More on pseudodifferential calculus norms

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Conference in honor of B. Helffer
Nantes, april 2019

Parts of the talk.

- Norms of pseudodiff. operators.
- Formulas for the Weyl and anti-Wick symbols.
- Examples in Physics.

References for Section 1.

[1] L. Amour, L. Jager, J. Nourrigat, *On bounded pseudodifferential operators in Wiener spaces.* J. Funct. Anal. **269** (2015), no. 9, 2747-2812.

[2] L. Amour, R. Lascar, J. Nourrigat, *Weyl calculus in Wiener spaces and in QED,* Journal of Pseudo-Diff. Op. and appl. **10** (2019) no 1, p. 1-47.
DOI 10.1007/s11868-018-0269-5. (Sections 1 and 2)

Sect. 1. Weyl quantization.

- For each operator A from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, the Weyl symbol $\sigma^{weyl}(A)$ is the element of $\mathcal{S}'(\mathbb{R}^{2n})$ related to the distribution kernel K_A by:

$$K_A(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma^{weyl}(A) \left(\frac{x+y}{2}, \xi \right) d\xi$$

$$\sigma^{weyl}(A)(x,\xi) = \int_{\mathbb{R}^n} e^{-it\cdot\xi} K_A(x+(t/2), x-(t/2)) dt$$

- Given F in $\mathcal{S}'(\mathbb{R}^{2n})$, let $Op^{weyl}(F)$ the operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ whose Weyl symbol is F.

Sect. 1. Class $S^m(H^2, Q)$ of pseudodiff. symbols.

- Let H be a Hilbert space.
- Let Q be a positive quadratic form on H^2.
  Let $S^m(H^2, Q)$ $(m \in \mathbf{N} \bigcup \infty)$ the space of $F \in C^m(H^2)$ s.t, for some $C_m(f) \geq 0$:

  $$|(d^k F)(X)(U_1, \dots, U_k)| \leq C_m(f) Q(U_1)^{1/2} \dots Q(U_k)^{1/2}$$

  for all $k \leq m$, for $X, U_1, \dots U_k$ in H^2.
  The best $C_m(f)$ is denoted by $\|F\|_{m,Q}$ or $\|F\|_{\infty,Q}$.
- It is well known (Calderón Vaillancourt) that, if F is in $S^{4n}(\mathbb{R}^{2n}, Q)$, then $Op^{weyl}(F)$ is bounded in $L^2(\mathbb{R}^n)$.

Sect. 1. Notations for the norm estimate.

- The quadratic form Q defining the class is:

$$Q(X) = Q_A(X) = < AX, X > \qquad X \in \mathbb{R}^{2n}$$

- Set $\mathcal{F}(x, \xi) = (-\xi, x)$.
- The product $\mathcal{F}A$ is called "fundamental matrix".
- Let $|\mathcal{F}A|_A$ the absolute value of $\mathcal{F}A$ (for the scalar product of Q).
  - $(\mathcal{F}A)^\star$ adjoint of $\mathcal{F}A$ for the scalar product of Q. We have $(\mathcal{F}A)^\star = -\mathcal{F}A$
  - $|\mathcal{F}A|_A$ is the positive self-adjoint operator (for the scalar product of Q), whose square is $(\mathcal{F}A)^\star\mathcal{F}A$.

Sect. 1. Our norm estimate.

Theorem

If $F \in S^{4n}(\mathbb{R}^{2n}, Q)$*, one has:*

$$\|Op^{weyl}(F)\| \leq \|F\|_{4n,Q_A} \ \left[\det\Big(I + 81\pi K|\mathcal{F}A|_A\Big)\right]^{1/2}$$

where

$$K = \max(1, \|\mathcal{F}A\|_{Q_A})$$

and $\|\mathcal{F}A\|_{Q_A}$ *is the norm of* $\mathcal{F}A$ *for the norm* $Q_A^{1/2}$.

Sect. 1. The usual hypothesis of L. Hörmander.

- The dual form of Q_A w.r.t. the symplectic form σ is:

$$Q_A^\sigma(X) = \sup_Y \frac{|\sigma(X,Y)|^2}{Q_A(Y)} = (A^{-1}\mathcal{F}X)\cdot(\mathcal{F}X),$$

- The usual hypothesis $Q_A \le Q_A^\sigma$ reads as $\|\mathcal{F}A\|_{Q_A} \le 1$.
- With this hypothesis, we have $K = \max(1, \|\mathcal{F}A\|_{Q_A}) = 1$.

L. Hörmander, *The analysis of linear partial differential operators,* Volume III, Springer, 1985.

Sect. 1. Toward infinite dimension (1).

We are given a Hilbert space H and:

- A function $F \in C^\infty(H^2)$.
- A positive quadratic form Q on H^2.
- For each finite dim. subspace $E \subset H$, let F_E be the restriction of F to E^2.
- We define an op. $Op^{weyl}(F_E)$, bounded in $L^2(E)$.

Question When is the norm of $Op^{weyl}(F_E)$ bounded independently of E?

Sect. 1. Toward infinite dimension(2)

It suffices that:

- We have $F \in S^{\infty}(H^2, Q)$. Hence F is analytic.
- We have $Q(X) = < AX, X >$, with A trace class. Because:

$$\det\Big(I + 81\pi Kh|\mathcal{F}A|_A\Big) \leq e^{81\pi Kh \mathrm{Tr}(|\mathcal{F}A|_A)} \leq e^{81\pi Kh \mathrm{Tr}(A)}$$

Sect. 1. Base of the proof: result with L. Jager.

Let $H = \mathbb{R}^n$. Let (x, ξ) the variable of H^2.

Theorem

Let $F : H^2 \to \mathbf{C}$ s. t.

$$|\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \leq M \prod_{j \geq 1} \varepsilon_j^{\alpha_j + \beta_j}$$

for each (α, β) s.t. $0 \leq \alpha_j \leq 2$ and $0 \leq \beta_j \leq 2$ for all j.
Then:

$$\| Op^{weyl}(F) \| \leq M \prod_{j \geq 1} (1 + 81 \pi S \varepsilon_j^2)$$

where $S = \mathrm{sup}_j \mathrm{max}(1, \varepsilon_j^2)$.

Sect. 1. Proof of the result with L. Jager.

Let F as above. For each subset $I \subset \{1, \dots, n\}$, set:

$$F_I = \prod_{j \in I} \left(I - e^{\frac{1}{4}\Delta_j} \right) \prod_{k \notin I} e^{\frac{1}{4}\Delta_k}$$

By results in dimension 1, we have:

$$\| Op_h^{weyl}(F) \| \leq M \prod_{j \in I} (81 \pi S \varepsilon_j^2)$$

We have:

$$F = \sum_{I \subset \{1, \dots n\}} F_I$$

The result follows.

Sect. 1. Main Proof (1).

There exist a symplectic linear map χ such that:

$$(Q_A \circ \chi)(x, \xi) = \sum_{j=1}^{d} \lambda_j (x_j^2 + \xi_j^2)$$

where the λ_j are the eigenvalues of the operator $|\mathcal{F}A|_A$.
There is a metaplectic unitary transform U_χ satisfying:

$$U_\chi^\star Op^{weyl}(F) U_\chi = Op^{weyl}(F \circ \chi).$$

For any $F \in S_{4n}(H^2, Q_A)$, one has:

$$\|Op^{weyl}(F)\| = \|U_\chi^\star Op^{weyl}(F) U_\chi\| = \|Op^{weyl}(F \circ \chi)\|.$$

The symbol $F \circ \chi$ belongs to $S_{4n}(H^2, Q_A \circ \chi)$.

Section 1. Main Proof (2).

Therefore we have, if $|\alpha + \beta| \leq 4d$:

$$|\partial_x^\alpha \partial_\xi^\beta (F \circ \chi)(x, \xi)| \leq \|F\|_{4n, Q_A} \prod_j \lambda_j^{(1/2)(\alpha_j + \beta_j)},$$

In particular true if $\alpha_j \leq 2$ and $\beta_j \leq 2$ for all j.

Sect. 1. Main Proof (3).

By the thm with L. Jager, with $\varepsilon_j = \sqrt{\lambda_j}$, one has

$$\|Op^{weyl}(F \circ \chi)\| \leq \|F\|_{4n,Q_A} \prod_{j=1}^{n} (1 + 81\pi S \lambda_j),$$

with

$$S = \sup_j \max(1, \lambda_j) = \max(1, \|\mathcal{F}A\|_{H^2,q_A}) = K$$

Therefore:

$$\|Op^{weyl}(F)\| \leq \|F\|_{4n,Q_A} \prod_{j=1}^{d} (1 + 81\pi S \lambda_j).$$

$$= \prod_{j=1}^{n} (1 + 81\pi S \lambda_j) = \det(I + \ 81\pi K \ |\mathcal{F}A|_A)^{1/2}$$

(each λ_j) is double).

Sect. 2. References for Section 2.

[4] L. Amour, J. Nourrigat, *Integral formulas for the Weyl and anti-Wick symbols*, To be published in: Journal de Math. pures et Appl.

Sect. 2. Weyl symbol and coherent states.

Coherent states.

$$\Psi_{x,\xi}(u) = \pi^{-n/4} e^{-\frac{1}{2}|u-x|^2 + iu\cdot\xi - \frac{i}{2}x\cdot\xi}$$

Main property. For f in $\mathcal{S}(\mathbb{R}^n)$:

$$f = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} < f, \Psi_X > \Psi_X dX$$

We want a formula for the Weyl symbol of A using the $< A\Psi_X, \Psi_Y >$ (see Unterberger characterization).

A. Unterberger, *Les opérateurs métadifférentiels,* in Lecture Notes in Physics 126 (1980) 205-241.

Sect. 2. Formula for the Weyl symbol.

Theorem

Let A be a bounded operator in $L^2(\mathbb{R}^n)$. Assume that:

$$\sup_{X \in \mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \frac{< A\Psi_{X+Z}, \Psi_{X-Z} >}{< \Psi_{X+Z}, \Psi_{X-Z} >} \right| \frac{e^{-|Z|^2} dZ}{\pi^n} < \infty$$

Then the Weyl symbol F of A is continuous and:

$$F(X) = \int_{\mathbb{R}^{2n}} \frac{< A\Psi_{X+Z}, \Psi_{X-Z} >}{< \Psi_{X+Z}, \Psi_{X-Z} >} \frac{e^{-|Z|^2} dZ}{\pi^n}$$

Sjöstrand Wiener.

Sect. 2. Weyl symbol formula: proof (1).

(Berezin, Folland) For each operator A bounded in $L^2(\mathbb{R}^n)$, there exists a function Φ holomorphic in $\mathbf{C}^{2n}$ s.t, for each $X = (x, \xi) = x + i\xi$ and $Y = (y, \eta) = y + i\eta$:

$$\frac{< A\Psi_X, \Psi_Y >}{< \Psi_X, \Psi_Y >} = \Phi(x + i\xi, y - i\eta)$$

The above hypothesis reads:

$$\sup_{X \in \mathbb{R}^{2n}} \pi^{-n} \int_{\mathbb{R}^{2n}} \left| \Phi(X + Z, \overline{X} - \overline{Z}) \right| e^{-|Z|^2} dZ < \infty$$

The Weyl symbol F must satisfy:

$$e^{\frac{1}{4}\Delta} F(x, \xi) = \Phi(x + i\xi, x - i\xi)$$

Sect. 2. Weyl symbol formula: proof (2).

Let Φ, holomorphic in $\mathbf{C}^{2n}$, satisfying the above condition.
For $\lambda > 0$, set

$$F_\lambda(X) = \int_{\mathbb{R}^{2n}} \Phi(X + Z, \overline{X - Z}) \frac{e^{-\frac{|Z|^2}{2\lambda}} dZ}{(2\pi\lambda)^n}$$

Then:

$$e^{\frac{\lambda}{2}\Delta} F(x, \xi) = \Phi(x + i\xi, x - i\xi)$$

Thus, $F_{1/2}$ is the Weyl symbol.

Sect. 2. Weyl-Campbell-Hausdorff formula.

Theorem

The Weyl symbol satisfies:

$$F(X) = \pi^{-n} \int_{\mathbb{R}^{2n}} \left\langle e^{-\Phi_S(Z)} A e^{\Phi_S(Z)} \Psi_X, \Psi_X \right\rangle e^{-|Z|^2} dZ$$

where

$$\Phi_S(Z) = \sum_{j=1}^{n} \left(z_j u_j + \zeta_j \frac{1}{i} \frac{\partial}{\partial u_j} \right)$$

Sect. 2. Weyl-Campbell-Hausdorff. Proof.

First, we prove that:

$$e^{\Phi_S(Z)}\Psi_X = e^{\frac{1}{2}|Z|^2+Z\cdot X-\frac{i}{2}\sigma(Z,X)}\Psi_{X+Z}$$

Hence:

$$\frac{< A\Psi_{X+Z}, \Psi_{X-Z} >}{< \Psi_{X+Z}, \Psi_{X-Z} >} = \left\langle e^{-\Phi_S(Z)}Ae^{\Phi_S(Z)}\Psi_X, \Psi_X \right\rangle$$

The theorem follows.

Sect. 2. Wick and anti-Wick symbols.

- Wick symbol of an operator A.

$$\sigma^{Wick}(A)(X) = < A\Psi_X, \Psi_X > \qquad X = (x, \xi)$$

- Anti-Wick operator $Op^{AW}(F)$ associated with F.

$$< Op^{AW}(F)f, g > = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} F(Z) < f, \Psi_Z > < \Psi_Z, g > dZ$$

Sect. 2. The anti-Wick symbol always makes sense.

Without any hypothesis, the Weyl symbol of $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is well-defined as a distribution.
The anti-Wick symbol exists, no more as a distribution, but as a "generalized function" of Gelfand Shilov.
Discussions with L. Amour and N. Lerner.
More precisely, if $\lambda > 0$ and $0 < \mu < 1/2$, the AW symbol is a continuous linear form on the space:

$$S(\lambda, \mu) = \Big\{ \varphi \in C^\infty(\mathbb{R}^{2n}), \quad \exists A > 0, \quad |x^\alpha \partial^\beta \varphi(x)| \leq$$

$$\dots \leq A^{|\alpha|+|\beta|} (\alpha!)^\lambda (\beta!)^\mu \Big\}$$

Sect. 2. Formula for the anti-Wick symbol.

Theorem

Let A be a bounded operator in $L^2(\mathbb{R}^n)$ s.t.

$$\sup_{X\in\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \frac{< A\Psi_{X+Z}, \Psi_{X-Z} >}{< \Psi_{X+Z}, \Psi_{X-Z} >} \right| \frac{e^{-\frac{|Z|^2}{2}} dZ}{(2\pi)^n} < \infty$$

Then, there exists a continuous bounded function G in $\mathbb{R}^{2n}$, s.t. $A = Op^{AW}(G)$ and:

$$G(X) = \int_{\mathbb{R}^{2n}} \frac{< A\Psi_{X+Z}, \Psi_{X-Z} >}{< \Psi_{X+Z}, \Psi_{X-Z} >} \frac{e^{-\frac{1}{2}|Z|^2} dZ}{(2\pi)^n}$$

Sect. 3. Modelization of Nuclear Magnetic Resonance.

- Classical model: Spin $S(t)$ = vector in $\mathbb{R}^3$ satisfying

$$S'(t) = B \wedge S(t)$$

F. Bloch, *Nuclear Induction*, Physical Review **70** 460-473, (1946).

Almog, Grebenkov, Helffer: fluid with density of spin.

- QED model: Spin $S(t) = 3$ operators in a Hilbert space $\mathcal{H}$. Evolution given by a Hamiltonian $H(h)$ in $\mathcal{H}$.

F. A. Reuse, *Electrodynamique et Optique Quantiques*, Presses Polytechniques et Universitaires Romandes, Lausanne, 2007.

References for Section 3.

- Slide 1.

[A] L. Amour, R. Lascar, J. Nourrigat, *Weyl calculus in QED I. The unitary group.* J. Math. Phys. **58** (2017), no. 1, 013501.

[B] L. Amour, R. Lascar, J. Nourrigat, *Weyl calculus in Wiener spaces and in QED*, Journal of Pseudo-Diff. Op. and appl. **10** (2019) no 1, p. 1-47. DOI 10.1007/s11868-018-0269-5. (Section 7)

- Slide 2.

[C] L. Amour, L. Jager, J. Nourrigat, *Infinite dimensional semiclassical analysis and applications to a model in NMR*, , preprint, arXiv:1705.07097.
(Int. Congress in Math. Physics, Montreal, 2018.)

Sect. 3. The symbol class $S^\infty(H^2, Q)$ and NMR.

Theorem

The Reuse Hamiltonian $H(h)$ satisfies:

$$e^{-itH(h)} = U_h(t) Op_h^{AW}(F_t))$$

where

- *$U_h(t)$ is a metaplectic (or Bogoliubov) operator.*
- *F_t is a fonction in $S^\infty(H^2, Q_t)$, where H^2 is the phase space.*
- *The quadratic form Q_t has the following form:*

$$Q_t(X) = \sum_{j=1}^{m} \int_0^t < B_j(s), X >^2 ds$$

Sect. 3. The symbol class $S^\infty(H^2, Q)$ and NMR.

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The evolving components of the spin operator

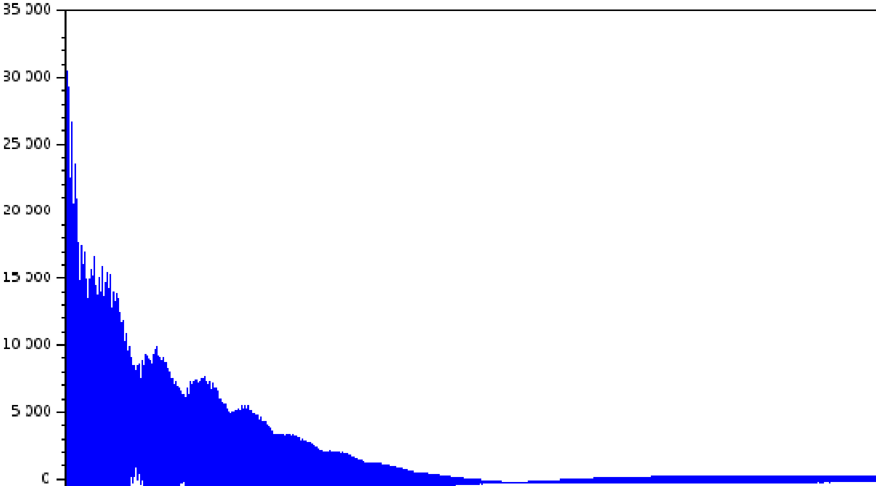
  $$S_j(t) = e^{itH(h)}(I \otimes \sigma_j)e^{-itH(h)}$$

  can be written

  $$S_j(t) = Op_h^{AW}(F_j(t))$$

  where $F_j(t) \in S^\infty(H^2, Q_t)$.
- The Wick symbol has an asyptotic expansion when $h \to 0$. The first term follows Bloch Equations.
- We have also an asympt. exp. for the number of photons emitted at time t.

Sect. 3. Open problems: Large time behaviour (here ibuprophen).



Sect. 3. Spin relaxation: references.

[6] M. Hübner, H. Spohn, *Radiative decay: nonperturbative approaches.* Rev. Math. Phys. **7** (1995), no. 3, 363-387.

[7] W. De Roeck, A. Kupiainen, *Approach to ground state and time-independent photon bound for massless spin-boson models.* Ann. Henri Poincaré **14** (2013), no. 2, 253-311.

THANKS