

# Courant-sharp Robin eigenvalues for the square

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Joint work with Bernard Helffer (Université de Nantes)

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# Dirichlet eigenvalues

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a connected, open set with  $|\Omega| < \infty$ .

Laplace operator  $\Delta : C^2(\mathbb{R}^n) \rightarrow C^0(\mathbb{R}^n)$ , where  $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ .

**Dirichlet** eigenvalues of the Laplacian on  $\Omega$ ,  $\lambda_k(\Omega) \in \mathbb{R}$ , satisfy

$$\begin{cases} -\Delta u_k(x) = \lambda_k(\Omega) u_k(x) & x \in \Omega, \\ u_k(x) = 0 & x \in \partial\Omega, \end{cases}$$

where  $u_k \in H_0^1(\Omega)$ , and form a non-decreasing sequence, counted with multiplicities,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \dots$$

where  $\lambda_k(\Omega) > 0$  for all  $k \in \mathbb{N}_{>0}$ .

# Courant-sharp Dirichlet eigenvalues

Nodal set of  $u_k$  is defined as  $\{x \in \Omega : u_k(x) = 0\}$ .

Nodal domains of  $u_k$  are the components of  $\Omega \setminus \{x \in \Omega : u_k(x) = 0\}$ .

**Sturm's Oscillation theorem** For a bounded interval, the eigenfunction corresponding to the  $k$ -th Dirichlet eigenvalue has  $k$  nodal domains.

**Courant's Nodal Domain theorem** Any eigenfunction corresponding to  $\lambda_k(\Omega)$  has at most  $k$  nodal domains.

If  $u_k$  is an eigenfunction corresponding to  $\lambda_k(\Omega)$  with  $k$  nodal domains, then we call  $(u_k, \lambda_k(\Omega))$  a Courant-sharp eigenpair.

# Number of Courant-sharp Dirichlet eigenvalues

Let  $\nu(u_k)$  denote the number of nodal domains of the eigenfunction  $u_k$  which corresponds to  $\lambda_k(\Omega)$ .

## Theorem (Pleijel's Theorem)

Let  $\Omega$  be an open, connected set in  $\mathbb{R}^n$ ,  $n \geq 2$ , with finite Lebesgue measure. Then

$$\limsup_{k \rightarrow \infty} \frac{\nu(u_k)}{k} \leq \gamma_n,$$

where

$$\gamma_n = \frac{(2\pi)^n}{\omega_n^2} (\lambda_1(\mathcal{B}_n))^{-n/2} < 1,$$

$\mathcal{B}_n$  is a ball of radius 1 in  $\mathbb{R}^n$  and  $\omega_n$  is the volume of  $\mathcal{B}_n$ .

# Upper bounds for Courant-sharp Dirichlet eigenvalues

Theorem (Bérard & Helffer, 2016)

Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected domain. Then, there exists a constant  $\beta(\Omega) > 0$  depending only on the geometry of  $\Omega$ , such that any Courant-sharp eigenvalue  $\lambda_k(\Omega)$  satisfies

$$k \frac{\lambda_1(\mathbb{D}_1)}{|\Omega|} \leq \lambda_k(\Omega) \leq \beta(\Omega),$$

where  $\mathbb{D}_1$  is the disc of unit area.

If  $\Omega$  is regular,

$$\lambda_k(\Omega) \leq \frac{2}{|\Omega|^3} \left( \frac{24\pi\lambda_1(\mathbb{D}_1)}{\lambda_1(\mathbb{D}_1) - 4\pi} \right)^4 \left( \max \left\{ \frac{|\Omega|}{\varepsilon_0(\Omega)}, 2\ell(\partial\Omega) \right\} \right)^4.$$

# Upper bounds for Courant-sharp Dirichlet eigenvalues

For  $\epsilon \geq 0$  and  $|\Omega| < \infty$ , define

$$\mu_\Omega(\epsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}|,$$

and

$$\epsilon(\Omega) = \inf\{\epsilon : \mu_\Omega(\epsilon) \geq 2^{-1}(1 - \gamma_n)|\Omega|\}.$$

## Theorem (van den Berg & G, 2016)

Let  $\Omega$  be an open, connected set in  $\mathbb{R}^n$  with finite Lebesgue measure. If  $\lambda_k(\Omega)$  is Courant-sharp, then

(i)

$$\lambda_k(\Omega) \leq \left( \frac{2\pi n^2}{(1 - \gamma_n)\epsilon(\Omega)} \right)^2.$$

(ii)

$$k \leq \frac{\omega_n}{(1 - \gamma_n)^n} (n^3(n+2))^{n/2} \frac{|\Omega|}{\epsilon(\Omega)^n}.$$

## Examples

**Helffer, Hoffmann-Ostenhof & Terracini (2009):**

The Courant-sharp Dirichlet eigenvalues of the disc in  $\mathbb{R}^2$  are  $\lambda_1, \lambda_2, \lambda_4$ .

**Pleijel (1956), Bérard & Helffer (2015):**

The Courant-sharp Dirichlet eigenvalues of the square in  $\mathbb{R}^2$  are  $\lambda_1, \lambda_2$  and  $\lambda_4$ .

Dirichlet eigenvalues of  $S := [-\frac{\pi}{2}, \frac{\pi}{2}]^2$  are

$$\lambda_k(S) = \lambda_{m,n}(S) = m^2 + n^2,$$

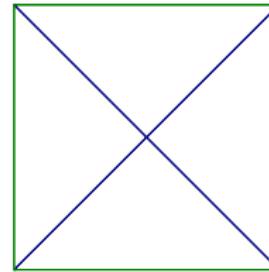
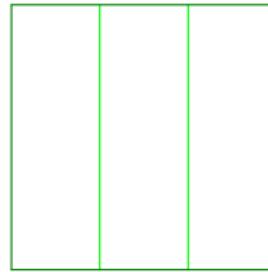
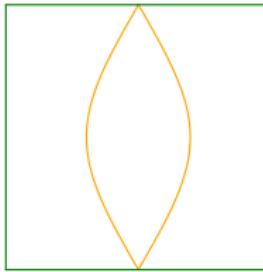
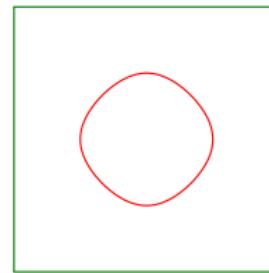
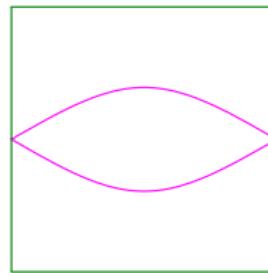
where  $m, n \in \mathbb{N}_{>0}$ , with corresponding eigenfunctions

$$u_{m,n}(x, y) = \sin\left(\frac{m\pi x}{\pi} + \frac{m\pi}{2}\right) \sin\left(\frac{n\pi y}{\pi} + \frac{n\pi}{2}\right).$$

# Nodal domains: fifth Dirichlet eigenfunction on the square

In the eigenspace corresponding to  $\lambda_5(S)$ , interested in

$$\cos \theta u_{1,3}(x, y) + \sin \theta u_{3,1}(x, y).$$



# Neumann eigenvalues

Let  $\Omega \subset \mathbb{R}^n$  be a bounded, connected, open set with Lipschitz boundary.

**Neumann** eigenvalues of the Laplacian on  $\Omega$ ,  $\mu_k(\Omega) \in \mathbb{R}$ , satisfy

$$\begin{cases} -\Delta u_k(x) = \mu_k(\Omega)u_k(x) & x \in \Omega, \\ \frac{\partial u_k(x)}{\partial \vec{n}} = 0 & x \in \partial\Omega, \end{cases}$$

$u_k \in H^1(\Omega)$ , where  $\vec{n}$  is the outward pointing unit normal vector to  $\partial\Omega$ , and form a non-decreasing sequence counted with multiplicities

$$\mu_1(\Omega) < \mu_2(\Omega) \leq \dots \leq \mu_k(\Omega) \leq \dots$$

with  $\mu_1(\Omega) = 0$ .

# Courant-sharp Neumann eigenvalues

**Courant's Nodal Domain theorem** Any eigenfunction corresponding to  $\mu_k(\Omega)$  has at most  $k$  nodal domains.

If  $u_k$  is an eigenfunction corresponding to  $\mu_k(\Omega)$  with  $k$  nodal domains, then we call  $(u_k, \mu_k(\Omega))$  a **Courant-sharp eigenpair**.

**Pleijel** (1956). The square in  $\mathbb{R}^2$  has finitely many Courant-sharp Neumann eigenvalues.

# Courant-sharp Neumann eigenvalues

Let  $\nu(u_k)$  denote the number of nodal domains of  $u_k$  where  $u_k$  is an eigenfunction corresponding to  $\mu_k(\Omega)$ .

**Polterovich** (2008). A bounded, connected planar domain with piecewise analytic boundary has finitely many Courant-sharp Neumann eigenvalues,

$$\limsup_{k \rightarrow +\infty} \frac{\nu(u_k)}{k} \leq \gamma_2.$$

**Léna** (2016). A bounded, connected, open set in  $\mathbb{R}^n$ ,  $n \geq 2$ , with  $C^{1,1}$  boundary has finitely many Courant-sharp Neumann eigenvalues,

$$\limsup_{k \rightarrow +\infty} \frac{\nu(u_k)}{k} \leq \gamma_n.$$

# Examples

**Helffer & Persson-Sundqvist (2016):**

The Courant-sharp Neumann eigenvalues of the disc in  $\mathbb{R}^2$  are  $\mu_1, \mu_2, \mu_4$ .

**Helffer & Persson-Sundqvist (2015):**

The Courant-sharp Neumann eigenvalues of the square in  $\mathbb{R}^2$  are  $\mu_1, \mu_2, \mu_4, \mu_5$  and  $\mu_9$ .

Neumann eigenvalues of  $S := [-\frac{\pi}{2}, \frac{\pi}{2}]^2$  are

$$\mu_k(S) = \mu_{m,n}(S) = m^2 + n^2,$$

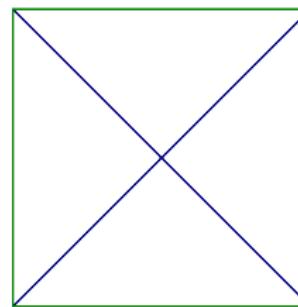
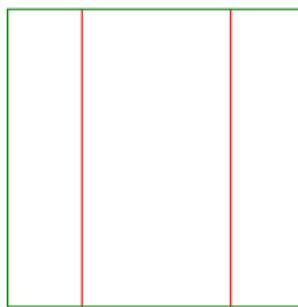
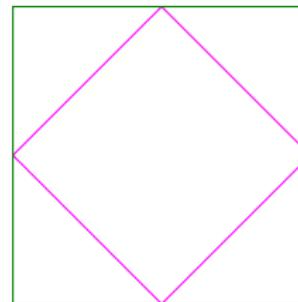
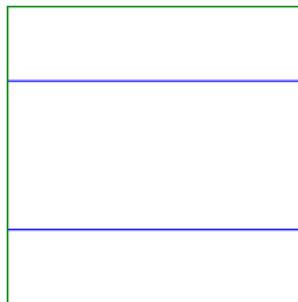
where  $m, n \in \mathbb{N}$ , with corresponding eigenfunctions

$$u_{m,n}(x, y) = \cos\left(\frac{m\pi x}{\pi} + \frac{m\pi}{2}\right) \cos\left(\frac{n\pi y}{\pi} + \frac{n\pi}{2}\right).$$

# Nodal domains: fifth Neumann eigenfunction on the square

In the eigenspace corresponding to  $\mu_5(S)$ , interested in

$$\cos \theta u_{0,2}(x, y) + \sin \theta u_{2,0}(x, y).$$



# Robin eigenvalues

Let  $\Omega \subset \mathbb{R}^n$  be a bounded, connected, open set with Lipschitz boundary.

**Robin** eigenvalues of the Laplacian on  $\Omega$ ,  $\lambda_{k,h}(\Omega) \in \mathbb{R}$ , with  $h \geq 0$  satisfy

$$\begin{cases} -\Delta u_k(x) = \lambda_{k,h}(\Omega) u_k(x) & x \in \Omega, \\ \frac{\partial u_k(x)}{\partial \vec{n}} + h u_k(x) = 0 & x \in \partial\Omega, \end{cases}$$

$u_k \in H^1(\Omega)$ , where  $\vec{n}$  is the outward pointing unit normal vector to  $\partial\Omega$ , and form a non-decreasing sequence counted with multiplicities

$$\lambda_{1,h}(\Omega) \leq \lambda_{2,h}(\Omega) \leq \dots \leq \lambda_{k,h}(\Omega) \leq \dots$$

Robin eigenvalues interpolate between Neumann and Dirichlet eigenvalues:

$$\mu_k(\Omega) \leq \lambda_{k,h}(\Omega) \leq \lambda_k(\Omega).$$

# Courant-sharp Robin eigenvalues

**Courant's Nodal Domain theorem** holds for Robin eigenvalues.

If  $u_k$  is an eigenfunction corresponding to  $\lambda_{k,h}(\Omega)$  with  $k$  nodal domains, then we call  $(u_k, \lambda_{k,h}(\Omega))$  a **Courant-sharp eigenpair**.

**Léna** (2016): A bounded, connected, open set in  $\mathbb{R}^n$ ,  $n \geq 2$ , with  $C^{1,1}$  boundary has finitely many Courant-sharp Robin eigenvalues ( $h \geq 0$ ).

# Robin eigenvalues of the square

Robin eigenfunctions on  $S := [-\frac{\pi}{2}, \frac{\pi}{2}]^2$  are

$$u_{m,n}(x, y) = u_m(x)u_n(y),$$

$m, n \in \mathbb{N}$ , where

$$u_p(x) = \frac{1}{\sin \frac{\alpha_p(h)}{2}} \cos \left( \frac{\alpha_p(h)x}{\pi} \right), p \in \mathbb{N}, p \text{ even},$$

and

$$u_p(x) = \frac{1}{\cos \frac{\alpha_p(h)}{2}} \sin \left( \frac{\alpha_p(h)x}{\pi} \right), p \in \mathbb{N}_{>0}, p \text{ odd},$$

and  $\alpha_p := \alpha_p(h)$  is the solution in  $[p\pi, (p+1)\pi)$  of

$$\frac{2\alpha_p}{h\pi} \cos \alpha_p + \left( 1 - \frac{(\alpha_p)^2}{h^2\pi^2} \right) \sin \alpha_p = 0.$$

Corresponding eigenvalues:

$$\lambda_{k,h}(S) = \lambda_{m,n,h}(S) = \pi^{-2}(\alpha_m(h)^2 + \alpha_n(h)^2).$$

# Courant-sharp Robin eigenvalues of the square

$\lambda_{3,h}(S) = \lambda_{2,h}(S)$  so  $\lambda_{3,h}(S)$  is not Courant-sharp for any  $h \geq 0$ .

$\lambda_{4,h}(S)$  corresponds to  $(m, n) = (1, 1)$  and

$$u_{1,1}(x, y) = \frac{1}{\cos^2 \frac{\alpha_1}{2}} \sin\left(\frac{\alpha_1 x}{\pi}\right) \sin\left(\frac{\alpha_1 y}{\pi}\right).$$

$\lambda_{4,h}(S)$  is Courant-sharp for all  $h \geq 0$ .

## Theorem (G & Helffer, 2018)

Let  $h \geq 0$ . If  $\lambda_{k,h}(S)$  is an eigenvalue of  $S$  with  $k \geq 520$ , then it is not Courant-sharp.

## Pleijel's strategy

Let  $(u_k, \lambda_{k,h})$  be a Courant-sharp eigenpair of  $\Omega$ . Let  $\Omega_j^{\text{inn}}$  be an “interior” nodal domain of  $u_k$ . Then by the Faber-Krahn inequality:

$$\lambda_{k,h} = \lambda_1(\Omega_j^{\text{inn}}) \geq \frac{\lambda_1(\mathbb{D}_1)}{|\Omega_j^{\text{inn}}|}.$$

If there is  $\Omega_j^{\text{inn}}$  such that  $|\Omega_j^{\text{inn}}| \leq |\Omega|/k$ , then

$$\frac{|\Omega|}{k\lambda_1(\mathbb{D}_1)} \geq \frac{1}{\lambda_{k,h}}.$$

Together with the bound on the counting function:

$$k > \frac{\pi}{4}\lambda_{k,h} - 2\sqrt{\lambda_{k,h}} + 2,$$

this gives  $\lambda_{k,h} \leq 50$ .

## Pleijel's strategy ( $h$ large)

Otherwise, there is  $\Omega_j^{\text{out}}$  a “boundary” nodal domain s.t.  $|\Omega_j^{\text{out}}| \leq |\Omega|/k$ .

By monotonicity,  $\lambda_{k,h} \geq \lambda_{1,h}(\Omega_j^{\text{out}})$ .

**Bossel** (1986) - **Daners** (2006):  $\lambda_{1,h}(\Omega) \geq \lambda_{1,h}(D)$ , where  $D \subset \mathbb{R}^2$  is a disc with  $|D| = |\Omega|$ .

Rescaling  $D$  to obtain  $\mathbb{D}_1$  gives

$$\lambda_{1,h}(D) = \lambda_{1,h|\Omega|^{1/2}}(\mathbb{D}_1)|\Omega|^{-1}.$$

Since  $\lambda_{k,h}$  is Courant-sharp,  $k \leq 520$  and we can thus show there exists a constant  $c > 0$  such that  $|\Omega_j^{\text{out}}| \geq c$ .

For  $h$  large enough,  $\lambda_{k,h} \leq 50$ .

By continuity of the Robin eigenvalues with respect to  $h$ , for  $h$  large enough, this leaves the remaining candidates  $k = 1, 2, 4, 5, 7, 9$ .

# Ninth Robin eigenvalue of the square

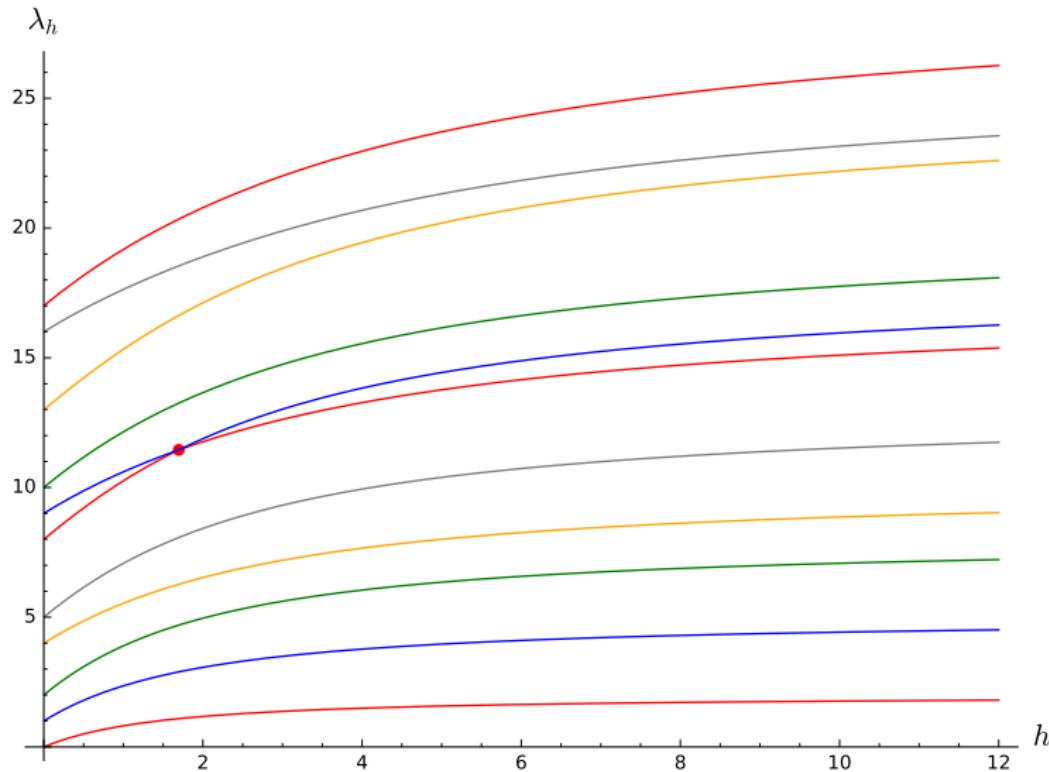
- Does there exist  $\underline{h}_9^*$  such that  $\lambda_{9,h}(S)$  is Courant-sharp for  $h < \underline{h}_9^*$ ?
- Does there exist  $\overline{h}_9^*$  such that  $\lambda_{9,h}(S)$  is not Courant-sharp for  $h > \overline{h}_9^*$ ?
- Is  $\underline{h}_9^* = \overline{h}_9^*$ ?

Proposition (G & Helffer, 2018)

There exists  $h_9^* > 0$  such that  $\lambda_{9,h}$  is Courant-sharp for  $0 \leq h \leq h_9^*$  and is not Courant-sharp for  $h > h_9^*$ .

Numerically:  $h_9^* \approx 1.6967$ .

# Ninth Robin eigenvalue of the square



# The case $h$ large

## Proposition (G & Helffer, 2018)

Let  $\Omega \subset \mathbb{R}^2$  be a bounded connected set with piecewise  $C^{2,\alpha}$  boundary ( $\alpha > 0$ ). Let  $h_0 > 0$  and  $M > 0$ . For  $h \in I \subset [h_0, +\infty)$ , let  $\Phi_{h,\theta}$  denote a smooth family of Robin eigenfunctions on  $\Omega$  corresponding to  $\lambda(\Omega) \leq M$ . Then

- ① Any nodal domain of  $\Phi_h$  satisfies the Robin Faber-Krahn inequality.
- ② There exists  $\varepsilon_0 > 0$  s.t. no nodal domain of  $\Phi_h$  has area less than  $\varepsilon_0$ .

If  $\tilde{\Omega}$  is a nodal domain of  $\Phi_h$ , then

$$M \geq \lambda(h) \geq \lambda(h_0) \geq \lambda_{1,h_0}(D_{\tilde{\Omega}}) = \lambda_{1,h_0|\tilde{\Omega}|^{\frac{1}{2}}}(\mathbb{D}_1)|\tilde{\Omega}|^{-1} \sim d h_0 / |\tilde{\Omega}|^{\frac{1}{2}},$$

where  $d > 0$  is a constant.

## The case $h$ large

**Leydold (1989):** the number of nodal domains can only change if there are interior critical points or if the number of boundary points changes.

- ① Starting from the nodal structure for the Dirichlet case, the number of nodal domains cannot increase under a small perturbation of  $h$ .
- ② If  $\lambda_k$  is not Courant-sharp, then  $\lambda_{k,h}$  is not Courant-sharp for  $h$  large enough.

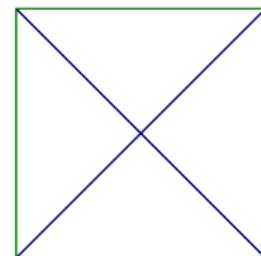
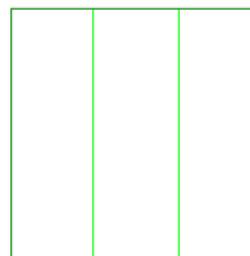
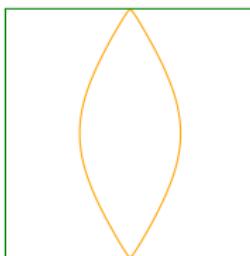
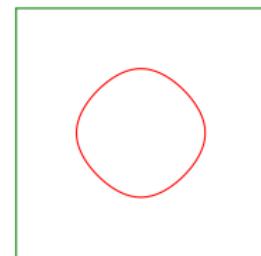
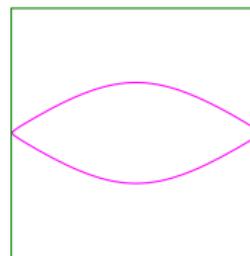
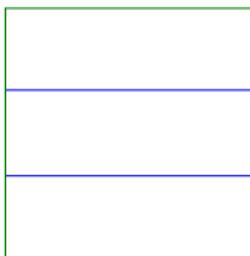
Theorem (G & Helffer, 2018)

*There exists  $h_1 > 0$  such that for  $h \geq h_1$ , the Courant-sharp cases for the Robin problem are the same as those for  $h = +\infty$ .*

# Fifth Robin eigenvalue of the square (h large)

Proposition (G & Helffer, 2018)

*There exists  $h_1 > 0$  such that for any  $h > h_1$ , any eigenfunction corresponding to  $\lambda_{5,h}(S)$  **is not** Courant-sharp.*



## The case $h$ small

- ① For  $k \geq 209$  and  $h$  small enough,  $\lambda_{k,h}$  is not Courant-sharp.
- ② Symmetry arguments reduce the number of potential candidates.
- ③ Starting from the nodal structure for the Neumann case\*, the number of nodal domains cannot increase under a small perturbation of  $h$ .
- ④ If  $\mu_k$  is not Courant-sharp, then  $\lambda_{k,h}$  is not Courant-sharp for  $h$  small enough.
- ⑤ One candidate for which previous arguments don't apply.
- ⑥  $\mu_5$  is Courant-sharp.

Theorem (G & Helffer, 2019)

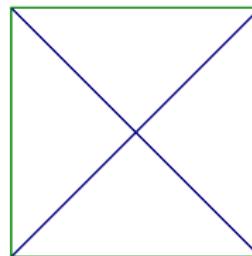
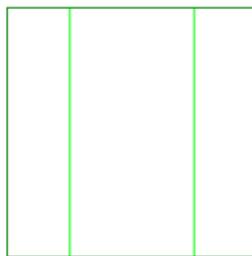
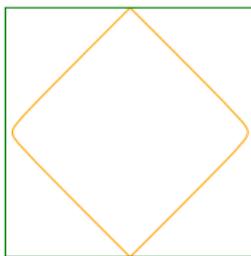
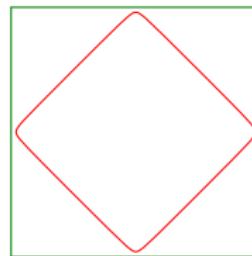
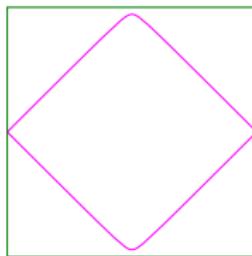
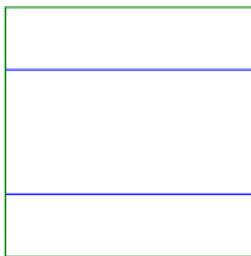
*There exists  $h_0 > 0$  such that for  $0 < h \leq h_0$ , the Courant-sharp cases for the Robin problem are the same, except the fifth one, as those for  $h = 0$ .*

Indeed,  $\lambda_{5,h}$  is not Courant-sharp for  $0 < h \leq h_0$ .

# Fifth Robin eigenvalue of the square (h small)

Proposition (G & Helffer, 2019)

*There exists  $h_0 > 0$  such that for any  $0 < h \leq h_0$ , any eigenfunction corresponding to  $\lambda_{5,h}(S)$  **is not** Courant-sharp.*



## Further questions

**Question:** Which Robin eigenvalues of the square are Courant-sharp for  $h \in (h_0, h_1)$ ?

**Question:** For other domains, is it possible to follow the Courant-sharp Neumann eigenvalues to Courant-sharp Dirichlet eigenvalues?

**Question:** What happens in the case where  $h < 0$ ?

**Question:** What is the optimal Pleijel constant?

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