Magnetic model operators

A short review and something new

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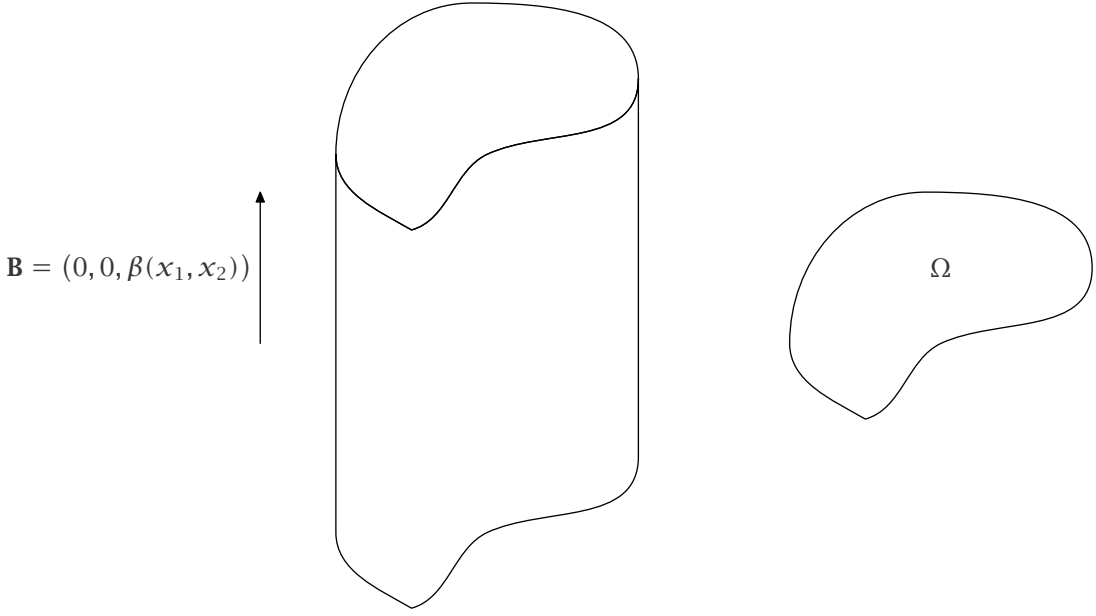
Introduction

The domain

$\Omega \subset \mathbb{R}^2$ with (piecewise) smooth boundary. Corners allowed, but no cusps.



Magnetic field and magnetic vector potential

Magnetic field:

$$\mathbf{B} = (0, 0, \beta(x_1, x_2))$$

Magnetic vector potential:

$$\mathbf{A} = (A_1, A_2), \qquad \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \beta(x_1, x_2)$$

Let us start by assuming that $\beta(x_1, x_2) > 0$ and that β is smooth.

The Neumann magnetic Schrödinger operator

Let $B > 0$ be a parameter. We let

$$\mathcal{H}_\Omega^N(B) = (i\nabla + B\mathbf{A})^2$$

be the self-adjoint realization in $L^2(\Omega)$ corresponding to Neumann boundary conditions (ν is an outward unit normal to $\partial\Omega$)

$$\nu \cdot (i\nabla + B\mathbf{A})\,\psi = 0 \quad (\forall \psi)$$

Usually defined via the Friedrichs extension of the quadratic form

$$\psi \mapsto \int_\Omega |(i\nabla + B\mathbf{A})\psi|^2 \, dx,$$

first defined on $C^{+\infty}(\overline{\Omega})$.

The bottom of the spectrum

We denote the bottom of the spectrum of $\mathcal{H}_\Omega^N(B)$ by

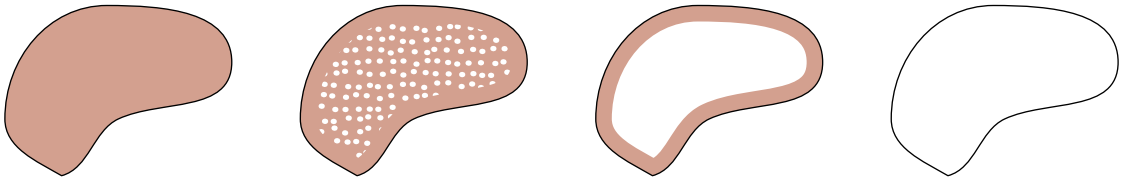
$$\lambda_{\Omega,1}^N(B)$$

If Ω is bounded this is an eigenvalue.

QUESTION Will $\lambda_{\Omega,1}^N(B)$ be strictly increasing if B is large?

Main motivation to study this question

THEOREM If $\lambda^N_{\Omega,1}(B)$ is strictly increasing for large B, then the so-called third critical field H_{C_3} in the Ginzburg–Landau theory of superconductivity is well-defined. Moreover, the main term in the asymptotic expansion of H_{C_3} is expressed in terms of the main term in the asymptotic expansion of $\lambda^N_{\Omega,1}(B)$.



$H < H_{C_1}$ $\quad$ $H_{C_1} < H < H_{C_2}$ $\quad$ $H_{C_2} < H < H_{C_3}$ $\quad$ $H > H_{C_3}$

Large magnetic fields vs. Semi-classical analysis

Let $h = 1/B$. Then

$$\frac{1}{B}\left(i\nabla + B\mathbf{A}\right)^2 = \frac{1}{h}\left(ih\nabla + \mathbf{A}\right)^2 .$$

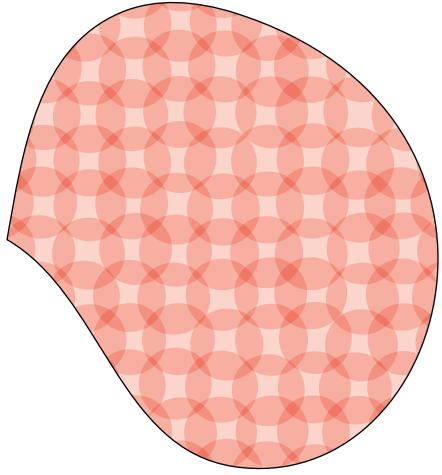
This enables the machinery from semi-classical analysis, starting with

- (Helffer & Sjöstrand, 1984)
- (Simon, 1983)

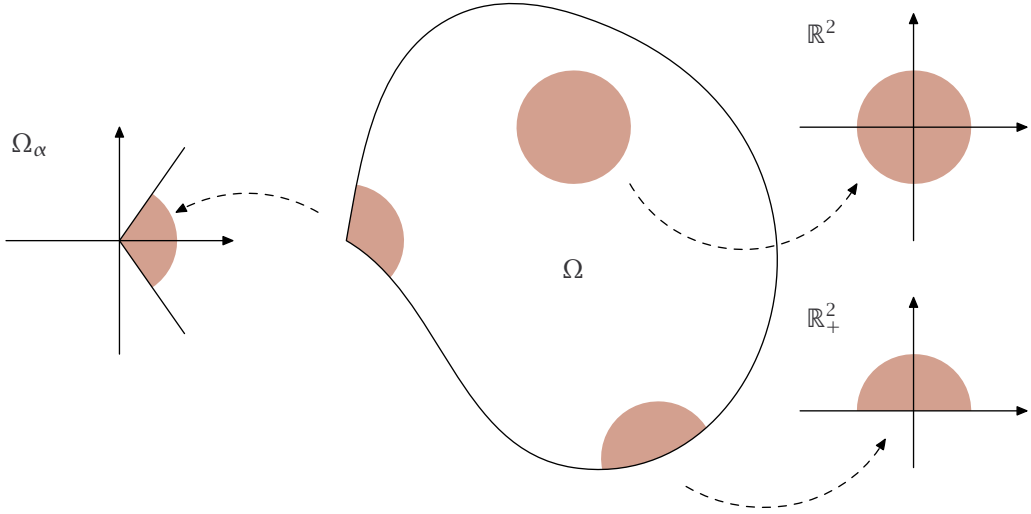
In particular it enables *localization* and reduction to *effective operators*.

Nonnegative and smooth magnetic fields

Localization



Localization



The operator in $\mathbb{R}^2$ (Fock, 1928; Landau, 1930)

Assume $\beta \equiv 1$. Spectrum consists of so-called Landau levels,

$$\Lambda_k = B(2k+1) \qquad (k \in \mathbb{Z},\ k \geq 0)$$

Perhaps most easily proved using annihilation and creation operators

$$\mathcal{Q} = -2ie^{-\Psi}\,\overline{\partial}\,e^{\Psi} \quad \text{and} \quad \mathcal{Q}^* = -2ie^{\Psi}\partial e^{-\Psi}.$$

Here $\Psi(z) = \frac{1}{4}B\,|z|^2$ satisfied $\Delta\Psi = B$.

$$\mathcal{H}_{\mathbb{R}^2} = \mathcal{Q}^*\mathcal{Q} + B = \mathcal{Q}\mathcal{Q}^* - B.$$

Lowest eigenspace

$$\mathcal{L}_0 = \left\{u \in L^2(\mathbb{R}^2)\ :\ e^{\Psi}u \text{ entire}\right\}.$$

Higher eigenspaces

$$\mathcal{L}_k = (\mathcal{Q}^*)^k\mathcal{L}_0.$$

The operator in $\mathbb{R}^2_+$

With $\beta \equiv 1$ and $\mathbf{A} = (-x_2, 0)$, one is lead to

$$\mathcal{H}^N_{\mathbb{R}^2_+}(B) = (i\partial_{x_1} - Bx_2)^2 - \partial^2_{x_2}.$$

After a partial Fourier transform (in x_1) and a dilation one meets the family

$$\mathcal{h}^N(\xi) = -\frac{d^2}{dt^2} + (t-\xi)^2 \quad \text{in } L^2(\mathbb{R}^+).$$

In particular, with $\mu(\xi)$ the bottom of spectrum of $\mathcal{h}^N(\xi)$,

$$\lambda^N_{\mathbb{R}^2_+,1}(B) = B \inf_{\xi\in\mathbb{R}} \mu(\xi).$$

The de Gennes operator (Fournais & Helffer, 2010)

If u_ξ denotes the normalized groundstate of $\mathcal{h}^N(\xi)$, then one has

- Feynman–Hellmann formula:

$$\frac{d}{d\xi}\mu^N(\xi) = -2\int_0^{+\infty}(t-\xi)u_\xi^2\,dt$$

- Bolley–Dauge–Helffer formula:

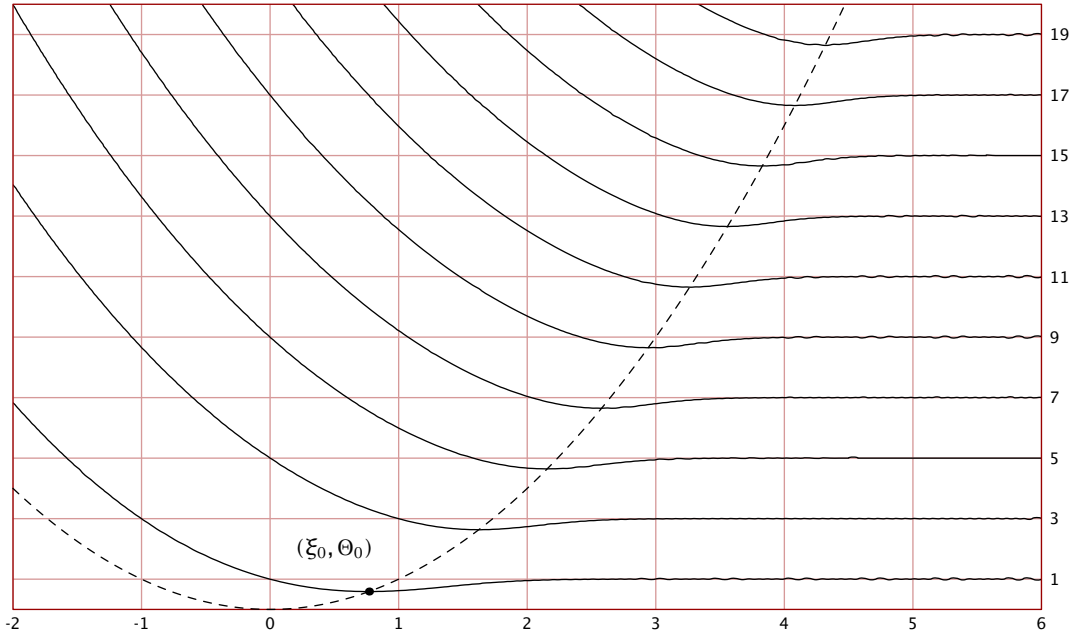
$$\frac{d}{d\xi}\mu^N(\xi) = u_\xi(0)^2(\xi^2 - \mu^N(\xi)).$$

$\xi \mapsto \mu^N(\xi)$ has a unique minimum at $\xi_0 > 0$,

$$\Theta_0 = \min_{\xi\in\mathbb{R}}\mu(\xi) = \mu(\xi_0) = \xi_0^2.$$

The constant $C_1 = \frac{1}{3}u_{\xi_0}(0)^2$ reappears later.

The de Gennes operator (cont.)



The number Θ_0

$\Theta_0 = \xi_0^2$, where ξ_0 is the smallest positive solution of (here D_ν denotes a certain Parabolic cylinderfunction, satisfying $y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}t^2\right) y = 0$)

$$-\xi D_{(\xi^2-1)/2}(-\sqrt{2}\xi) - \sqrt{2} D_{(\xi^2+1)/2}(-\sqrt{2}\xi) = 0$$

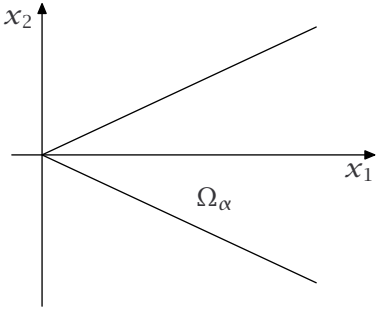
In a fraction of a second we can get over 200 decimals:

$$\begin{aligned}\Theta_0 = {} & 0.59010612495023412872815716628408 66\\ & 75175991713693791792854882140410 20\\ & 34244734903426846814806875563827 84\\ & 08573170527463507394041434271181 68\\ & 59746960145623921662859691305933 35\\ & 13257475810804991668726469110244 46\ldots\end{aligned}$$

The red figures are correct according to (Bonnaillie-Noël, 2012), where upper and lower bounds are given.

Infinite sectors Ω_α

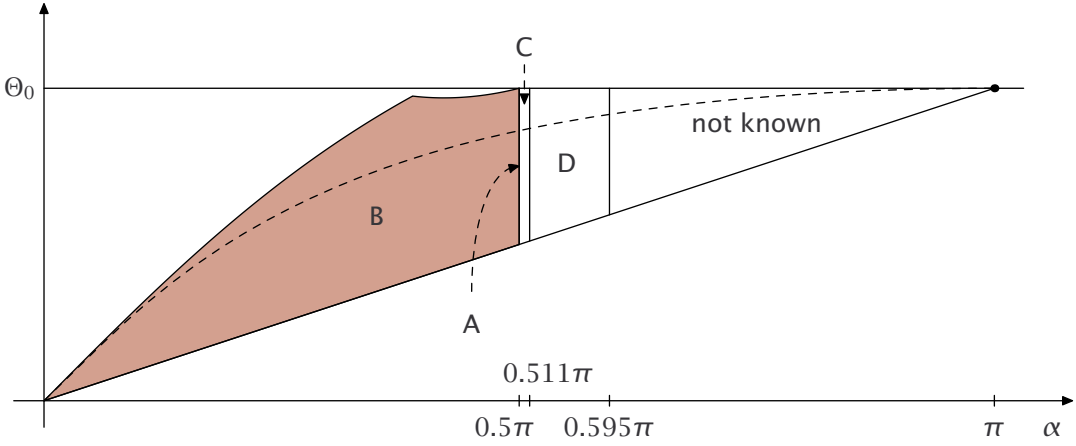
$$\Omega_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2 \ -\alpha/2 < \arg(x_1 + ix_2) < \alpha/2\right\}.$$



THEOREM ((BONNAILLIE, 2005)) Assume $\alpha \in (0, 2\pi)$ and that $\beta(x_1, x_2) = 1$. Then the bottom of the essential spectrum of $\mathcal{H}^N_{\Omega_\alpha}(B)$ is given by $\Theta_0 B$.

QUESTION Does there exist eigenvalues below $\Theta_0 B$?

The eigenvalue problem for infinite sectors



A. (Jadallah, 2001)
B. (Bonnaillie, 2005)
C. (Bonnaillie, 2003)
D. (Exner, Lotoreichik, & Pérez-Obiol, 2018)

Asymptotic results in (Bonnaillie, 2005; Bonnaillie-Noël & Dauge, 2006).

Trial state (Exner, Lotoreichik, & Pérez-Obiol, 2018)

They used trial states in the form

$$u(r,\theta) = e^{-cr^2/2} \exp\left(i \sum_{k=1}^{N} r^k g_k(\theta) \right).$$

Optimal choice of g_k is given by some complicated linear system:

$$\mathsf{K}(c)\mathbf{g}'' = \mathsf{L}(c)\mathbf{g},$$

where $\mathbf{g} = (g_1, g_2, \ldots, g_N)$ and $\mathsf{K}(c)$ and $\mathsf{L}(c)$ are $N \times N$ matrices.

Conclusion, large field asymptotics

THEOREM ((HELFFER & MORAME, 2001), (BONNAILLIE, 2005)) Assume that Ω is bounded and smooth except for a corner at $s \in \partial\Omega$ with opening angle α. Also, assume that the magnetic field β is positive, and that

$$b = \min_{x \in \overline{\Omega}} \beta(x), \quad \text{and} \quad b' = \min_{x \in \partial\Omega} \beta(x).$$

Also, let $\Lambda = \min\left(\lambda^N_{\Omega_\alpha,1}(1)\beta(s), b, \Theta_0 b'\right)$. Then, as $B \to +\infty$,

$$\lambda^N_1(B) = \Lambda B + \mathcal{O}(B^{3/4}).$$

THEOREM ((DEL PINO, FELMER & STERNBERG), (LU & PAN), (HELFFER & MORAME)) If $\beta \equiv 1$, there are no corners of Ω, and k_0 is the maximum of the curvature of the boundary,

$$\lambda^N_1(B) = \Theta_0 B - C_1 k_0 B^{1/2} + \mathcal{O}(B^{1/3}) \quad (B \to +\infty),$$

Monotonicity for large B

THEOREM Assume that $\beta \equiv 1$. Then there exists B_0 such that

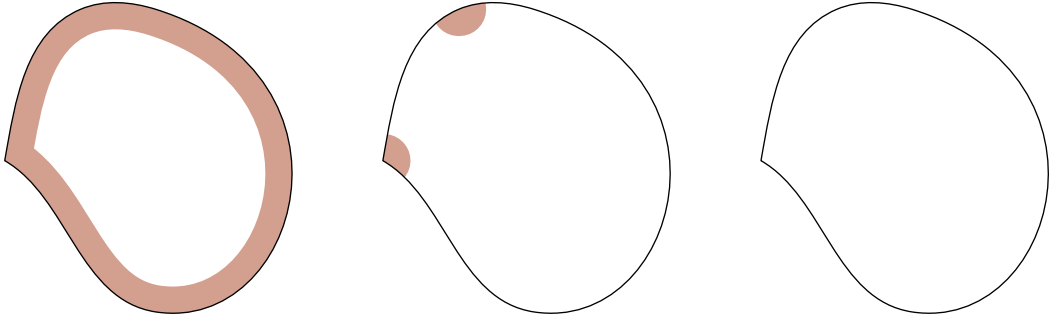
$$B \mapsto \lambda^N_{\Omega,1}(B)$$

is strictly increasing for $B > B_0$.

The proof is different in the cases that

- Ω is generic (Fournais & Helffer, 2006),
- Ω is a disk (Fournais & Helffer, 2007; 2010),
- Ω has corners (Bonnaillie-Noël & Fournais, 2007).

Where does superconductivity survive longest?



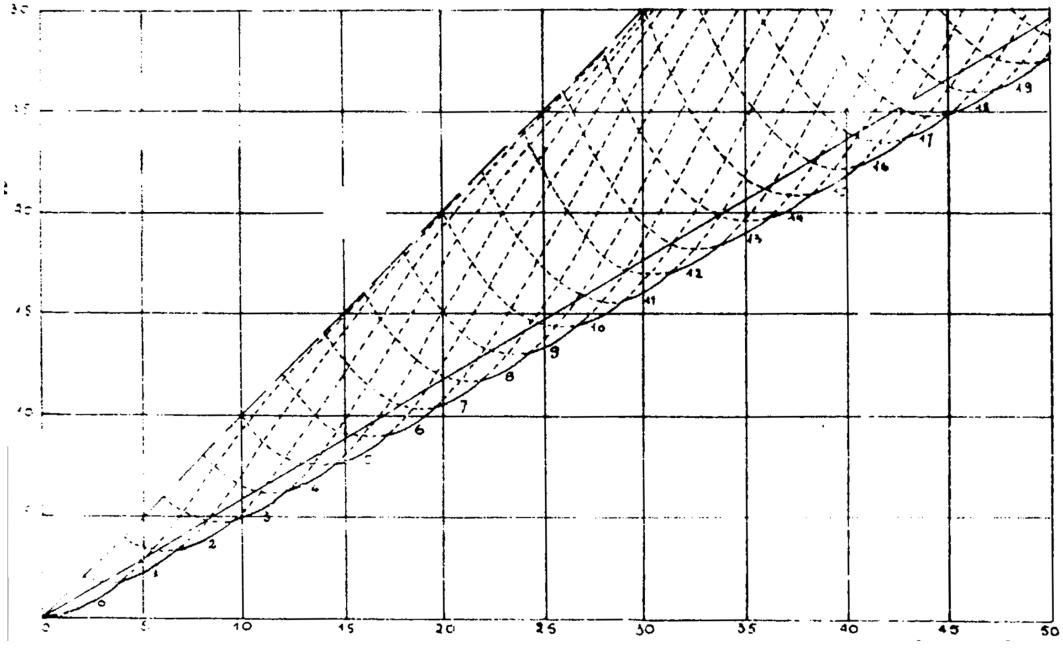
The operator(s) in the unit disk

Assume that $\beta \equiv 1$. After a separation of variables, one is lead to the study of the family of operators

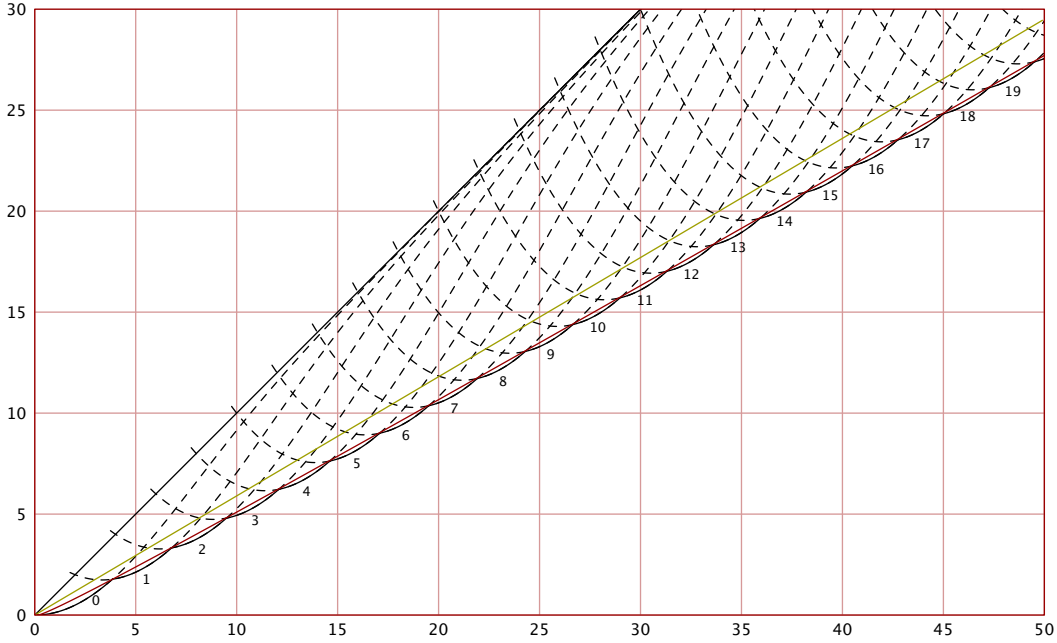
$$\mathcal{H}_m = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \left(\frac{m}{r} - \frac{Br}{2}\right)^2 \qquad (m \in \mathbb{Z})$$

in $L^2((0,1), r\,dr)$.

The unit disk (Neumann) (Saint-James, 1965)



The unit disk (Neumann)



The unit disk (Neumann)

THEOREM ((FOURNAIS & HELFFER, 2010)) Assume that $\beta \equiv 1$. Then

$$\lambda^N_{D(0,1),1}(B) = \Theta_0 B - C_1\sqrt{B} + \mathcal{O}(1) \qquad (B \to +\infty)$$

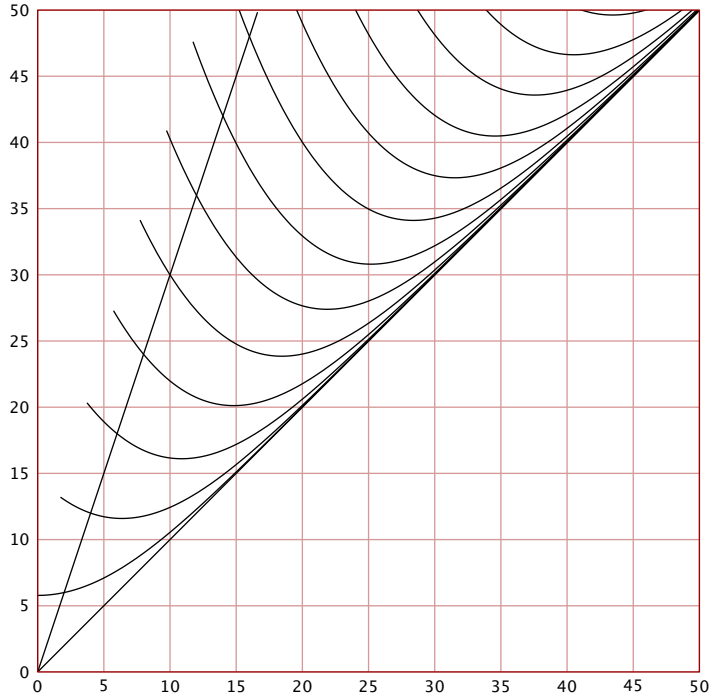
A more careful study of the bounded term implies that

THEOREM ((FOURNAIS & HELFFER, 2007; 2010)) There exists B_0 such that

$$B \mapsto \lambda^N_{D(0,1),1}(B) \qquad (B > B_0)$$

is strictly increasing.

The unit disk (Dirichlet)



The unit disk (Dirichlet)

THEOREM ((ERDŐS, 1996; HELFFER & MORAME, 2001))

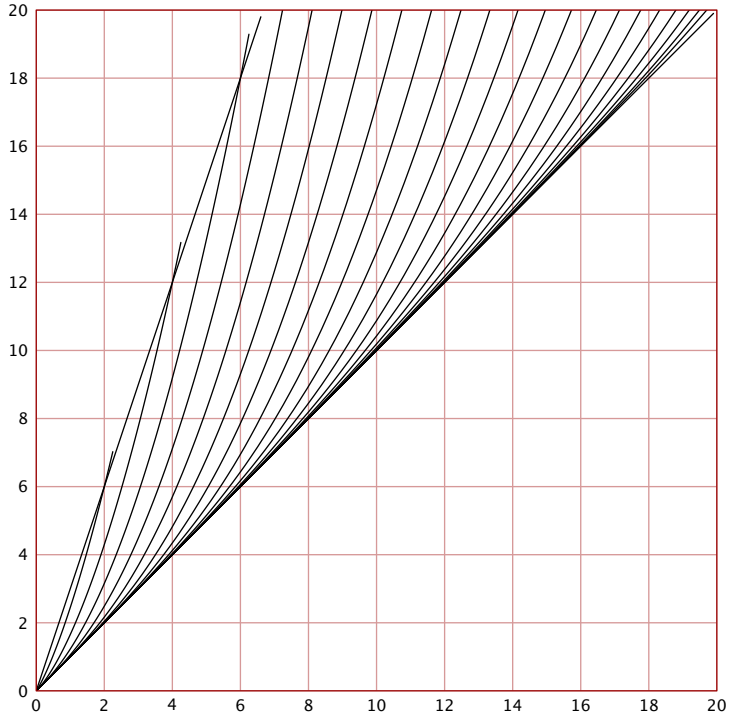
$$\lambda^D_{D(0,1),1}(B) - B \sim \sqrt{\frac{8B^3}{\pi}} \exp(-B/2) \qquad (B \to +\infty)$$

This in fact an estimate of the lowest eigenvalue of the Pauli operator. Other recent works on the lowest eigenvalue of the Pauli operator are

- (Ekholm, Kovařík, & Portmann, 2016),
- (Helffer & S, 2017a), (Helffer & S, 2017b), (Helffer, Kovařík, & S, 2019)

In (Barbaroux, Treust, Raymond, & Stockmeyer, 2018) estimates for higher eigenvalues $\{\lambda^D_{D(0,1),k}(B)\}_{k\geq 2}$ were given.

The complement of the unit disk (Dirichlet)



The complement of the unit disk (Dirichlet)

Assume $\beta \equiv 1$. Enumerate the eigenvalues of $\mathcal{H}^D_{D(0,1)}(B)$ in the gap between the Landau levels $\Lambda_0 = B$ and $\Lambda_1 = 3B$ as

$$\lambda_1 \geq \lambda_2 \geq \ldots$$

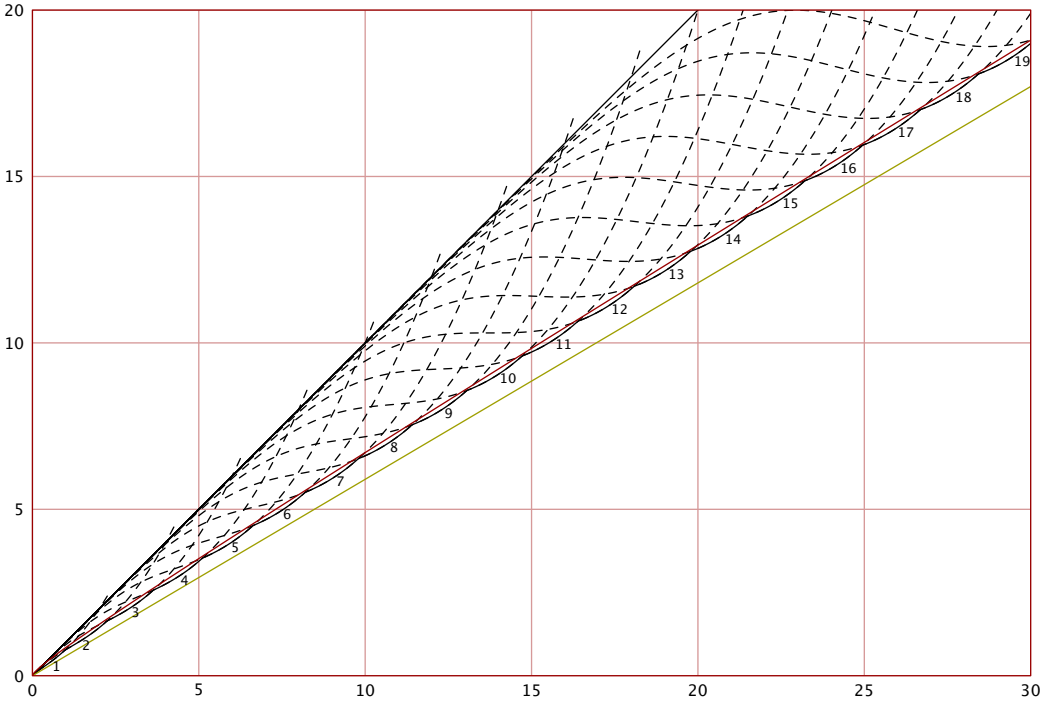
THEOREM ((PUSHNITSKI & ROZENBLUM, 2007)) Assume that $\beta \equiv 1$ and that $B > 0$ is fixed. Then

$$\lim_{k \to +\infty} [k!\,(\lambda_k - \Lambda_0)]^{1/k} = \frac{B}{2}.$$

Similar statements hold for eigenvalues between higher Landau levels.

One should mention in this context works by Filonov & Pushnitski, Raikov and Bruneau.

The complement of the unit disk (Neumann)



The complement of the unit disk (Neumann)

THEOREM ((FOURNAIS & HELFFER, 2010)) Assume that $\beta \equiv 1$. Then

$$\lambda_1(B) = \Theta_0 B + C_1\sqrt{B} + \mathcal{O}(1) \qquad (B \to +\infty)$$

Assume that $\beta \equiv 1$, and enumerate the eigenvalues of $\mathcal{H}(B)$ below the Landau level $\Lambda_0 = B$ as

$$\lambda_1 \leq \lambda_2 \leq \ldots$$

THEOREM ((GOFFENG, KACHMAR, & S, 2016))
Assume that $\beta \equiv 1$ and that $B > 0$ is fixed. Then

$$\lim_{k\to+\infty} [k!\,(\Lambda_1 - \lambda_k)]^{1/k} = \frac{B}{2}.$$

Similar statements hold for eigenvalues between higher Landau levels.

Magnetic fields vanishing along a curve

Operator introduced in (Montgomery, 1995), with the amusing title *Hearing the zero locus of a magnetic field*. Let

$$\widetilde{\mathcal{H}}(B) = -\frac{\partial^2}{\partial t^2} + \left(i\frac{\partial}{\partial s} - Bt^2/2\right)^2.$$

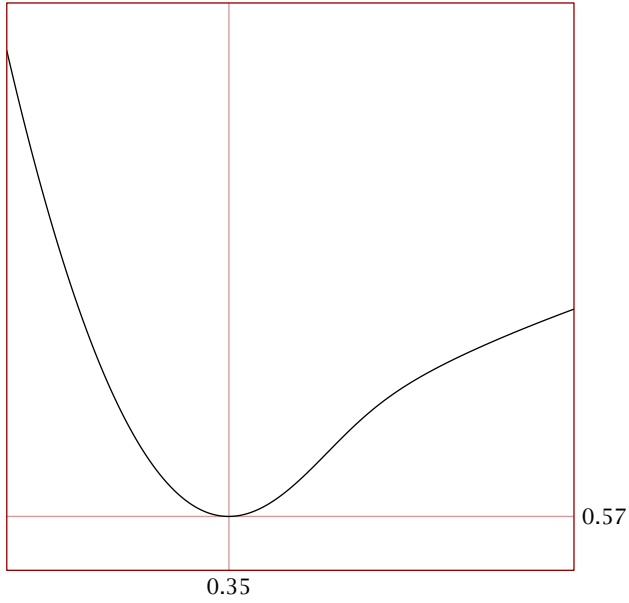
THEOREM ((PAN & KWEK, 2002), (HELFFER, 2010)) The lowest eigenvalue of the operator

$$-\frac{d^2}{dt^2} + \left(\frac{t^2}{2} - \alpha\right)^2$$

in $L^2(\mathbb{R})$ admits a unique, non-degenerate minimum Λ as α varies in $\mathbb{R}$. In this case,

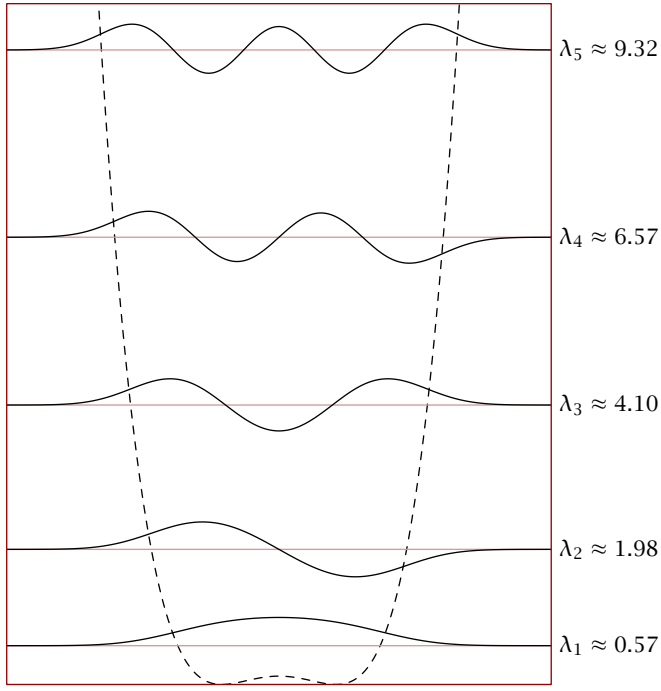
$$\widetilde{\lambda}_1(B) = \Lambda B^{2/3}$$

Magnetic fields vanishing along a curve ($k = 1$)



Numerics done with a method proposed in (Korsch & Glück, 2002).

Magnetic fields vanishing along a curve ($k = 1$)



Magnetic fields vanishing along a curve

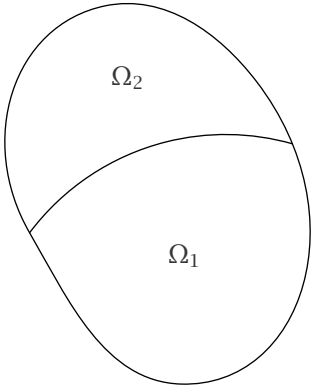
THEOREM ((HELFFER & S, 2010), (FOURNAIS & S, 2015))
Let $k \in \mathbb{N}$, $k \geq 2$. The lowest eigenvalue of the operator

$$-\frac{d^2}{dt^2} + \left(\frac{t^{k+1}}{k+1} - \alpha\right)^2$$

in $L^2(\mathbb{R})$ admits a unique, non-degenerate minimum as α varies in $\mathbb{R}$.

A non-continuous (and sign changing) magnetic field

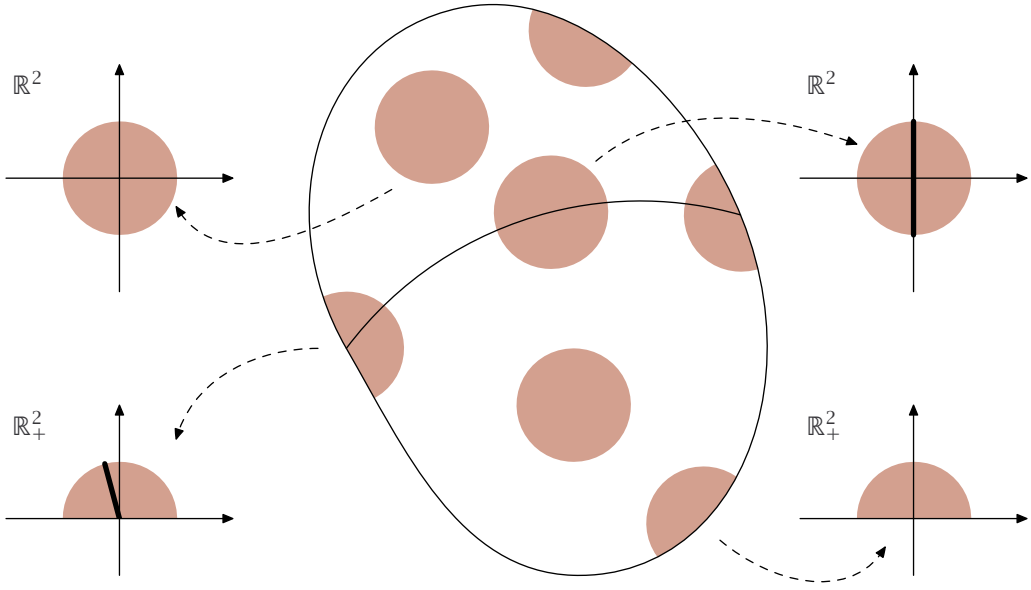
A piecewise constant magnetic field



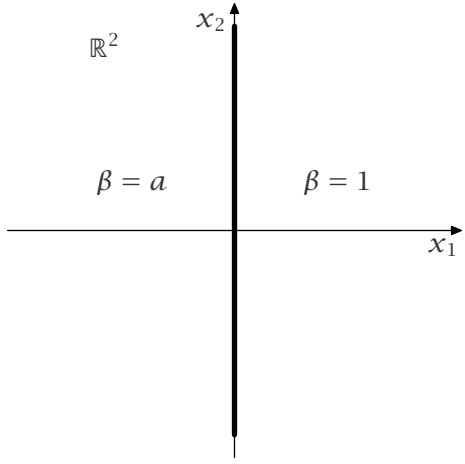
$$\beta(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in \Omega_1 \\ a, & (x_2, x_2) \in \Omega_2 \end{cases}$$

By scaling, we can assume that $a \in [-1, 1]$.

Localization



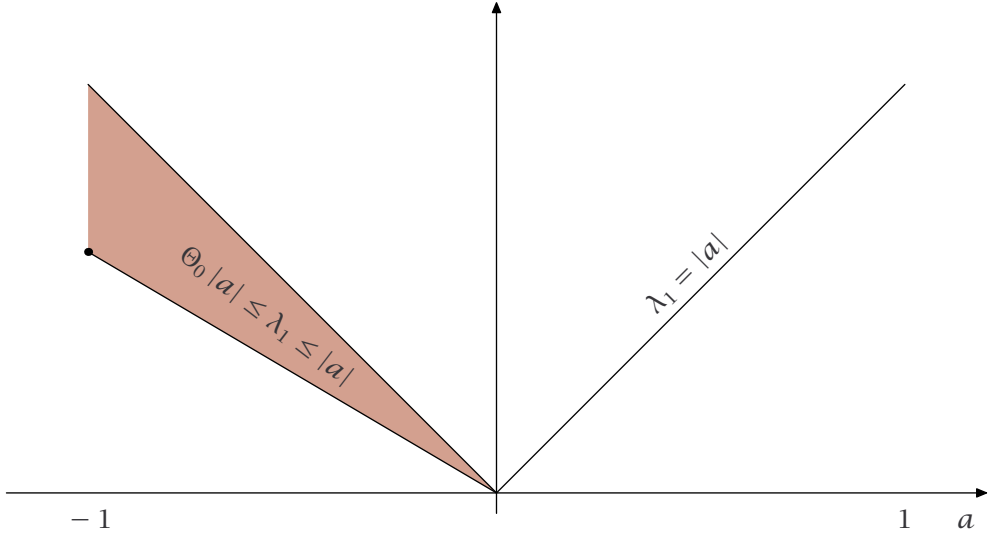
The step model in the plane



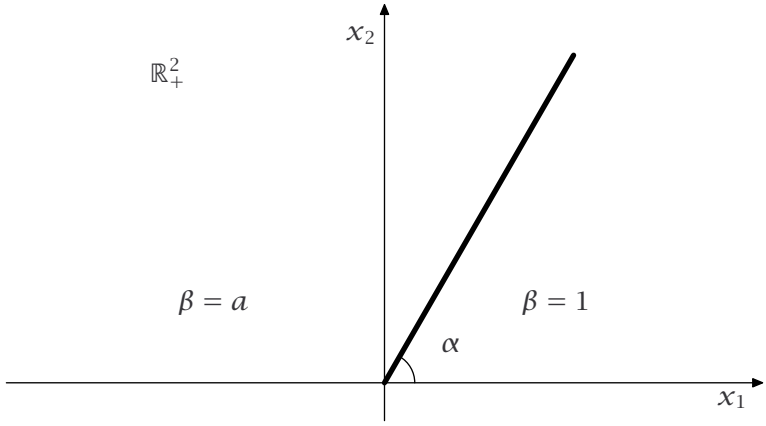
$$\beta(x_1, x_2) = \begin{cases} 1, & x_1 > 0 \\ a, & x_1 < 0 \end{cases} \qquad (-1 < a < 1)$$

The step model in the plane

(Iwatsuka, 1985), (Hislop & Soccorsi, 2015), (Hislop, Popoff, Raymond, & S, 2016), (Assaad, Kachmar, & S, 2019)



The step model in the half plane



$$\beta(x_1, x_2) = \begin{cases} 1, & 0 < \arg(x_1 + ix_2) < \alpha \\ a, & \alpha < \arg(x_1 + ix_2) < \pi \end{cases}$$

The step model in the half plane

THEOREM ((ASSAAD, 2019)) The bottom of the essential spectrum is $\Theta_0 |a|$.

The proof is as usual based on (Persson, 1960).

QUESTION Does there exist eigenvalues below $\Theta_0 |a|$?

Trial state (Assaad, 2019)

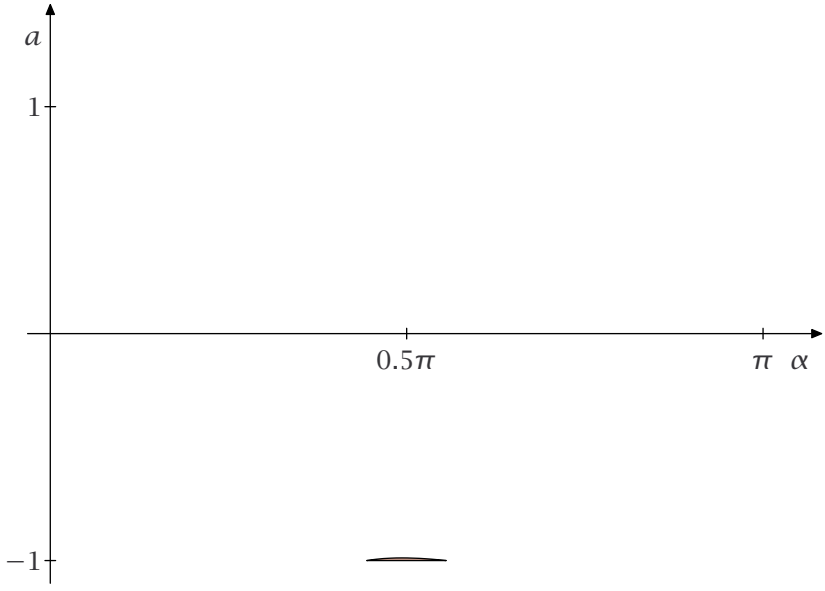
Essentially: Glue two trial-states from (Exner, Lotoreichik, & Pérez-Obiol, 2018), continuously over the barrier,

$$u(r,\theta) = e^{-cr^2/2} \exp\left(irg(\theta)\right),$$

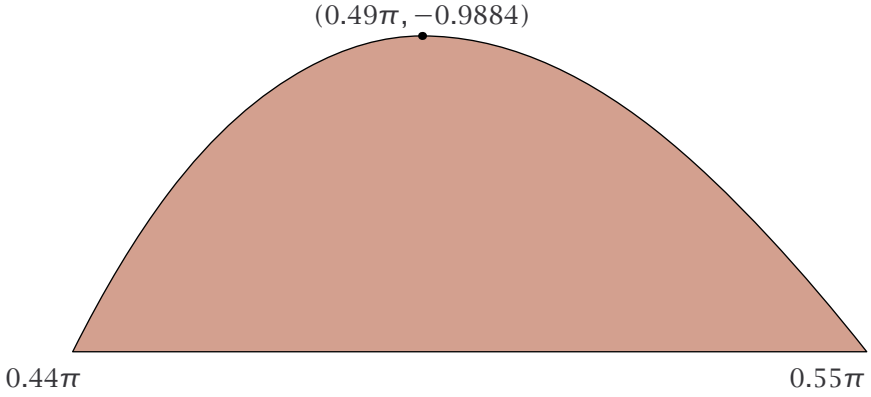
with

$$g(\theta) = \begin{cases} c_1 e^{\theta} + c_2 e^{-\theta} & 0 < \theta < \alpha \\ c_3 e^{\theta} + c_4 e^{-\theta} & \alpha < \theta < \pi \end{cases}$$

Eigenvalue below $|a|\Theta_0$, (Assaad, 2019)



Eigenvalue below $|a|\Theta_0$, (Assaad, 2019)



Continuity, (Assaad, 2019)

Let $\mu(\alpha, a)$ denote the ground state energy of the magnetic step operator in the half plane.
The regularity of $(\alpha, a) \mapsto \mu(\alpha, a)$ is not easy to prove.

THEOREM Fix $\alpha \in (0, \pi)$. Then the mapping

$$a \mapsto \mu(\alpha, a)$$

is continuous for $a \in [-1, 1]$, $a \neq 0$.

Consequences for superconductivity, (Assaad, 2019)

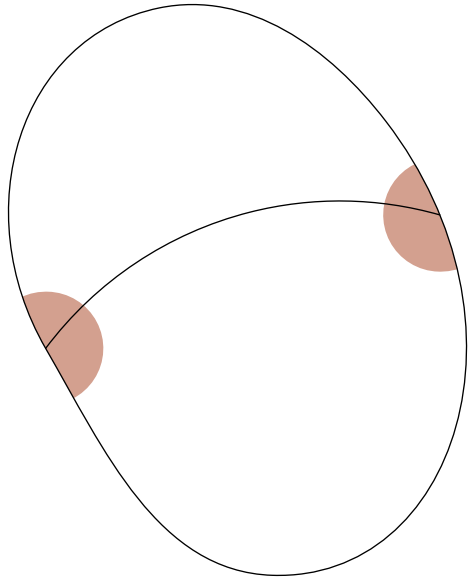
THEOREM Assume that $\Lambda = \min_{j\in\{1,\dots,n\}} \mu(\alpha_j, a)$. Assume that $\Lambda < \Theta_0 |a|$. There exists κ_0 such that for $\kappa \geq \kappa_0$ the equation

$$\lambda_{\Omega,1}(\kappa H) = \kappa^2$$

admits a unique solution $H = H_{C_3}$, with the estimate

$$H_{C_3}(\kappa) = \frac{\kappa}{\Lambda} + \mathcal{O}(\kappa^{1/2}) \quad (\kappa \to +\infty).$$

Last surviving superconductivity



Happy birthday, Bernard!

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