rho-\rho_0)d\rho_0\right)d\rho\right) U_N\left(\frac{r}{N}\right).$$

For our purpose, we would like to verify that it remains a positive quantization procedure with a nice structure. To see this, we introduce the following periodization operators on $\mathcal{S}(\mathbb{R}^{2d})$:

$$\forall\, \theta \in \mathbb{R}^{2d},\ \forall\, F \in \mathcal{S}(\mathbb{R}^{2d}),\ T_\theta(F)(\rho) := \sum_{r\in\mathbb{Z}^{2d}} F\left(\rho + r - \frac{J\theta}{2N}\right) e^{2\imath\pi\langle\rho + r, \theta\rangle}. \tag{19}$$

This definition makes sense for a function in the Schwartz class $\mathcal{S}(\mathbb{R}^{2d})$ that would depend on $\hbar$. We also underline that in every case, the observable $T_\theta(F)$ is $\mathbb{Z}^{2d}$-periodic, i.e. for every $r_0 \in \mathbb{Z}^{2d}$, $T_\theta(F)(\rho + r_0) = T_\theta(F)(\rho)$. In particular, $T_0(G_\hbar^{\rho_0}\sharp G_\hbar^{\rho_0})$ is smooth, $\mathbb{Z}^{2d}$-periodic and in the class $S^k_{\frac{1}{2}}(1)$ for some positive k. We can use this function to rewrite

$$\mathrm{Op}_N^+(a) = \int_{\mathbb{T}^{2d}} a(\rho_0)\mathrm{Op}_N^w\left(T_0(G_\hbar^{\rho_0}\sharp G_\hbar^{\rho_0})\right) d\rho_0.$$

The translation operators T_θ satisfy the following property:

Proposition 4.2. *Fix $\hbar > 0$. Let F_1 and F_2 be two elements in $\mathcal{S}(\mathbb{R}^{2d})$ (depending eventually on $\hbar$). One has*

$$Op_\hbar^w(T_0(\overline{F}_1\sharp F_2)) = \int_{\mathbb{T}^{2d}} Op_\hbar^w(T_\theta F_1)^* \circ Op_\hbar^w(T_\theta F_2) d\theta.$$

We postpone the proof of this lemma (which is just a careful application of the Poisson formula) to appendix B. This proposition provides an alternative form for our quantization procedure, i.e.

$$\mathrm{Op}_N^+(a) = \int_{\mathbb{T}^{2d}} a(\rho_0) \int_{\mathbb{T}^{2d}} \mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))^* \circ \mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0})) d\theta d\rho_0. \tag{20}$$

In particular, it implies that Op_N^+ is a nonnegative quantization procedure. We also underline that we have the following resolution of identity:

$$\mathrm{Id}_{\mathcal{H}_N} = \int_{\mathbb{T}^{2d}} \int_{\mathbb{T}^{2d}} \mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))^* \circ \mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0})) d\theta d\rho_0. \tag{21}$$

Remark. These last two formulas are the analogues of the ones obtained for the anti-Wick quantization. The expressions seems more complicated but we will see that it is more adapted to the dynamics induced by A.

4.4. **Long times Egorov property.** In this last paragraph, we show that as the anti-Wick procedure, the quantization procedure Op_N^+ satisfies an Egorov property until times of order $T_E(N) := \frac{\log N}{2\lambda_{\max}}$. We fix some positive $\epsilon \ll \min(\epsilon_0, \lambda_1)$ and define the Ehrenfest time

$$m_E(N) := \left[\frac{1-\epsilon}{2\lambda_{\max}} \log N \right], \tag{22}$$

The parameter ϵ will be kept fixed (until the end of the proof of theorem 1.1). In order to state our result, we denote μ^N the measure associated to a unit eigenvector ψ_N of $M_N(A)$, i.e.

$$\mu^N(a) := \langle \psi_N | \mathrm{Op}_N^+(a) | \psi_N \rangle_{\mathcal{H}_N} = \int_{\mathbb{T}^{2d}} a(\rho_0) \int_{\mathbb{T}^{2d}} \| \mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))\psi_N \|_{\mathcal{H}_N}^2 \, d\theta d\rho_0.$$

One can show the following (pseudo)-invariance property of the measures μ^N until time $m_E(N)$:

Proposition 4.3. *Let $(\psi_N)_N$ be a sequence of unit eigenvectors of $M_N(A)$ in $\mathcal{H}_N$ and μ^N the associated sequence of measures. Then, for every positive ϵ, one has*

$$\forall a \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C}), \ \forall |t| \leq m_E(N), \ \mu^N(a \circ A^t) = \mu^N(a) + o_{a,\epsilon}(1), \tag{23}$$

where the constant in the remainder depends only on a and ϵ.

Proof. We have an exact Egorov property for the Weyl quantization. In particular, it tells us that, for every integer t,

$$\left\| \mathrm{Op}_N^+(a \circ A^t) - \mathrm{Op}_N^+(a)(t) \right\|_{\mathcal{L}(\mathcal{H}_N)} = \left\| \mathrm{Op}_N^+(a \circ A^t) - \mathrm{Op}_N^w(a \circ A^t) \right\|_{\mathcal{L}(\mathcal{H}_N)} + O_a(N^{-\gamma}),$$

where $\mathrm{Op}_N^+(a)(t) := M_N(A)^{-t} \mathrm{Op}_N^+(a) M_N(A)^t$. From the decomposition of the space $L^2(\mathbb{R}^d)$ along the spaces $\mathcal{H}_N(\kappa)$, we know that

$$\left\| \mathrm{Op}_N^+(a \circ A^t) - \mathrm{Op}_N^w(a \circ A^t) \right\|_{\mathcal{L}(\mathcal{H}_N)} \leq \left\| \mathrm{Op}_\hbar^+(a \circ A^t) - \mathrm{Op}_\hbar^w(a \circ A^t) \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}.$$

Recall that we know that, for a bounded symbol b, $\mathrm{Op}_\hbar^+(b)$ is equal to the operator $\mathrm{Op}_\hbar^w(b \star (G_\hbar \sharp G_\hbar))$ and that $b \star (G_\hbar \sharp G_\hbar)(\rho) = \int_{\mathbb{R}^{2d}} b(\rho + B(\hbar)^{-1}\rho_0) K_\hbar(\rho_0) d\rho_0$ (see paragraph 4.2 and appendix A). We write this formula for $b = a \circ A^t$ and combine it with the Taylor formula. We find that

$$(a \circ A^t) \star (G_\hbar \sharp G_\hbar)(\rho) = a \circ A^t(\rho) + \int_{\mathbb{R}^{2d}} K_\hbar(\rho_0) \int_0^1 \left(d_{\rho + sB(\hbar)^{-1}\rho_0} a \right) . (A^t B(\hbar)^{-1}) \rho_0 ds d\rho_0.$$

Appendix A gives us an exact expression for $K_\hbar$. We can compute the derivatives of the second term of the sum and according to Calderón-Vaillancourt theorem (see paragraph 2.2), we finally find that

$$\left\| \mathrm{Op}_N^+(a \circ A^t) - M_N(A)^{-t} \mathrm{Op}_N^+(a) M_N(A)^t \right\|_{\mathcal{L}(\mathcal{H}_N)} = O_a\left(\|A^t B(\hbar)^{-1}\|_\infty \right), \tag{24}$$

where $\|A^t B(\hbar)^{-1}\|_\infty$ is the supremum of the moduli of the coefficients of $A^t B(\hbar)^{-1}$. By construction, $B(\hbar)$ was constructed to be adapted to the classical dynamics induced by A and we know that, using the notations of section 3,

$$A^t B(\hbar) = Q \left((A_0^t \hbar^{\frac{\epsilon_0}{2\lambda_{\max}}}) \diamond (A_1^t \hbar^{\frac{\lambda_1}{2\lambda_{\max}}}) \diamond \cdots \diamond (A_0^t \hbar^{\frac{\lambda_1}{2\lambda_{\max}}}) \right) Q^{-1}.$$

We can verify that these two last equalities allows to conclude the proof of proposition 4.3.□

5. Proof of theorem 1.1

We consider a semiclassical measure μ. Without loss of generality, we can suppose that it is constructed from Op_N^+ and that it is associated to a sequence of eigenvectors ψ_{N_k} of $M_{N_k}(A)$ in $\mathcal{H}_{N_k}$ where $(N_k)_k$ is an increasing sequence of integers. Precisely, we have

$$\forall\ a \in \mathcal{C}^\infty(\mathbb{T}^2,\mathbb{C}),\ \mu(a) = \lim_{k\to+\infty}\langle\psi_{N_k}|\mathrm{Op}_{N_k}^+(a)|\psi_{N_k}\rangle_{\mathcal{H}_{N_k}}.$$

We recall that we have denoted $\mu^{N_k}(a) = \langle\psi_{N_k}|\mathrm{Op}_{N_k}^+(a)|\psi_{N_k}\rangle_{\mathcal{H}_{N_k}}$. To simplify notations, we will not mention k in the following of this article. We start our proof by fixing a finite measurable partition $\mathcal{Q} := \{Q_1,\cdots,Q_K\}$ of small diameter δ_0 whose boundary is not charged[3] by μ (paragraph 2.2.8 in [3]). We denote $\eta(x) := -x\log x$ (with the convention $0\log 0 = 0$). We recall that the Kolmogorov-Sinai entropy of the measure μ for the partition $\mathcal{Q}$ can be defined as [29]

$$h_{KS}(\mu,A,\mathcal{Q}) := \lim_{m\to+\infty}\frac{1}{2m}\sum_{|\alpha|=2m}\eta\left(\mu\left(A^mQ_{\alpha_{-m}}\cdots\cap A^{-(m-1)}Q_{\alpha_{m-1}}\right)\right),$$

where α_j varies in $\{1,\cdots,K\}$ (K is the cardinal of $\mathcal{Q}$).

5.1. **Using the entropic uncertainty principle.** Our quantization is defined for smooth observables on the torus. So we start by defining a smoothing of the partition $\mathcal{Q}$: it is defined by a family $(P_i)_{i=1}^K$ of smooth observables in $\mathcal{C}^\infty(\mathbb{T}^{2d},[0,1])$ (of small support of diameter less than 2δ) that satisfies the following property of partition of $\mathbb{T}^{2d}$:

$$\forall\rho\in\mathbb{T}^{2d},\ \sum_{i=1}^K P_i^2(\rho) = 1. \tag{25}$$

Mimicking the definition of Kolmogorov-Sinai entropy, we define the quantum entropy of ψ_N with respect to $\mathcal{P}$:

$$h_{2m}(\psi_N,\mathcal{P}) := -\sum_{|\alpha|=2m}\mu^N(\mathbf{P}_\alpha^2)\log\mu^N(\mathbf{P}_\alpha^2), \tag{26}$$

where $\mathbf{P}_\alpha := \prod_{j=-m}^{m-1}P_{\alpha_j}\circ A^j$ for $\alpha := (\alpha_{-m},\cdots,\alpha_{m-1})$. One can verify that for any fixed m, we have

$$h_{2m}(\mu,\mathcal{P}) := -\sum_{|\alpha|=2m}\mu(\mathbf{P}_\alpha^2)\log\mu(\mathbf{P}_\alpha^2) = \lim_{N\to\infty}h_{2m}(\psi_N,\mathcal{P}). \tag{27}$$

So, for a fixed m and as $N\to\infty$, the quantum entropy we have just defined tends to the usual entropy of μ at time $2m$ (with the notable difference that we consider smooth partitions). Our crucial observation to apply the entropic uncertainty principle is that we have the following partition of identity for $\mathcal{H}_N$:

$$\sum_{|\alpha|=2m}\int_{\mathbb{T}^{2d}}\int_{\mathbb{T}^{2d}}\mathbf{P}_\alpha^2(\rho_0)\mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))^*\circ\mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))d\theta d\rho_0 = \mathrm{Id}_{\mathcal{H}_N}. \tag{28}$$

This partition of identity is derived from equation (21) and is crucial to apply the entropic uncertainty principle (6). Moreover, this partition looks more like a classical partition as it is defined by the quantity $\mathbf{P}_\alpha^2$ and it will also make the computation in the entropic uncertainty principle easier. This uncertainty principle can be applied for $\mathcal{H} = L^2(\mathbb{T}^{4d},\mathcal{H}_N)$ and $\tilde{\mathcal{H}} = \mathcal{H}_N$. For (θ,ρ_0) in $\mathbb{T}^{4d}$ and ψ in $\mathcal{H}_N$, we define

$$\pi_\alpha|\psi\rangle(\theta,\rho_0) := \mathbf{P}_\alpha(\rho_0)\mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))|\psi\rangle.$$

This defines a linear application from $\mathcal{H}_N$ to $L^2(\mathbb{T}^{4d},\mathcal{H}_N)$ and its adjoint is given by

$$\pi_\alpha^\dagger f := \int_{\mathbb{T}^{4d}}\mathbf{P}_\alpha(\rho_0)\mathrm{Op}_N^w(T_\theta(G_\hbar^{\rho_0}))^*f(\theta,\rho_0)d\theta d\rho_0,$$

[3]The parameter δ_0 is small and fixed for all the article: it has no vocation to tend to 0.

for f in $L^2(\mathbb{T}^{4d},\mathcal{H}_N)$. It defines a quantum partition of identity as it satisfies the relation $\sum_{|\alpha|=2m}\pi_\alpha^\dagger\pi_\alpha=\mathrm{Id}_{\mathcal{H}_N}$. Applying the entropic uncertainty principle for this partition and $U=M_N(A)^n$, we bound $\|\pi_\alpha M_N(A)^n\pi_\beta^\dagger\|_{\mathcal{L}(L^2(\mathbb{T}^{4d},\mathcal{H}_N))}$ and derive the following corollary:

Corollary 5.1. *Using the previous notations, one has*

$$\forall n\in\mathbb{N},\ \forall m\in\mathbb{N},\ h_{2m}(\psi_N,\mathcal{P})\geq-\log\sup_{|\alpha|=2m}\{Leb(\mathbf{P}_\alpha^2)\}-\log c(A,n),$$

where $c(A,n):=\sup_{\theta,\theta',\rho_0,\rho_0'\in\mathbb{T}^{2d}}\left\{\left\|Op_N^w(T_\theta(G_\hbar^{\rho_0}))M_N(A)^n\,Op_N^w(T_{\theta'}(G_\hbar^{\rho_0'}))^*\right\|_{\mathcal{L}(\mathcal{H}_N)}\right\}.$

5.2. **Estimate of $c(A,n)$.** In section 6, we will prove the following theorem:

Theorem 5.2. *Let A be a* quantizable *matrix and let ϵ be some (small) positive number. For every positive δ (small enough), there exists a constant C such that, for $n:=n_E(\hbar)=[(1-\epsilon)|\log\hbar|/\lambda_{\max}]$,*

$$c(A,n)\leq C|\det B(\hbar)|\hbar^{-\delta-\frac{\Lambda_+}{\lambda_{\max}}\epsilon}e^{-n_E(\hbar)\Lambda_0},$$

where Λ_+ and Λ_0 depend on the Lyapunov exponents of A and were defined in section 3 relations (15) and (16)).

Recall that we have the relation $2\pi\hbar N=1$. If we consider $\delta\ll\epsilon$, then the quantum entropy at time $2m$ is bounded from below as follows:

$$\forall m\geq1,\ h_{2m}(\psi_N,\mathcal{P})\geq\frac{\log N}{\lambda_{\max}}\left((\Lambda_0-\Lambda_+)(1-2\epsilon)-d_0\epsilon_0\right)-\log\sup_{|\alpha|=2m}\{\mathrm{Leb}(\mathbf{P}_\alpha^2)\}+\tilde{C}. \tag{29}$$

The quantity $\frac{\log N}{\lambda_{\max}}((\Lambda_0-\Lambda_+)(1-2\epsilon)-d_0\epsilon_0)$ comes from the $\hbar$ term in the upper bound of theorem 5.2. This estimate is our main simplification compared with [3] as it will only use estimates of gaussian integrals. We underline that this theorem plays a crucial role in our proof[4] as it replaces all the discussion of section 3 in [3].

5.3. **Subadditivity of the quantum entropy.** Now, we have to find a time m for which inequality (29) is optimal. It will depend on N and the last difficulty is that if $m(N)$ grows too fast with N, $h_{2m(N)}(\psi_N,\mathcal{P})$ has no particular reason to tend to $h_{KS}(\mu,A,\mathcal{P})$ in the semiclassical limit. We have to be careful and we first verify that classical arguments from ergodic theory (subadditivity of the entropy) can be adapted for the quantum entropy as long as $m\leq\log N/(2\lambda_{\max})$. In particular, we prove that the sequence $\frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})$ is 'almost' decreasing until the Ehrenfest time (see appendix C):

Lemma 5.3. *We fix an integer m_0. We denote $m_E(N)=[(1-\epsilon)\log N/(2\lambda_{\max})]$ and we have then*

$$\frac{1}{2m_E(N)}h_{2m_E(N)}(\psi_N,\mathcal{P})\leq\frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})+R(m_0,N),$$

where $R(m_0,N)$ is a remainder that satisfies $\lim_{N\to\infty}R(m_0,N)=0$.

Combining this lemma with the entropic estimation (29), we have, for every fixed $m_0>0$,

$$\frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})+\tilde{R}(m_0,N)\geq\left((\Lambda_0-\Lambda_+)-\frac{d_0\epsilon_0}{1-2\epsilon}\right)-\frac{1}{2m_E(N)}\log\sup_{|\alpha|=2m_E(N)}\{\mathrm{Leb}(\mathbf{P}_\alpha^2)\}. \tag{30}$$

where $\tilde{R}(m_0,N)$ is a remainder that satisfies $\forall m_0\in\mathbb{N},\ \lim_{N\to\infty}\tilde{R}(m_0,N)=0$.

[4] The other term on the lower bound will be estimate thanks to the computation of the entropy of the Lebesgue measure.

5.4. **The conclusion.** To conclude, it remains to bound the quantity $\sup_{|\alpha|=2m_E(N)}\{\text{Leb}(\mathbf{P}_\alpha^2)\}$. To do this, we underline that, for each α of length $2m$,

$$\forall x\in\text{supp}(\mathbf{P}_\alpha^2),\ \text{Leb}(\mathbf{P}_\alpha^2)\leq\text{Leb}(\text{supp}(\mathbf{P}_\alpha^2))\leq\text{Leb}(B(x,2\delta_0,2m)),$$

where $B(x,2\delta_0,2m)$ is the Bowen ball given by $\{y\in\mathbb{T}^{2d}:\forall j\in[-m,m-1],\ d(A^jx,A^jy)<2\delta_0\}$, where d is the metric induced on $\mathbb{T}^{2d}$ by the Euclidean norm on $\mathbb{R}^{2d}$. By induction and using the invariance of the metric d, we know that for every x in $\mathbb{T}^{2d}$ and for every k in $\mathbb{Z}$, $A^{-k}B(A^kx,2\delta_0)=x+A^{-k}B(0,2\delta_0)$. Then, using the invariance by translation of the Lebesgue measure, we know that for every x in $\mathbb{T}^{2d}$, $\text{Leb}(B(x,2\delta_0,2m))=\text{Leb}(B(0,2\delta_0,2m))$. Combining [8] and theorem 8.15 from [29], we know that $\text{Leb}(B(0,2\delta_0,2m))\leq C_{\delta_0}e^{-2m(\Lambda_+-\epsilon)}$ (to avoid too many small parameters, we choose the same ϵ as before). We use this last inequality and we make N tends to infinity in (30). It gives, for every positive m_0,

$$\frac{1}{2m_0}h_{2m_0}(\mu,\mathcal{P})\geq\Lambda_0-\frac{d_0\epsilon_0}{1-2\epsilon}-\epsilon.$$

This last inequality holds for all (small enough) smoothing $\mathcal{P}$ of the partition $\mathcal{Q}$. The lower bound does not depend on the derivatives of $\mathcal{P}$ so we we can replace the smooth partition $\mathcal{P}$ by the true partition $\mathcal{Q}$ in the definition of $h_{2m_0}(\mu,\mathcal{P})$. We let m_0 tends to infinity, then ϵ to 0 and finally ϵ_0 to 0. We find

$$h_{KS}(\mu,A)\geq h_{KS}(\mu,A,\mathcal{Q})\geq\Lambda_0.\square$$

6. The main estimate: proof of theorem 5.2

In this section, we want to prove theorem 5.2, i.e. give an estimate of $c(A,n)$. We underline that the spirit of the proof will be similar to the proof of estimates on the propagation of coherent states under the quantum propagator [5], [13].

First, we use exact Egorov property for the Weyl quantization and we find

$$c(A,n):=\sup_{\theta,\theta',\rho_0,\rho_0'\in\mathbb{T}^{2d}}\left\{\left\|\text{Op}_N^w\left(T_\theta(G_\hbar^{\rho_0})\circ A^{\frac{n}{2}}\right)\text{Op}_N^w\left(\overline{T_{\theta'}(G_\hbar^{\rho_0'})}\circ A^{-\frac{n}{2}}\right)\right\|_{\mathcal{L}(\mathcal{H}_N)}\right\}.$$

As $\text{Op}_N^w(a)$ is the restriction of $\text{Op}_\hbar^w(a)$ to $\mathcal{H}_N$ and using the decomposition of $L^2(\mathbb{R}^d)$ along the $\mathcal{H}_N(\kappa)$, we know that

$$c(A,n)\leq\sup_{\theta,\theta',\rho_0,\rho_0'\in\mathbb{T}^{2d}}\left\{\left\|\text{Op}_\hbar^w\left(T_\theta\left(G_\hbar^{\rho_0}\right)\circ A^{\frac{n}{2}}\right)\text{Op}_\hbar^w\left(\overline{T_{\theta'}(G_\hbar^{\rho_0'})}\circ A^{-\frac{n}{2}}\right)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\right\}.\tag{31}$$

6.1. **Strategy of the proof.** For the sake of simplicity, we draw our strategy in the case where $\theta=\theta'=\rho_0=\rho_0'=0$. In this case, we have to consider two symbols which are defined as infinite sums over $\mathbb{Z}^{2d}$

$$T_0(G_\hbar)\circ A^{\frac{n}{2}}(\rho)=\sum_{r\in\mathbb{Z}^{2d}}G_\hbar(A^{\frac{n}{2}}\rho+r)\text{ and }T_0(G_\hbar)\circ A^{-\frac{n}{2}}(\rho)=\sum_{r\in\mathbb{Z}^{2d}}G_\hbar(A^{-\frac{n}{2}}\rho+r).$$

In the case where $n=0$, we can observe that these two observables are in the class $S_{\frac{1}{2}}^{(\Lambda_++d_0\epsilon_0)/(2\lambda_{\max})}(1)$ and so they are amenable to standard symbol calculus. The situation becomes more complicated for n large of order $|\log\hbar|/\lambda_{\max}$. The observables $T_0(G_\hbar)\circ A^{\frac{n}{2}}$ and $T_0(G_\hbar)\circ A^{-\frac{n}{2}}$ are sums of Gaussian observables centered on ellipsoids whose lengths in the different directions are in an interval $[\hbar^{1-\nu},\hbar^\nu]$ with $\nu\ll1$. So the symbols we consider are not nice symbols amenable to the usual symbol calculus of [11] as derivatives can explode with a rate of order $\hbar^{-1+\nu}$ in some directions.

Moreover, $T_0(G_\hbar)\circ A^{\frac{n}{2}}$ and $T_0(G_\hbar)\circ A^{-\frac{n}{2}}$ are defined as sums of Gaussian observables centered on the lattice $\mathbb{Z}^{2d}$. Thanks to these two observations, when we will do the product of the two corresponding operators, two terms of the sum which are not centered on the same element of $\mathbb{Z}^{2d}$

will have a contribution $\mathcal{O}(\hbar^{\infty})$ (section 6.3). The main contribution in the norm of the operator will then come from the operator

$$\sum_{r\in\mathbb{Z}^{2d}} U_{\hbar}(r)^*\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{\frac{n}{2}}\right)\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{-\frac{n}{2}}\right)U_{\hbar}(r).$$

Again, this operator is defined as an infinite sum and each operator in the sum is located at a different point of the lattice $\mathbb{Z}^{2d}$. Moreover, the symbols of these operators are located in ellipsoids of small size $\hbar^{\nu}$. Combining these two observations to the Coltar-Stein theorem (section 6.4), we will prove that the main contribution in the norm of the operator is given by the norm of $\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{\frac{n}{2}}\right)\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{-\frac{n}{2}}\right)$. The reduction of the proof to the estimation of this operator norm is explained in sections 6.3 and 6.4. The strategy follows standard estimations from semi-classical analysis [11], [12]. The main difference with usual computations comes from the fact that we have to deal with functions that do not belong to nice class of symbols and we have to verify that the particular form of these functions allows to draw the same conclusions.

So the main difficulty will be to give an upper bound on $\left\|\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{\frac{n}{2}}\right)\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\circ A^{-\frac{n}{2}}\right)\right\|$ (section 6.2). To do this, we will do a precise analysis of the kernel of this operator. Precisely, we can introduce the function

$$\mathcal{K}_{\hbar}^{(n)}(x,y,z):=\frac{1}{(2\pi\hbar)^d}\left(\int_{\mathbb{R}^d}e^{\frac{\imath}{\hbar}\langle x-y,\xi\rangle}G_{\hbar}\circ A^{\frac{n}{2}}\left(\frac{x+y}{2},\xi\right)d\xi\right)\left(\int_{\mathbb{R}^d}e^{\frac{\imath}{\hbar}\langle y-z,\eta\rangle}G_{\hbar}\circ A^{-\frac{n}{2}}\left(\frac{y+z}{2},\eta\right)d\eta\right).$$

We have to understand the norm of the operator of kernel $\int_{\mathbb{R}^d}\mathcal{K}_{\hbar}^{(n)}(x,y,z)dy$. This can be done by a careful analysis of $\mathcal{K}_{\hbar}^{(n)}(x,y,z)$ (i.e. understand where it is negligible or not): it will be the subject of section 6.2.

6.2. **The leading term.** We start our estimate by giving a bound on the term centered on $(0,0)$ in $\mathbb{Z}^{4d}$. It means that we will look at the norm of the operator

(32)
$$\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\left(A^{\frac{n}{2}}\bullet-\pi\hbar J\theta-\rho_0\right)e^{2\imath\pi\langle A^{\frac{n}{2}}\bullet-\rho_0|\theta\rangle}\right)\mathrm{Op}_{\hbar}^w\left(G_{\hbar}\left(A^{-\frac{n}{2}}\bullet-\pi\hbar J\theta'-\rho_0'\right)e^{2\imath\pi\langle A^{-\frac{n}{2}}\bullet-\rho_0'|\theta'\rangle}\right)^*,$$

where $\theta,\theta',\rho_0,\rho_0'\in\mathbb{T}^{2d}$. Precisely, we will prove in this paragraph the following proposition:

Proposition 6.1. *Let θ, θ', ρ_0 and ρ_0' be elements in $\mathbb{T}^{2d}$. Let ϵ be some (small) fixed positive number. Then, for every positive δ and for $n=n_E(\hbar):=[(1-\epsilon)|\log\hbar|/\lambda_{\max}]$, one has*

$$\|(32)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C|\det B(\hbar)|\hbar^{-\delta-\epsilon\frac{\Lambda_+}{\lambda_{\max}}}\exp\left(-\sum_{i:2\lambda_i^+-\lambda_{\max}>0}d_i\left(\lambda_i^+-\frac{\lambda_{\max}}{2}\right)n_E(\hbar)\right),$$

where $\Lambda_+:=\sum_{i=0}^r d_i\lambda_i^+$ and where the constant C is uniform for θ, θ', ρ_0 and ρ_0' in $\mathbb{T}^{2d}$

Remark. We underline that this estimate is exactly the one of theorem 5.2. In the following sections (6.3 and 6.4), we will verify that the main contribution comes from the term centered in $(0,0)$ only if we restrict ourselves to $0\leq n\leq n_E(\hbar)$. Moreover, it will be clear in the proof that the bound is the smallest possible for $n=n_E(\hbar)$ if we only consider the range of times $0\leq n\leq n_E(\hbar)$.

6.2.1. *First observations.* For simplicity of notations, we introduce two auxiliary matrices

$$A_+(n,\hbar):=Q^{-1}\sqrt{\hbar}B(\hbar)A^{\frac{n}{2}}Q \text{ and } A_-(n,\hbar):=Q^{-1}\sqrt{\hbar}B(\hbar)A^{-\frac{n}{2}}Q.$$

Recall that, using the notations of section 3, we have

$$A_+(n,\hbar)=\left(\hbar^{\frac{\lambda_{\max}-\epsilon_0}{2\lambda_{\max}}}A_0^{\frac{n}{2}}\right)\diamond\left(\hbar^{\frac{\lambda_{\max}-\lambda_1^+}{2\lambda_{\max}}}A_1^{\frac{n}{2}}\right)\cdots\diamond\left(\hbar^{\frac{\lambda_{\max}-\lambda_r^+}{2\lambda_{\max}}}A_r^{\frac{n}{2}}\right)$$

and

$$A_-(n,\hbar)=\left(\hbar^{\frac{\lambda_{\max}-\epsilon_0}{2\lambda_{\max}}}A_0^{-\frac{n}{2}}\right)\diamond\left(\hbar^{\frac{\lambda_{\max}-\lambda_1^+}{2\lambda_{\max}}}A_1^{-\frac{n}{2}}\right)\cdots\diamond\left(\hbar^{\frac{\lambda_{\max}-\lambda_r^+}{2\lambda_{\max}}}A_r^{-\frac{n}{2}}\right).$$

We would like now to write the norm we have to estimate into a simpler form using these new notations. First, we underline that if we define

$$V_\hbar u(x) := \hbar^{\frac{d}{4}} u(\sqrt{\hbar}x),$$

then, for any bounded operators $\mathrm{Op}_\hbar^w(a)$ and $\mathrm{Op}_\hbar^w(b)$ on $L^2(\mathbb{R}^d)$, one has

$$\|\mathrm{Op}_\hbar^w(a)\mathrm{Op}_\hbar^w(b)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \left\|\mathrm{Op}_1^w(a\circ(\sqrt{\hbar}\mathrm{Id}_{\mathbb{R}^{2d}}))\mathrm{Op}_1^w(b\circ(\sqrt{\hbar}\mathrm{Id}_{\mathbb{R}^{2d}}))\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}.$$

Moreover, we can use that the matrix Q is an element of $Sp(2d,\mathbb{R})$. In particular, its quantization $M(Q)$ (via the metaplectic representation) satisfies an exact Egorov property [11]. Using these two observations and defining $\tilde{\Gamma}_\theta(w) := G\circ Q(w)e^{2\imath\pi\langle w|\theta\rangle}$, we can deduce that the norm of the operator (32) is bounded by

(33)
$$2^d|\det B(\hbar)|\sup_{\theta,\theta'\in[-M,M]^{2d};\rho_0,\rho_0'\in\mathbb{R}^{2d}}\left\|\mathrm{Op}_1^w\left(\tilde{\Gamma}_\theta\left(A_+(n,\hbar)\bullet-\rho_0\right)\right)\mathrm{Op}_1^w\left(\tilde{\Gamma}_{\theta'}\left(A_-(n,\hbar)\bullet-\rho_0'\right)\right)^*\right\|_{\mathcal{L}(L(\mathbb{R}^d))},$$

where M is a constant depending only on Q. Initially, we underline that the parameter θ was varying in $[0,1]^{2d}$ and with our change of variables, it has to vary in $Q^*[0,1]^{2d}$ which remains in a fixed compact set $[-M,M]^{2d}$. This is important to underline that this parameter θ can not be arbitrarly large in $\hbar$ as when we will derive $\tilde{\Gamma}_\theta$, the upper bounds in the norm of the derivatives will depend on the norm of θ.

6.2.2. *Study of the evolution for positive times.* To study the norm of the previous operator, we will first rewrite the operator $\mathrm{Op}_1^w\left(\tilde{\Gamma}_\theta\left(A_+(n,\hbar)\bullet-\rho_0\right)\right)$ under a more compact form. To do this, define now the Fourier transform of $\tilde{\Gamma}_\theta(A_+(n,\hbar)\bullet-\rho_0)$ along the impulsion variable, i.e. for $\theta:=(\theta^1,\theta^2)\in[-M,M]^{2d}$ and $\rho_0:=(\rho_0^1,\rho_0^2)\in\mathbb{R}^{2d}$,

$$\Gamma_{\theta,\rho_0}^{n,+}(x,\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\tilde{\Gamma}_\theta\left(A_+(n,\hbar)\begin{pmatrix} x\\ y\end{pmatrix}-\rho_0\right)e^{\imath\langle\xi|y\rangle}dy.$$

With this notation, we can rewrite

$$\forall u\in L^2(\mathbb{R}^d),\ \mathrm{Op}_1^w\left(\tilde{\Gamma}_\theta\left(A_+(n,\hbar)\bullet-\rho_0\right)\right)u(x) = \int_{\mathbb{R}^d}\Gamma_{\theta,\rho_0}^{n,+}\left(\frac{x+y}{2},x-y\right)u(y)dy.$$

For future purpose, we need to have a precise estimate on the kernel of this operator. Using the Oseledets decomposition of section 3, we introduce the notation $(x,\xi):=(\tilde{x}_0,\cdots,\tilde{x}_r,\tilde{\xi}_0,\cdots,\tilde{\xi}_r)\in\mathbb{R}^{2d}$ where $(\tilde{x}_i,\tilde{\xi}_i)$ is an element of $\mathbb{R}^{2d_i}$. Recall also that the matrix A_i that appears in the $\diamond$-decomposition of A is of the form $\mathrm{diag}(D_i,D_i^{-1*})$ for $1\leq i\leq r$. Define now, for $1\leq i\leq r$,

$$\tilde{A}_{i,+}(n,\hbar) := \mathrm{diag}\left(\hbar^{\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{\frac{n}{2}},\hbar^{-\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{\frac{n}{2}}\right)$$

and

$$\tilde{A}_{0,+}(n,\hbar) := \mathrm{diag}\left(\hbar^{\frac{1}{2}}\mathrm{Id}_{\mathbb{R}^{d_0}},\hbar^{-\frac{1}{2}}\mathrm{Id}_{\mathbb{R}^{d_0}}\right).$$

In the case $i\geq 1$, these matrices differ from the ones used to define $A_+(n,\hbar)$ as the lower block has been inversed. In the case $i=0$, they are also different because we used Id instead of A_0. They allow us to bound the kernel of the operator and to precise where it is large (or not):

Lemma 6.2. *Let L be a positive integer and δ' be some positive real number. There exists a constant $C_{L,\delta'}>0$ such that for every θ in $[-M,M]^{2d}$ and every $\rho_0:=(\tilde{\rho}_0^{1,0},\cdots,\tilde{\rho}_0^{1,r},\tilde{\rho}_0^{2,0},\cdots,\tilde{\rho}_0^{2,r})\in\mathbb{R}^{2d}$, one has, for every $0\leq n\leq(1-\epsilon)|\log\hbar|/\lambda_{\max}$ and every $(x,\xi)\in\mathbb{R}^{2d}$,*

$$\prod_{i=0}^r\left(1+\left\|\tilde{A}_{i,+}(n,\hbar)\begin{pmatrix}\tilde{x}_i-\tilde{\rho}_0^{1,i}(\hbar)\\ \tilde{\xi}_i\end{pmatrix}\right\|^2\right)^L\left|\Gamma_{\theta,\rho_0}^{n,+}(x,\xi)\right|\leq C_{L,\delta'}\hbar^{-\frac{d+\delta'}{2}}\exp\left(\frac{1}{2}\left(n+\frac{\log\hbar}{\lambda_{\max}}\right)\Lambda_+\right),$$

where $\Lambda_+:=\sum_{i=1}^r d_i\lambda_i^+$, $\tilde{\rho}_0^{1,i}(\hbar)=\hbar^{-\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{-\frac{n}{2}}\tilde{\rho}_0^{1,i}$ for $1\leq i\leq r$ and $\tilde{\rho}_0^{1,0}(\hbar)=0$.

Proof. In order to prove this lemma, one starts from the explicit form of $\Gamma^{n,+}_{\theta,\rho_0}$ and in particular, we write that, for $i\geq 1$,

$$\hbar^{\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}A_i^{\frac{n}{2}}\left(\begin{array}{c}\tilde{x}_i\\ \tilde{y}_i\end{array}\right)-\left(\begin{array}{c}\tilde{\rho}_0^{1,i}\\ \tilde{\rho}_0^{2,i}\end{array}\right)=\left(\begin{array}{c}\hbar^{\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{\frac{n}{2}}\left(\tilde{x}_i-\tilde{\rho}_0^{1,i}(\hbar)\right)\\ \hbar^{\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{-\frac{n}{2}*}\tilde{y}_i\end{array}\right)-\left(\begin{array}{c}0\\ \tilde{\rho}_0^{2,i}\end{array}\right).$$

Regarding this expression, we make the change of variables $\tilde{y}_i'=\hbar^{\frac{\lambda_{\max}-\lambda_i^+}{2\lambda_{\max}}}D_i^{-\frac{n}{2}*}\tilde{y}_i$ in the integral defining $\Gamma^{n,+}_{\theta,\rho_0}$. In the case $i=0$, we put $\tilde{y}_0'=\hbar^{\frac{1}{2}}\tilde{y}_0$. As the function $\tilde{\Gamma}_\theta$ (used in the integral defining $\Gamma^{n,+}_{\theta,\rho_0}$) is in the Schwartz class $\mathcal{S}(\mathbb{R}^{2d})$ with semi-norms uniformly bounded (as θ vary in a uniform compact set), we obtain the following upper bound on the quantity we want to bound in the lemma:

$$C_L\hbar^{-\frac{d}{2}}\exp\left(\frac{1}{2}\left(n+\frac{\log\hbar}{\lambda_{\max}}\right)\Lambda_+\right)\int_{\mathbb{R}^d}\left(1+\left\|A_0^{\frac{n}{2}}\left(\begin{array}{c}\hbar^{\frac{1}{2}}\tilde{x}_0\\ \tilde{y}_0\end{array}\right)\right\|^2\right)^{-2d}\prod_{i=1}^r(1+\|\tilde{y}_i\|^2)^{-2d}dy.$$

In order to bound this integral independently of n, we can use that for every $\epsilon'>0$, there exists a constant $C_{\epsilon'}>0$ such that, for every $(\tilde{x}_0,\tilde{\xi}_0)$ in $\mathbb{R}^{2d_0}$, one has

$$\forall n\geq 0,\ C_{\epsilon'}^{-1}e^{-n\epsilon'}\|(\tilde{x}_0,\tilde{y}_0)\|\leq\left\|A_0^{\frac{n}{2}}\left(\begin{array}{c}\tilde{x}_0\\ \tilde{y}_0\end{array}\right)\right\|\leq C_{\epsilon'}e^{n\epsilon'}\|(\tilde{x}_0,\tilde{y}_0)\|.$$

These estimates allow us to obtain the bounds we need (using also the fact that we restrict ourselves to times n that are at most logarithmic in $\hbar$).□

6.2.3. *Study of the evolution for negative times.* In the previous paragraph, we studied the norm of the operator for positive times. We can also rewrite

$$\forall u\in L^2(\mathbb{R}^d),\ \mathrm{Op}_1^w\left(\tilde{\Gamma}_{\theta'}\left(A_-(n,\hbar)\bullet-\rho_0'\right)\right)u(x)=\int_{\mathbb{R}^d}\overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{x+y}{2},y-x\right)u(y)dy.$$

If we define $\tilde{A}_{i,-}(n,\hbar):=\tilde{A}_{i,+}(-n,\hbar)$ for $0\leq i\leq r$, we also have the following lemma:

Lemma 6.3. *Let L be a positive integer and δ' be some positive real number. There exists a constant $C_{L,\delta'}>0$ such that for every θ in $[-M,M]^{2d}$ and every $\rho_0:=(\tilde{\rho}_0^{1,0},\cdots,\tilde{\rho}_0^{1,r},\tilde{\rho}_0^{2,0},\cdots,\tilde{\rho}_0^{2,r})\in\mathbb{R}^{2d}$, one has, for every $0\leq n\leq(1-\epsilon)|\log\hbar|/\lambda_{\max}$ and every $(x,\xi)\in\mathbb{R}^{2d}$,*

$$\prod_{i=0}^r\left(1+\left\|\tilde{A}_{i,-}(n,\hbar)\left(\begin{array}{c}\tilde{x}_i-\tilde{\rho}_0^{1,i}\\ \tilde{\xi}_i\end{array}\right)\right\|^2\right)^L\left|\Gamma^{-n,+}_{\theta,\rho_0}(x,\xi)\right|\leq C_{L,\delta'}\hbar^{-\frac{d+\delta'}{2}}\exp\left(\frac{1}{2}\left(\frac{\log\hbar}{\lambda_{\max}}-n\right)\Lambda_+\right),$$

where $\Lambda_+:=\sum_{i=1}^r d_i\lambda_i^+$.

6.2.4. *Estimate of the norm.* In order to compute a bound on the norm of the operator, we consider two elements ϕ_1 and ϕ_2 in $L^2(\mathbb{R}^d)$ and we want to estimate

$$C_{\phi_1,\phi_2}(n):=\left\langle\phi_1,\mathrm{Op}_1^w\left(\tilde{\Gamma}_\theta\left(A_+(n,\hbar)\bullet-\rho_0\right)\right)\mathrm{Op}_1^w\left(\tilde{\Gamma}_{\theta'}\left(A_-(n,\hbar)\bullet-\rho_0'\right)\right)^*\phi_2\right\rangle_{L^2(\mathbb{R}^d)}.$$

With the notations of the previous paragraphs, one has

$$C_{\phi_1,\phi_2}(n):=\int_{\mathbb{R}^{3d}}\Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2},x-y\right)\overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2},y-z\right)\overline{\phi}_1(x)\phi_2(z)dxdydz.$$

Fix now some positive (small) number δ and introduce the two following subsets of $\mathbb{R}^{3d}$:

$$X_\delta^+(n):=\left\{(x,y,z):\forall 0\leq i\leq r\left\|\tilde{A}_{i,+}(n,\hbar)\left(\begin{array}{c}\frac{\tilde{x}_i+\tilde{y}_i}{2}-\tilde{\rho}_0^{1,i}\\ \tilde{x}_i-\tilde{y}_i\end{array}\right)\right\|^2\leq\hbar^{-\delta}\right\}$$

and

$$X_\delta^-(n):=\left\{(x,y,z):\forall 0\leq i\leq r\left\|\tilde{A}_{i,-}(n,\hbar)\left(\begin{array}{c}\frac{\tilde{z}_i+\tilde{y}_i}{2}-\tilde{\rho}_0^{'1,i}\\ \tilde{y}_i-\tilde{z}_i\end{array}\right)\right\|^2\leq\hbar^{-\delta}\right\}.$$

This defines two "cylinders": the first one is along the z-direction and the second one along the x-direction. We also define the intersection of these two "cylinders" $X_\delta(n) := X_\delta^+(n) \cap X_\delta^-(n)$ which is of finite volume. Thanks to lemmas 6.2 and 6.3, we know that outside the set $X_\delta(n)$, the kernel of the operator is small in $\hbar$. Precisely, we can prove the following lemma:

Lemma 6.4. *Let δ be a (small) positive real number. Let ϕ_1 and ϕ_2 be two elements in $L^2(\mathbb{R}^d)$. One has, for $0 \leq n \leq (1-\epsilon)|\log \hbar|/\lambda_{\max}$,*

$$\int_{\mathbb{R}^{3d}\backslash X_\delta(n)} \Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right) \overline{\phi}_1(x)\phi_2(z)dxdydz = \mathcal{O}(\hbar^\infty)\|\phi_1\|_{L^2(\mathbb{R}^d)}\|\phi_2\|_{L^2(\mathbb{R}^d)},$$

where the remainder is uniform for $\theta, \theta' \in [-M, M]^{2d}$ and $\rho_0, \rho_0' \in \mathbb{R}^{2d}$.

Proof. Thanks to the Cauchy Schwarz inequality, it is sufficient to give an estimate on

$$\int_{\mathbb{R}^{3d}\backslash X_\delta(n)} \left|\Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right)\right| \left|\overline{\phi}_1(x)\right|^2 dxdydz.$$

According to lemmas 6.2 and 6.3, we know that for every integer L, there exists a constant $C_L > 0$ such that, for every $0 \leq n \leq |\log \hbar|/\lambda_{\max}$ and every (x, y, z) in $\mathbb{R}^{3d}$

$$\left|\Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right)\right|$$

$$\leq C_L \hbar^{-d-\delta} \prod_{i=0}^r \left(1 + \left\|\tilde{A}_{i,+}(n,\hbar)\begin{pmatrix} \frac{\tilde{x}_i+\tilde{y}_i}{2} - \tilde{\rho}_0^{1,i} \\ \tilde{x}_i - \tilde{y}_i \end{pmatrix}\right\|^2\right)^{-L} \prod_{i=0}^r \left(1 + \left\|\tilde{A}_{i,-}(n,\hbar)\begin{pmatrix} \frac{\tilde{z}_i+\tilde{y}_i}{2} - \tilde{\rho}_0'^{1,i} \\ \tilde{y}_i - \tilde{z}_i \end{pmatrix}\right\|^2\right)^{-L}.$$

Under the extra assumption that (x, y, z) in $\mathbb{R}^{3d}\backslash X_\delta$, one knows that

$$\left|\Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right)\right|$$

$$\leq C_L \hbar^{\delta L-d} \prod_{i=0}^r \left(1 + \left\|\tilde{A}_{i,+}(n,\hbar)\begin{pmatrix} \frac{\tilde{x}_i+\tilde{y}_i}{2} - \tilde{\rho}_0^{1,i} \\ \tilde{x}_i - \tilde{y}_i \end{pmatrix}\right\|^2\right)^{-d} \prod_{i=0}^r \left(1 + \left\|\tilde{A}_{i,-}(n,\hbar)\begin{pmatrix} \frac{\tilde{z}_i+\tilde{y}_i}{2} - \tilde{\rho}_0'^{1,i} \\ \tilde{y}_i - \tilde{z}_i \end{pmatrix}\right\|^2\right)^{-d}.$$

In the allowed range of times n one can check that, for some uniform constant D, one has

$$\int_{\mathbb{R}^{2d}} \prod_{i=0}^r \left(1 + \left\|\tilde{A}_{i,-}(n,\hbar)\begin{pmatrix} \frac{\tilde{z}_i+\tilde{y}_i}{2} - \tilde{\rho}_0'^{1,i} \\ \tilde{y}_i - \tilde{z}_i \end{pmatrix}\right\|^2\right)^{-d} dydz = \mathcal{O}(\hbar^{-D}).$$

Combining these last two estimates, we find that, for every $L > 0$ (that can be chosen independently of δ),

$$\int_{\mathbb{R}^{3d}\backslash X_\delta(n)} \left|\Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right)\right| \left|\overline{\phi}_1(x)\right|^2 dxdydz = \mathcal{O}(\hbar^{\delta L-D-d}).\square$$

6.2.5. *The conclusion.* According to the previous paragraph, we know that, modulo a remainder of order $\mathcal{O}(\hbar^\infty)\|\phi_1\|_{L^2(\mathbb{R}^d)}\|\phi_2\|_{L^2(\mathbb{R}^d)}$, the quantity $C_{\phi_1,\phi_2}(n)$ is equal to

$$\int_{X_\delta(n)} \Gamma^{n,+}_{\theta,\rho_0}\left(\frac{x+y}{2}, x-y\right) \overline{\Gamma}^{-n,+}_{\theta',\rho_0'}\left(\frac{z+y}{2}, y-z\right) \overline{\phi}_1(x)\phi_2(z)dxdydz. \tag{34}$$

For the sake of simplicity, we now restrict ourselves to the case[5] $n = n_E(\hbar) = [(1-\epsilon)|\log \hbar|]$. According to lemmas 6.2 and 6.3, one knows that there exists a constant $C > 0$ such that

$$|(34)| \leq C\hbar^{-d-\delta+\frac{\Lambda_+}{\lambda_{\max}}} \int_{X_\delta(n_E(\hbar))} |\phi_1(x)\phi_2(z)|dxdydz.$$

We have then

$$|(34)| \leq C\hbar^{-d-\delta+\frac{\Lambda_+}{\lambda_{\max}}} \left(\int_{X_\delta(n_E(\hbar))} |\phi_1(x)|^2 dxdydz\right)^{\frac{1}{2}} \left(\int_{X_\delta(n_E(\hbar))} |\phi_2(z)|^2 dxdydz\right)^{\frac{1}{2}}, \tag{35}$$

[5]The reader can check that the bound is optimal in this case.

We will estimate these two integrals and to do this, we will distinguish two cases:

- the indices i such that $2\lambda_i^+ - \lambda_{\max} > 0$;
- the indices i such that $2\lambda_i^+ - \lambda_{\max} \leq 0$.

The case $2\lambda_i^+ - \lambda_{\max} > 0$**.** Define, for i such that $2\lambda_i^+ - \lambda_{\max} > 0$ and $\tilde{x}_i$ in $\mathbb{R}^{d_i}$, the sets

$$X_\delta^+(\tilde{x}_i, n_E(\hbar)) := \left\{ (\tilde{y}_i, \tilde{z}_i) : \left\| \tilde{A}_{i,+}(n_E(\hbar), \hbar) \begin{pmatrix} \frac{\tilde{x}_i + \tilde{y}_i}{2} - \tilde{\rho}_0^{1,i} \\ \tilde{x}_i - \tilde{y}_i \end{pmatrix} \right\|^2 \leq \hbar^{-\delta} \right\}$$

and

$$X_\delta^-(\tilde{x}_i, n_E(\hbar)) := \left\{ (\tilde{y}_i, \tilde{z}_i) : \left\| \tilde{A}_{i,-}(n_E(\hbar), \hbar) \begin{pmatrix} \frac{\tilde{z}_i + \tilde{y}_i}{2} - \tilde{\rho}_0^{'1,i} \\ \tilde{y}_i - \tilde{z}_i \end{pmatrix} \right\|^2 \leq \hbar^{-\delta} \right\}.$$

The set $X_\delta^+(\tilde{x}_i, n_E(\hbar))$ is of infinite volume and the set $X_\delta^-(\tilde{x}_i, n_E(\hbar))$ has a larger volume than the intersection $X_\delta^+(\tilde{x}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{x}_i, n_E(\hbar))$. In particular, for every $\tilde{x}_i$ in $\mathbb{R}^{d_i}$, the optimal bound on the volume is obtained by using the relations on $\tilde{x}_i - \tilde{y}_i$ and on $\tilde{y}_i - \tilde{z}_i$ and it is given by

$$\mathrm{Vol}\left(X_\delta^+(\tilde{x}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{x}_i, n_E(\hbar))\right) \leq \tilde{C}\hbar^{-d_i\delta}\hbar^{d_i\left(1 - \frac{\lambda_i^+}{\lambda_{\max}}\right)},$$

where $\tilde{C}$ is some uniform constant. If we do the same thing but exchange the roles played by $\tilde{x}_i$ and $\tilde{z}_i$, we can introduce the sets

$$X_\delta^+(\tilde{z}_i, n_E(\hbar)) := \left\{ (\tilde{x}_i, \tilde{y}_i) : \left\| \tilde{A}_{i,+}(n_E(\hbar), \hbar) \begin{pmatrix} \frac{\tilde{x}_i + \tilde{y}_i}{2} - \tilde{\rho}_0^{1,i} \\ \tilde{x}_i - \tilde{y}_i \end{pmatrix} \right\|^2 \leq \hbar^{-\delta} \right\}$$

and

$$X_\delta^-(\tilde{z}_i, n_E(\hbar)) := \left\{ (\tilde{x}_i, \tilde{y}_i) : \left\| \tilde{A}_{i,-}(n_E(\hbar), \hbar) \begin{pmatrix} \frac{\tilde{z}_i + \tilde{y}_i}{2} - \tilde{\rho}_0^{'1,i} \\ \tilde{y}_i - \tilde{z}_i \end{pmatrix} \right\|^2 \leq \hbar^{-\delta} \right\}.$$

We verify that the optimal bound on the volume is given by the volume of $X_\delta^+(\tilde{z}_i, n_E(\hbar))$, i.e.

$$\mathrm{Vol}\left(X_\delta^+(\tilde{z}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{z}_i, n_E(\hbar))\right) \leq \tilde{C}\hbar^{-\frac{d_i\delta}{2}} |\det \tilde{A}_{i,+}(n_E(\hbar), \hbar)|^{-1} = \tilde{C}\hbar^{-d_i\delta}\hbar^{d_i\frac{\lambda_i^+}{\lambda_{\max}}}.$$

It is the optimal bound we can get as we have $2\lambda_i^+ - \lambda_{\max} > 0$ (the radii of the "cylinder" $X_\delta^-(\tilde{z}_i, n_E(\hbar))$ explode with $\hbar$ and so provide worst upper bounds on the volume). These estimates will allow us to treat the variables corresponding to indices i such that $2\lambda_i^+ - \lambda_{\max} > 0$ in the right hand side of (35).

The case $2\lambda_i^+ - \lambda_{\max} \leq 0$**.** We now treat the case of the other variables. We fix such a i. We also use the same auxiliary sets for $\tilde{x}_i$ and $\tilde{z}_i$ in $\mathbb{R}^{d_i}$. For every $\tilde{x}_i$ in $\mathbb{R}^{d_i}$, we can verify that, as in the previous case, the optimal bound on the volume is given by

$$\mathrm{Vol}\left(X_\delta^+(\tilde{x}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{x}_i, n_E(\hbar))\right) \leq \tilde{C}\hbar^{-d_i\delta}\hbar^{d_i\left(1 - \frac{\lambda_i^+}{\lambda_{\max}}\right)},$$

where $\tilde{C}$ is some uniform constant. The difference with the previous case is that, as $2\lambda_i^+ - \lambda_{\max} \leq 0$, we can not obtain a better bound in the case of $\tilde{z}_i$. It means that we have, for every $\tilde{z}_i$, the optimal bound on the volume is given by

$$\mathrm{Vol}\left(X_\delta^+(\tilde{z}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{z}_i, n_E(\hbar))\right) \leq \tilde{C}\hbar^{-d_i\delta}\hbar^{d_i\left(1 - \frac{\lambda_i^+}{\lambda_{\max}}\right)}.$$

Combining the different estimates. Using the previous definitions, we have the following inequality

$$\int_{\tilde{X}_\delta(n_E(\hbar))} |\phi_1(x)|^2 dxdydz \leq \int_{\tilde{X}_\delta(n_E(\hbar))} |\phi_1(x)|^2 \prod_{i=0}^{r} \mathrm{Vol}\left(X_\delta^+(\tilde{x}_i, n_E(\hbar)) \cap X_\delta^-(\tilde{x}_i, n_E(\hbar))\right) d\tilde{x}_0 \cdots d\tilde{x}_r.$$

With our previous estimates, we find that

$$\int_{\tilde{X}_\delta(n_E(\hbar))} |\phi_1(x)|^2 dxdydz \leq \tilde{C}^2\hbar^{-d\delta + d - \frac{\Lambda_+}{\lambda_{\max}}} \|\phi_1\|^2_{L^2(\mathbb{R}^d)}.$$

where we recall that $\Lambda_+ := \sum_{i=1}^r d_i\lambda_i^+$. For the other integral, we find that

$$\int_{\tilde{X}_\delta(n_E(\hbar))} |\phi_2(z)|^2 dxdydz \leq \tilde{C}^2\hbar^{-d\delta}\hbar^{\sum_{i:2\lambda_i^+-\lambda_{\max}\leq 0} d_i\left(1-\frac{\lambda_i^+}{\lambda_{\max}}\right)}\hbar^{\sum_{i:2\lambda_i^+-\lambda_{\max}>0} d_i\frac{\lambda_i^+}{\lambda_{\max}}}\|\phi_2\|^2_{L^2(\mathbb{R}^d)}.$$

Finally, using (35), we find that, for $n = n_E(\hbar)$,

$$|(34)| \leq C\tilde{C}^2\hbar^{-(d+1)\delta-\epsilon\frac{\Lambda_+}{\lambda_{\max}}}\exp\left(\sum_{i:2\lambda_i^+-\lambda_{\max}>0} d_i\left(\lambda_i^+ - \frac{\lambda_{\max}}{2}\right)\frac{\log\hbar}{\lambda_{\max}}\right)\|\phi_1\|_{L^2(\mathbb{R}^d)}\|\phi_2\|_{L^2(\mathbb{R}^d)}.\square$$

6.3. **Negligible terms.** In the previous section, we have estimated the term that is supposed to be the leading term in the operator norm. As explained in the strategy of the proof, we will prove in this section that terms not centered at the same point of $\mathbb{Z}^{2d}$ are negligible when estimating the operator norm. First, for simplicity of notations, we introduce the following notations, for $w \in \mathbb{R}^{2d}$,

$$F_+^{(n)}(w) := G\left(B(\hbar)A^{\frac{n}{2}}(w-\rho_0-\pi\hbar\theta)\right)e^{2\imath\pi\langle A^{\frac{n}{2}}w|\theta\rangle}$$

and

$$F_-^{(n)}(w) := G\left(B(\hbar)A^{-\frac{n}{2}}(w-\pi\hbar\theta')\right)e^{-2\imath\pi\langle A^{-\frac{n}{2}}w|\theta'\rangle}.$$

We underline that we have taken $\rho_0' = 0$ without loss of generality (see the expression (31) we want to estimate). Moreover, we can also suppose that ρ_0 is an element in $[-1/2, 1/2]^{2d}$. We now estimate the norm of two translated operators with $r \neq r'$. To do this, we write the exact formula for the Moyal product (see [12]-chapter 4), for r and r' in $\mathbb{Z}^{2d}$,

$$A_{r,r'}(w) := F_+^{(n)}(\bullet+r)\sharp F_-^{(n)}(\bullet+r')(w) = \int_{\mathbb{R}^{4d}} F_+^{(n)}(w+w_1+r)F_-^{(n)}(w+w_2+r')e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}\frac{dw_1dw_2}{(\pi\hbar)^{2d}}.$$

Let $\chi(w_1,w_2)$ be a smooth function on $\mathbb{R}^{4d}$ compactly supported in a small neighborhood of 0. We fix some small positive number ϵ' and we suppose that χ is equal to 1 on the set $\{\|w_1\|_2 \leq \epsilon'$ and $\|w_2\|_2 \leq \epsilon'\}$ and to 0 outside $\{\|w_1\|_2 \leq 2\epsilon'$ and $\|w_2\|_2 \leq 2\epsilon'\}$. Using this cutoff, we can split the integral in two parts

$$A^1_{r,r'}(w) := \int_{\mathbb{R}^{4d}} \chi(w_1,w_2)F_+^{(n)}(w+w_1+r)F_-^{(n)}(w+w_2+r')e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}\frac{dw_1dw_2}{(\pi\hbar)^{2d}}$$

and

$$A^2_{r,r'}(w) := \int_{\mathbb{R}^{4d}} (1-\chi(w_1,w_2))F_+^{(n)}(w+w_1+r)F_-^{(n)}(w+w_2+r')e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}\frac{dw_1dw_2}{(\pi\hbar)^{2d}}.$$

We will now prove that these two symbols are in the class $S^{-\infty}(1)$ with an explicit control on the norm of the derivatives depending on r and r'.

6.3.1. *Class of $A^2_{r,r'}$.* We know that the integral defining $A^2_{r,r'}$ is over variables (w_1,w_2) that satisfy

$$\|w_1\|_2 > \epsilon' \text{ or } \|w_2\|_2 > \epsilon'.$$

Thanks to this last property, we are able to use the (non)-stationary phase property. To do this, we introduce the operators

$$L := \frac{\hbar}{2\imath}\left\langle\frac{w_1}{\|w_1\|_2^2}, Jd_{w_2}\right\rangle \text{ or } L' := -\frac{\hbar}{2\imath}\left\langle\frac{Jw_2}{\|w_2\|_2^2}, d_{w_1}\right\rangle.$$

Using the fact that $L(e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}) = L'(e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}) = e^{-\frac{2\imath}{\hbar}\langle w_1,Jw_2\rangle}$ and performing integration by parts, we find that the observable $A^2_{r,r'}(w)$ is a $\mathcal{O}(\hbar^\infty)$ as long as $0 \leq n \leq \frac{1-\epsilon}{\lambda_{\max}}|\log\hbar|$ (the derivatives of $F_+^{(n)}$ and $F_-^{(n)}$ are bounded by some $\mathcal{O}(\hbar^{-1+\frac{\epsilon}{2}})$ for this range of times). Moreover, we can make other integrations by parts using the operators

$$L_r := \frac{1+\frac{\hbar}{2\imath}\langle w+r, Jd_{w_2}\rangle}{1+\|w+r\|_2^2} \text{ and } L'_{r'} := \frac{1-\frac{\hbar}{2\imath}\langle w+r', d_{w_1}\rangle}{1+\|w+r'\|_2^2}.$$

We verify then that, for every M in $\mathbb{N}$, there exists a constant C_M such that

$$\forall r \neq r' \in \mathbb{Z}^{2d},\ \forall w \in \mathbb{R}^{2d},\ |A^2_{r,r'}(w)| \leq \frac{C_M \hbar^M}{(1+\|w+r'\|_2)^{2d}(1+\|w+r\|_2)^{2d}}.$$

Making the same computations, we find the same properties hold for any derivative of $A^2_{r,r'}$. In particular, we know that the symbol $\sum_{r\neq r'} A^2_{r,r'}$ is in the class $S^{-\infty}(1)$, as long as $0 \leq n \leq \frac{1-\epsilon}{\lambda_{\max}}|\log \hbar|$.

6.3.2. *Class of $A^1_{r,r'}$.* For $\hbar$ small enough and for w_1 on the support of χ, we know that the observable $F^{(n)}_+(w+w_1+r)$ is gaussian and centered on a point in the ball $B(r, 3\epsilon'+1/2)$. Moreover, for w_2 on the support of χ, the other observable $F^{(n)}_-(w+w_2+r')$ is gaussian and centered on a point in the ball $B(r', 3\epsilon')$ (again when $\hbar$ is small enough). As we made the assumption that $r \neq r'$, we also know that $\|r-r'\|_2 \geq 1$. If we restrict ouselves to $0 \leq n \leq (1-\epsilon)|\log\hbar|/\lambda_{\max}$, the variance of the two gaussian observables is of order at most $\mathcal{O}(\hbar^{\epsilon})$. These different observations tell us that the observable $F^{(n)}_+(w+w_1+r)$ is exponentially small in $\hbar$ when $F^{(n)}_-(w+w_2+r')$ is large. The converse is also true. In particular, we know that $|A^1_{r,r'}(w)| = \mathcal{O}(\hbar^{\infty})$ (uniformly for w in $\mathbb{R}^{2d}$). In fact, we can even be more precise and we can verify that, for every $L>0$,

$$(1+\|w+r'\|_2)^{2d}(1+\|w+r\|_2)^{2d}|A^1_{r,r'}(w)| = \mathcal{O}(\hbar^L),$$

where the constant involved is uniform for $r \neq r'$ in $\mathbb{Z}^{2d}$, w in $\mathbb{R}^{2d}$ and $0 \leq n \leq (1-\epsilon)|\log\hbar|/\lambda_{\max}$. Finally, we underline that the same method allows to derive the same on the derivatives of $A^1_{r,r'}$. In particular, the symbol $\sum_{r\neq r'} A^1_{r,r'}$ is in the class $S^{-\infty}(1)$.

6.3.3. *Applying Calderón Vaillancourt theorem.* Using the two previous paragraphs, we know that the symbol $\sum_{r\neq r'} A_{r,r'}$ is in the class $S^{-\infty}(1)$. Thanks to the Calderón Vaillancourt theorem (see equation (11)), we know that, as long as $n \leq \frac{1-\epsilon}{\lambda_{\max}}|\log\hbar|$,

$$\left\|\mathrm{Op}_{\hbar}\left(\sum_{r\neq r'} A_{r,r'}\right)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \mathcal{O}(\hbar^{\infty}).$$

Finally, we can derive that

$$\begin{aligned}&\left\|\mathrm{Op}^w_{\hbar}(T_{\theta}(G_{\hbar}\circ A^{\frac{n}{2}})^{\rho_0})\mathrm{Op}^w_{\hbar}(T_{\theta'}(G_{\hbar}\circ A^{-\frac{n}{2}})^{\rho_0'})^*\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\\ &\qquad = 2^d|\det B(\hbar)|\left\|\mathrm{Op}^w_{\hbar}\left(T_0\left(F^{(n)}_+\sharp F^{(n)}_-\right)\right)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} + \mathcal{O}(\hbar^{\infty}).\end{aligned}$$

6.4. **Applying Coltar-Stein theorem.** To summarize, we have shown that in order to prove theorem 5.2, we only need to get an estimate on the norm of the operator

$$\mathrm{Op}^w_{\hbar}(T_0(F^{(n)}_+\sharp F^{(n)}_-)) = \sum_{r\in\mathbb{Z}^{2d}} U_{\hbar}(r)\mathrm{Op}^w_{\hbar}(F^{(n)}_+\sharp F^{(n)}_-)U_{\hbar}(r)^*,$$

where we used the notations of the previous section. Moreover, proposition 6.1 shows that the norm of $\mathrm{Op}^w_{\hbar}(F_+\sharp F_-)$ is bounded by the expected quantity. It remains to show that these two properties are sufficient to prove the main theorem. To do this, we define $\mathbf{A}_r := U_{\hbar}(r)\mathrm{Op}^w_{\hbar}(F^{(n)}_+\sharp F^{(n)}_-)U_{\hbar}(r)^*$. Our goal is to give a bound on the two following quantities:

$$\sup_r \sum_{r'\in\mathbb{Z}^{2d}} \|\mathbf{A}_r^*\mathbf{A}_{r'}\|^{\frac{1}{2}}_{\mathcal{L}(L^2(\mathbb{R}^d))} \text{ and } \sup_r \sum_{r'\in\mathbb{Z}^{2d}} \|\mathbf{A}_r\mathbf{A}_{r'}^*\|^{\frac{1}{2}}_{\mathcal{L}(L^2(\mathbb{R}^d))}.$$

If we are able to prove that both quantities are bounded by the same quantity C, Coltar-Stein theorem will tell us that C is a bound on the norm of $\mathbf{A} := \sum_{r\in\mathbb{Z}^{2d}} \mathbf{A}_r$ [11]. Regarding this goal, we write

$$\mathbf{A}_r^*\mathbf{A}_{r'} = \mathrm{Op}^w_{\hbar}\left((F^{(n)}_-)^r\right)^* \mathrm{Op}^w_{\hbar}\left(\overline{(F^{(n)}_+)}^r\sharp(F^{(n)}_+)^{r'}\right)\mathrm{Op}^w_{\hbar}\left((F^{(n)}_-)^{r'}\right),$$

where $(F_+^{(n)})^r(\rho) := F_+^{(n)}(\rho+r)$. Proceeding as in paragraph 6.3 and applying Calderón-Vaillancourt theorem, we find that, for every M in $\mathbb{N}$, there exists a constant C_M, such that

$$\forall\ r \neq r',\ \ \|\mathbf{A}_r^*\mathbf{A}_{r'}\|_{\mathcal{L}(L^2(\mathbb{R}^d))}^{\frac{1}{2}} \leq C_M\hbar^M(1+\|r-r'\|_2)^{-2d}.$$

In particular, it implies that, for every $r \in \mathbb{Z}^{2d}$,

$$\sum_{r'\in\mathbb{Z}^{2d}} \|\mathbf{A}_r^*\mathbf{A}_{r'}\|_{\mathcal{L}(L^2(\mathbb{R}^d))}^{\frac{1}{2}} = \left\|\mathrm{Op}_\hbar^w(F_-^{(n)})^*\mathrm{Op}_\hbar^w(F_+^{(n)})^*\mathrm{Op}_\hbar^w(F_+^{(n)})\mathrm{Op}_\hbar^w(F_-^{(n)})\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}^{\frac{1}{2}} + \mathcal{O}(\hbar^\infty).$$

Proposition 6.1 gives a bound on the norm $\mathrm{Op}_\hbar^w(F_+^{(n)})\mathrm{Op}_\hbar^w(F_-^{(n)})$. We find then that, for $n = n_E(\hbar) := [(1-\epsilon)|\log\hbar|/\lambda_{\max}]$,

$$\forall r \in \mathbb{Z}^{2d},\ \sum_{r'\in\mathbb{Z}^{2d}} \|\mathbf{A}_r^*\mathbf{A}_{r'}\|_{\mathcal{L}(L^2(\mathbb{R}^d))}^{\frac{1}{2}} \leq C\hbar^{-\delta-\epsilon\frac{\Lambda_+}{\lambda_{\max}}}\exp\left(-\sum_{i:2\lambda_i^+-\lambda_{\max}>0} d_i\left(\lambda_i^+ - \frac{\lambda_{\max}}{2}\right)n_E(\hbar)\right).$$

By Coltar-Stein theorem (lemma 7.10 in [11]), we can deduce that, for $n = n_E(\hbar)$,

$$\left\|\mathrm{Op}_\hbar^w(T_0(F_+^{(n)}\sharp F_-^{(n)}))\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{-\delta-\epsilon\frac{\Lambda_+}{\lambda_{\max}}}\exp\left(-\sum_{i:2\lambda_i^+-\lambda_{\max}>0} d_i\left(\lambda_i^+ - \frac{\lambda_{\max}}{2}\right)n_E(\hbar)\right).\square$$

Appendix

Appendix A. Proof of lemma 4.1

In this appendix, we give a proof of lemma 4.1. Precisely, we have to verify that the symbols a and $a \star (G_\hbar\sharp G_\hbar)$ have the same principal symbol. Using the definition of the Moyal product [12], we can compute an exact expression of the symbol

$$\begin{aligned} a\star(G_\hbar\sharp G_\hbar)(\rho) &= \int_{\mathbb{R}^{2d}} a(\rho_0)\int_{\mathbb{R}^{4d}} e^{-\frac{2\imath}{\hbar}\sigma(w_1,w_2)}G_\hbar(\rho-\rho_0+w_1)G_\hbar(\rho-\rho_0+w_2)\frac{dw_1dw_2}{(\pi\hbar)^{2d}}d\rho_0 \\ &= \int_{\mathbb{R}^{2d}} a(\rho_0)\int_{\mathbb{R}^{4d}} e^{-2\imath\pi\sigma(w_1,w_2)}G_\hbar(\rho-\rho_0+\sqrt{\pi\hbar}w_1)G_\hbar(\rho-\rho_0+\sqrt{\pi\hbar}w_2)dw_1dw_2d\rho_0 \\ &= \int_{\mathbb{R}^{2d}} a(\rho+B(\hbar)^{-1}\rho_0)K_\hbar(\rho_0)d\rho_0, \end{aligned}$$

where

$$K_\hbar(\rho_0) := \frac{1}{|\det B(\hbar)|}\int_{\mathbb{R}^{4d}} e^{-2\imath\pi\sigma(w_1,w_2)}G_\hbar(\sqrt{\pi\hbar}w_1 - B(\hbar)^{-1}\rho_0)G_\hbar(\sqrt{\pi\hbar}w_2 - B(\hbar)^{-1}\rho_0)dw_1dw_2.$$

We start by computing $\int_{\mathbb{R}^{2d}} e^{-2\imath\pi\sigma(w_1,w_2)}G(\sqrt{\pi\hbar}B(\hbar)w_1-\rho_0)dw_1$. Changing the variables, we find that it is equal to

$$\frac{e^{-2\imath\pi\left\langle Jw_2,(\pi\hbar)^{-\frac{1}{2}}B(\hbar)^{-1}\rho_0\right\rangle}}{|\det B(\hbar)|(\pi\hbar)^d}\int_{\mathbb{R}^{2d}} e^{-2\imath\pi\left\langle w_1,(\pi\hbar)^{-\frac{1}{2}}B(\hbar)^{-1*}Jw_2\right\rangle}G(w_1)dw_1.$$

We find then

$$K_\hbar(\rho_0) = \frac{2^d}{|\det B(\hbar)|}\int_{\mathbb{R}^{2d}} e^{-2\imath\pi\left\langle Jw_2,B(\hbar)^{-1}\rho_0\right\rangle}G\left(B(\hbar)^{-1*}Jw_2\right)G\left(\pi\hbar B(\hbar)w_2-\rho_0\right)dw_2$$

We make a change of variables to find that

$$K_\hbar(\rho_0) = 2^d\int_{\mathbb{R}^{2d}} e^{-2\imath\pi\langle\rho_1,\rho_0\rangle}G\left(\rho_1\right)G\left(\pi\hbar B(\hbar)JB(\hbar)^*\rho_1-\rho_0\right)d\rho_1.$$

This integral defines an observable with a Gaussion shape that could be made explicit using for instance the appendix of [15] but this explicit form is not really simple. In order to verify that a and $a\star(G_\hbar\sharp G_\hbar)$ have the same principal symbol, we write the Taylor formula with integral remainder at the point ρ and find that

$$a\star(G_\hbar\sharp G_\hbar)(\rho) = a(\rho)\int_{\mathbb{R}^{2d}} K_\hbar(\rho_0)d\rho_0 + \int_{\mathbb{R}^{2d}} K_\hbar(\rho_0)\int_0^1\left(d_{\rho+tB(\hbar)^{-1}\rho_0}a\right).(B(\hbar)^{-1})\rho_0dtd\rho_0.$$

We can verify that $\int_{\mathbb{R}^{2d}} K_\hbar(\rho_0)d\rho_0 = 1$. In fact, one has

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} e^{-2\imath\pi\langle\rho_1,\rho_0\rangle} G(\rho_1)\, G(\pi\hbar B(\hbar)JB(\hbar)^*\rho_1-\rho_0)\, d\rho_1 d\rho_0 = \int_{\mathbb{R}^{2d}} G(\rho_1)^2 e^{2\imath\pi\langle\rho_1,A(\hbar)\rho_1\rangle}d\rho_1,$$

where $A(\hbar) := \pi\hbar B(\hbar)JB(\hbar)^*$. We note that $A(\hbar)$ is antisymmetric and we find that $\int_{\mathbb{R}^{2d}} K_\hbar(\rho_0)d\rho_0 = 1$.

Finally, we recall that we have that $\|B(\hbar)^{-1}\|_\infty = O(\hbar^\gamma)$. We have to check that for a polynomial $P(\rho_0)$ independent of $\hbar$, the term $\int_{\mathbb{R}^{2d}} |P|(\rho_0)|K_\hbar|(\rho_0)d\rho_0$ is uniformly bounded independently of $\hbar$. This quantity is bounded by

$$2^d\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} G(\rho_1)\, G(\pi\hbar B(\hbar)JB(\hbar)^*\rho_1-\rho_0)\,|P|(\rho_0)|d\rho_0 d\rho_1.$$

Using the fact that $\|\pi\hbar B(\hbar)JB(\hbar)^*\|_\infty$ is uniformly bounded (as $\lambda_i^+ \leq \lambda_{\max}$), we have the expected property. In particular, we can use Calderón-Vaillancourt theorem (property (11)) to derive that

$$\|\mathrm{Op}_\hbar^w(a) - \mathrm{Op}_\hbar^+(a)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O_a(\hbar^\gamma).\square$$

Appendix B. Proof of proposition 4.2

In this appendix, we prove proposition 4.2 on our quantization Op_κ^+. This proposition was crucial in our proof as it ensures that Op_κ^+ is nonnegative and has the same nice ('product') structure as the anti-Wick quantization. We start the proof of this proposition by computing the n-th Fourier coefficient of $T_0(\overline{F}_1\sharp F_2)$ (where $\sharp$ is the Moyal product of two observables [11]). We show that:

Lemma B.1. *Let F_1 and F_2 be two elements in $\mathcal{S}(\mathbb{R}^{2d})$. Then, we have, for any n in $\mathbb{Z}^{2d}$,*

$$(T_0(\overline{F}_1\sharp F_2))_n := \int_{\mathbb{T}^{2d}} e^{2\imath\pi\langle\rho,Jn\rangle}T_0(\overline{F}_1\sharp F_2)(\rho)d\rho = \left(\int_{\mathbb{R}^{2d}} e^{\frac{\imath\pi}{N}\langle n,\rho\rangle}\hat{\overline{F}}_1(-Jn+\rho)\,\hat{F}_2(-\rho)\,d\rho\right),$$

where $\hat{F}_*(\rho) := \int_{\mathbb{R}^{2d}} F_*(w)e^{-2\imath\pi\langle\rho,w\rangle}dw$ is the standard Fourier transform of F_*.

Proof. Using exact expression of the Moyal product from [12] (see also [11]), we write

$$T_0(\overline{F}_1\sharp F_2)(\rho) = \sum_{r\in\mathbb{Z}^{2d}}\int\int_{\mathbb{R}^{4d}} e^{-2\imath\pi\langle\rho_1,J\rho_2\rangle}\overline{F}_1\left(\frac{\rho_1}{\sqrt{2N}}+\rho+r\right)F_2\left(\frac{\rho_2}{\sqrt{2N}}+\rho+r\right)d\rho_1 d\rho_2.$$

Using Poisson formula, we find that

$$T_0(\overline{F}_1\sharp F_2)(\rho) = \sum_{r\in\mathbb{Z}^{2d}}\left(\int\int\int_{\mathbb{R}^{6d}} e^{-2\imath\pi(\langle\rho_1,J\rho_2\rangle+\langle r,\rho'\rangle)}\overline{F}_1\left(\frac{\rho_1}{\sqrt{2N}}+\rho'\right)F_2\left(\frac{\rho_2}{\sqrt{2N}}+\rho'\right)d\rho_1 d\rho_2 d\rho'\right)e^{2\imath\pi\langle r,\rho\rangle}.$$

We recall that we are interested in the (Jn)-th Fourier coefficient of $T_0(\overline{F}_1\sharp F_2)$. Under the previous form, we immediatly check that

$$(T_0(\overline{F}_1\sharp F_2))_n = \left(\int\int\int_{\mathbb{R}^{6d}} e^{-2\imath\pi(\langle\rho_1,J\rho_2\rangle-\langle Jn,\rho'\rangle)}\overline{F}_1\left(\frac{\rho_1}{\sqrt{2N}}+\rho'\right)F_2\left(\frac{\rho_2}{\sqrt{2N}}+\rho'\right)d\rho_1 d\rho_2 d\rho'\right).$$

We first make the integration into the ρ_2 variable and we find that

$$(T_0(\overline{F}_1\sharp F_2))_n = \left(\int\int_{\mathbb{R}^{4d}} e^{2\imath\pi(\langle Jn,\rho'\rangle-\langle\sqrt{2N}J\rho_1,\rho'\rangle)}\overline{F}_1\left(\frac{\rho_1}{\sqrt{2N}}+\rho'\right)(2N)^d\hat{F}_2\left(-\sqrt{2N}J\rho_1\right)d\rho_1 d\rho'\right).$$

Then, making the integration against the ρ' variable, we find that

$$(T_0(\overline{F}_1\sharp F_2))_n = \left(\int_{\mathbb{R}^{2d}} e^{\frac{2\imath\pi}{\sqrt{2N}}\langle\sqrt{2N}J\rho-Jn,\rho\rangle}\hat{\overline{F}}_1\left(-Jn+\sqrt{2N}J\rho\right)(2N)^d\hat{F}_2\left(-\sqrt{2N}J\rho\right)d\rho\right).$$

An obvious change of variables allows to find

$$(T_0(\overline{F}_1\sharp F_2))_n = \left(\int_{\mathbb{R}^{2d}} e^{\frac{\imath\pi}{N}\langle n,\rho\rangle}\hat{\overline{F}}_1(\rho-Jn)\,\hat{F}_2(-\rho)\,d\rho\right).\square$$

Proof of proposition 4.2. Under the previous form, we can verify that

$$(T_0(\overline{F}_1\sharp F_2))_n = \sum_{r\in\mathbb{Z}^{2d}} \left(\int_{\mathbb{T}^{2d}} e^{\frac{\imath\pi}{N}\langle n,Jr\rangle} e^{\frac{\imath\pi}{N}\langle n-r,\rho\rangle} \hat{\overline{F}}_1\left(\rho - J(n-r)\right) e^{\frac{\imath\pi}{N}\langle r,\rho\rangle} \hat{F}_2\left(-\rho - Jr\right) d\rho\right).$$

We introduce the $\mathbb{Z}^{2d}$ periodic function

$$\tilde{T}_\theta(F_2)(\rho) := \sum_{r\in\mathbb{Z}^{2d}} e^{\frac{\imath\pi}{N}\langle r,\theta\rangle} \hat{F}_2\left(-\theta - Jr\right) e^{-2\imath\pi\langle Jr,\rho\rangle}.$$

Using the Poisson formula, it verifies also

$$\tilde{T}_\theta(F_2)(\rho) = T_\theta(F_2)(\rho) = \sum_{r\in\mathbb{Z}^{2d}} F_2\left(r+\rho-\frac{J\theta}{2N}\right) e^{2\imath\pi\langle r+\rho,\theta\rangle}. \tag{36}$$

With these definitions, we have

$$\overline{T_\theta(F_1)}(\rho) := \sum_{r\in\mathbb{Z}^{2d}} \overline{F}_1\left(r+\rho-\frac{J\theta}{2N}\right) e^{-2\imath\pi\langle r+\rho,\theta\rangle}.$$

Using these new notations, we have shown the following equality which is exactly proposition 4.2:

$$\mathrm{Op}_\hbar^w(T_0(\overline{F}_1\sharp F_2)) = \int_{\mathbb{T}^{2d}} \mathrm{Op}_\hbar^w(T_\theta(F_1))^* \circ \mathrm{Op}_\hbar^w(T_\theta(F_2)) d\theta. \square$$

Appendix C. Proof of lemma 5.3

To complete the proof of theorem 1.1, it remains to prove lemma 5.3. To prove this lemma, we use classical properties of the entropy of a partition [29] (chapter 4) that we briefly prove here (see theorem 4.3 and 4.9 in [29] for details). We fix three integers p, n and m. To simplify our notations, we define the p-translated entropy as follows:

$$h_{2m}^p(\psi_N,\mathcal{P}) := \sum_{|\alpha|=2m} \eta\left(\mu^N(\mathbf{P}_\alpha^2\circ A^p)\right).$$

Mimicking the usual proof for the subadditivity of the entropy of a partition [29] (chapter 4), we write

$$\begin{aligned} h_{2(n+m)}^p(\psi_N,\mathcal{P}) &= -\sum_{|\alpha|=2(n+m)} \mu^N\left(\prod_{j=-m-n}^{n+m-1} P_{\alpha_j}^2\circ A^{j+p}\right) \log \mu^N\left(\prod_{j=-m+n}^{m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right) \\ &+ \sum_{|\alpha|=2(n+m)} \eta\left(\frac{\mu^N\left(\prod_{j=-m-n}^{m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right)}{\mu^N\left(\prod_{j=-m+n}^{m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right)}\right) \mu^N\left(\prod_{j=-m+n}^{m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right). \end{aligned}$$

Using the concavity of the function η and the property of partition of identity (25), we can write the following inequality:

$$h_{2(n+m)}^p(\psi_N,\mathcal{P}) \leq \sum_{|\alpha|=2m} \eta\left(\mu^N\left(\prod_{j=-m+n}^{m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right)\right) + \sum_{|\alpha|=2n} \eta\left(\mu^N\left(\prod_{j=-m-n}^{-m+n-1} P_{\alpha_j}^2\circ A^{j+p}\right)\right).$$

Under a more compact form, it can be reformulated as follows:

Lemma C.1. *Using previous notations, one has*

$$\forall p\in\mathbb{N},\ \forall n\geq 0,\ \forall m\geq 0,\ h_{2(n+m)}^p(\psi_N,\mathcal{P}) \leq h_{2m}^{n+p}(\psi_N,\mathcal{P}) + h_{2n}^{-m+p}(\psi_N,\mathcal{P}). \tag{37}$$

We fix now two integers $m_0 < m$ and write the Euclidean division $m = qm_0 + r$ where $0\leq r < m_0$. We use inequality (37) to derive

$$h_{2m}(\psi_N,\mathcal{P}) \leq h_{2qm_0}^r(\psi_N,\mathcal{P}) + h_{2r}^{-qm_0}(\psi_N,\mathcal{P}).$$

We apply one more time inequality (37) to find

$$h_{2m}(\psi_N,\mathcal{P}) \leq h_{2(q-1)m_0}^{r+m_0}(\psi_N,\mathcal{P}) + h_{2m_0}^{-(q-1)m_0+r}(\psi_N,\mathcal{P}) + h_{2r}^{-qm_0}(\psi_N,\mathcal{P}).$$

By induction, we finally have the following corollary:

Corollary C.2. *Using previous notations, one has*

$$h_{2m}(\psi_N,\mathcal{P}) \leq h_{2r}^{-qm_0}(\psi_N,\mathcal{P}) + \sum_{j=1}^{q} h_{2m_0}^{-(q+1-2j)m_0+r}(\psi_N,\mathcal{P}). \tag{38}$$

Proof of lemma 5.3. This last inequality is true for any integers (m, m_0, r) satisfying $m = qm_0 + r$. We can now give the proof of lemma 5.3. To do this, we fix a positive integer m_0 and consider (q, r) in $\mathbb{N}\times\mathbb{N}$ satisfying $qm_0 + r = m_E(N)$ where $0 \leq r < m_0$. Recall that according to Egorov property (proposition 4.3), one has, for every a in $\mathcal{C}^\infty(\mathbb{T}^2)$,

$$\forall\ |t| \leq m_E(N),\ \mu^N\left(a\circ A^t\right) = \mu^N(a) + o_a(1),\ \text{as } N \to +\infty.$$

We underline that the remainder tends to 0 uniformly for t in the allowed interval. We now apply this property to $\mathbf{P}_\alpha^2$ where $|\alpha| = 2m_0$. Using the continuity of η, we find that

$$\forall\ |t| \leq m_E(N),\ \eta\left(\mu^N\left(\mathbf{P}_\alpha^2\circ A^t\right)\right) = \eta\left(\mu^N(\mathbf{P}_\alpha^2)\right) + o_\alpha(1),\ \text{as } N \to +\infty.$$

As m_0 is fixed, we can deduce from the definition of $h_{2m_0}^p(\psi_N,\mathcal{P})$ that

$$\forall\ |p| \leq m_E(N),\ h_{2m_0}^p(\psi_N,\mathcal{P}) = h_{2m_0}(\psi_N,\mathcal{P}) + o_{m_0}(1),\ \text{as } N \to +\infty.$$

We can apply this result in inequality (38). In this case, one has that $p = -(q+1-2j)m_0+r$ belongs to $[-m_E(N), m_E(N)]$. As $|qm_0| \leq m_E(N)$, we can also write $h_{2r}^{-qm_0}(\psi_N,\mathcal{P}) = h_{2r}(\psi_N,\mathcal{P}) + o_r(1)$ as N tends to infinity. Finally, we find that

$$h_{2m_E(N)}(\psi_N,\mathcal{P}) \leq h_{2r}(\psi_N,\mathcal{P}) + qh_{2m_0}(\psi_N,\mathcal{P}) + (q+1)R'(m_0,N),$$

where $R'(m_0,N)$ is a remainder that satisfies $\forall m_0 \in \mathbb{N},\ \lim_{N\to\infty} R'(m_0,N) = 0$. The conclusion of the lemma follows from this last statement.□

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