Discrete geometry and isotropic surfaces

a glimpse of piecewise linear symplectic geometry

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Background – Symplectic topology

(M, ω) symplectic manifolds and $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian transformations.

We want to understand Lagrangian submanifolds of M and, more generally isotropic submanifolds of M.

- Lagrangian sub. are invariant under the action of Ham.
- The question of classification of Lagrangians upto Hamiltonian isotopy remains open in many cases.
- Example: Closed Lagrangian surfaces of $\mathbb{R}^4$, with its standard symplectic form, must be diffeomorphic to a torus. But the problem of classification is open in this case.

Examples of Lagrangian submanifolds:

- $M = T^*N$ carries a symplectic form $\omega = d\lambda$.
- The zero section L of $T^*N \to N$ is a Lagrangian submanifold.
- Any closed 1-form define a Lagrangian deformation of L.
- Any exact 1-form define a Hamiltonian deformation of L.

By the Lagrangian neigbohood theorem, the above remark provides general deformations for any Lagrangian submanifold.

Background – Symplectic geometry

Lagrangian submanifolds are too flexible.
Can we find canonical representatives in a given isotopy class? (Oh)

- Consider a Kähler manifold (M, ω, g, J).
- Stationary Lagrangian submanifold are the critical points for the volume in a Ham-orbit.
- Theory analogue to minimal submanifolds.
- Idea: define a version of the mean curvature flow in this context.
- Such flow is complicated... Numerical approach ?
- Big problem: almost no examples of discrete Lagrangian surfaces in $\mathbb{R}^4$. No deformation theory.

Main result

Theorem 1 (Jauberteau-Rollin-Tapie)

Let $\ell : \Sigma \to \mathbb{R}^{2n}$ be a smooth isotropic immersion, with $n \geq 2$ and Σ a surface diffeomorphic to a torus. Then for every $\varepsilon > 0$, there exists an isotropic piecewise linear map $\hat{\ell} : \Sigma \to \mathbb{R}^{2n}$, such that for every $x \in \Sigma$

$$\|\ell(x) - \hat{\ell}(x)\| \leq \varepsilon.$$

If $n \geq 3$, we may assume that $\hat{\ell}$ is an immersion. If $n = 2$, we may assume that $\hat{\ell}$ is an immersion away from a finite union of embedded circles in Σ.

Corollary 2

Every smoothly immersed isotropic 2-torus of $\mathbb{R}^{2n}$, where $n \geq 3$, admits arbitrarily close approximations by isotropic piecewise linear immersed tori, in Hausdorff distance.

Rough idea of the proof

Start with an isotropic immersion $\ell : \Sigma \to \mathbb{R}^{2n}$, then

- Pick a triangulation of Σ with very small faces.
- The corresponding Eulidean triangles of $\mathbb{R}^{2n}$ are almost isotropic.
- Perturb the triangular mesh to obtain an isotropic mesh.

Problems:

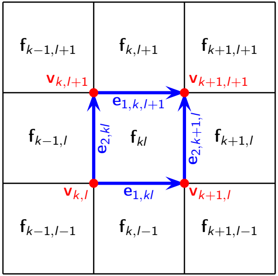
- In which direction should we move the vertices?
- The problem ill posed. The linearized problem comes with an almost kernel whose dimension goes to infinity with the number of faces.

Quadrangulations of $\mathbb{R}^2$

For a positive integer N, we define the lattice $\Lambda_N \subset \mathbb{R}^2$ and the checkers graph sublattice $\Lambda_N^{ch} \subset \Lambda_N$

$$\Lambda_N = \mathbb{Z}\frac{e_1}{N} \oplus \mathbb{Z}\frac{e_2}{N}, \quad \Lambda_N^{ch} = \mathbb{Z}\frac{e_1+e_2}{N} \oplus \mathbb{Z}\frac{e_2-e_1}{N}.$$

The points $\mathbf{v}_{kl} \in \Lambda_N$ are the vertices of a familiar square grid of $\mathbb{R}^2$ understood as a quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ of the plane, tiled by squares of size N^{-1}.



Quadrangulations of the torus

The immersion $\ell : \Sigma \to \mathbb{R}^{2n}$ induces a conformal structure $g_\Sigma = \ell^* g$ on Σ. By the uniformization theorem, there exists a conformal covering map

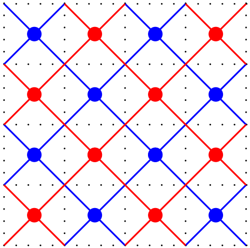
$$p : \mathbb{R}^2 \to \Sigma$$

with group of deck transformations $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$. Then $g_\sigma = p_* g_{euc}$ satisfies $g_\Sigma = \theta g_\sigma$.

- Pick $\gamma_i^N \in \Lambda_N^{ch}$ a best approximation of γ_i.
- Put $\Gamma_N = \mathbb{Z}\gamma_1^N \oplus \mathbb{Z}\gamma_2^N \subset \Lambda_N^{ch}$.
- Define $U_N : \mathbb{R}^2 \to \mathbb{R}^2$ by $U_N(\gamma_i^N) = \gamma_i$.
- Then U_N induces a diffeo $\mathbb{R}^2/\Gamma_N \to \mathbb{R}^2/\Gamma \simeq \Sigma$.
- $\mathcal{Q}_N(\mathbb{R}^2)$ descends to the quotient as a quadrangulation $\mathcal{Q}_N(\Sigma)$.

Checkers graph

One can associate a checkers graph $\mathscr{G}_N(\mathbb{R}^2)$ to the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ as follows



The graph splits into two connected components

$$\mathscr{G}_N(\mathbb{R}^2) = \mathscr{G}_N^+(\mathbb{R}^2) \cup \mathscr{G}_N^-(\mathbb{R}^2).$$

$\mathscr{G}_N(\mathbb{R}^2)$ is acted on by $\Gamma_N \subset \Lambda_N^{ch}$, which preserves the connected components. We obtain quotient checkers graphs

$$\mathscr{G}_N(\Sigma) = \mathscr{G}_N^+(\Sigma) \cup \mathscr{G}_N^-(\Sigma).$$

Discrete functions on Σ

The space of discrete functions is by definition:

$$C^2(\mathcal{Q}_N(\Sigma)) \simeq C^0(\mathscr{G}_N(\Sigma)).$$

The splitting $\mathscr{G}_N(\Sigma) = \mathscr{G}_N^+(\Sigma) \cup \mathscr{G}_N^-(\Sigma)$, provides a direct sum decomposition

$$C^0(\mathscr{G}_N(\Sigma)) = C^0(\mathscr{G}_N^+(\Sigma)) \oplus C^0(\mathscr{G}_N^-(\Sigma))$$

and accordingly

$$\psi = \psi^+ + \psi^-.$$

Discrete analysis

- We define finite differences $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ on $C^0(\mathscr{G}^+(\mathbb{R}^2))$.
- This leads to discrete $\mathscr{C}^k$-norm on $C^0(\mathscr{G}^+(\mathbb{R}^2))$.
- Similarly, we defines discrete $\mathscr{C}^{k,\alpha}$-Hölder norm $\|\psi^+\|_{\mathscr{C}^{k,\alpha}}$ on $C^0(\mathscr{G}^+(\mathbb{R}^2))$ and also on $C^0(\mathscr{G}^+(\Sigma))$ by pullback.
- We define a weak discrete Hölder norm

$$\|\psi\|_{\mathscr{C}_w^{k,\alpha}} := \|\psi^+\|_{\mathscr{C}^{k,\alpha}} + \|\psi^-\|_{\mathscr{C}^{k,\alpha}}$$

- There is a notion of convergence $\psi_N^+ \to \phi$ for a sequence of discrete functions ψ_N^+ toward a function $\phi : \Sigma \to \mathbb{R}$.
- There is version of the Ascoli-Arzela theorem: if $\|\psi_N^+\|_{\mathscr{C}^{0,\alpha}} < c$, we may extract a converging sequence.
- A Theorem of Thomée ('68) shows that finite difference elliptic operators on $C^0(\mathscr{G}_N^+(\Sigma))$ satisfy uniform Schauder estimates under some mild assumptions.

Quadranglar meshes

A quadrangular mesh τ is an element of the moduli space

$$\tau \in \mathscr{M}_N = C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$$

Definition 3

The symplectic area of an oriented quadrilateral of $\mathbb{R}^{2n}$ is the integral of the Liouville form λ along the quadrialteral.
Thus we define

$$\mu_N^r : \mathscr{M}_N \to C^2(\mathcal{Q}_N(\Sigma))$$

by $\langle \mu_N^r(\tau), \mathbf{f} \rangle = N^{-2} S(\mathbf{f}, \tau)$, where S is the symplectic area of the quadrilateral associated to a face $\mathbf{f}$ via τ.
We say that a quadrangular mesh τ is isotropic if $\mu_N^r(\tau) = 0$.

Existence of isotropic meshes

Given a smooth isotropic immersion $\ell : \Sigma \to \mathbb{R}^{2n}$, we define its samples $\tau_N \in \mathscr{M}_N$ by

$$\langle \tau_N, \mathbf{v} \rangle = \ell(\mathbf{v}).$$

Since ℓ is isotropic, it follows that

$$\|\mu^r_N(\tau_N)\|_{\mathscr{C}^k_w} = \mathcal{O}(N^{-1}).$$

Theorem 1 is a consequence of the following perturbation theorem, under an assumption of non degeneracy for the pair (p, ℓ), which is always satisfied for a suitable choice of cover p.

Theorem 4 (Jauberteau-Rollin-Tapie)

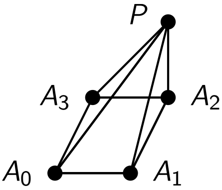
For every sufficiently large N, there exists isotropic quadrangular meshes $\rho_N \in \mathscr{M}_N$ such that

$$\max_{\mathbf{v}} \|\rho_N(\mathbf{v}) - \tau_N(\mathbf{v})\| = \mathcal{O}(N^{-1}).$$

From quadrangulations to triangulations

We give a sketch of proof for Theorem 4 $\Rightarrow$ Theorem 1.

- For each isotropic quadrilateral of $\mathbb{R}^{2n}$, there exists an isotropic pyramid obtained by adding an apex to the quadrilateral. This is a linear problem and the space of apexes is generically $2n-3$-dimensional.



- We may replace an isotropic quadrangular mesh with an isotropic triangular mesh, which defines an isotropic piecewise linear map $\hat{\ell} : \Sigma \to \mathbb{R}^{2n}$.
- The $\mathscr{C}^0$-estimate on $\hat{\ell}$ is obtained by showing that the apex P does not escape to infinity.
- The fact that ℓ can be chosen as an immersion relies on the degrees of freedom for the choice of the apex P and the shear action.

Donaldson moment map geometry

Assume that (Σ, σ) is a closed surface with area form σ.
The moduli space

$$\mathscr{M} = \{f : \Sigma \to \mathbb{R}^{2n}\}$$

is endowed with a natural Kähler structure $(\mathfrak{J}, \Omega, \mathfrak{g})$, given by

$$\mathfrak{J}V = JV, \quad \Omega(V, W) = \int_\Sigma \omega(V, W)\sigma, \quad \mathfrak{g}(V, W) = \int_\Sigma g(V, W)\sigma$$

for every $V, W : \Sigma \to \mathbb{R}^{2n} \in T_f\mathscr{M}$.
The group $\mathrm{Ham}(\Sigma, \sigma)$ act on $\mathscr{M}$ and preserves the Kähler structure. Its action is Hamiltonian, with moment map

$$\mu : \mathscr{M} \to C_0^\infty(\Sigma) \simeq \mathrm{Lie}(\mathrm{Ham}(\Sigma, \sigma))^*$$

given by $\mu(f) = \frac{f^*\omega}{\sigma}$.

Elliptic equation in the infinite dimensional world

We define an operator

$$\delta_f : T_f\mathscr{M} \to C^\infty_0(\Sigma) \quad \text{by} \quad \delta_f V = -D\mu|_f \circ \mathfrak{J} V$$

and its adjoint

$$\delta_f^\star : C^\infty_0(\Sigma) \to T_f\mathscr{M} \quad \text{defined by} \quad \langle\langle \delta_f V, \psi \rangle\rangle = \mathfrak{g}(V, \delta_f^\star \psi).$$

Proposition 5

The operator $\Delta_\ell = \delta_f \delta_f^\star$ is elliptic of order 2 at an immersion ℓ, with kernel reduced to constants.

Let X_h be the Hamiltonian vector field of (Σ, σ) such that $dh = \iota_{X_h}\sigma$. The fundamental vector field $Y_n = f_* X_h \in T_f\mathscr{M}$ satisfies

$$Y_h = \delta_f^\star h.$$

In particular, by the IFT, the equation $\mu(f - \mathfrak{J}\delta_f^\star h) = 0$ admits a unique solution $h \in C^\infty_0(\Sigma)$ for every $f \in \mathscr{M}$ sufficiently close to an isotropic immersion ℓ.

Moment map flow

Such general setting gives rise to a moment map flow, which converge (conjecturally) toward a zero of the moment map.

$$\frac{df}{dt} = \mathfrak{J}\delta_f^\star \mu(f) = -\frac{1}{2}\mathrm{grad}\|\mu\|^2$$

A finite dimensional approximation of this flow provides an evolution equation for quadrangular meshes. The flow is an ODE on $\mathscr{M}_N$. Its flow lines are approximated by the Euler method in the DMMF program.

Transplantation in the finite dimensional case

For $\tau \in \mathscr{M}_N$ and $V \in T_\tau \mathscr{M}_N = C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$, put

$$\delta_\tau V = -D\mu^r_N|_\tau \circ JV.$$

We define a discrete analogue of the L^2-inner product by

$$\langle\langle \mathbf{f}_1^*, \mathbf{f}_2^* \rangle\rangle = N^{-2}\delta_{\mathbf{f}_1, \mathbf{f}_2},$$

which induces the adjoint defined by $\langle\langle \delta_\tau^\star \psi, V\rangle\rangle = \langle\langle \psi, \delta_\tau V\rangle\rangle$.
We consider the equation

$$\mu^r_N(\tau_N - J\delta^\star_N \psi) = 0, \tag{1}$$

which is solved via the fixed point principle.

Linear theory

The linearization of Equation (1) gives the operator

$$\Delta_N : C^2(\mathcal{Q}_N(\Sigma)) \to C^2(\mathcal{Q}_N(\Sigma)) \quad \text{given by} \quad \Delta_N\psi = \delta_N\delta_N^\star\psi.$$

Proposition 6

Suppose that $\psi_N \in C^2(\mathcal{Q}_N(\Sigma))$ converge up to order k towards a pair a functions (ϕ^+, ϕ^-). Then $\Delta_N\psi_N$ converge up to order $k-2$ towards a pair of functions given by

$$\Xi(\phi^+,\phi^-) = \Big(\theta\Delta_\sigma\phi^+ - g_\sigma(d\phi^+, d\theta) + (K+E)(\phi^+ - \phi^-),$$
$$\theta\Delta_\sigma\phi^- - g_\sigma(d\phi^-, d\theta) + (K+E)(\phi^- - \phi^+)\Big)$$

where $\ell^* g = g_\Sigma = \theta g_\sigma$, K is the Gauß curvature of g_Σ and

$$E = \left\| \mathrm{I\!I}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) \right\|_g^2$$

Spectral gap

If $E = 0$, we say that the pair (p, ℓ) is degenerate.

Lemma 7

- *There exists a rotation r of $\mathbb{R}^2$ such that $(p \circ r, \ell)$ is non degenerate.*
- *If (p, ℓ) is non degenerate, then $\ker \Xi$ consists of functions such that $\phi^+ = \phi^- = cste$.*

Discrete analysis + Limit operator Ξ + Thomée $\Rightarrow$

Proposition 8

If (p, ℓ) is non degenerate, there exists $c > 0$ such that for every N sufficiently large and $\psi \in C^2(\mathcal{Q}_N(\Sigma))$ with $\langle\langle \psi, \mathbf{1} \rangle\rangle = 0$, we have

$$\|\Delta_N \psi\|_{\mathscr{C}_w^{0,\alpha}} \geq c \|\psi\|_{\mathscr{C}_w^{2,\alpha}}.$$

Proof of Lemma 7

- If II vanishes identically on every orthonormal basis, this implies that $K \geq 0$ by Gauß Theorema Egregium.
- Since Σ is a torus $\Rightarrow K = 0$ by Gauß-Bonnet.
- By Theorema Egregium be deduce that $\mathrm{II} = 0 \Rightarrow \ell(\Sigma)$ totally geodesic.
- This is impossible since Σ is closed.

We have the formulae

$$d^{*\sigma}\theta df = \theta\Delta_\sigma f - g_\sigma(df, d\theta) and$$

$$\theta d^{*\sigma}\theta^{-1}d\theta f = \theta\Delta_\sigma f - g_\sigma(df, d\theta) + 2Kf,$$

(using $2K = \theta\Delta_\sigma \log\theta$)
$\Xi(\phi^+, \phi^-) = 0$ implies $\phi^+ + \phi^- = c_0$ by the first formula.
The second formula implies $\phi^+ - \phi^- = c_1\theta^{-1}$ if $E = 0$
and $\phi^+ = \phi^-$ otherwise.

Fixed point principle

Theorem 9

Assume that (p,ℓ) is non degenerate. Then for $\varepsilon > 0$ sufficiently small and for every sufficiently large N, there exists a unique $\psi_N \in C^2(\mathcal{Q}_N(\Sigma))$, such that

- *ψ_N is orthogonal to constants,*
- *$\|\psi_N\|_{\mathscr{C}^{2,\alpha}_w} < \varepsilon$ and*
- *$\mu^r_N(\tau_N - J\delta^\star_N\psi_N) = 0$.*

Furthermore, we have $\|\psi_N\|_{\mathscr{C}^{2,\alpha}_w} = \mathcal{O}(N^{-1})$.

This result proves Theorem 4 with $\rho_N = \tau_N - J\delta^\star_N\psi_N$.

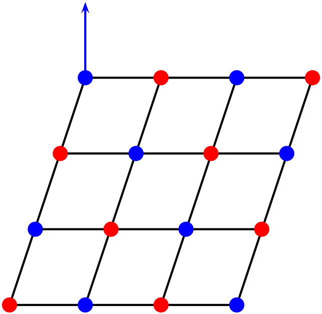
Discretization and artifacts

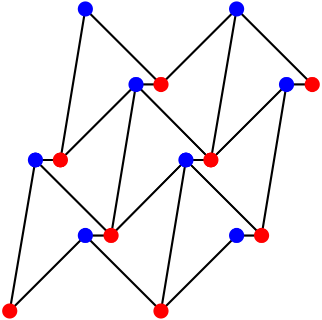
It seems impossible to get better estimates than the weak Hölder estimates.
We have some sort of geometrical evidence for this given by the shear action on $\mathscr{M}_N$.
The vertices of $\mathcal{Q}_N(\Sigma)$ are acted on by Λ_N^{ch} and we have two equivalence classes of vertices, say red and blue.
For $T = (T_+, T_-) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, we define

$$\langle T \cdot \tau, \mathbf{v} \rangle = \begin{cases} \langle \tau, \mathbf{v} \rangle + T_+ & \text{if } \mathbf{v} \text{ is blue} \\ \langle \tau, \mathbf{v} \rangle + T_- & \text{if } \mathbf{v} \text{ is red} \end{cases}$$

The symplectic area of a quarilateral of $\mathbb{R}^{2n}$ with diagonals D_0, D_1 is given by $\frac{1}{2}\omega(D_0, D_1)$.
In particular the symplectic area remains invariant when translating two opposite vertices simulatneously.
$\Rightarrow$ isotropic quadrangular meshes are invariant under the shear action.

Wild isotropic meshes





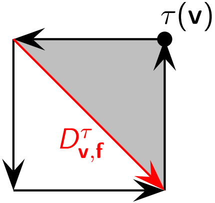
Discrete moment map flow

We mimic the moment map flow in the infinite dimensional setting and define the ODE on $\mathscr{M}_N$ by

$$\frac{d\tau}{dt} = J\delta_\tau^\star \mu_N^r(\tau) = -\frac{1}{2}\mathrm{grad}\|\mu_N^r\|^2.$$

The operators involved here are all explicit. In particular

$$\delta_\tau^\star \psi = \frac{N^2}{2}\sum_{\mathbf{v},\mathbf{f}} \psi(\mathbf{f}) D^\tau_{\mathbf{v},\mathbf{f}} \otimes \mathbf{v}^*$$



Open questions

- Is there a converse to Theorem 1 ?
- Are the weak Hölder controls inherent to our construction or geometrically signigicant ?
- Short time existence of the infinite dimensional flow ?
- Long time existence of the finite dimensional flow ?
- Convergence of the flows as $N \to \infty$?
- Our construction provides a map $\mathscr{M} \to \mathscr{M}_N$ defined along the zero sets of μ and μ_N^r. Is there an nice interpretation of this map in terms of GIT ?

Toward quantization

Infinite dimensional case	finite dimensional case
Area form σ on Σ	Quadrangulation $\mathcal{Q}_N(\Sigma)$
$\mathscr{M} = \{f : \Sigma \to \mathbb{R}^{2n}\}$	$\mathscr{M}_N = C^0(\mathcal{Q}_N(\Sigma) \otimes \mathbb{R}^{2n}$
Canonical Kähler structure	Canonical Kähler structure
$\mathrm{Ham}(\Sigma, \sigma)$-action	???
Fundamental V.F $Y_h(f) = \delta_f^\star h$	$\delta_\tau^\star \phi$
A moment map $\mu : \mathscr{M} \to C^\infty(\Sigma)$	$\mu_N^r : \mathscr{M}_N \to C^2(\mathcal{Q}_N(\Sigma))$
The moment map flow	The discrete flow

Conjecture: the right column is (in some sense) a quantization of the left column.

Enjoy the DMMF program available on my webpage:
http://www.math.sciences.univ-nantes.fr/~rollin

Thanks for you attention !