SPECTRUM OF A NON-SELFADJOINT QUANTUM STAR GRAPH

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ABSTRACT. We study the spectrum of a quantum star graph with a non-selfadjoint Robin condition at the central vertex. We first prove that, in the high frequency limit, the spectrum of the Robin Laplacian is close to the usual spectrum corresponding to the Kirchhoff condition. Then, we describe more precisely the asymptotics of the difference in terms of the Barra-Gaspard measure of the graph. This measure depends on the arithmetic properties of the lengths of the edges. As a by-product, this analysis provides a Weyl Law for non-selfadjoint quantum star graphs and it gives the asymptotic behaviour of the imaginary parts of the eigenvalues.

1. INTRODUCTION

1.1. **Quantum star graphs.** We consider a compact metric star graph Γ. It is defined, for some integer $N \geqslant 2$, by a set of N edges $e_1, \ldots, e_N$ of respective lengths $\ell_1, \ldots, \ell_N \in \mathbb{R}_+^* = (0, +\infty)$, and by $(N+1)$ vertices $v, v_1, \ldots, v_N$. We set $\ell = (\ell_1, \ldots, \ell_N)$. For each $j \in [\![1, N]\!]$ (we denote by $[\![1, N]\!]$ the set $\{1, 2, \ldots, N\}$), the edge e_j is identified with the interval $[0, \ell_j]$, the end of e_j parametrized by 0 is identified with v_j and the end parametrized by ℓ_j is identified with v. Then all the egdes have a common end (the vertex v).

A quantum star graph is a metric star graph on which we consider a differential operator. One typically considers the Laplacian on each edge with some boundary conditions at the vertices. In this article, we aim at understanding the spectrum of these operators when we impose boundary conditions of *non-selfadjoint* type. More precisely, let $\alpha \in \mathbb{C}$. We consider the set $\mathsf{Dom}(H_\alpha)$ of functions $u = (u_j)_{1 \leqslant j \leqslant N}$ belonging to the Sobolev space $\mathcal{H}^2(\Gamma)$ (each u_j belongs to $\mathcal{H}^2(0, \ell_j)$, see (2.1) below) and satisfying the following conditions (see Figure 1):

(i) u vanishes on $v_1, \ldots, v_N$:
$$\forall j \in [\![1, N]\!], \quad u_j(0) = 0. \tag{1.1}$$

(ii) u is continuous at the vertex v:
$$\forall j, k \in [\![1, N]\!], \quad u_j(\ell_j) = u_k(\ell_k). \tag{1.2}$$

(iii) At v we have a Robin-type condition on the derivatives of u:
$$\sum_{j=1}^{N} u_j'(\ell_j) + \alpha u(v) = 0, \tag{1.3}$$

where $u(v)$ is the common value of $u_j(\ell_j)$, $1 \leqslant j \leqslant N$.

Notice that for $\alpha = 0$ we recover the usual Kirchhoff (or Neumann) condition in (1.3). Then, for $\alpha \in \mathbb{C}$, we define on the domain $\mathsf{Dom}(H_\alpha)$ the operator H_α which maps $u = (u_j)_{1 \leqslant j \leqslant N}$ to

$$H_\alpha u = (-u_j'')_{1 \leqslant j \leqslant N}. \tag{1.4}$$

A more complete definition of this operator will be given in Section 2.1 below. The purpose of this paper is to investigate the spectral properties of H_α. Note that H_α can also be viewed as a Schrödinger type operator with a (possibly complex valued) Dirac potential at the central vertex.
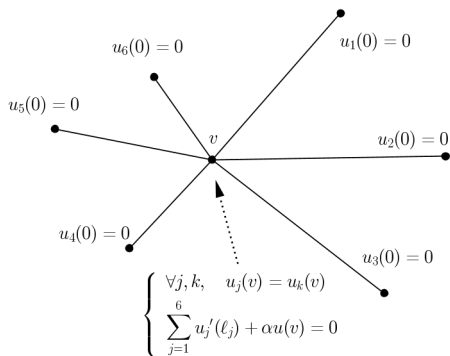

FIGURE 1. The quantum star graph with $N = 6$ edges.

1.2. **Main results.** It is known (see [BK13] for a general overview about quantum graphs) that the spectrum of the selfadjoint operator H_0 is given by a sequence of positive eigenvalues. We denote them by

$$0 < \lambda_1(0) \leqslant \lambda_2(0) \leqslant \ldots \leqslant \lambda_n(0) \leqslant \ldots$$

where each eigenvalue is repeated according to its multiplicity. Then, for $n \in \mathbb{N}^*$, we denote by τ_n the positive square root of $\lambda_n(0)$.

We recall (see for instance [BK13, Lemma 3.7.4] or Remark 2.8 below) that we have the following Weyl Law. If for $R_1 < R_2$ we set

$$\mathcal{N}_0(R_1, R_2) = \#\{n : R_1 < \tau_n \leqslant R_2\},$$

then there exists $C_\Gamma > 0$ independant of R_1 and R_2 such that

$$\left| \mathcal{N}_0(R_1, R_2) - \frac{|\Gamma|}{\pi}(R_2 - R_1) \right| \leqslant C_\Gamma, \tag{1.5}$$

where we have set

$$|\Gamma| = \sum_{j=1}^{N} \ell_j.$$

This gives in particular

$$\frac{\tau_n}{n} \xrightarrow[n\to+\infty]{} \frac{\pi}{|\Gamma|}. \tag{1.6}$$

These properties are also known for any $\alpha \in \mathbb{R}$, but the corresponding proofs strongly rely on the selfadjointness of H_α. Our first purpose here is to verify that the spectrum of the operator H_α has in fact similar general properties for any $\alpha \in \mathbb{C}$. We begin with the nature of the spectrum for a general $\alpha \in \mathbb{C}$:

Proposition 1.1. *For all $\alpha \in \mathbb{C}$, the spectrum of H_α consists of infinitely many isolated eigenvalues with finite multiplicities. Moreover, the geometric multiplicity of each eigenvalue coincides with its algebraic multiplicity.*

Then we turn to the Weyl Law for H_α. The eigenvalues of H_α are denoted by $\lambda_n(\alpha)$, $n \in \mathbb{N}^*$. They are repeated according to their algebraic multiplicities and ordered with the convention that

$$\forall n \in \mathbb{N}^*, \quad \mathrm{Re}(\lambda_n(\alpha)) \leqslant \mathrm{Re}(\lambda_{n+1}(\alpha)).$$

Then, for $n \in \mathbb{N}^*$, we denote by $z_n(\alpha)$ a square root of $\lambda_n(\alpha)$ with non-negative real part. In particular, $z_n(0) = \tau_n$.

For $R_1, R_2 \in \mathbb{R}$ with $R_1 < R_2$ we set

$$\mathcal{N}_\alpha(R_1, R_2) = \#\{n : R_1 < \mathrm{Re}(z_n(\alpha)) \leqslant R_2\}.$$

Note that the $z_n(\alpha)$, $n \in \mathbb{N}^*$, are not necessarily ordered by non-decreasing real parts, but we will see that this is in fact the case for n large enough. The next result shows that we recover for any $\alpha \in \mathbb{C}$ the same Wayl Law as in the selfadjoint case. For $r > 0$, we denote by $D(r)$ the closed disk centered at 0 and with radius r in $\mathbb{C}$.

Proposition 1.2 (Weyl Law). *Let $r > 0$. Then there exists $C > 0$ such that for $\alpha \in D(r)$ and $R_1, R_2 \in \mathbb{R}$ with $R_1 < R_2$, we have*

$$\left| \mathcal{N}_\alpha(R_1, R_2) - \frac{|\Gamma|}{\pi}(R_2 - R_1) \right| \leqslant C.$$

The proof of Proposition 1.2 will rely on the fact that for large n the eigenvalue $\lambda_n(\alpha)$ is in fact, in a suitable sense, close to $\lambda_n(0)$. In the following, we aim at describing more precisely the asymptotic behaviour of the difference between $\lambda_n(0)$ and $\lambda_n(\alpha)$. For $n \in \mathbb{N}^*$, we set

$$\delta_n(\alpha) = \lambda_n(\alpha) - \lambda_n(0). \tag{1.7}$$

Our main result is the following description of the asymptotics of $\delta_n(\alpha)$:

Theorem 1.3. *There exists a probability measure μ_ℓ supported in $[0, 2|\Gamma|^{-1}]$ such that, for every $\alpha \in \mathbb{C}$ and for every continuous function f on $\mathbb{C}$,*

$$\frac{1}{n}\sum_{k=1}^n f(\delta_k(\alpha)) \xrightarrow[n\to+\infty]{} \int_0^{\frac{2}{|\Gamma|}} f(s\alpha)\, d\mu_\ell(s).$$

Moreover,

(i) *if $\ell_j/\ell_1 \in \mathbb{Q}$ for all $j \in [\![1, N]\!]$, then μ_ℓ is a linear combination of Dirac masses, one of them being carried by $s = 0$;*

(ii) *if $\kappa \cdot \ell \neq 0$ for all $\kappa \in \mathbb{Z}^N \backslash \{0\}$, then the measure μ_ℓ is absolutely continuous with respect to the Lebesgue measure and its support is exactly $[0, 2|\Gamma|^{-1}]$.*

Note that $\mathsf{Im}(\lambda_n(\alpha)) = \mathsf{Im}(\delta_n(\alpha))$, so the asymptotic properties of $\delta_n(\alpha)$ give in particular the asymptotic properties of the imaginary parts of the eigenvalues of H_α.

The measure which appears in Theorem 1.3 will be defined in terms of the measure introduced by Barra and Gaspard in [BG00] to describe the asymptotics of $\lambda_{n+1}(0) - \lambda_n(0)$. It is in fact more specifically connected to the measure appearing in the related result of Keating, Marklof and Winn on spectral determinants [KMW03, Th. 3]. As in this reference, it turns out that the size of $\delta_n(\alpha)$ is in fact closely related to the distance between $\lambda_n(0)$ and the rest of the spectrum of H_0. In particular we have the following estimate:

Proposition 1.4. *Let $r > 0$. Then there exists $n_r \in \mathbb{N}^*$ such that for all $\alpha \in D(r)$ and $n \geqslant n_r$ we have*

$$|\delta_n(\alpha)| \leqslant \mathsf{dist}\big(\lambda_n(0), \mathsf{Sp}(H_0) \backslash \{\lambda_n(0)\}\big).$$

With the Weyl Law (1.5), this implies in particular that the sequence $(\delta_n(\alpha))_{n \in \mathbb{N}^*}$ is bounded (uniformly in $\alpha \in D(r)$).

In the proofs of Proposition 1.2 and Theorem 1.3, we will be more precise about the properties of $\delta_n(\alpha)$ in different directions, which we describe in the next results. The following result is already known for α real:

Proposition 1.5. *Assume that there exist $j, k \in [\![1, N]\!]$ such that $j \neq k$ and $\ell_j/\ell_k \in \mathbb{Q}$. Let*

$$\tau \in \frac{\pi \mathbb{Z}}{\ell_j} \cap \frac{\pi \mathbb{Z}}{\ell_k}.$$

Then τ^2 is an eigenvalue of H_α for any $\alpha \in \mathbb{C}$. Moreover, its multiplicity does not depend on α.

As we shall see in the proof of Theorem 1.3, 0 is always in the support of the measure μ_ℓ. This means that one can always find a subsequence of eigenvalues $(\lambda_{n_k}(\alpha))_{k \geqslant 1}$ such that $n_k \to +\infty$ and $\delta_{n_k}(\alpha) \to 0$. We will in fact be slightly more precise on that issue:

Proposition 1.6. *Let $\alpha \in \mathbb{C}$. Then, there exists an increasing sequence $(n_k)_{k \in \mathbb{N}^*}$ in $\mathbb{N}^*$ such that*

$$\delta_{n_k}(\alpha) = \mathop{\mathcal{O}}_{k \to +\infty} \left(\frac{1}{\lambda_{n_k}(0)} \right).$$

Observe that, under the assumption of Proposition 1.5, one has in fact $\delta_{n_k}(\alpha) = 0$ along an increasing subsequence. Hence, Proposition 1.6 is meaningful in the case where $\kappa \cdot \ell \neq 0$ for every $\kappa \in \mathbb{Z}^N \backslash \{0\}$. In that case, Theorem 1.3 also implies that for every $s \in [0, 2|\Gamma|^{-1}]$ there exists a subsequence of eigenvalues $(\lambda_{n_k}(\alpha))_{k \geqslant 1}$ such that $n_k \to +\infty$ and $\delta_{n_k}(\alpha) \to s\alpha$. Under appropriate Diophantine assumptions on ℓ, our proof is slightly more quantitative on that particular question:

Proposition 1.7. *There exists $\Omega \subset (\mathbb{R}_+^*)^N$ such that $(\mathbb{R}_+^*)^N \backslash \Omega$ has null Lebesgue measure and if $\ell \in \Omega$ then the following assertion holds. Let $\varepsilon > 0$ and $\alpha \in \mathbb{C}$. Let $s \in [0, 2\,|\Gamma|^{-1}]$. Then there exists an increasing sequence $(n_k)_{k \in \mathbb{N}^*}$ in $\mathbb{N}^*$ such that*

$$\delta_{n_k}(\alpha) = s\alpha + \underset{k \to +\infty}{\mathcal{O}} \left(\frac{1}{\lambda_{n_k}(0)^{\frac{1}{2N} - \varepsilon}} \right).$$

1.3. **Related literature on non-selfadjoint problems.** The properties of selfadjoint operators on compact graphs are now well understood (see [BK13]), but much less seems to be known on the spectrum of non-selfadoint Laplacian. Yet, some authors have already considered the related problem of the damped wave equation on graphs, which motivated our analysis. More precisely, if we consider on Γ the problem

$$\begin{cases} \partial_t^2 u_j(t, x_j) - \partial_{x_j}^2 u_j(t, x_j) = 0, & \forall t \geqslant 0, \forall j \in [\![1, N]\!], \forall x_j \in (0, \ell_j), \\ u_j(t, 0) = 0 & \forall t \geqslant 0, \forall j \in [\![1, N]\!], \\ u_j(t, \ell_j) = u_k(t, \ell_k), & \forall t \geqslant 0, \forall j, k \in [\![1, N]\!], \\ \sum_{j=1}^N \partial_{x_j} u_j(t, \ell_j) + a\partial_t u(t, v) = 0, & \forall t \geqslant 0, \end{cases} \tag{1.8}$$

with some initial conditions $(u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{H}^1(\Gamma) \times L^2(\Gamma)$. The first results on that direction concern the stabilization of coupled strings joint by a damper (this can be seen as a star-graph with $N = 2$ edges). This is related to the accumulation of eigenvalues on the real axis. In [CCW87] it is shown that, depending on the damper and boundary conditions, we may or may not have exponential decay to equilibrium. In [Liu88, GZ97], the lengths are still equal to 1 but the wave speeds can be different on the two edges. The asymptotic properties of a wave (the energy does not go to 0, it goes strongly to 0, it decays uniformly exponentially to 0) are then described, depending on the arithmetic properties of the ratio of these wave speeds. Similar stabilization results for the wave equation were also obtained on star graphs [AJ04, AJK05] (see also [NV07]) and on more general graphs [VZ09, Zua13].

The spectrum of the damped wave equation has been studied in [AMN13] (star-graph) and [FL17] (graph with general topology). In both cases, the lengths of the edges are rationally dependant. As in the present paper, the result in [FL17] is the description of the distribution of high frequency eigenvalues. However, the damping is on each edge, and not at the vertices as in the present work or as in (1.8). In these references, it is proved that this asymptotics only depends on the mean value of the damping on each edge. Moreover, there are only a finite number of high frequency spectral abscissa. With the notation of our Theorem 1.3, this means that the measure μ_ℓ is given by a finite sum of Dirac measures. This is exactly what we recover with a Robin vertex condition for rationnaly dependant lengths (case (i) of Theorem 1.3). We also refer to [GX09] for a discussion about the basis properties of the (generalized) eigenfunctions.

We expect our study to be a first a step towards the understanding of the spectrum of the damped wave equation (1.8) on graphs. Here we consider a Laplacian with a fixed Robin coefficient. The operator corresponding to the wave equation with damping at the central vertex would be of the same form with a frequency dependent Robin coefficient.

Indeed, if we look for stationary solutions to the damped wave equation (1.8) of the form $u_j(t, x_j) = e^{-itz}w_j(x_j)$, then $w = (w_j)_{1 \leqslant j \leqslant N}$ has to be a solution of the problem

$$H_{-iaz}w = z^2 w.$$

This kind of problems are left for further works. Other classical and related non-selfadjoint problems on graphs are the ones concerned with the distribution of resonances on non-compact quantum graphs, see [EL10, DEL10, DP11, LZ16, CT18] and references therein. For the wave equation on some non-compact star-graph, we also refer to [AJK16].

Our results can also be compared to questions that appear in the context of the Laplace operator on a compact Riemannian manifold (M, g). In the selfadjoint case, we can mention the study of the Schrödinger operator $-\Delta_{\mathbb{S}^n} + a$, on the sphere $\mathbb{S}^n$ endowed with its canonical metric where a is a smooth real-valued function. In that case, eigenvalues are organized in a sequence of clusters and the deviation between the eigenvalues for $a \equiv 0$ and for a general a was described by Weinstein in [Wei77] in terms of the Liouville measure. When $\alpha \in \mathbb{R}$, our Theorem 1.3 can be thought as an analogue of this result where the potential would be a Dirac mass at the central vertex of the star graph. Yet, as we shall see in our proof, the measure μ_ℓ that we obtain in the case of a graph is of completely different nature.

The spectral properties of the damped wave equation have also attracted a lot of attention on compact manifolds. See for instance the results of Sjöstrand [Sjö00], Hitrik [Hit02], Asch-Lebeau [AL03] and Hitrik-Sjöstrand [HS04]. Sjöstrand proved that the complex eigenvalues of the damped wave operator verify a Weyl Law in the high frequency limit and he considered the distribution of the imaginary parts of their square roots. He showed that any limit distribution is supported on an interval $[a_-, a_+]$ where $a_\pm$ corresponds to the infimum (resp. supremum) of the average of the (smooth) damping function $-a$ along a typical orbit of the geodesic flow. In particular, in the case of flat tori or of negatively curved manifolds, both quantities are equal to the average of $-a$ against the normalized volume on M. In other words, the limit distribution of the imaginary parts is the Dirac mass at a_+. For more general manifolds, identifying the limit measure remains an open problem – see for instance [Hit02, AL03, HS04] for results in the case of the sphere. These results of Sjöstrand remain true for non-selfadjoint Schrödinger operators with a weaker non-selfadjoint perturbation, e.g. $\Delta_g + ia$. In the irrational case (case (ii) of Theorem 1.3), our limit measure is supported on an interval and it is in some sense opposite to the case of flat tori and of negatively curved manifolds, where the limit measure is carried by a single point.

1.4. **The case of more general graphs.** In principle, the results presented in this article could be extended to more general graphs that are not star-shaped. Yet, we emphasize that such an extension would a priori require to make some restriction on the choice of the metric (i.e. on the choice of ℓ) we put on the graph. In fact, to every choice of graph and to every choice of boundary conditions for the Laplace operator at the vertices of the graph, one can associate an analytic function F_{Γ, H_0} defined on the N-dimensional torus $\mathbb{T}^N$, where N is the number of edges of the graph. This function does not depend on the

choice of ℓ and the vanishing locus of F_{Γ,H_0} plays a central role when studying the spectral properties of H_0 [BK13]. Our argument is based on ideas that go back to the works of Barra and Gaspard studying the level spacings on quantum graphs [BG00]. More precisely, it consists in analyzing the ergodic properties of the map induced by the flow

$$\varphi_\ell^t : y \in \mathbb{T}^N \mapsto y + t\ell$$

on the vanishing locus of F_{Γ,H_0}. To exploit this idea of Barra and Gaspard, it is necessary that the orbit of 0 under this flow intersects the vanishing locus of F_{Γ,H_0} on a nice enough subset, typically a smooth submanifold. For instance, this can be achieved by considering a quantum graph H_0 with Neumann boundary conditions at each vertex (for which F_{Γ,H_0} is nice enough thanks to [Fri05]) and by picking a *generic* metric ℓ (typically such that φ_ℓ^t is uniquely ergodic). This was the approach followed in earlier works of Berkolaiko-Winn on level spacings [BW10] or of Colin de Verdière on the failure of quantum ergodicity for quantum graphs [Col15]. See also [KMW03, BKW04] for earlier results on these problems by Keating et al. in the case of star graphs (still for a generic ℓ). Here, we chose to work only on star-graphs for which the expression of F_{Γ,H_0} is explicit enough to determine all the singularities of the vanishing locus and to handle *any* choice of metric ℓ. This will illustrate the difference of behaviour depending on our choice of metric ℓ.

Finally, in order to extend the results to more general graphs, we would need to define an appropriate equivalent of $\delta_n(\alpha)$ (for instance, the imaginary part of $\lambda_n(\alpha)$ when $\alpha \notin \mathbb{R}$) in order to state an analogue of Theorem 1.3. Typically, the proof of Proposition 4.2 below, which gives Proposition 1.4 and then Theorem 1.3, relies on explicit computations.

1.5. **Organization of the paper.** In Section 2.1 we define the operator H_α and give some basic spectral properties. In particular, we compute the secular equation on which all the subsequent analysis relies. In Section 3, we focus on the distribution of the eigenvalues of H_0. In particular, we introduce the corresponding dynamical system on $\mathbb{T}^N$ and the Barra-Gaspard measure. In Section 4 we prove that, for high frequencies, the perturbation of the Robin parameter does not move the spectrum so much. So each eigenvalue of H_α remains close, in a suitable sense, to the corresponding eigenvalue of H_0. This will finish the proof of Proposition 1.1 and prove Propositions 1.2, 1.4 and 1.5. Finally, the last section is devoted to the proofs of Theorem 1.3 and Propositions 1.6 and 1.7.

1.6. **Notation.** We denote by $\mathbb{N}$ the set of non-negative integers, and $\mathbb{N}^*$ stands for $\mathbb{N} \setminus \{0\}$. As already mentioned, we also write $[\![1, N]\!]$ for $\{1, \cdots, N\}$. For $z_0 \in \mathbb{C}$ and $r > 0$ we set $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$ and $C(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. As said above, we write $D(r)$ for $D(0, r)$.

2. Spectral framework and first properties

In this section, we define the operator H_α on a suitable Hilbert space on Γ. Then we derive some basic properties for its spectrum. In particular, we give some rough localization for the eigenvalues, and we derive the secular equation, that will be used all along the article.

2.1. **The Schrödinger operator with Robin condition.** The metric graph Γ is endowed with the Hilbert structure of

$$L^2(\Gamma) = \bigoplus_{j=1}^{N} L^2(0, \ell_j).$$

In particular, for $u = (u_j)_{1 \leqslant j \leqslant N} \in L^2(\Gamma)$, we have

$$\|u\|_{L^2(\Gamma)}^2 = \sum_{j=1}^{N} \|u_j\|_{L^2(0,\ell_j)}^2 .$$

For $m \in \mathbb{N}$, we similarly define the Sobolev space

$$\mathcal{H}^m(\Gamma) = \bigoplus_{j=1}^{N} \mathcal{H}^m(0, \ell_j). \tag{2.1}$$

In the introduction, we have denoted by $\mathsf{Dom}(H_\alpha)$ the set of $u \in \mathcal{H}^2(\Gamma)$ such that (1.1), (1.2) and (1.3) hold, and for $u \in \mathsf{Dom}(H_\alpha)$ we have defined $H_\alpha u \in L^2(\Gamma)$ by (1.4). For later purpose, we also define $\mathsf{Dom}(H_\infty)$ as the set of $u = (u_j)_{1 \leqslant j \leqslant N} \in \mathcal{H}^2(\Gamma)$ such that

$$\forall j \in [\![1, N]\!], \quad u_j(0) = u_j(\ell_j) = 0. \tag{2.2}$$

Then, for $u \in \mathsf{Dom}(H_\infty)$, we define $H_\infty u \in L^2(\Gamma)$ as in (1.4).

It is known that, for $\alpha \in \mathbb{R} \cup \{\infty\}$, the operator H_α is selfadjoint and bounded from below (it is even non-negative for $\alpha \geqslant 0$ or $\alpha = \infty$). In particular, since $\mathcal{H}^2(\Gamma)$ is compactly embedded in $L^2(\Gamma)$, the spectrum of H_α consists of a sequence of isolated real eigenvalues going to $+\infty$. When α is not real, the operator H_α is no longer symmetric. However, we prove in Proposition 2.1 that H_α is at least maximal sectorial. We recall that an operator T on some Hilbert space $\mathcal{H}$ is said to be sectorial (with vertex $\gamma \in \mathbb{R}$ and semi-angle $\theta \in [0, \frac{\pi}{2}[$) if the numerical range of T,

$$\left\{ \langle T\varphi, \varphi \rangle_{\mathcal{H}} , \varphi \in \mathsf{Dom}(T), \|\varphi\|_{\mathcal{H}} = 1 \right\},$$

is a subset of the sector

$$\Sigma_{\gamma,\theta} = \left\{ \zeta \in \mathbb{C} : |\arg(\zeta - \gamma)| \leqslant \theta \right\}.$$

Then T is said to be maximal sectorial if $(T - \zeta)$ has a bounded inverse on $\mathcal{H}$ for some (and hence any) ζ in $\mathbb{C} \backslash \Sigma_{\gamma,\theta}$. Similarly, we say that a quadratic form q on $\mathcal{H}$ with domain $\mathsf{Dom}(q)$ is sectorial (with vertex $\gamma \in \mathbb{R}$ and semi-angle $\theta \in [0, \frac{\pi}{2}[$) if for all $\varphi \in \mathsf{Dom}(q)$ with $\|\varphi\|_{\mathcal{H}} = 1$ we have

$$q(\varphi) \in \Sigma_{\gamma,\theta}.$$

In this paper, we study the properties of H_α for some fixed $\alpha \in \mathbb{C}$. However, at some point (see the proof of Proposition 4.4 below), we will deduce some properties of H_α from the corresponding properties of H_0 by a regularity argument with respect to the *parameter* α. This is why we need uniform estimates on $H_{\tilde\alpha}$ for $\tilde\alpha$ lying on a disk containing 0 and α. Thus, we fix $r > 0$ and give the following results with constants uniform with respect to α in the disk $D(r)$.

Proposition 2.1. *Let $r > 0$. There exist $\gamma \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right[$ such that for any $\alpha \in D(r)$ the operator H_α is maximal sectorial with vertex γ and semi-angle θ.*

Proof. • Let $\alpha \in D(r)$. We denote by $\mathsf{Dom}(Q_\alpha)$ the set of functions $u \in \mathcal{H}^1(\Gamma)$ which satisfy (1.1) and (1.2) (notice that this set does not depend on α). Then, for $u, w \in \mathsf{Dom}(Q_\alpha)$, we set

$$Q_\alpha(u, w) = \sum_{j=1}^N \int_0^{\ell_j} u_j'(x_j)\overline{w_j'(x_j)}\, dx_j + \alpha u(v)\overline{w(v)}.$$

In particular, $\mathsf{Dom}(H_\alpha) \subset \mathsf{Dom}(Q_\alpha)$ and, for $u \in \mathsf{Dom}(H_\alpha)$ and $w \in \mathsf{Dom}(Q_\alpha)$, we have

$$\langle H_\alpha u, w\rangle_{L^2(\Gamma)} = Q_\alpha(u, w). \tag{2.3}$$

• For $u \in \mathsf{Dom}(Q_\alpha) = \mathsf{Dom}(Q_0)$ and $j \in [\![1, N]\!]$ we have

$$
\begin{aligned}
|(Q_\alpha - Q_0)(u)| = |\alpha|\,|u(v)|^2 &\leqslant 2\,|\alpha| \int_0^{\ell_j} |u_j(x_j)|\,|u_j'(x_j)|\ dx_j \\
&\leqslant \frac{1}{4}\,\|u_j'\|_{L^2(0,\ell_j)}^2 + 4\,|\alpha|^2\,\|u_j\|_{L^2(0,\ell_j)}^2 \\
&\leqslant \frac{1}{4}Q_0(u) + 4r^2\,\|u\|_{L^2(\Gamma)}^2.
\end{aligned}
$$

Since the form Q_0 is symmetric, bounded from below and closed, we deduce that the form Q_α is sectorial (uniformly in $\alpha \in D(r)$) and closed [Kat80, Th. VI.1.33].
• By the Representation Theorem [Kat80, Th. VI.2.1], there exists a unique maximal sectorial operator $\tilde{H}_\alpha$ on $L^2(\Gamma)$ such that $\mathsf{Dom}(\tilde{H}_\alpha) \subset \mathsf{Dom}(Q_\alpha)$ and

$$\forall u \in \mathsf{Dom}(\tilde{H}_\alpha), \forall w \in \mathsf{Dom}(Q_\alpha), \quad \left\langle \tilde{H}_\alpha u, w\right\rangle_{L^2(\Gamma)} = Q_\alpha(u, w).$$

Moreover, $\mathsf{Dom}(\tilde{H}_\alpha)$ is the set of $u \in \mathsf{Dom}(Q_\alpha)$ for which there exists $f \in L^2(\Gamma)$ such that

$$\forall w \in \mathsf{Dom}(Q_\alpha), \quad Q_\alpha(u, w) = \langle f, w\rangle_{L^2(\Gamma)},$$

and, in this case, $\tilde{H}_\alpha u = f$.
• By (2.3), we have $\mathsf{Dom}(H_\alpha) \subset \mathsf{Dom}(\tilde{H}_\alpha)$ and $\tilde{H}_\alpha$ coincides with H_α on $\mathsf{Dom}(H_\alpha)$. Now let $u = (u_j)_{1\leqslant j\leqslant N} \in \mathsf{Dom}(\tilde{H}_\alpha)$. Let us prove that $u \in \mathsf{Dom}(H_\alpha)$. There exists $f = (f_j)_{1\leqslant j\leqslant N} \in L^2(\Gamma)$ such that, for all $w \in \mathsf{Dom}(Q_\alpha)$, we have

$$Q_\alpha(u, w) = \langle f, w\rangle_{L^2(\Gamma)}.$$

Let $k \in [\![1, N]\!]$. Considering all the test functions $w = (w_j)_{1\leqslant j\leqslant N}$ such that $w_k \in C_0^\infty((0, \ell_j))$ and $w_j = 0$ for $k \neq j$, we obtain that u_k belongs to $\mathcal{H}^2(0, \ell_k)$ and $-u_k'' = f_k$. Then

$u \in \mathcal{H}^2(\Gamma)$. Now, for all $w \in \mathsf{Dom}(Q_\alpha)$, we have

$$
\begin{aligned}
Q_\alpha(u, w) &= \sum_{j=1}^{N} \langle u_j', w_j' \rangle_{L^2(0,\ell_j)} + \alpha u(v)\overline{w(v)} \\
&= \sum_{j=1}^{N} \left(u_j'(\ell_j)\overline{w_j(\ell_j)} - \langle u_j'', w_j \rangle_{L^2(0,\ell_j)} \right) + \alpha u(v)\overline{w(v)} \\
&= \langle f, w \rangle_{L^2(\Gamma)} + \left(\sum_{j=1}^{N} u_j'(\ell_j) + \alpha u(v) \right) \overline{w(v)}.
\end{aligned}
$$

Choosing $w \in \mathsf{Dom}(Q_\alpha)$ such that $w(v) \neq 0$ yields (1.3). This proves that $u \in \mathsf{Dom}(H_\alpha)$, and hence $H_\alpha = \tilde{H}_\alpha$ is maximal sectorial. □

Remark 2.2. With the same proof we recover that H_α is selfadjoint if α is real, and non-negative if $\alpha \geqslant 0$. Notice that [BK13, Th. 1.4.11] provides a similar result for more general *selfadjoint* vertices conditions and more general graphs.

2.2. **First properties of the spectrum.** In this paragraph we begin our description of the localization of the spectrum with some rough properties.

Proposition 2.3. *Let $r > 0$. Let γ and θ be given by Proposition 2.1. Let $\alpha \in D(r)$. Then the spectrum of the operator H_α is included in $\Sigma_{\gamma,\theta}$ and consists of isolated eigenvalues with finite multiplicities.*

Proof. By Proposition 2.1, the operator H_α is maximal sectorial with vertex $\gamma \in \mathbb{R}$ and semi-angle $\theta \in \left(0, \frac{\pi}{2}\right)$, and hence its spectrum is a subset of $\Sigma_{\gamma,\theta}$ [Kat80, §V.3.10]. In particular, the resolvent set of H_α is not empty. Moreover, $\mathsf{Dom}(H_\alpha)$ is compactly embedded in $L^2(\Gamma)$, so the operator H_α has a compact resolvent. Thus, its spectrum consists of isolated eigenvalues with finite multiplicities. □

Notice that we have not proved yet that H_α has an infinite number of eigenvalues. We will get this as a byproduct of the more refined analysis of Section 4.

Remark 2.4. Even if we are mostly interested in the large eigenvalue limit of the spectrum, we note that 0 is an eigenvalue of H_α if and only if

$$
\alpha = \sum_{j=1}^{N} \frac{1}{\ell_j},
$$

and that in this case $\mathsf{ker}(H_\alpha)$ is generated by $u = (u_j)_{1 \leqslant j \leqslant N}$ such that $u_j(x_j) = x_j/\ell_j$ for all $j \in [\![1, N]\!]$ and $x_j \in [0, \ell_j]$.

Now we consider non-zero eigenvalues. Let $z \in \mathbb{C}^*$. For $u = (u_j)_{1 \leqslant j \leqslant N} \in \mathcal{H}^2(\Gamma)$, we have $-u_j'' = z^2 u_j$ for all $j \in [\![1, N]\!]$ if and only if there exist constants $(\beta_j, \tilde{\beta}_j)_{1 \leqslant j \leqslant N}$ such that, for $j \in [\![1, N]\!]$ and for $x \in [0, \ell_j]$,

$$
u_j(x) = \beta_j \sin(zx) + \tilde{\beta}_j \cos(zx).
$$

Such a function u belongs to $\mathsf{Dom}(H_\alpha)$ (and then $H_\alpha u = z^2 u$) if and only if the vertices conditions (1.1), (1.2) and (1.3) are satisfied. The Dirichlet condition (1.1) at the external vertices gives

$$\forall j \in [\![1, N]\!], \quad \tilde{\beta}_j = 0, \tag{2.4}$$

so that

$$u_j(x) = \beta_j \sin(zx). \tag{2.5}$$

With this simple calculation, we can already prove the following rough result.

Proposition 2.5.   (i) *Let $\alpha \in \mathbb{C}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of H_α then $\mathsf{Im}(\lambda)$ and $\mathsf{Im}(\alpha)$ have the same sign (recall that the spectrum of H_α is real if α is real, on the other hand there can be real eigenvalues even if α is not real).*
  (ii) *Let $r > 0$. There exists $c_r > 0$ such that, if $\lambda \in \mathbb{C}$ is an eigenvalue of H_α for some $\alpha \in D(r)$, then*

$$\mathsf{Re}(\lambda) \geqslant -c_r \quad and \quad |\mathsf{Im}(\lambda)| \leqslant c_r.$$

Moreover, we have

$$\limsup_{R \to +\infty} \sup_{\substack{\lambda \in \mathsf{Sp}(H_\alpha) \\ \mathsf{Re}(\lambda) \geqslant R}} |\mathsf{Im}(\lambda)| \leqslant \frac{2\,|\mathsf{Im}(\alpha)|}{|\Gamma|}.$$

Remark 2.6. The last statement should be compared with the support of μ_ℓ in Theorem 1.3. In particular, this is an equality when $\kappa \cdot \ell \neq 0$ for every $\kappa \in \mathbb{Z}^N \backslash \{0\}$.

Proof. Let $z = \tau + i\eta \in \mathbb{C}^*$ (with $\tau, \eta \in \mathbb{R}$) and assume that $\lambda = z^2$ is an eigenvalue of H_α. Let $u \in \mathsf{ker}(H_\alpha - z^2) \backslash \{0\}$, normalized by $\|u\|_{L^2(\Gamma)} = 1$. Let $\beta_1, \ldots, \beta_N \in \mathbb{C}$ be such that (2.5) holds. Taking the imaginary parts in the equality

$$\lambda = \langle H_\alpha u, u \rangle = Q_\alpha(u, u)$$

gives

$$\mathsf{Im}(\lambda) = \mathsf{Im}(\alpha)\,|u(v)|^2. \tag{2.6}$$

This implies in particular that $\mathsf{Im}(\lambda)$ and $\mathsf{Im}(\alpha)$ have the same sign.

Then we compute

$$1 = \|u\|_{L^2(\Gamma)}^2 = \sum_{j=1}^N \int_0^{\ell_j} |\beta_j \sin(zx_j)|^2 \, dx_j = \sum_{j=1}^N |\beta_j|^2 \left(\frac{\mathrm{sh}(2\eta\ell_j)}{4\eta} - \frac{\sin(2\tau\ell_j)}{4\tau} \right). \tag{2.7}$$

Let $(\lambda_m)_{m\in\mathbb{N}}$ be a sequence of complex numbers such that $\mathsf{Re}(\lambda_m)$ goes to $+\infty$ and, for all $m \in \mathbb{N}$, λ_m is an eigenvalue of H_{α_m} for some $\alpha_m \in D(r)$. For $m \in \mathbb{N}$ we consider $\tau_m \geqslant 0$ and $\eta_m \in \mathbb{R}$ such that $\lambda_m = (\tau_m + i\eta_m)^2$. Then τ_m goes to $+\infty$ as m goes to $+\infty$. Assume for instance that $\mathsf{Im}(\alpha) \geqslant 0$. After extracting a subsequence if necessary, we can assume that the sequence $(\eta_m)_{m\in\mathbb{N}}$ has a limit η_∞ in $[0, +\infty]$. For $m \in \mathbb{N}$ we consider $u_m \in \mathsf{ker}(H_{\alpha_m} - \lambda_m)$ such that $\|u_m\|_{L^2(\Gamma)} = 1$ and we denote by $\beta_{1,m}, \ldots, \beta_{N,m} \in \mathbb{C}$ the corresponding coefficients in (2.4).

Assume that $\eta_\infty = +\infty$. Then, by (2.7), we have

$$\sum_{j=1}^{N} \frac{|\beta_{j,m}|^2 \, e^{2\eta_m \ell_j}}{4\eta_m} \xrightarrow[m \to +\infty]{} 1.$$

Then

$$|u(v)|^2 = |\beta_{1,m}|^2 \, |\sin(z_m \ell_1)|^2 \leqslant |\beta_{1,m}|^2 \, e^{2\eta_m \ell_1} = \mathcal{O}(\eta_m),$$

and hence, by (2.6),

$$2\tau_m \eta_m = \mathsf{Im}(\lambda_n) \leqslant r \, |u(v)|^2 = \mathcal{O}(\eta_m).$$

This gives a contradiction.

We similarly get a contradiction if we assume that $\eta_\infty \in (0, +\infty)$ so, finally, η_m goes to 0 as m goes to $+\infty$. (2.7) now gives

$$\sum_{j=1}^{N} \frac{|\beta_{j,m}|^2 \, \ell_j}{2} \xrightarrow[m \to +\infty]{} 1$$

so

$$\limsup_{m \to +\infty} \frac{|\Gamma| \, |u_m(v)|^2}{2} = \limsup_{m \to +\infty} \sum_{j=1}^{N} \frac{|\beta_{j,m}|^2 \, \ell_j}{2} \, |\sin(z_m \ell_j)|^2 \leqslant 1.$$

Then (2.6) gives

$$\limsup_{m \to +\infty} |\mathsf{Im}(\lambda_n)| \leqslant |\mathsf{Im}(\alpha)| \, |u_m(v)|^2 \leqslant \frac{2 \, |\mathsf{Im}(\alpha)|}{|\Gamma|}.$$

We conclude with Proposition 2.3.   □

2.3. Comparison with the eigenvalues of the Dirichlet problem. The spectrum of H_∞ (see (2.2)) is completely explicit. If we set

$$\mathcal{T}_\infty = \bigcup_{j=1}^{N} \frac{\pi \mathbb{Z}}{\ell_j},$$

then

$$\mathsf{Sp}(H_\infty) = \left\{ \tau^2, \tau \in \mathcal{T}_\infty \right\}.$$

Moreover, if τ^2 is an eigenvalue of H_∞, then its multiplicity is given by

$$\# \left\{ j \in [\![1, N]\!] : \tau \ell_j \in \pi \mathbb{Z} \right\}.$$

In particular, if the lengths of the edges are pairwise incommensurable, then all the eigenvalues of H_∞ are simple. On the contrary, if all the lengths are equal, then all the eigenvalues of H_∞ have multiplicity $N - 1$.

In the selfadjoint case ($\alpha \in \mathbb{R}$) it is known (see [BK13, Th. 3.1.8] or Remark 2.8 below) that between two consecutive eigenvalues of H_∞ there is exactly one simple eigenvalue of H_α. In the degenerate situation where $\lambda \in \mathbb{R}$ is an eigenvalue of H_∞ of multiplicity $m \geqslant 2$, then λ is for all $\alpha \in \mathbb{R}$ an eigenvalue of multiplicity $m - 1$ for H_α. The first statement has no meaning for the possibly non-real eigenvalues of H_α, $\alpha \notin \mathbb{R}$, but the second still holds in general.

Lemma 2.7. *Let $\alpha \in \mathbb{C}$.*

(i) *If $\tau \in \mathcal{T}_\infty$, then*

$$\dim(\ker(H_\alpha - \tau^2)) = \dim(\ker(H_\infty - \tau^2)) - 1.$$

(ii) *If $z \in \mathbb{C}\backslash\mathcal{T}_\infty$, then*

$$\dim(\ker(H_\alpha - z^2)) \leqslant 1.$$

Proof. Assume that τ^2 is an eigenvalue of H_∞. There exists $j \in [\![1, N]\!]$ such that $\tau\ell_j/\pi$ is an integer. Let $u \in \ker(H_\alpha - \tau^2)$. By (2.5) we have $u(v) = u_j(\tau\ell_j) = 0$. This implies that $\ker(H_\alpha - \tau^2) \subset \ker(H_\infty - \tau^2)$. Moreover, the map $u \mapsto \sum_{j=1}^N u'_j(\ell_j)$ is a non-zero linear form on $\ker(H_\infty - \tau^2)$, which yields the first part of the lemma.

Now let $z \in \mathbb{C}\backslash\mathcal{T}_\infty$. If $u \in \ker(H_\alpha - z^2)$ is such that $u(v) = 0$, then it belongs to $\ker(H_\infty - z^2)$, so $u = 0$. This means that the linear form $u \mapsto u(v)$ is injective on $\ker(H_\alpha - z^2)$ and proves the second statement. $\qquad\square$

2.4. **The secular equation.** In Paragraph 2.2, we have only used the vertex condition (1.1). For a more refined analysis, we have to take into account (1.2) and (1.3).

Let $z \in \mathbb{C}^*$ and $u \in \ker(H_\alpha - z^2)$. Let $\beta \in \mathbb{C}^N$ be such that (2.5) holds. Then the conditions (1.2) and (1.3) at the vertex v read

$$\beta_1 \sin(z\ell_1) = \beta_2 \sin(z\ell_2) = \ldots = \beta_N \sin(z\ell_N) \tag{2.8}$$

and

$$z \sum_{j=1}^N \beta_j \cos(z\ell_j) + \alpha\beta_N \sin(z\ell_N) = 0. \tag{2.9}$$

We can divide this last equality by z. Conversely, if β satisfies (2.8)-(2.9) and $u \in L^2(\Gamma)$ is defined by (2.5) then $u \in \mathsf{Dom}(H_\alpha)$ and $H_\alpha u = z^2 u$. Thus, $z^2 \neq 0$ is an eigenvalue of H_α if and only if

$$\det\left(A\left(z\ell, \frac{\alpha}{z}\right)\right) = 0,$$

where, for $y = (y_1, \ldots, y_N) \in \mathbb{C}^N$ and $\eta \in \mathbb{C}$,

$$A(y, \eta) = \begin{pmatrix} \sin(y_1) & -\sin(y_2) & 0 & \ldots & 0 \\ 0 & \sin(y_2) & -\sin(y_3) & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \sin(y_{N-1}) & -\sin(y_N) \\ \cos(y_1) & \cos(y_2) & \ldots & \ldots & \cos(y_N) + \eta\sin(y_N) \end{pmatrix}.$$

Moreover, since the map

$$(\beta_j)_{1\leqslant j \leqslant N} \in \ker\left(A\left(z\ell, \frac{\alpha}{z}\right)\right) \mapsto u \in \ker\left(H_\alpha - z^2\right),$$

defined by (2.5), is an isomorphism, we have

$$\dim\left(\ker\left(A\left(z\ell, \frac{\alpha}{z}\right)\right)\right) = \dim\left(\ker\left(H_\alpha - z^2\right)\right).$$

By a straightforward computation we observe that

$$\det(A(y,\eta)) = F_N(y) + \eta F_D(y), \tag{2.10}$$

where

$$F_N(y_1, \ldots, y_N) = \sum_{j=1}^{N} \cos(y_j) \prod_{k \neq j} \sin(y_k) \tag{2.11}$$

and

$$F_D(y_1, \ldots, y_N) = \prod_{j=1}^{N} \sin(y_j). \tag{2.12}$$

Note that $F_N = \det(A(\cdot,0))$ is the determinant associated with the Neumann (or Kirchhoff) problem ($\alpha = 0$) and F_D is the determinant corresponding to the Dirichlet problem, i.e. z^2 is an eigenvalue of H_∞ if and only if $F_D(z\ell) = 0$.

For $y \in \mathbb{C}^N$ such that $F_D(y) \neq 0$ we set

$$\Psi(y) = -\frac{F_N(y)}{F_D(y)} = -\sum_{j=1}^{N} \cotan(y_j). \tag{2.13}$$

Remark 2.8. Let $\tau \in \mathbb{R}_+ \backslash \mathcal{T}_\infty$. Then τ^2 is an eigenvalue of H_0 is and only if $\Psi(\tau\ell) = 0$. We have

$$\frac{d}{d\tau}\Psi(\tau\ell) = \sum_{j=1}^{N} \ell_j\big(1 + \cotan(\tau\ell_j)^2\big) > 0.$$

Since $|\Psi(\tau\ell)|$ goes to $+\infty$ when τ approaches $\mathcal{T}_\infty$, we see that in each connected component I of $\mathbb{R}_+\backslash\mathcal{T}_\infty$ (which does not contain 0) there exists exactly one $n \in \mathbb{N}^*$ such that $\tau_n \in I$. See Figure 2 in Section 4 below. On the other hand, we deduce from the discussion of Paragraph 2.3 that the operator H_∞ satisfies a Weyl Law as in (1.5). Then, combining Lemma 2.7 and the previous remark, we recover (1.5) for H_0. In fact, the same applies to H_α for any $\alpha \in \mathbb{R}$ if we observe that on $\mathbb{R}_+\backslash\mathcal{T}_\infty$ we also have

$$\frac{d}{d\tau}\big(\tau\Psi(\tau\ell)\big) > 0.$$

3. Eigenvalues of H_0 and the Barra-Gaspard measure

In this section, we review a few facts on a natural Radon measure associated to our metric star graph (Γ, ℓ). This measure was defined by Barra and Gaspard in [BG00] to study the level spacings of the operator H_0 (see also [BW10]) and further used to study the distribution of the eigenmodes of H_0 [KMW03, BKW04, Col15] or resonances on noncompact graphs [CT18]. The main differences with these references is that we use the explicit structure of the graph to handle *any* choice of metric on Γ.

3.1. **A stratified manifold associated to the eigenvalues of H_0.** We have said in the previous section that τ^2 is an eigenvalue of H_0 if and only if $F_N(\tau\ell) = 0$, where F_N is defined by (2.11). We set

$$Z = \left\{ y \in \mathbb{R}^N : F_N(y) = 0 \right\}.$$

This defines a stratified submanifold, which can be splitted as

$$Z = \bigsqcup_{J \subset [\![1,N]\!]} Z_J,$$

where, for $J \subset [\![1, N]\!]$,

$$Z_J = \{ y \in Z : y_j = 0 \bmod \pi \Leftrightarrow j \in J \}.$$

With this definition of Z, we see that τ^2 is an eigenvalue of H_0 if and only if $\tau\ell \in Z$. More precisely,

(i) if $\tau\ell \in Z_\varnothing$ then $F_D(\tau\ell) \neq 0$ so, by Lemma 2.7, τ^2 is a simple eigenvalue of H_0 ;

(ii) if $\tau\ell \in Z_J$ for some $J \subset [\![1, N]\!]$ with $\#J \geqslant 2$, then τ^2 is an eigenvalue of H_∞ of multiplicity $\#J$ and, again by Lemma 2.7, τ^2 is an eigenvalue of H_0 of multiplicity $\#J - 1$.

Note that Z_J is empty if $\#J = 1$ and $Z_\varnothing = \Psi^{-1}(\{0\})$ (see (2.13)). Then for $J \subset [\![1, N]\!]$ we set

$$m_J = \begin{cases} 1 & \text{if } J = \varnothing, \\ \#J - 1 & \text{if } \#J \geqslant 2. \end{cases} \tag{3.1}$$

Now we check that each stratum of Z is a submanifold of $\mathbb{R}^N$ to which the vector ℓ is transverse.

Lemma 3.1. (i) $Z_\varnothing$ is a submanifold of $\mathbb{R}^N$ of dimension $N - 1$, and for all $y \in Z_\varnothing$ we have $\nabla\Psi(y) \cdot \ell \neq 0$.

(ii) Let $J \subset [\![1, N]\!]$ with $\#J \geqslant 2$. Then Z_J is a submanifold of $\mathbb{R}^N$ of dimension $N - (\#J)$ and its boundary $\partial Z_J = \overline{Z_J} \backslash Z_J$ satisfies

$$\partial Z_J = \bigcup_{J \subsetneq J' \subset [\![1,N]\!]} Z_{J'}.$$

Moreover, ℓ is transverse to Z_J.

Proof. •   Let $y \in Z_\varnothing$. We have

$$\nabla\Psi(y) \cdot \ell = \sum_{j=1}^N \ell_j \left(1 + \cotan(y_j)^2\right) > 0.$$

In particular, $\nabla\Psi(y) \neq 0$. By the Implicit Function Theorem, this proves that $Z_\varnothing = \Psi^{-1}(\{0\})$ is a submanifold of dimension $(N - 1)$ in $\mathbb{R}^N$, and ℓ is transverse to $Z_\varnothing$.

•   Now we consider $J \subset [\![1, N]\!]$ with $\#J \geqslant 2$. Then any $y \in \mathbb{R}^N$ such that $y_j = 0 \bmod \pi$ for all $j \in J$ belongs to Z, so

$$Z_J = \left\{ y \in \mathbb{R}^N : y_j = 0 \bmod \pi \Leftrightarrow j \in J \right\}.$$

This defines a submanifold of dimension $N - (\#J)$. On the other hand, it is not difficult to check that
$$\overline{Z_J} = \left\{ y \in \mathbb{R}^N : y_j = 0 \bmod \pi, \ \forall j \in J \right\},$$
and the statement about ∂Z_J follows. Finally, ℓ is transverse to each Z_J because all its components are positive. $\qquad\square$

3.2. **Restriction to a dynamical system on a torus.** Since F_N is 2π-periodic with respect to each variable, the condition $F_N(\tau \ell) = 0$ can be seen as an equation on the torus $\mathbb{T}^N = \mathbb{R}^N/(2\pi\mathbb{Z}^N)$. We set
$$\Lambda_\ell = \left\{ k \in \mathbb{Z}^N : k \cdot \ell = 0 \right\}$$
and
$$\Lambda_\ell^\perp = \left\{ \xi \in \mathbb{R}^N : \forall k \in \Lambda_\ell, \ k \cdot \xi = 0 \right\}.$$
In particular, $\tau \ell$ belongs to $\Lambda_\ell^\perp$ for all $\tau \in \mathbb{R}$. We denote by N_ℓ the dimension of $\Lambda_\ell^\perp$.

Remark 3.2. N_ℓ is the smallest integer for which there exist $\tilde{\ell}_1, \ldots, \tilde{\ell}_{N_\ell} \in (0, +\infty)$ such that for all $j \in [\![1, N]\!]$ we have
$$\ell_j = \sum_{k=1}^{N_\ell} m_{j,k} \tilde{\ell}_k,$$
for some $m_{j,1}, \ldots, m_{j,N_\ell} \in \mathbb{N}^*$.

Noticing that $2\pi\mathbb{Z}^N \cap \Lambda_\ell^\perp$ is a lattice of $\Lambda_\ell^\perp$, we define the torus
$$\mathbb{T}_\ell = \Lambda_\ell^\perp/(2\pi\mathbb{Z}^N \cap \Lambda_\ell^\perp).$$
It can be identified with a subset of $\mathbb{T}^N$. We denote by $\mu_{\mathbb{T}_\ell}$ the Lebesgue measure on $\mathbb{T}_\ell$ (inherited from the Lebesgue measure on $\Lambda_\ell^\perp$) and we set $|\mathbb{T}_\ell| = \mu_{\mathbb{T}_\ell}(\mathbb{T}_\ell)$.

For $y \in \mathbb{T}^N$ and $t \geqslant 0$ we set
$$\varphi_\ell^t(y) = y + t\ell \mod 2\pi.$$
From the unique ergodicity of this flow on $\mathbb{T}_\ell$, one has, for every continuous function f on $\mathbb{T}^N$ and uniformly for $y \in \mathbb{T}_\ell$,
$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T (f \circ \varphi_\ell^t)(y) dt = \frac{1}{|\mathbb{T}_\ell|} \int_{\mathbb{T}_\ell} f \, d\mu_{\mathbb{T}_\ell} \tag{3.2}$$
(this can be computed explicitely if f is of the form $y \mapsto e^{i\kappa \cdot y}$ for some $\kappa \in \mathbb{Z}^N$, and the general case follows by decomposing f in Fourier series). This implies in particular that any orbit of the flow φ_ℓ^t starting in $\mathbb{T}_\ell$ is dense in $\mathbb{T}_\ell$.

Since Z (and each Z_J, $J \subset [\![1, N]\!]$) is $(2\pi\mathbb{Z}^N)$-periodic, we can consider
$$\mathcal{Z} = (Z \cap \Lambda_\ell^\perp)/(2\pi\mathbb{Z}^N \cap \Lambda_\ell^\perp) = \bigsqcup_{J \subset \{1,\ldots,N\}} \mathcal{Z}_J,$$
where
$$\mathcal{Z}_J = (Z_J \cap \Lambda_\ell^\perp)/(2\pi\mathbb{Z}^N \cap \Lambda_\ell^\perp).$$

Taking the convention that $\dim \mathcal{Z}_J = -\infty$ when $\mathcal{Z}_J$ is empty, we have results analogous to Lemma 3.1:

Lemma 3.3. (i) $\mathcal{Z}_\varnothing$ is a submanifold of $\mathbb{T}_\ell$ of dimension $N_\ell - 1$ to which ℓ is transverse.
   (ii) Let $J \subset [\![1, N]\!]$ with $\#J \geqslant 2$. Then $\mathcal{Z}_J$ is a submanifold of $\mathbb{T}_\ell$ of dimension not greater than $N_\ell - 1$ and

$$\partial \mathcal{Z}_J := \overline{\mathcal{Z}_J} \backslash \mathcal{Z}_J \subset \bigcup_{J'} \mathcal{Z}_{J'},$$

where the union is over the sets $J' \subset [\![1, N]\!]$ such that $\dim(\mathcal{Z}_{J'}) < N_\ell - 1$. Moreover, ℓ is transverse to $\mathcal{Z}_J$.

Proof. Since ℓ is transverse to $Z_\varnothing = \Psi^{-1}(\{0\})$ and belongs to $\Lambda_\ell^\perp$, $Z_\varnothing \cap \Lambda_\ell^\perp$ is a submanifold of $\Lambda_\ell^\perp$ of dimension $N_\ell - 1$ to which ℓ is transverse. After taking the quotient, the analogous properties hold for $\mathcal{Z}_\varnothing$.

Now let $J \subset [\![1, N]\!]$ with $\#J \geqslant 2$. Since Z_J coincides around each of its points with an affine subspace of $\mathbb{R}^N$ (with constant tangent space), the same holds for $Z_J \cap \Lambda_\ell^\perp$. Since ℓ belongs to $\Lambda_\ell^\perp$ and not to $T Z_J$, the dimension of $Z_J \cap \Lambda_\ell^\perp$ is not greater than $N_\ell - 1$ and ℓ is a transverse vector. All this then holds for $\mathcal{Z}_J$.

We have $\partial(Z_J \cap \Lambda_\ell^\perp) \subset \partial Z_J \cap \Lambda_\ell^\perp$. Let $y_0 \in \partial(Z_J \cap \Lambda_\ell^\perp)$ and $J' \subset [\![1, N]\!]$ such that $y_0 \in Z_{J'} \cap \Lambda_\ell^\perp$ and $J \subsetneq J'$. Let $\mathcal{V}$ be a neighborhood of y_0 in $Z_{J'} \cap \Lambda_\ell^\perp$ such that $\mathcal{V}$ is an open subset of an affine subspace of $\mathbb{R}^N$. If we denote by $(f_1, \ldots, f_N)$ the canonical basis of $\mathbb{R}^N$, there exists $j \in J' \backslash J$ such that $y_0 + t f_j$ belongs to $Z_J \cap \Lambda_\ell^\perp$ for $t > 0$ small enough. Then for all $y \in \mathcal{V}$ and $t > 0$ small enough we have $y + t f_j \in Z_J \cap \Lambda_\ell^\perp$. This implies that

$$\dim(Z_{J'} \cap \Lambda_\ell^\perp) < \dim(Z_J \cap \Lambda_\ell^\perp) \leqslant N_\ell - 1.$$

The same conclusion holds for $\mathcal{Z}_J$. $\qquad\square$

3.3. **Distribution of the eigenvalues by ergodic averaging.** We now study the distribution of the eigenvalues of H_0 following the seminal work of Barra and Gaspard [BG00]. The results in this paragraph were already presented under some genericity assumptions on ℓ for quantum star graphs [KMW03, §3] and for general quantum graphs in [BW10, §4.3] and [Col15, §3]. We briefly recall these arguments to see that the case of a general metric ℓ can be handled similarly for quantum star graphs using the conventions of the above paragraphs. For later purpose, note also that, compared with [KMW03, §3], we will also allow test functions which are not necessarily compactly supported in $\mathcal{Z}_\varnothing$. To that aim, we set

$$\mathcal{J}_\ell = \left\{ J \subset [\![1, N]\!] \ : \ \dim(\mathcal{Z}_J) = N_\ell - 1 \right\}.$$

For $J \in \mathcal{J}_\ell$ we denote by μ_J the Lebesgue measure on $\mathcal{Z}_J$ and by ν a unit vector normal to $\mathcal{Z}_J$ in $\mathbb{T}_\ell$.

Lemma 3.4. Let g be a continuous function on $\mathcal{Z}$. We assume that g is compactly supported on $\mathcal{Z}_J$ for some $J \in \mathcal{J}_\ell$. Then, one has

$$\frac{1}{n} \sum_{k=1}^n (g \circ \varphi_\ell^{\tau_k})(0) \xrightarrow[n \to +\infty]{} \frac{\pi m_J}{|\Gamma| \, |\mathbb{T}_\ell|} \int_{\mathcal{Z}_J} g(y) |\ell \cdot \nu(y)| \, d\mu_J(y).$$

In the case of a generic ℓ, see [KMW03, Th. 8] for star graphs and [BW10, Prop. 4.4] and [Col15, Lemma 3.1] for the case of more general graphs.

Proof. Let $\mathcal{V}$ be a neighborhood of the support of g in $\mathcal{Z}_J$ such that $\overline{\mathcal{V}} \subset \mathcal{Z}_J$. Since ℓ is transverse to $\mathcal{Z}_J$, there exists $\delta \in]0, \tau_1[$ such that the map

$$\mathcal{T} : \begin{cases}]-2\delta, 2\delta[\times \mathcal{V} & \to & \mathbb{T}_\ell \\ (t, y) & \mapsto & y + t\ell \end{cases}$$

is injective with an image which does not intersect $\mathcal{Z} \backslash \mathcal{Z}_J$. Let $\chi \in C_0^\infty(]-\delta, \delta[, [0, 1])$ be equal to 1 near 0 and such that $\int_{\mathbb{R}} \chi(t) dt = \delta$. We define a continuous function $\tilde{g}$ on $\mathbb{T}_\ell$ by setting

$$\tilde{g}(y + t\ell) = \chi(t) g(y)$$

for all $t \in]-\delta, \delta[$ and $y \in \mathcal{V}$, and $\tilde{g} = 0$ outside the image of $\mathcal{T}$. Then

$$\frac{1}{n} \sum_{k=1}^n (g \circ \varphi_\ell^{\tau_k})(0) = \frac{\tau_n + \delta}{n} \frac{1}{\tau_n + \delta} \frac{m_J}{\delta} \int_0^{\tau_n + \delta} \tilde{g} \circ \varphi_\ell^t(0) dt.$$

By (1.6) and unique ergodicity (3.2) of the flow φ_ℓ^t on $\mathbb{T}_\ell$, one finds that

$$\frac{1}{n} \sum_{k=1}^n (g \circ \varphi_\ell^{\tau_k})(0) \xrightarrow[n \to +\infty]{} \frac{\pi m_J}{|\Gamma| \, |\mathbb{T}_\ell|} \frac{1}{\delta} \int_{\mathbb{T}_\ell} \tilde{g} \, d\mu_{\mathbb{T}_\ell}.$$

Since the Jacobian of $\mathcal{T}$ at (t, y) is $|\ell \cdot \nu(y)|$, one gets

$$\frac{1}{\delta} \int_{\mathbb{T}_\ell} \tilde{g} \, d\mu_{\mathbb{T}_\ell} = \int_{\mathcal{Z}_J} |\ell \cdot \nu(y)| g(y) \, d\mu_J(y),$$

and the conclusion follows. $\square$

For $\varepsilon > 0$ we introduce

$$\mathcal{Z}(\varepsilon) = \{y \in \mathcal{Z} : \exists J \text{ s.t. } \dim(\mathcal{Z}_J) < N_\ell - 1 \text{ and } d(y, \mathcal{Z}_J) \leqslant \varepsilon\}. \tag{3.3}$$

In other words, we consider an ε-neighborhood of the strata of $\mathcal{Z}$ whose dimensions are strictly smaller than $N_\ell - 1$. In particular, the $(N_\ell - 1)$-dimensional Hausdorff measure of this set is $\mathcal{O}(\varepsilon)$.

Lemma 3.5. *We have*

$$\limsup_{n \to +\infty} \frac{\#\{k \leqslant n : \varphi_\ell^{\tau_k}(0) \in \mathcal{Z}_\ell(\varepsilon)\}}{n} \underset{\varepsilon \to 0}{=} \mathcal{O}(\varepsilon).$$

See [Col15, p. 354] for the case of a generic metric on a general quantum graph.

Proof. Let

$$B(\varepsilon) = \{y + t\ell : -\varepsilon \leqslant t \leqslant \varepsilon \text{ and } y \in \mathcal{Z}(\varepsilon)\}.$$

Since this set is contained in an $\mathcal{O}(\epsilon)$ neighborhood of the strata of $\mathcal{Z}$ having dimensions not greater than $N_\ell - 2$, its Lebesgue measure is of size $\mathcal{O}(\varepsilon^2)$ as ε tends to 0.

Let $n \in \mathbb{N}^*$ and $k \leqslant n$ such that $\varphi_\ell^{\tau_k}(0) \in \mathcal{Z}(\varepsilon)$. Then $\varphi_\ell^t(0)$ stays inside $B(\varepsilon)$ for $t \in [\tau_n - \varepsilon, \tau_n + \varepsilon]$. Moreover, by the Weyl Law, the number (counted with multiplicities)

of square roots of eigenvalues of H_0 lying in an interval of length 1 is uniformly bounded by some constant M. Hence, for $\varepsilon > 0$ small enough, one has

$$\frac{\#\{k \leqslant n : \varphi_\ell^{\tau_k}(0) \in \mathcal{Z}_\ell(\varepsilon)\}}{n} \leqslant \frac{M}{2\varepsilon} \int_0^{\tau_n + \varepsilon} (\mathbf{1}_{B(\varepsilon)} \circ \varphi_\ell^t)(0) dt.$$

Hence, by letting n tend to $+\infty$ and by combining (3.2) with [Wal82, p.149, Rk. 3], one finds that

$$\limsup_{n \to +\infty} \frac{\#\{k \leqslant n : \varphi_\ell^{\tau_k}(0) \in \mathcal{Z}_\ell(\varepsilon)\}}{n} \leqslant \frac{M}{2\varepsilon} \frac{\mu_{\mathbb{T}_\ell}(B(\varepsilon))}{|\mathbb{T}_\ell|},$$

and the conclusion follows. $\qquad\square$

We define the *Barra-Gaspard measure* on $\mathcal{Z}$ by

$$\mu_{\mathsf{BG}} = \sum_{J \in \mathcal{J}_\ell} \mu_{\mathsf{BG}, J}, \tag{3.4}$$

where, for $J \in \mathcal{J}_\ell$, we have set

$$\mu_{\mathsf{BG}, J} = m_J |\ell \cdot \nu(y)| \mu_J.$$

Finally, we obtain the following averaging property:

Proposition 3.6. *Let g be a continuous function on $\mathcal{Z}$. Then, one has*

$$\frac{1}{n} \sum_{k=1}^n (g \circ \varphi_\ell^{\tau_k})(0) \xrightarrow[n \to +\infty]{} \frac{\pi}{|\Gamma| \, |\mathbb{T}_\ell|} \int_{\mathcal{Z}} g \, d\mu_{\mathsf{BG}}.$$

Again, we refer to [KMW03, BW10, Col15] for earlier versions of this result for quantum graphs endowed with Kirchhoff conditions and with a generic metric ℓ. In particular, this proposition implies

$$\mu_{\mathsf{BG}}(\mathcal{Z}) = \frac{|\Gamma| \, |\mathbb{T}_\ell|}{\pi}. \tag{3.5}$$

Proof. Let $\varepsilon > 0$. We introduce a family of nonnegative continuous functions $\tilde{\chi}$ and $(\chi_J)_{J \in \mathcal{J}_\ell}$ on $\mathcal{Z}$ such that

$$\tilde{\chi} + \sum_{J \in \mathcal{J}_\ell} \chi_J = 1,$$

$\tilde{\chi}$ is supported in $\mathcal{Z}(\varepsilon)$ and χ_J is compactly supported in $\mathcal{Z}_J$. By Lemmas 3.4 and 3.5 applied to $\chi_J g$, $J \in \mathcal{J}_\ell$, and $\tilde{\chi} g$, respectively, we have

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^n (g \circ \varphi_\ell^{\tau_k})(0) = \sum_{J \in \mathcal{J}_\ell} \int_{\mathcal{Z}_J} \chi_J g \, d\mu_{\mathsf{BG}, J} + \mathcal{O}(\varepsilon).$$

We let ε go to 0, and by the dominated convergence Theorem we obtain the expected result. $\qquad\square$

4. SEPARATION AND LOCALIZATION OF THE EIGENVALUES

In this section, we will give an acccurate description of the localization of the eigenvalues of H_α when $\alpha \in \mathbb{C}$. This will rely on a careful analysis that will allow us to view in a quantitative manner the eigenvalues of H_α as a perturbation of the eigenvalues of H_0 as the real part of the spectral parameter tends to $+\infty$. We proceed in two steps. First, we localize a sequence of eigenvalues of H_α near the eigenvalues of H_0. Then, we show that we have indeed found all the eigenvalues. All this analysis is summarized in Proposition 4.4 which is the main result of this section and from which Propositions 1.1, 1.2, 1.4 and 1.5 follow.

To that aim, given $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}^*\backslash\mathcal{T}_\infty$, we set

$$\psi_\alpha(z) = \Psi(z\ell) - \frac{\alpha}{z} = -\sum_{j=1}^N \cot an(z\ell_j) - \frac{\alpha}{z}.$$

The functions ψ_α, $\alpha \in \mathbb{C}$, are holomorphic on $\mathbb{C}^*\backslash\mathcal{T}_\infty$ and z^2 is an eigenvalue of H_α inside $\mathbb{C}^*\backslash\mathcal{T}_\infty$ if and only if $\psi_\alpha(z) = 0$ (see the discussion of Paragraph 2.4). In particular, the positive roots of ψ_0 outside $\mathcal{T}_\infty$ are given by the τ_n such that τ_n^2 is not an eigenvalue of H_∞.

4.1. Upper bound on the set of eigenvalues. We set

$$\mathcal{N} = \{n \in \mathbb{N}^* : \tau_n \notin \mathcal{T}_\infty\},$$

and, for $n \in \mathcal{N}$,

$$\rho_n = \frac{1}{\psi_0'(\tau_n)} = \frac{1}{\nabla_{\tau_n\ell}\Psi \cdot \ell}.$$

Then, the following holds:

Lemma 4.1.   (i) *For $n \in \mathcal{N}$, one has*

$$0 < \rho_n \leqslant \min\left\{|\Gamma|^{-1}, \frac{\sin(\tau_n\ell_1)^2}{\ell_1}, \dots, \frac{\sin(\tau_n\ell_N)^2}{\ell_N}\right\}. \tag{4.1}$$

(ii) *Let $\gamma_0 = \frac{1}{16eN}$. Then, for $n \in \mathcal{N}$, one has $D(\tau_n, \gamma_0\rho_n) \cap \mathcal{T}_\infty = \varnothing$ and, for $\gamma \in [0, \gamma_0]$ and $z \in D(\tau_n, \gamma\rho_n)$,*

$$\left|\psi_0(z) - \frac{z - \tau_n}{\rho_n}\right| \leqslant \frac{\gamma}{2}. \tag{4.2}$$

Proof. •   For $z \in \mathbb{C}^*\backslash\mathcal{T}_\infty$, one has

$$\psi_0'(z) = \sum_{j=1}^N \frac{\ell_j}{\sin(\ell_j z)^2} = \sum_{j=1}^N \ell_j\big(1 + \cot an^2(z\ell_j)\big), \tag{4.3}$$

and

$$\psi_0''(z) = -2\sum_{j=1}^N \frac{\ell_j^2 \cos(z\ell_j)}{\sin(z\ell_j)^3}. \tag{4.4}$$

In particular, (4.3) gives (4.1).

- Let $\gamma \in]0, \gamma_0]$, $n \in \mathcal{N}$, $j \in [\![1, N]\!]$ and $z \in D(\tau_n, \gamma \rho_n)$. Since $\gamma \leqslant 1$ and $\ell_j \rho_n \leqslant 1$ one has

$$|\cos(z\ell_j)| \leqslant e^{\ell_j \gamma \rho_n} \leqslant e,$$

and hence, since we also have $\gamma \leqslant 1/2e$,

$$|\sin(z\ell_j)| \geqslant |\sin(\tau_n \ell_j)| - |z - \tau_n| \ell_j e \geqslant \ell_j^{\frac{1}{2}} \rho_n^{\frac{1}{2}} - \gamma \rho_n \ell_j e \geqslant \frac{\ell_j^{\frac{1}{2}} \rho_n^{\frac{1}{2}}}{2}.$$

In particular, $F_D(z\ell) \neq 0$, so $D(\tau_n, \gamma \rho_n) \cap \mathcal{T}_\infty = \varnothing$. Then, by (4.4) we have

$$|\psi_0''(z)| \leqslant \frac{16e}{\rho_n^{3/2}} \sum_{j=1}^{N} \sqrt{\ell_j}. \tag{4.5}$$

From this one can infer that, for every (z, z_1) in $D(\tau_n, \gamma \rho_n)$,

$$\frac{|z - \tau_n|^2}{2} |\psi_0''(z_1)| \leqslant 8\gamma^2 e \sum_{j=1}^{N} \sqrt{\rho_n \ell_j} \leqslant \frac{\gamma}{2},$$
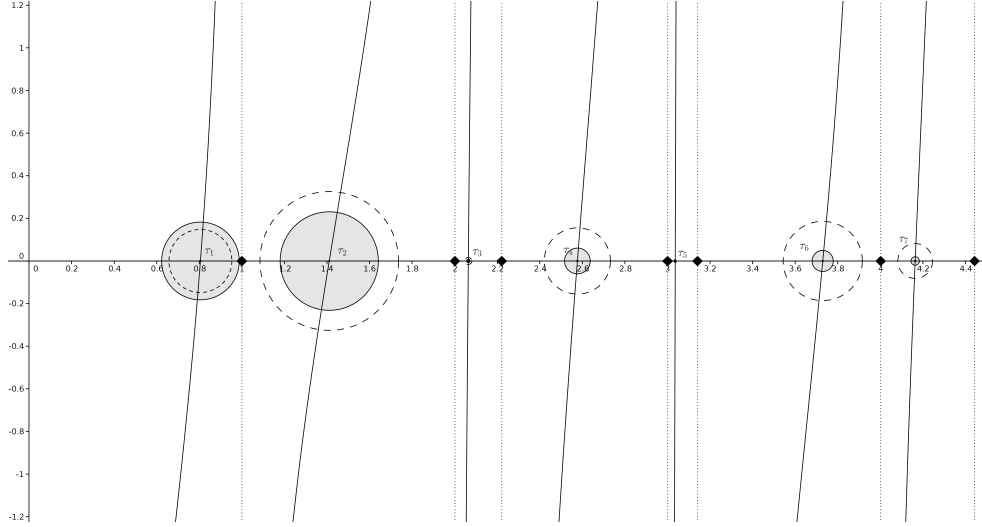
and the result follows by Taylor expansion. $\qquad\square$

FIGURE 2. The graph of Ψ for $\ell = (1, \sqrt{2}, \pi)$, with the elements of $\mathcal{T}_\infty$ (diamonds), τ_1 to τ_7, and the corresponding disks of radii $2\rho_n/\tau_n$ (filled) and $2\rho_n$ (dotted).

Proposition 4.2. *Let $\gamma_0 > 0$ be given by Lemma 4.1. Let $r > 0$. There exists $n_r \in \mathbb{N}^*$ such that, for every $n \in \mathcal{N}$ with $n \geqslant n_r$ and for every $\alpha \in D(r)$, there is a unique*

$z_n(\alpha) \in D(\tau_n, \gamma_0\rho_n)$ *such that* $z_n(\alpha)^2$ *is an eigenvalue of* H_α. *Moreover,* $z_n(\alpha)$ *depends analytically on* $\alpha \in D(r)$, $z_n(\alpha)^2$ *is an algebraically simple eigenvalue of* H_α, *and*

$$|z_n(\alpha) - \tau_n| \leqslant \frac{8r\rho_n}{\tau_n}. \tag{4.6}$$

Proof. For $\gamma \in]0, \gamma_0]$, $n \in \mathcal{N}$ and z on the circle $C(\tau_n, \gamma\rho_n)$, one has by (4.2)

$$|\psi_0(z)| \geqslant \frac{\gamma}{2}.$$

On the other hand, since $\rho_n \leqslant |\Gamma|^{-1}$, one has, for n large enough,

$$\frac{|\alpha|}{|z|} \leqslant \frac{r}{\tau_n - \gamma\rho_n} \leqslant \frac{2r}{\tau_n}.$$

Thus, there exists $n_r \in \mathbb{N}^*$ such that for $n \in \mathcal{N}$ with $n \geqslant n_r$, $\gamma \in \left[\frac{8r}{\tau_n}, \gamma_0\right]$ and $z \in C(\tau_n, \gamma\rho_n)$ we have

$$|\psi_\alpha(z)| \geqslant |\psi_0(z)| - \frac{|\alpha|}{|z|} \geqslant \frac{2r}{\tau_n} > 0.$$

This proves that, for every $\alpha \in D(r)$ and every $\gamma \in \left[\frac{8r}{\tau_n}, \gamma_0\right]$, the operator H_α has no eigenvalue on the set $\{z^2, z \in C(\tau_n, \gamma\rho_n)\}$. By continuity of the spectrum, the number of eigenvalues counted with multiplicity in $\{z^2, z \in D(\tau_n, \gamma\rho_n)\}$ does not depend on $\alpha \in D(r)$.

By Remark 2.8, τ_n is the only τ in $D(\tau_n, \gamma\rho_n)$ such that τ^2 is an eigenvalue of H_0. Thus, for $\alpha \in D(r)$, there is exactly one $z_n(\alpha) \in D(\tau_n, \gamma\rho_n)$ such that $z_n(\alpha)^2$ is an eigenvalue of H_α. The analyticity of this eigenvalue comes from the usual perturbation theory for parameter dependant operators (see [Kat80]). $\qquad\square$

From this point on, we set, for $r > 0$,

$$\mathcal{N}_r = \{n \in \mathcal{N} : n \geqslant n_r\}.$$

4.2. **Lower bound on the set of eigenvalues.**
At this stage, *we have identified some eigenvalues of* H_α. If τ^2 is an eigenvalue of H_0 which is also an eigenvalue of H_∞ then it is also an eigenvalue of H_α, and the multiplicities of τ^2 as an eigenvalue of H_0 or H_α coincide. Otherwise, τ^2 is a simple eigenvalue of H_0, and around τ^2 (in a sense made precise by Proposition 4.2), there is a unique simple eigenvalue of H_α. *Our purpose is now to prove that we have in fact found all the eigenvalues of* H_α. For this we will need the following lemma.

Lemma 4.3. *We have*

$$\limsup_{\substack{n \in \mathcal{N} \\ n \to +\infty}} \rho_n > 0.$$

Proof. Let χ be a continuous function from $\mathcal{Z}$ to $[0, 1]$ compactly supported in $\mathcal{Z}_\varnothing$ and equal to 1 on some open subset thereof. Then the function

$$f : y \mapsto \frac{\chi(y)}{\nabla_y \Psi \cdot \ell}$$

can be seen as a continuous function on $\mathcal{Z}$, compactly supported in $\mathcal{Z}_\varnothing$. By Lemma 3.4 we have

$$\frac{1}{n} \sum_{k=1}^{n} (f \circ \varphi_\ell^{\tau_k})(0) \xrightarrow[n \to +\infty]{} \frac{\pi}{|\Gamma|\,|\mathbb{T}_\ell|} \int_{\mathcal{Z}_\varnothing} f \, d\mu_{\mathsf{BG},\varnothing} > 0.$$

This implies in particular that ρ_n does not go to 0 as $n \in \mathcal{N}$ goes to $+\infty$. $\square$

Now we can state the main result of this section which compares the set of eigenvalues of H_α with the ones of H_0.

Proposition 4.4. *Let $r > 0$ and $n_r \in \mathbb{N}^*$ be given by Proposition 4.2. Then, for every $\alpha \in D(r)$, the spectrum of H_α is given by a sequence of eigenvalues $(z_n(\alpha)^2)_{n \in \mathbb{N}^*}$ (repeated according to algebraic multiplicities) such that the following assertions hold for $n \geqslant n_r$.*

(i) If $n \notin \mathcal{N}$ then $z_n(\alpha) = \tau_n$.

(ii) If $n \in \mathcal{N}$ then $z_n(\alpha)$ is given by Proposition 4.2. In particular, there exists a disk in $\mathbb{C}$ centered at τ_n which contains $z_n(\alpha)$ but does not contain τ_m or $z_m(\alpha)$ for any $m \neq n$.

(iii) The functions $\alpha \mapsto z_n(\alpha)$ are analytic,

(iv) The geometric and algebraic multiplicity of the eigenvalue $z_n(\alpha)^2$ are equal, and in particular this multiplicity is 1 if $n \in \mathcal{N}$.

Propositions 1.1, 1.2, 1.4 and 1.5 are consequences of the previous proposition combined with Remark 2.8 (for the proof of Proposition 1.2).
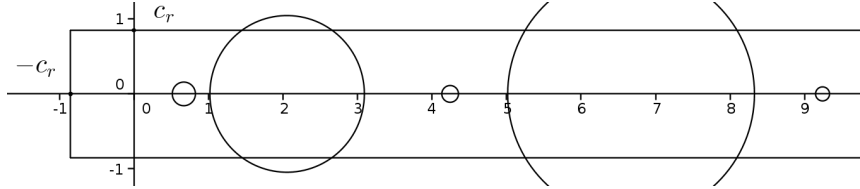

FIGURE 3. The sets $\{z^2 : |z - \tau_n| = \gamma_0 \rho_n\}$ for $\ell = (1, \sqrt{2}, \pi)$. They are close to the disks centered at τ_n^2 with radius $2\gamma_0 \tau_n \rho_n$. By Lemma 4.3, this radius is regularly greater than c_r.

Proof. Let c_r be given by Proposition 2.5. We set

$$\Omega_r = \{\lambda \in \mathbb{C} : \mathsf{Re}(\lambda) \geqslant -c_r, |\mathsf{Im}(\lambda)| \leqslant c_r\} \setminus \bigcup_{n \in \mathcal{N}_r} \left\{ z^2 : \frac{8r\rho_n}{\tau_n} < |z - \tau_n| < \gamma_0 \rho_n \right\}.$$

By Propositions 2.5 and 4.2, we have $\mathsf{Sp}(H_\alpha) \subset \Omega_r$ for all $\alpha \in D(r)$. By Lemma 4.3, all the connected components of Ω_r are bounded (see Figure 3). By the perturbation theory of Kato [Kat80], the number of eigenvalues of H_α (counted with algebraic multiplicities) in each of these connected components does not depend on α. Choosing n_r larger if necessary, we can assume that there exists a finite union $\tilde{\Omega}_r$ of connected components of Ω_r which contains τ_n^2 if and only if $n < n_r$. Then, for all $\alpha \in D(r)$, $\tilde{\Omega}_r$ contains exactly $n_r - 1$ eigenvalues of H_α (counted with multiplicities), which can be denoted by $z_n(\alpha)^2$,

$1 \leqslant n < n_r$. The other connected components of Ω_r contain eigenvalues which have already been identified. If such a connected component contains τ_n^2 for some $n \in \mathcal{N}$, $n \geqslant n_r$, it contains exactly one eigenvalue of H_α, which we have denoted by $z_n(\alpha)^2$. We also know that for $n \notin \mathcal{N}$, the eigenvalue τ_n^2 remains an eigenvalue of H_α with the same multiplicity. Since the number of eigenvalue in each connected components is fixed, there cannot be any other eigenvalue for H_α in Ω_r. All the properties enumerated in the proposition then follow from Lemma 2.7 and Proposition 4.2. $\qquad\square$

In this proof, it is important that all the connected components of Ω_r are bounded. Otherwise, even with the continuity of the spectrum, an eigenvalue could disappear (by escaping to infinity) or be created (by coming from infinity).

5. Proofs of the main results

Let $\alpha \in \mathbb{C}$. We have seen that for n large enough, the eigenvalue $z_n(\alpha)^2$ of H_α is either equal to τ_n^2, or close to τ_n^2 in the sense of Proposition 4.2. Now we prove that the difference $\delta_n(\alpha)$ (see (1.7)) is, at first order, a simple function of τ_n, and we use the results of Section 3.3 to prove Theorem 1.3. Finally, as a by-product of our analysis, we will prove Propositions 1.6 and 1.7.

5.1. The limit measure.

Lemma 5.1. *Let $r > 0$ and let $n_r \in \mathbb{N}^*$ be given by Proposition 4.2. Then, for $n \in \mathcal{N}_r$, we have*

$$\delta_n(\alpha) = 2\alpha\rho_n + \underset{n\to+\infty}{\mathcal{O}}(n^{-1}),$$

where the rest is uniform with respect to $\alpha \in D(r)$.

This asymptotic expression will allow us to relate the distribution of $\delta_n(\alpha)$ for large n to the Barra-Gaspard measure, as in [KMW03, Th. 3] where the asymptotic distribution of $\psi_0'(\tau_n) = \rho_n^{-1}$ on compact intervals of $\mathbb{R}$ was described for $\Lambda_\ell = \{0\}$ (in that case, $\mathcal{N} = \mathbb{N}^*$).

Proof. For $\alpha \in D(r)$ and $n \in \mathcal{N}_r$, we set

$$\eta_n(\alpha) = z_n(a) - \tau_n.$$

By (4.6) and (4.1), we have

$$\eta_n(\alpha) = \mathcal{O}(\rho_n \tau_n^{-1}) = \mathcal{O}(\tau_n^{-1}).$$

Then, by (4.5) and (4.6),

$$|\psi_0(z_n(\alpha)) - \eta_n(\alpha)\psi_0'(\tau_n)| \leqslant \frac{|\eta_n(\alpha)|^2}{2} \sup_{|z-\tau_n|\leqslant|\eta_n(\alpha)|} |\psi_0''(z)| \lesssim \frac{\rho_n^{1/2}}{\tau_n^2} \lesssim \frac{1}{\tau_n^2}.$$

We have

$$0 = \psi_0\big(\tau_n + \eta_n(\alpha)\big) - \frac{\alpha}{\tau_n + \eta_n(\alpha)} = \eta_n(\alpha)\psi_0'(\tau_n) - \frac{\alpha}{\tau_n} + \mathcal{O}(\tau_n^{-2}).$$

Since ρ_n is bounded, this gives

$$\eta_n(\alpha) = \frac{\alpha \rho_n}{\tau_n} + \mathcal{O}(\tau_n^{-2}), \tag{5.1}$$

and then

$$\delta_n(\alpha) = 2\tau_n \eta_n(\alpha) + \eta_n(\alpha)^2 = 2\alpha \rho_n + \mathcal{O}(\tau_n^{-1}). \tag{5.2}$$

We conclude with (1.6). $\qquad\square$

We consider on $\mathcal{Z}$ the continuous function defined by

$$\Phi(y) = \frac{2}{\nabla_y \Psi \cdot \ell} = \frac{2}{\sum_{j=1}^{N} \ell_j (1 + \cotan^2(y_j))}, \qquad \text{if} \quad y \in \mathcal{Z}_\varnothing,$$

and $\Phi(y) = 0$ otherwise.

Proposition 5.2. *The first statement of Theorem 1.3 holds if we set*

$$\mu_\ell = \frac{\pi}{|\Gamma| \, |\mathbb{T}_\ell|} \left(\Phi_* \mu_{\mathsf{BG},\varnothing} + \delta_0 \sum_{J \in \mathcal{J}_\ell, J \neq \varnothing} \mu_{\mathsf{BG},J}(\mathcal{Z}_J) \right), \tag{5.3}$$

where $\Phi_* \mu_{\mathsf{BG},\varnothing}$ is the pushforward of the measure $\mu_{\mathsf{BG},\varnothing}$ by Φ.

Proof. We first observe that the image of Φ is included in $[0, 2\,|\Gamma|^{-1}]$ so, by (3.5), the right-hand side of (5.3) defines a probability measure on $[0, 2\,|\Gamma|^{-1}]$.

Let $\alpha \in \mathbb{C}$. Let f be a continuous function on $\mathbb{C}$. By Lemma 5.1 we have

$$\frac{1}{n} \sum_{k=1}^{n} f(\delta_n(\alpha)) = \frac{1}{n} \sum_{k=1}^{n} f(\alpha \Phi(\varphi_\ell^{\tau_n}(0))) + \underset{n \to +\infty}{\mathcal{O}}(n^{-1}).$$

Then, by Proposition 3.6 applied to $g = f \circ (\alpha \Phi)$,

$$\frac{1}{n} \sum_{k=1}^{n} f(\delta_n(\alpha)) \xrightarrow[n \to +\infty]{} \frac{\pi}{|\Gamma| \, |\mathbb{T}_\ell|} \left(\int_{\mathcal{Z}_\varnothing} (f \circ (\alpha \Phi)) \, d\mu_{\mathsf{BG},\varnothing} + f(0) \sum_{J \in \mathcal{J}_\ell, J \neq \varnothing} \mu_{\mathsf{BG},J}(\mathcal{Z}_J) \right).$$

The proposition follows. $\qquad\square$

We observe that if $\ell_j/\ell_1 \in \mathbb{Q}$ for all $j \in [\![1, N]\!]$ then $\Lambda_\ell^\perp$ is of dimension 1. So, by Lemma 3.3, $\mathcal{Z}$ consists of isolated points, and Φ only takes a finite number of values (including 0, at the point 0). This implies that μ_ℓ is a finite sum of Dirac masses, including the Dirac mass at 0.

To conclude the proof of Theorem 1.3, it remains to prove the last statement. It is given by the following proposition.

Proposition 5.3. *Suppose that $\Lambda_\ell = \{0\}$. Then the measure μ_ℓ is absolutely continuous with respect to the Lebesgue measure and its support is the interval $[0, 2|\Gamma|^{-1}]$.*

Proof. We first observe that when $N_\ell = N$ we have $\mathcal{J}_\ell = \{\varnothing\}$. Then, for $f \in C_c^0(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f \, d\mu_\ell = \frac{\pi}{|\Gamma|(2\pi)^N} \int_{\mathcal{Z}_\varnothing} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing}.$$

The image of Φ is included in $[0, 2\,|\Gamma|^{-1}]$, and considering the point $\left(\theta, -\theta, \frac{\pi}{2}, \ldots, \frac{\pi}{2}\right)$ (modulo 2π) for θ going from $\frac{\pi}{2}$ to 0, we see that the image of Φ is in fact exactly $[0, 2\,|\Gamma|^{-1}]$.

We extend Φ to a map $\tilde{\Phi} : y \mapsto 2(\nabla_y \Psi \cdot \ell)^{-1}$ on an open neighborhood of $\mathcal{Z}_\varnothing$ in $\mathbb{T}^N$. For $y \in \mathcal{Z}_\varnothing$ we have

$$\nabla_y \tilde{\Phi} = \Phi(y)^2 \left(\frac{\ell_1 \cos(y_1)}{\sin^3(y_1)}, \ldots, \frac{\ell_N \cos(y_N)}{\sin^3(y_N)} \right)$$

and

$$\nabla_y \Psi = \left(\frac{1}{\sin^2(y_1)}, \ldots, \frac{1}{\sin^2(y_N)} \right).$$

If these two vectors are colinear we have

$$\ell_1 \cotan(y_1) = \cdots = \ell_N \cotan(y_N).$$

Since $\Psi(y) = 0$, this only happens when $y_j = \frac{\pi}{2} \mod \pi$ for all $j \in [\![1, N]\!]$. Then, for $t \in (0, 2|\Gamma|^{-1})$, the subset $\Phi^{-1}(\{t\})$ is a smooth submanifold of codimension 1 in $\mathcal{Z}_\varnothing$.

Let $f \in \mathcal{C}_c^\infty((0, 2|\Gamma|^{-1}))$. Applying the co-area formula (see [Fed96]), we can write

$$\int_{\mathcal{Z}_\varnothing} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing} = \int_0^{2|\Gamma|^{-1}} f(t) \left(\int_{\Phi^{-1}(\{t\})} \frac{|\nu(y) \cdot \ell|}{\|\nabla \Phi\|} \, d\mu_{\Phi^{-1}(\{t\})} \right) dt,$$

where $\mu_{\Phi^{-1}(\{t\})}$ is the Lebesgue measure on $\Phi^{-1}(\{t\})$. Hence, the measure μ_ℓ is absolutely continuous with respect to the Lebesgue measure on the interval $(0, 2|\Gamma|^{-1})$. We can also note that its density is positive almost everywhere since $|\nu(y) \cdot \ell| > 0$ on $\mathcal{Z}_\varnothing$ and $\Phi^{-1}(\{t\})$ has positive measure for all $t \in]0, |\Gamma|^{-1}[$. Since Φ does not vanish on $\mathcal{Z}_\varnothing$, the measure μ_ℓ does not put any mass at $t = 0$, and $\Phi^{-1}(\{2|\Gamma|^{-1}\})$ is reduced to 2^N points so $\mu_{\mathsf{BG},\varnothing}\left(\Phi^{-1}(\{|\Gamma|^{-1}\})\right) = 0$. $\qquad\square$

Remark 5.4. More generally, we can consider the case where $1 < N_\ell \leqslant N$. We denote by P_ℓ the orthogonal projection of $\mathbb{R}^N$ on the vector space $\Lambda_\ell^\perp$. Then, we write down the partition

$$\mathcal{Z}_\varnothing = \mathcal{Z}_\varnothing^{\mathrm{reg}} \sqcup \mathcal{Z}_\varnothing^{\mathrm{sing}},$$

where

$$\mathcal{Z}_\varnothing^{\mathrm{reg}} = \{ y \in \mathcal{Z}_\varnothing : \mathrm{rk}(P_\ell \nabla_y \Psi, P_\ell \nabla_y \tilde{\Phi}) = 2 \},$$

and

$$\mathcal{Z}_\varnothing^{\mathrm{sing}} = \{ y \in \mathcal{Z}_\varnothing : \mathrm{rk}(P_\ell \nabla_y \Psi, P_\ell \nabla_y \tilde{\Phi}) = 1 \}.$$

We can decompose the integral following this partition of $\mathcal{Z}_\varnothing$:

$$\int_{\mathcal{Z}_\varnothing} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing} = \int_{\mathcal{Z}_\varnothing^{\mathrm{reg}}} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing} + \int_{\mathcal{Z}_\varnothing^{\mathrm{sing}}} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing}.$$

To the first term, we can again apply the co-area formula

$$\int_{\mathcal{Z}_\varnothing^{\mathrm{reg}}} (f \circ \Phi) \, d\mu_{\mathsf{BG},\varnothing} = \int_0^{2|\Gamma|^{-1}} f(t) \left(\int_{\Phi^{-1}(\{t\}) \cap \mathcal{Z}_\varnothing^{\mathrm{reg}}} \frac{|\nu(y) \cdot \ell|}{\|\nabla \Phi\|} \, d\mu_{\Phi^{-1}(\{t\})} \right) dt,$$

which gives as above a contribution to the absolutely continuous part of the the measure μ_ℓ. In particular, if

$$\mu_{\mathcal{Z}_\varnothing}(\mathcal{Z}_\varnothing^{\mathrm{sing}}) = 0,$$

then the limit measure is a linear combination of a Dirac mass at 0 and of an absolutely continuous measure.

5.2. **Rates of convergence.** In this paragraph we prove Propositions 1.6 and 1.7.

Proof of Proposition 1.6. We already know that if two lengths are commensurable then $\lambda_n(\alpha) = \lambda_n(0)$ (and hence $\delta_n(\alpha) = 0$) for an infinite number of $n \in \mathbb{N}^*$. Now assume that, for instance, ℓ_1/ℓ_2 does not belong to $\mathbb{Q}$.

We have seen in Proposition 4.2 that $\delta_n(\alpha)$ is small when ρ_n is small. This happens when τ_n is close to $\mathcal{T}_\infty$. By the Dirichlet Approximation Theorem, there exists a sequence $((p_k, q_k))_{k \in \mathbb{N}}$ in $\mathbb{Z}_+^2$ such that q_k goes to $+\infty$ and, for all $k \in \mathbb{N}$,

$$\left| \frac{\ell_1}{\ell_2} - \frac{p_k}{q_k} \right| \leqslant \frac{1}{q_k^2}.$$

Thus, one has

$$\left| \frac{\pi q_k}{\ell_2} - \frac{\pi p_k}{\ell_1} \right| \leqslant \frac{\pi}{q_k \ell_1}.$$

Since $\pi q_k/\ell_1$ and $\pi p_k/\ell_2$ belong to $\mathcal{T}_\infty$, there exists $n_k \in \mathbb{N}^*$ such that τ_{n_k} is between them (see Remark 2.8). In particular,

$$|\tau_{n_k}\ell_2 - \pi q_k| = \ell_2 \left| \tau_{n_k} - \frac{\pi q_k}{\ell_2} \right| \leqslant \frac{\pi \ell_2}{q_k \ell_1},$$

so, by Lemma 4.1,

$$\rho_{n_k} \lesssim \sin(\tau_{n_k}\ell_2)^2 \lesssim \frac{1}{q_k^2} \lesssim \frac{1}{\tau_{n_k}^2}.$$

By Proposition 4.2, there exists an eigenvalue $\lambda_{n_k}(\alpha) = z_{n_k}(\alpha)^2$ of H_α such that

$$z_{n_k}(\alpha) \in D\left(\tau_{n_k}, \frac{8|\alpha|\rho_{n_k}}{\tau_{n_k}} \right).$$

Then

$$|\delta_{n_k}(\alpha)| \lesssim \tau_{n_k} |z_{n_k}(\alpha) - \tau_{n_k}| \lesssim \frac{1}{\tau_{n_k}^2},$$

and the conclusion follows. $\qquad\square$

Let $\gamma > 0$, we say that ℓ is γ-Diophantine (see [Pös01]) if there exists $C > 0$ such that, for every $\kappa \in \mathbb{Z}^N \backslash \{0\}$,

$$|\kappa \cdot \ell| \geqslant C\|\kappa\|^{-\gamma}, \tag{5.4}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^N$. In particular, $\kappa \cdot \ell \neq 0$ for all $\kappa \in \mathbb{Z}\backslash\{0\}$ in this case. According to [Pös01], the set of γ-Diophantine vectors has full Lebesgue measure for $\gamma > N - 1$ and is empty when $\gamma < N - 1$. The Diophantine properties of ℓ allow to be more precise on the unique ergodicity of the flow φ_ℓ^t:

Lemma 5.5. *Suppose that ℓ is γ-Diophantine for some $\gamma > 0$. Then there exists $C_1 > 0$ such that for $a \in \mathcal{C}^\infty(\mathbb{T}^N)$, $T > 0$ and $x \in \mathbb{T}^N$ one has*

$$\left| \frac{1}{T} \int_0^T a \circ \varphi_\ell^t(x) dt - \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} a(y) \, dy \right| \leqslant \frac{C_1}{T} \sum_{\kappa \in \mathbb{Z}^N \setminus \{0\}} \|\kappa\|^\gamma \left| \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} a(y) e^{-i\kappa \cdot y} \, dy \right|.$$

Proof. The proof is immediate once we have decomposed a in Fourier series. $\qquad\square$

We now proceed to the proof of Proposition 1.7.

Proof of Proposition 1.7. Let $\gamma \in]N - 1, N[$. Let $\Omega \subset (\mathbb{R}_+^*)^N$ be of full Lebesgue measure and such that any vector in Ω is γ-Diophantine. We assume that ℓ belongs to Ω. Let $\sigma \in]0, 2|\Gamma|^{-1}]$ and let y_0 in $\mathcal{Z}_\varnothing$ be such that

$$\Phi(y_0) = \sigma. \tag{5.5}$$

Let $\chi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be supported in the unit ball and be equal to 1 on a small neighborhood of 0. For $y \in \mathbb{R}^N$ and $r > 0$ we set

$$\chi_r(y) = \sum_{\kappa \in \mathbb{Z}^N} \chi\left(\frac{y - y_0 + 2\kappa\pi}{r} \right).$$

Here we have identified y_0 with a point in $\mathbb{R}^N$, and then χ_r can be seen as a function on $\mathbb{T}^N$. Let $\kappa \in \mathbb{Z}^N \setminus \{0\}$ and $j \in [\![1, N]\!]$ such that $|\kappa_j| = \max_{1 \leqslant k \leqslant N} |\kappa_k|$. Then, for $p \in \mathbb{N}$, we have

$$\left| \int_{\mathbb{T}^N} \chi_r(y) e^{-i\kappa \cdot y} \frac{dy}{(2\pi)^N} \right| = \frac{1}{\kappa_j^p} \left| \int_{\mathbb{T}^N} \partial_{y_j}^p \chi_r(y) e^{-i\kappa \cdot y} \frac{dy}{(2\pi)^N} \right| \leqslant C r^{N-p} \|\kappa\|^{-p}, \tag{5.6}$$

where C only depends on χ and p. Let $T > 0$ and $r \in (0, 1]$. We apply Lemma 5.5 with $a = \chi_r$ and $x = \varphi_\ell^T(0)$, and (5.6) with $p = 2N$. This gives

$$\left| \int_T^{2T} (\chi_r \circ \varphi_\ell^\tau)(0) \, d\tau - r^N T \int_{\mathbb{T}^N} \chi_1(y) \frac{dy}{(2\pi)^N} \right| \leqslant C r^{-N}.$$

Let $\varepsilon > 0$. For $T \geqslant 1$ we set

$$r_T = T^{-\frac{1}{2N} + 2\varepsilon}.$$

Then

$$\int_T^{2T} (\chi_{r_T} \circ \varphi_\ell^\tau)(0) \, d\tau \underset{T \to +\infty}{\sim} r_T^N T \int_{\mathbb{T}^N} \chi_1(y) \frac{dy}{(2\pi)^N}.$$

In particular, for T large enough the left-hand side is positive, so there exists $\tau \in [T, 2T]$ such that $\chi_{r_T}(\varphi_\ell^\tau(0)) \neq 0$, and in particular $|\varphi_\ell^\tau(0) - y_0| \leqslant r_T$. Now, as ℓ is transverse to $\mathcal{Z}_\varnothing$, there exists $c > 0$ independant of T large enough and $n_T \in \mathbb{N}^*$ such that $\tau_{n_T} \in \left[\frac{T}{2}, 3T \right]$ and $\left| \varphi_\ell^{\tau_{n_T}}(0) - y_0 \right| \leqslant c r_T$. By (5.2) and as Φ is of class $\mathcal{C}^1$ near y_0, we obtain

$$\delta_{n_T}(\alpha) = \frac{2\alpha}{\psi_0'(\tau_{n_T})} + \mathcal{O}(\tau_{n_T}^{-1}) = \alpha\Phi(y_0) + \mathcal{O}(r_T) + \mathcal{O}(\tau_{n_T}^{-1}) = \alpha\sigma + \mathcal{O}(\tau_{n_T}^{-\frac{1}{2N} + 2\varepsilon}).$$

This concludes the proof. $\qquad\square$

References

[AJ04]   K. Ammari and M. Jellouli. Stabilization of star-shaped networks of strings. *Differential and Integral Equations*, 17(11-12):1395–1410, 2004.

[AJK05] K. Ammari, M. Jellouli, and M. Khenissi. Stabilization of generic trees of strings. *Journal of Dynamical and Control Systems*, 11(2):177–193, 2005.

[AJK16] R. Assel, M. Jellouli, and M. Khenissi. Optimal decay rate for the local energy of a unbounded network. *Journal of Differential Equations*, 261(7):4030 – 4054, 2016.

[AL03]   M. Asch and G. Lebeau. The spectrum of the damped wave operator for a bounded domain in $\mathbb{R}^2$. *Exp. Math.*, 12(2):227–241, 2003.

[AMN13] F. Abdallah, D. Mercier, and S. Nicaise. Exponential stability of the wave equation on a star shaped network with indefinite sign damping. *Palest. J. Math.*, 2(2):113–143, 2013.

[BG00]   F. Barra and P. Gaspard. On the level spacing distribution in quantum graphs. *J. Stat. Phys.*, 101(1-2):283–319, 2000.

[BK13]   G. Berkolaiko and P. Kuchment. *Introduction to quantum graphs*. Mathematical surveys and monographs. American Mathematical Society, Providence, Rhode Island, 2013.

[BKW04] G. Berkolaiko, J.P. Keating, and B. Winn. No quantum ergodicity for star graphs. *Comm. Math. Phys.*, 250(2):259–285, 2004.

[BW10]   G. Berkolaiko and B. Winn. Relationship between scattering matrix and spectrum of quantum graphs. *Trans. Am. Math. Soc.*, 362(12):6261–6277, 2010.

[CCW87] G. Chen, M. Coleman, and H. H. West. Pointwise stabilization in the middle of the span for second order systems, nonuniform and uniform exponential decay of solutions. *SIAM J. Appl. Math.*, 47:751–780, 1987.

[Col15] Y. Colin de Verdière. Semi-classical measures on quantum graphs and the Gauß map of the determinant manifold. *Annales Henri Poincaré*, 16:347–364, 2015.

[CT18]   Yves Colin de Verdière and Françoise Truc. Topological resonances on quantum graphs. *Ann. Henri Poincaré*, 19(5):1419–1438, 2018.

[DEL10] E.B. Davies, P. Exner, and J. Lipovský. Non-Weyl asymptotics for quantum graphs with general coupling conditions. *J. Phys. A, Math. Theor.*, 43(47):16, 2010.

[DP11]   E.B. Davies and A. Pushnitski. Non-Weyl resonance asymptotics for quantum graphs. *Anal. PDE*, 4(5):729–756, 2011.

[EL10]   P. Exner and J. Lipovský. Resonances from perturbations of quantum graphs with rationally related edges. *J. Phys. A, Math. Theor.*, 43(10):21, 2010.

[Fed96] H. Federer. *Geometric measure theory*. Berlin: Springer-Verlag, repr. of the 1969 ed. edition, 1996.

[FL17]   P. Freitas and J. Lipovsky. Eigenvalue asymptotics for the damped wave equation on metric graphs. *Journal of Differential Equations*, (263):2780–2811, 2017.

[Fri05] L. Friedlander. Genericity of simple eigenvalues for a metric graph. *Isr. J. Math.*, 146:149–156, 2005.

[GX09]   Y. N. Guo and G. Q. Xu. Stability and Riesz basis property for general network of strings. *J. Dyn. Control Syst.*, 15(2):223–245, 2009.

[GZ97]   B.Z. Guo and W.D. Zhu. On the energy decay of two coupled strings through a joint damper. *Journal of Sounds and Vibration*, 203(3):447–455, 1997.

[Hit02] M. Hitrik. Eigenfrequencies for damped wave equations on Zoll manifolds. *Asymptotic Anal.*, 31(3-4):265–277, 2002.

[HS04]   M. Hitrik and J. Sjöstrand. Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions I. 5(1):1–73, 2004.

[Kat80] T. Kato. *Perturbation Theory for linear operators*. Classics in Mathematics. Springer, second edition, 1980.

[KMW03] J.P. Keating, J. Marklof, and B. Winn. Value distribution of the eigenfunctions and spectral determinants of quantum star graphs. *Comm. Math. Phys.*, 241(2-3):421–452, 2003.

[Liu88] K. Liu. Energy decay problems in the design of a point stabilizer for coupled string vibrating systems. *SIAM J. Control Optim.*, 26(6):1348–1356, 1988.

[LZ16]   M. Lee and M. Zworski. A Fermi golden rule for quantum graphs. *J. Math. Phys.*, 57(9), 2016.

[NV07]   S. Nicaise and J. Valein. Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks. *Netw. Heterog. Media*, 2(3):425–479, 2007.

[Pös01] J. Pöschel. A lecture on the classical KAM Theorem. *Proc. Symp. Pure Math.*, 69:707–732, 2001.

[Sjö00] J. Sjöstrand. Asymptotic distribution of eigenfrequencies for damped wave equations. *Publ. RIMS, Kyoto Univ.*, 36:573–611, 2000.

[VZ09]   J. Valein and E. Zuazua. Stabilization of the wave equation on 1-d networks. *SIAM Journal on control and Optimization*, 48(4):2771–2797, 2009.

[Wal82] P. Walters. *An introduction to ergodic theory*, volume 79. Springer, New York, NY, 1982.

[Wei77] A. Weinstein. Asymptotics of eigenvalue clusters for the Laplacian plus a potential. *Duke Math. J.*, 44:883–892, 1977.

[Zua13] E. Zuazua. Control and stabilization of waves on 1-d networks. In *Modelling and optimisation of flows on networks*, pages 463–493. Springer, 2013.

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