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Non-minimal scalar-flat Kähler surfaces and parabolic stability. (English summary)
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This paper presents a new construction of scalar-flat Kähler metrics on non-minimal ruled surfaces. The main theorem states that if Σ is a compact Riemann surface of genus g with a finite number of marked points $\{P_j\}_{j=1}^r$ and $\rho\colon \pi_1(\Sigma/\{P_1,\dots,P_r\})\to \mathrm{SU}(2)/(\mathbb{Z}/2\mathbb{Z})$ is a homomorphism such that

(1) if l_j is the homotopy class of a small loop around P_j, then $\rho(l_j)$ has finite order q_j,
(2) $2-2g-\sum_{j=1}^r(1-\frac{1}{q_j})<0$, and
(3) ρ defines an irreducible representation,

then there is a non-minimal ruled surface M_ρ associated to ρ which admits a scalar flat metric. Moreover, this metric is shown to be unique.

The assumptions on ρ are reformulated in terms of stable parabolic structures using a theorem of V. B. Mehta and C. S. Seshadri [Math. Ann. **248** (1980), no. 3, 205–239; MR0575939 (81i:14010)]. A parabolic structure on a ruled surface $\mathbb{P}(E)$ consists of a finite set of points $Q_1,\dots,Q_r$ in $\mathbb{P}(E)$ in distinct fibres, and for each Q_j an associated weight $\alpha_j\in(0,1)\cap\mathbb{Q}$. Then M_ρ is obtained as an iterated blowup of $\mathbb{P}(E)$ in such a way that the exceptional set is described by the Hirzebruch-Jung continued fraction of the α_j. This blowup is canonically associated to ρ; however, as noted by the authors, this map from parabolic structures to non-minimal ruled surfaces is not smooth in any obvious sense.

The paper includes many interesting applications of the theorem, including: (1) the complex plane $\mathbb{P}^2$ blown up at 10 suitably chosen points admits a scalar flat metric; (2) if $\mathbb{T}$ is a compact Riemann surface of genus 1 and L_1, L_2 are non-isomorphic line bundles of the same degree on $\mathbb{T}$, then there is a 2-point blowup of $\mathbb{P}(L_1\oplus L_2)$ that admits a scalar flat metric; (3) there is a 4-point blowup of $\mathbb{T}\times\mathbb{P}^1$ that admits a scalar flat metric.

The proof of the main theorem starts by associating to the representation ρ a natural complex orbifold $\overline{M}$. Using the assumptions on ρ, this orbifold admits a scalar flat orbifold Kähler metric by an argument analogous to that of D. M. Burns, Jr. and P. De Bartolomeis [Invent. Math. **92** (1988), no. 2, 403–407; MR0936089 (89d:53114)]. The manifold M_ρ is then the minimal resolution of $\overline{M}$. To construct the metrics on M_ρ the authors start with suitable local models of scalar flat metrics constructed by D. M. J. Calderbank and M. A. Singer which are ALE (asymptotically locally Euclidean) [Invent. Math. **156** (2004), no. 2, 405–443; MR2052611 (2005h:53064)]. The proof is completed by a sophisticated argument which glues these models into the orbifold $\overline{M}$ to produce the desired metric on M_ρ.

Reviewed by *Julius A. Ross*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.