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Rigidité d'Einstein du plan hyperbolique complexe. (French. English summary) [Einstein rigidity of the complex hyperbolic plane]
J. Reine Angew. Math. **567** (2004), 175–213.

This paper proves that the Bergman metric on the open unit ball $B \subset \mathbf{C}^2$ cannot be deformed as an Einstein metric without changing its conformal infinity. To state the result more precisely, denote by g^H the Bergman metric and suppose that g is another Einstein metric on B satisfying, away from some compact set,

$$g = g^H + e^{-\delta t} h,$$

where $\delta > 0$ is some real number, t denotes the distance from the center of B, and h has bounded C^∞ norm (in both cases, with respect to g^H). Then, there exists a diffeomorphism $f\colon B \to B$ such that $f^* g = g^H$.

The idea to prove the result is to deduce from the hypothesis that g is in fact complex hyperbolic, so that the standard uniqueness result applies (see, for example, Theorem 7.9 in Chapter IX of S. Kobayashi and N. Nomizu's book [*Foundations of differential geometry. Vol. II*, Reprint of the 1969 original, Wiley, New York, 1996; MR1393941 (97c:53001b)]). To check that g is complex hyperbolic, the author uses results of P. B. Kronheimer and T. S. Mrowka [Invent. Math. **130** (1997), no. 2, 209–255; MR1474156 (98h:57058)] to construct a solution of the Seiberg-Witten equations on B. (In fact, the author proves a more general existence result giving solutions of the SW equations on manifolds with boundary which carry an asymptotically complex hyperbolic Einstein metric.) Applying Lichnerowicz's formula, he deduces that g is Kähler-Einstein and self-dual. In this last step, in order to be able to integrate by parts, a detailed knowledge of the asymptotic behaviour of g is necessary, and the author relies on results of O. Biquard and M. Herzlich ["A Burns-Epstein invariant for ACHE 4-manifolds", preprint, arxiv.org/abs/math.DG/0111218]. Finally, since g is Kähler-Einstein and self-dual, it is also complex hyperbolic.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.