MR2199446 (2007e:57026) 57R17 (57M25 57M27 57R57)
Mrowka, Tomasz (1-MIT); **Rollin, Yann** (4-LNDIC)
Legendrian knots and monopoles. (English summary)
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In [P. B. Kronheimer and T. S. Mrowka, Invent. Math. **130** (1997), no. 2, 209–255; MR1474156 (98h:57058)] the Seiberg-Witten invariants were defined for a 4-manifold M with boundary Y having a fixed contact structure ξ. This invariant is a map $\mathrm{sw}_{m,\xi}$ from Spin$^c(M,\xi)$ to $\mathbb{Z}$ where Spin$^c(M,\xi)$ is the set of pairs $(\mathfrak{s},h)$, where $\mathfrak{s}$ is a Spin^c structure on M and h is an isomorphism between $\mathfrak{s}|_Y$ and the canonical spin^c structure associated to ξ. The first result in this paper generalizes a slice Bennequin type inequality originally proven for Stein surfaces in [S. Akbulut and R. Matveyev, Turkish J. Math. **21** (1997), no. 1, 47–53; MR1456158 (98d:57053)] and [P. Lisca and G. Matić, Topology Appl. **88** (1998), no. 1-2, 55–66; MR1634563 (99f:57037)]. More specifically, if K is a Legendrian knot in the contact manifold (Y,ξ) and Σ is a compact connected oriented surface in M with boundary K, then for every element $(\mathfrak{s},h)$ in Spin$^c(M,\xi)$ for which the Seiberg-Witten invariant is nonzero the inequality $\chi(\Sigma)+\mathrm{tb}(K)+|r(K)|\leq 0$ holds, where $\mathrm{tb}(K)$ measures the contact framing with respect to the framing induced by the homology class of Σ and $r(K)$ is a relative Chern class determined by $(\mathfrak{s},h)$ evaluated on Σ. As a corollary of this result one sees that the Seiberg-Witten invariants of a manifold M with overtwisted contact structure ξ on its boundary are identically zero. This in turn gives another proof of Eliashberg's result that a weakly fillable contact structure is not overtwisted. In fact, this result seems to generalize Eliashberg's result in that you do not really need a symplectic filling but merely a 4-manifold bounding your contact manifold so that the relative Seiberg-Witten invariant is not identically zero; however, it is currently unknown if there is such a 4-manifold that does not come from a symplectic filling.

The main result is proven using the following excision result for the Seiberg-Witten invariants. Let M,Y, and ξ be as above and let Z be a symplectic cobordism from (Y,ξ) to (Y',ξ') such that a neighborhood of (Y,ξ) in Z is symplectomorphic to the symplectization of ξ and the map induced by inclusion from $H^1(Z,Y')$ to $H^1(Y)$ is the zero map. If M' is the manifold obtained by gluing M and Z along Y then $\mathrm{sw}_{M,\xi}\circ j=\pm\mathrm{sw}_{M',\xi;'}$ where j is the canonical map from Spin$^c(M,\xi)$ to Spin$^c(M',\xi')$. With this result in hand the main theorem is proven by attaching a symplectic 2-handle to M along K to obtain a closed surface Σ' in the resulting manifold. Then, thanks to the excision result, the adjunction inequality from [P. B. Kronheimer and T. S. Mrowka, op. cit.] completes the proof.

Reviewed by *John B. Etnyre*

References

1. **S Akbulut, R Matveyev,** *Exotic structures and adjunction inequality,* Turkish J. Math. 21 (1997) 47–53 MR1456158 MR1456158 (98d:57053)

2. **Y Eliashberg,** *Filling by holomorphic discs and its applications*, from: ”Geometry of low-dimensional manifolds, 2 (Durham, 1989)”, London Math. Soc. Lecture Note Ser. 151, Cambridge Univ. Press, Cambridge (1990) 45–67 MR1171908 MR1171908 (93g:53060)
3. **Y Eliashberg,** *Topological characterization of Stein manifolds of dimension*

$$>$$

$2, Internat. J. Math. 1 (1990) 29--46 MR 1044658$ $MR1044658 (91k:32012)$
4. **Y Eliashberg,** *A few remarks about symplectic filling*, Geom. Topol. 8 (2004) 277–293 MR2023279 MR2023279 (2005a:57022b)
5. **J B Etnyre, K Honda,** *Tight contact structures with no symplectic fillings*, Invent. Math. 148 (2002) 609–626 MR1908061 MR1908061 (2003c:57025)
6. **P Gauduchon,** *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) 11 (1997) 257–288 MR1456265 MR1456265 (98c:53034)
7. **R E Gompf,** *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998) 619–693 MR1668563 MR1668563 (2000a:57070)
8. **K Honda,** *Gluing tight contact structures*, Duke Math. J. 115 (2002) 435–478 MR1940409 MR1940409 (2003i:53125)
9. **P B Kronheimer, T S Mrowka,** *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994) 797–808 MR1306022 MR1306022 (96a:57073)
10. **P B Kronheimer, T S Mrowka,** *Monopoles and contact structures*, Invent. Math. 130 (1997) 209–255 MR1474156 MR1474156 (98h:57058)
11. **P Lisca, G Matić,** *Stein* 4*-manifolds with boundary and contact structures*, Topology Appl. 88 (1998) 55–66 MR1634563 MR1634563 (99f:57037)
12. **P Lisca, A I Stipsicz,** *An infinite family of tight, not semi-fillable contact three-manifolds*, Geom. Topol. 7 (2003) 1055–1073 MR2026538 MR2026538 (2005a:57023)
13. **P Lisca, A I Stipsicz,** *Tight, not semi-fillable contact circle bundles*, Math. Ann. 328 (2004) 285–298 MR2030378 MR2030378 (2004i:57031)
14. **C H Taubes,** *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994) 809–822 MR1306023 MR1306023 (95j:57039)
15. **A Weinstein,** *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20 (1991) 241–251 MR1114405 MR1114405 (92g:53028)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.