A unified approach to three themes in harmonic analysis

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Three fundamental symmetries

Assume $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Take $1 \leq p \leq \infty$ and fix $x,\, x_0,\, \xi,\, \lambda \in \mathbb{R}$. We define the following classes of symmetries

- $T_{x_0} f(x) := f(x - x_0)$ - spatial translation (with x_0)
- $M_{\xi_0} f(x) := e^{2\pi i (x \cdot \xi_0)} f(x)$ - frequency modulation (with ξ_0)
- $D_\lambda^p f(x) := \frac{1}{\lambda^{\frac{1}{p}}} f(\frac{x}{\lambda})$ - L^p normalized dilation.

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Fourier transform - key properties

- For $f \in \mathcal{S}(\mathbb{R})$ we define the Fourier transform of f as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x} .$$

- $\mathcal{F}\, T_{x_0} = M_{-x_0}\, \mathcal{F}$;
- $\mathcal{F}\, M_{\xi_0} = T_{\xi_0}\, \mathcal{F}$;
- $\mathcal{F}\, D_{\lambda}^{p} = D_{\lambda^{-1}}^{p'}\, \mathcal{F}$, where here p, p' are Holder conjugates, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.
- Another two fundamental relations obeyed by the Fourier transform:
  - $\mathcal{F}\,(\frac{d}{dx} f)(\xi) = 2\pi\, i\, \xi\, \mathcal{F}(f)(\xi)$;
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$$\mathcal{F}^* g(x) := \check{g} = \int_{\mathbb{R}} g(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi$$

- Ex.1: Check that both $\mathcal{F}$ and $\mathcal{F}^*$ map the Schwartz class into the Schwartz class.
- Since $\mathcal{F}$ and $\mathcal{F}^*$ leave unaffected the Gaussian function $e^{-\pi |x|^2}$

  - same happens for any linear combination of Gaussians

  - but linear combinations of Gaussian are dense in the Schwartz class

  - hence we obtain the inversion formula:

$$\mathcal{F}^* \mathcal{F} f = f \quad \text{and} \quad \mathcal{F}\,\mathcal{F}^* g = g\,,$$

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- As a corollary we get the Parseval formula:

$$< f, g >=< \mathcal{F}^* \mathcal{F} f, g >=< \mathcal{F} f, \mathcal{F} g > \ .$$

- Hence we deduce Plancherel:

$$\|\mathcal{F} f\|_{L^2_\xi(\mathbb{R})} = \|f\|_{L^2_x(\mathbb{R})} \, .$$

- Now it is trivial to check that

$$\|\mathcal{F} f\|_{L^\infty_\xi(\mathbb{R})} \leq \|f\|_{L^1_x(\mathbb{R})} \, .$$

- Apply complex-interpolation to deduce the Hausdorff-Young ineq

$$\|\mathcal{F} f\|_{L^{p'}_\xi(\mathbb{R})} \leq \|f\|_{L^p_x(\mathbb{R})} \quad \text{for } 1 \leq p \leq 2 \, .$$

- **Ex. 2** Prove that Hausdorff-Young is sharp in the sense that it can not be extended in the range $2 < p < \infty$.

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What is time-frequency?

- The field of mathematics which, in order to establish qualitative and quantitative information about different categories of objects (functions, operators etc), analyzes both space and Fourier transform properties of the corresponding objects; [space/Fourier transform MATH - time/frequency PHS]
- Thus, this field is intimately connected to Fourier analysis and can be regarded as a development of the theory of trigonometric series initiated in the 19^{th} cent. by Fourier.
- Initial theme of research: understand the relation between

$$f(x) \in L^1(\mathbb{R}) \text{ and } \hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx\,, \;\; \xi \in \mathbb{R} \;\; (\text{cont})$$

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- Leitmotif:

$$f(x) \Rightarrow^{\text{decomposition}} \{\hat{f}(n)\}_n \Rightarrow^{\text{reconstruction}} \sum_n \hat{f}(n)\, e^{2\pi\, i\, n\, x}\,.$$

- Two fundamental facts (for "suitable objects"):
- **Inversion formula:** $f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i\, x\, \xi}\, d\xi\,;$
- smoothness (decay) $f \;\Leftrightarrow\;$ decay (smoothness) $\hat{f}$
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- **Parseval identity:** $\int f(x)\, \bar{g}(x)\, dx = \int \hat{f}(\xi)\, \bar{\hat{g}}(\xi)\, d\xi\,;$
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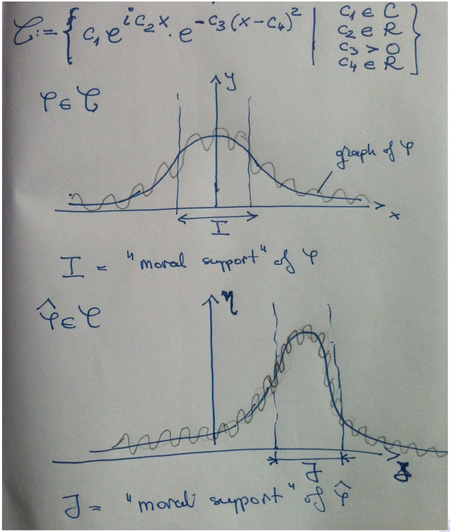
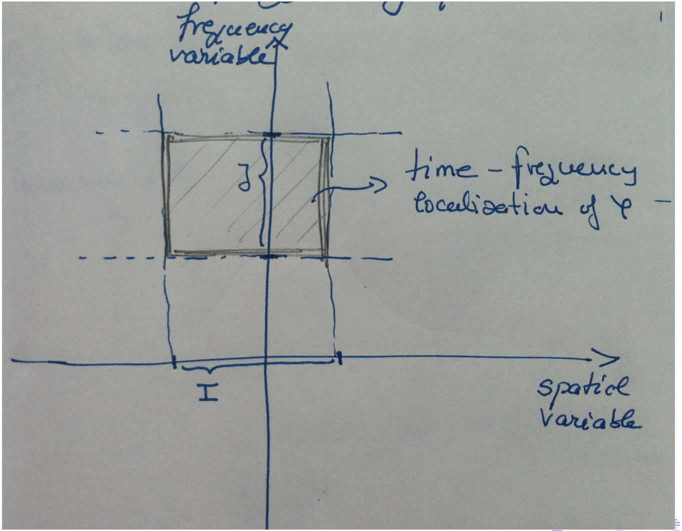
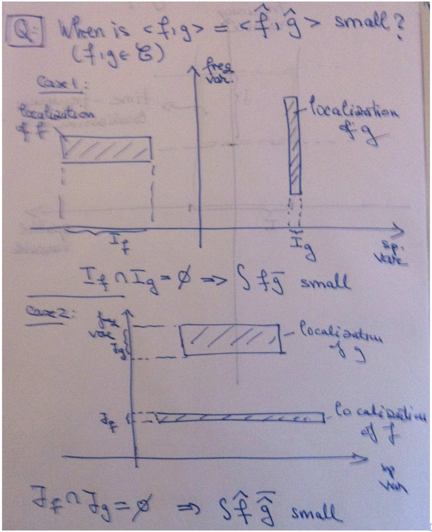
Parseval's story: space/Fourier analysis



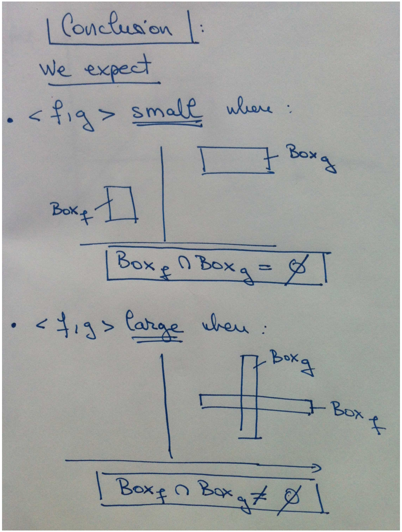
Parseval's story: time-frequency localization



Parseval's story: space/frequency cancelation



Parseval's story: summary



Fundamental philosophy: Understand f $\leftrightarrow$ information about localization & oscillation of f $\leftrightarrow$ localization of f, localization of $\hat{f}$ $\leftrightarrow$ localization of the pair $(f, \hat{f})$

The Hilbert transform

- We start our journey with the simplest fundamental object: the **Hilbert transform**

$$H:\ S(\mathbb{R}) \to S'(\mathbb{R}) \quad Hf(x) := \text{p.v.} \int_R f(x-t)\,\frac{dt}{t}\,.$$

- A celebrated result of M. Riesz (1928) states that H is a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for any $1 < p < \infty$.
- **Relevance**:
- H connects the real and imaginary parts of functions on $\mathbb{R}$ which are boundary restrictions of suitable holomorphic functions in the upper-half plane; this is realized via Cauchy-Riemann system and (conjugate) Poisson kernel.

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The Hilbert transform

- H is characterized (up to a constant multiple), by the following symmetry behavior:
  - H commutes with translations and dilations;
  - H anticommutes with reflections $f(x) \to f(-x)$;

  These facts are direct consequences of the homogeneity of the kernel $\frac{1}{t}$ or, equivalently, of the multiplier $\pi\, i \operatorname{sgn} \xi$.
- Ex.3 Prove that the Hilbert transform in the unique $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ linear bounded operator up to the identity operator that commutes with both translations and dilations.
- Serves as the main prototype for the theory of Calderon-Zygmund operators.

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  - H commutes with translations and dilations;
  - H anticommutes with reflections $f(x) \to f(-x)$;

  These facts are direct consequences of the homogeneity of the kernel $\frac{1}{t}$ or, equivalently, of the multiplier $\pi\, i\, \mathrm{sgn}\, \xi$.
- Ex.3 Prove that the Hilbert transform in the unique $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ linear bounded operator up to the identity operator that commutes with both translations and dilations.
- Serves as the main prototype for the theory of Calderon-Zygmund operators.

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Time-frequency decomposition Hilbert transform

- We split the kernel (Ex. 4)

$$\frac{1}{y} = \sum_{k\in\mathbb{N}} \psi_k(y)\,,$$

where $\psi \in C_0^\infty$ is an odd function supported away from the origin and $\psi_k(y) = 2^k\psi(2^k y)$, $k \in \mathbb{N}$.

- Next, for each scale k we take the collection $\{I_{k,j}\}_j$ of all dyadic intervals in $[0,1]$ of length 2^{-k} and write

$$Hf(x) = \sum_{k,j} H_{k,j}f(x) = \sum_{k,j} (\psi_k * f)(x)\chi_{I_{k,j}}(x)\,.$$

- Observe that each $H_{k,j}f$ has time support included in $I_{k,j}$ while on the frequency side it is "morally" supported near the origin, in an interval of length $|I_{k,j}|^{-1}$.

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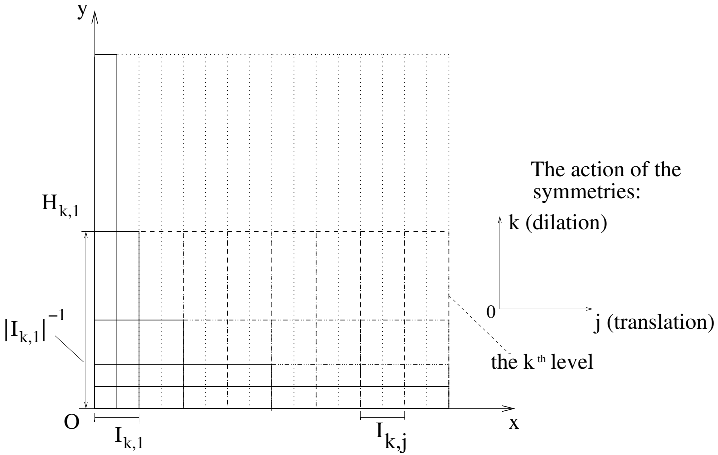
The time-frequency portrait Hilbert transform



Observe that the origin plays here a special role: each rectangle has its basis on the real axis.

The Hardy-Littlewood maximal function.

- Define the **Hardy-Littlewood maximal function** (operator) as
$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \,.$$
- M is a sublinear operator.
- M is trivially of strong type (∞, ∞).
- M is NOT of strong type $(1,1)$ as one can show that (Exercise)
$$\exists\ f \in L^1(\mathbb{R}) \ \ |Mf(x)| \gtrsim_f \frac{1}{1+|x|}\,.$$
- Dimensional analysis - dilation symmetry imposes that no weak or strong type (p,q) estimates are available off the diagonal case $p \neq q$. (Ex.5)

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- Let S_j for $j \in \mathbb{N}$ be the partial Fourier sum of order j attached to a function $f \in L^2(\Pi)$, hence

$$S_j f(x) = \sum_{k=-j}^{j} \hat{f}(k)\, e^{2\pi ikx} \,.$$

- We define the Carleson operator by (Ex.6)

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- On top of the previous symmetries for the Hilbert transform - dilations and translations - we are now dealing with an operator that has an extra modulation symmetry.
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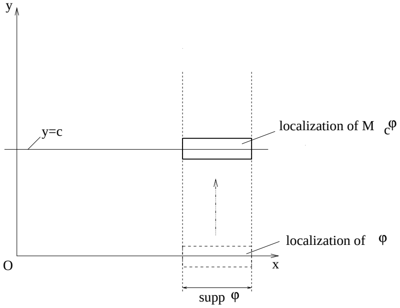
  First task: understand the time-frequency behavior of M_c.



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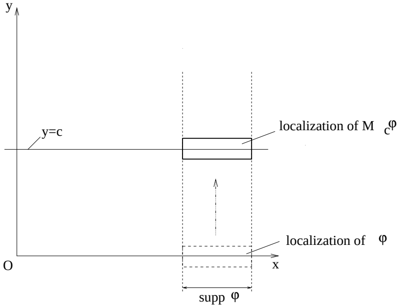

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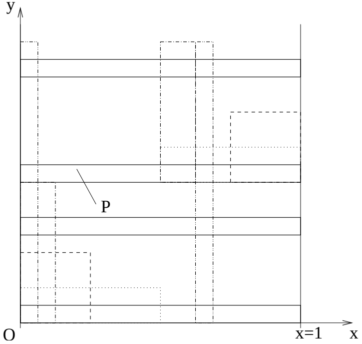

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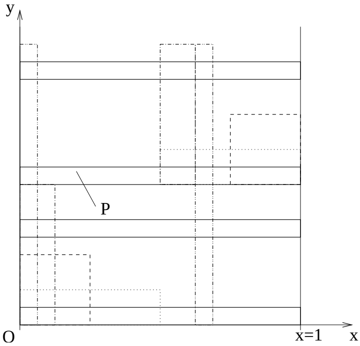

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Historical context

- (1913) Luzin conjectures that if f is square integrable then its Fourier series converges to f almost everywhere.
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Historical context

Theorem (Carleson, 1966)

The Carleson operator obeys the bound

$$\|Cf\|_{L^2(\mathbb{T})} \leq const \, \|f\|_{L^2(\mathbb{T})} \,,$$

where here const is a positive absolute constant.

Carleson's theorem - story

Discretization of the Carleson operator - an overview

- Let C be the Carleson operator

$$Cf(x) := \sup_{j\in\mathbb{N}} |S_j f(x)| \approx \sup_{j\in\mathbb{Z}} \left| \int_{\mathbb{T}} \frac{1}{y}\, e^{2\pi i j y}\, f(x-y)\, dy \right| .$$

- Write $\frac{1}{y} = \sum_{k\in\mathbb{N}} \psi_k(y)$ for $|y| < 1$ where $\psi_k(y) = 2^k\, \psi(2^k\, y)$ and $\psi \in C_0^\infty(\mathbb{R})$ odd.
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- Choose the canonical dyadic grids on $\mathbb{T} \times \mathbb{R}$ and partition the time-frequency plane in tiles of the form $P = [\omega, I]$ with $\omega \subset \mathbb{R}$, $I \subset \mathbb{T}$ dyadic intervals such that $|\omega| = |I|^{-1}$.

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$$Cf(x) := \sup_{j\in\mathbb{N}} |S_j f(x)| \approx \sup_{j\in\mathbb{Z}} \left| \int_{\mathbb{T}} \frac{1}{y}\, e^{2\pi\, i\, j\, y}\, f(x-y)\, dy \right| .$$

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where $N : \mathbb{T} \to \mathbb{Z}$ is a measurable function (Ex.6).

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Carleson's theorem - story

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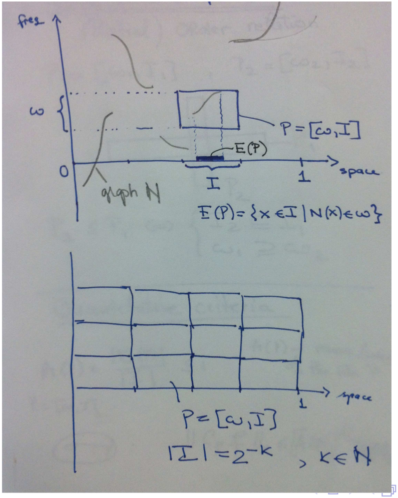
- For $P = [\omega,\ I] \in \mathbb{P}$ define $E(P) := \{x \in I \mid N(x) \in \omega\}$.



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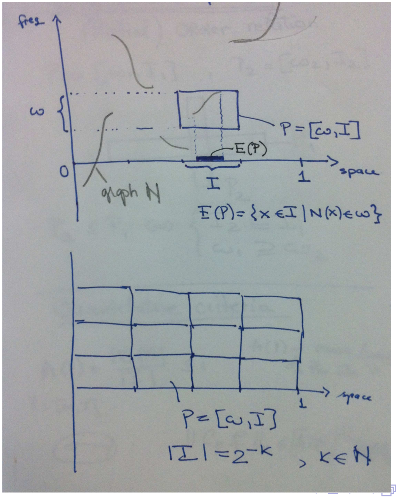

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Carleson's theorem - story

- Next, for $P=[\omega,\, I]\in\mathbb{P}$ with $|I|=2^{-k}$ $(k\in\mathbb{N})$ we set

$$C_P f(x) := \Big(\int_{\mathbb{T}} e^{2\pi\, i\, N(x)\, y}\, \psi_k(y)\, f(x-y)\, dy\Big)\ \chi_{E(P)}(x)\,.$$

- With this, conclude that

$$Cf(x)=\sum_{P\in\mathbb{P}} C_P f(x)\,.$$

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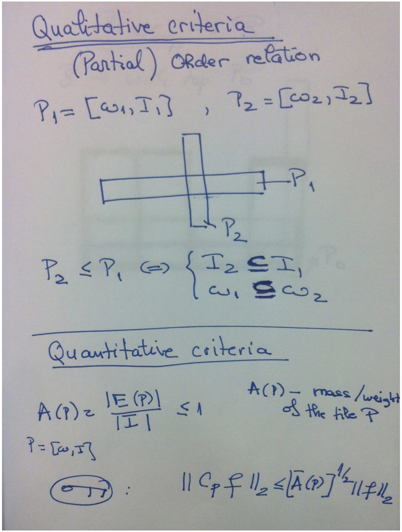


Qualitative criteria

(Partial) Order relation

$P_1 = [\omega_1, I_1]$, $P_2 = [\omega_2, I_2]$

$P_2 \le P_1 \iff \begin{cases} I_2 \subseteq I_1 \\ \omega_1 \subseteq \omega_2 \end{cases}$

Quantitative criteria

$A(P) = \frac{|E(P)|}{|I|} \le 1$ $A(P)$ – mass/weight of the tile P

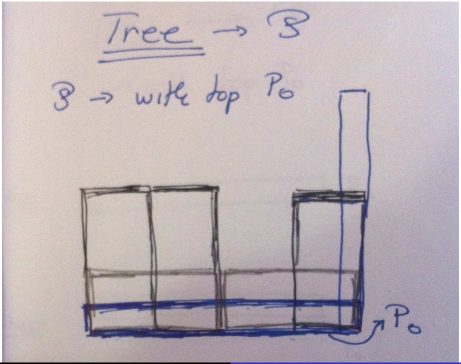
$P = [\omega, I]$

(OTT) : $\|C_P f\|_2 \le [A(P)]^{1/2} \|f\|_2$

Carleson's theorem - story: tree

Definition

A collection of tiles $\mathcal{P} \subset \mathbb{P}$ is called a **tree** with top P_0 iff
1) $\forall\, P \in \mathcal{P} \;\; \Rightarrow \;\; P \leq P_0$.
2) if $P_1,\, P_2 \in \mathcal{P}$ and $P_1 \leq P \leq P_2$ then $P \in \mathcal{P}$.



Carleson's theorem - story: tree

For $\mathcal{P}$ family of tiles set $C^{\mathcal{P}} := \sum_{P\in\mathcal{P}} C_P$. Using now the second criteria - the **mass/weight** of a tile - $A(P)$, we have

Proposition

Fix $n \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathbb{P}$ be a tree such that

$$A(P) \approx 2^{-n} \quad \forall \quad P \in \mathcal{P} .$$

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Carleson's story: The counting function of order n

- Break $\mathbb{P}$ into $\bigcup_{n=0}^{\infty} \mathbb{P}_n$ where

$$\mathbb{P}_n = \left\{P \in \mathbb{P} \,|\, 2^{-n-1} < A(P) \leq 2^{-n}\right\} .$$

- Fix $n \in \mathbb{N}$. We say that $P \in \mathbb{P}_n^{max}$ iff P is **maximal** relative to "$\leq$" and $P \in \mathbb{P}_n$.
- Define the **counting function of order** n as

$$\mathcal{N}_n(x) := \frac{1}{2^n} \sum_{P \in \mathbb{P}_n^{max}} \chi_{I_P}(x) .$$

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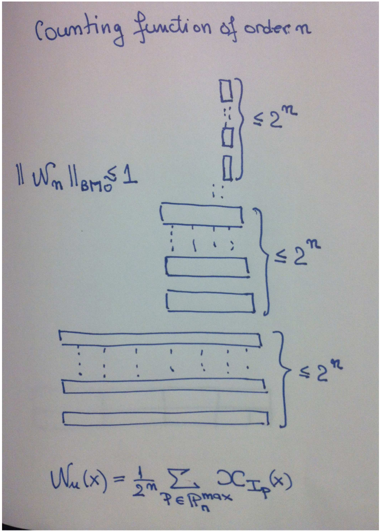
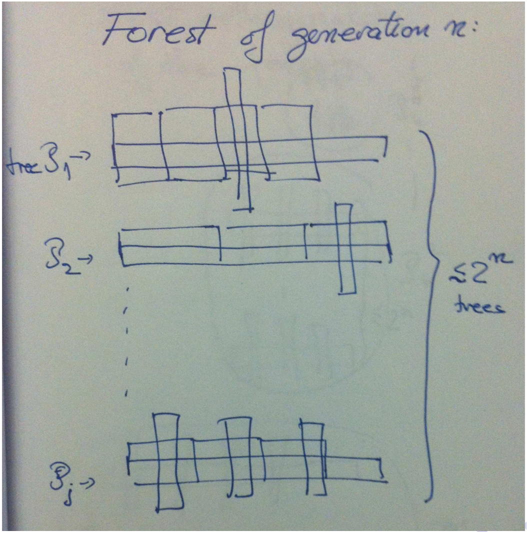


Carleson's theorem - story: Forest



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Let $\mathcal{P}$ be a forest of generation n as above.
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Key: Almost orthogonality of the trees inside $\mathcal{P}$.

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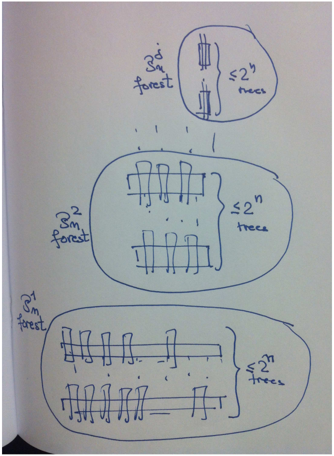
Proof of the "pointwise convergence"

- Recall $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n = \left\{ P \in \mathbb{P} \,|\, 2^{-n-1} < A(P) \leq 2^{-n} \right\}$.

- 

Proof of the "pointwise convergence"

- Now roughly
$$\mathbb{P}_n = \bigcup_k \mathcal{P}_n^k$$
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- For each point $x \in \mathbb{R}$ we associate a curve $\Gamma_x = (t, -\gamma_x(t))$ in the plane, where here $t \in \mathbb{R}$ and

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  is a real function obeying some "suitable" smoothness and non-zero curvature conditions in the t-parameter.
- Define now the variable family of curves in the plane $\Gamma \equiv \{\Gamma_x\}_{\{x \in \mathbb{R}\}}$.
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- **the linear Hilbert transform along Γ**

$$H_{\Gamma} \; : \; S(\mathbb{R}^2) \longrightarrow L^{\infty}(\mathbb{R}^2)\,,$$

$$H_{\Gamma}(f)(x,y) := \mathrm{p.v.} \int_{\mathbb{R}} f(x-t,\; y+\gamma(x,t))\,\frac{dt}{t}\;.$$

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- We consider in the previous definitions a generic class of curves with $\gamma(x,t)=\sum_{j=1}^{n} a_j(x)\,t^j$ and $\{a_j(\cdot)\}_j$ arbitrary real measurable functions. Then, one has
- the zero-curvature case; prototype: $n=1$, with $\gamma(x,t)=a_1(x)t$.
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- We consider here Polynomial Carleson-type operators, which - following Kolmogorov's linearization - can be written in the form

$$C_\gamma f(x) := \int_{\mathbb{R}} e^{i\,\gamma(x,t)}\, f(x-t)\, \frac{dt}{t}\ ,$$

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III. Bilinear Hilbert transform along curves

- Taking the generic case $\gamma(x,t)=\gamma(t)=\sum_{j=1}^{n} a_j\, t^j$ with $\{a_j\}_j$ real and $\Gamma=(t,-\gamma(t))$, we define

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- In the zero-curvature (flat) case all the above operators obey suitable invariance under modulation symmetry.
- *Consequence:* any method of proof requires an approach based on wave-packet analysis and thus in particular a time-frequency discretization of the corresponding operator.
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- More standard analysis can be performed on the object under study: TT^* (orthogonality methods), (non)stationary phase principle, Van der Corput estimates, Littlewood-Paley techniques, square-function arguments, etc.
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- While both situations are interesting and historically motivated, generically speaking the zero-curvature situation tends to be more difficult and accordingly most of the celebrated problems in this area - some of which remain open - regard precisely this case.
- The situation of nonzero curvature can also prove challenging, but to a lesser extent. In this context, while often regarded as model problems for the flat case, the corresponding non-flat case problems usually can only provide limited intuition, since, they require yet distinct methods of proof.

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- Natural Goal: *unify* the two approaches corresponding to the zero/non-zero curvature cases, and thus to provide a method of proof for the situation in which γ is given by a polynomial in t with the linear term included.
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Historical background and motivation (I): I.1. PDE

- **A. Constant coefficient elliptic differential operators**
- Model: Laplace/Poisson equation in $\mathbb{R}^d$, $d \geq 2$:

$$\triangle u = f\,.$$

- The fundamental solution $U^0(x)$ is given by

$$U^0(x) := -\frac{1}{2\pi}\,\log\frac{1}{|x|}\ \text{ if } d=2\,,$$

$$U^0(x) := \frac{1}{(d-2)\omega_d}\,|x|^{2-d}\ \text{ if } d>2,\ \ (\omega_d = Area(S^d))\,.$$

- Taking $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$ and letting

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  - K is homogeneous of degree $-d$, *i.e.* if $\delta_\alpha(x) = (\alpha x_1, \ldots, \alpha x_d)$ then $K(\delta_\alpha(x)) = \alpha^{-d} K(x)$, $\alpha > 0$.
  - K is C^∞ away from the origin;
  - $\int_{|x|=1} K(x)\, d\sigma(x) = 0$.
- Now the map $Tf := K * f$ represents a Calderon-Zygmund operator and hence

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- Model: The heat equation in $\mathbb{R}^{d+1}_{+} = \mathbb{R}^d \times R_{+}$, $d \geq 2$

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- For $t > 0$ and $x \in \mathbb{R}^d$ the fundamental solution $U^0(x,t)$ is

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- **C. Connections between the theme of constant coefficient parabolic differential operators and that of the Hilbert transform along curves.**
- By Plancherel the L^2-boundedness of $Tf(x,t) := K * f(x,t)$ follows from the L^∞ uniform boundedness in $0 < \epsilon < R$ of

$$\hat{K}_{\epsilon,R}(\xi,\eta) = \int_{\mathbb{R}^d} K(x,1) \int_{\epsilon}^{R} \frac{e^{i\xi s}\, e^{ix\cdot\eta s^{\frac{1}{2}}}}{s}\, ds\, dx\,.$$

- Based on our hypothesis on K the uniform boundedness of $\hat{K}_{\epsilon,R}(\xi,\eta)$ is essentially equivalent with the L^2-bdd of H_Γ along a parabola ($\gamma(x,y,t) = t^2$) since the corresponding multiplier for H_Γ is given by $m_{H_\Gamma}(\xi,\eta) = \int_{\mathbb{R}} \frac{e^{-i\xi t}\, e^{i\eta t^2}}{t}\, dt$.

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- The $L^2(\mathbb{R}^2)$-boundedness of the Hilbert transform along $\Gamma = (t, t^{\alpha})$ with $\alpha > 0$ and $\alpha \neq 1$ ($\gamma(x, y, t) = t^{\alpha}$) was proved by Fabes (1966) via complex integration methods.

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Historical background and motivation (I): I.2. Singular oscillatory integral operators

- Departing from the uniform estimates for the parabola case, E. Stein and S. Wainger initiated a systematic study of the singular oscillatory integral expressions/operators.
- One of their first results (1970): If $\{a_j\}_{j=1}^n \subset \mathbb{R}_+$ and $\{b_j\}_{j=1}^n \subset \mathbb{R}$, $n \in \mathbb{N}$ one has

$$\left| \int_{\mathbb{R}} e^{i\sum_{j=1}^n b_j\, t^{a_j}}\, \frac{dt}{t} \right| < K(a_1, \ldots, a_n)\,,$$

  with K independent of $\{b_j\}_{j=1}^n$. This result is based on Van der Corput estimates.
- Thus, they obtained the $L^2(\mathbb{R}^2)-$boundedness of H_Γ for $\Gamma = (t, \gamma(t))$ with $\gamma(t) = \sum_{j=1}^n b_j\, t^{a_j}$.

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- Departing from the uniform estimates for the parabola case, E. Stein and S. Wainger initiated a systematic study of the singular oscillatory integral expressions/operators.
- One of their first results (1970): If $\{a_j\}_{j=1}^n \subset \mathbb{R}_+$ and $\{b_j\}_{j=1}^n \subset \mathbb{R}$, $n \in \mathbb{N}$ one has

$$\left| \int_{\mathbb{R}} e^{i \sum_{j=1}^n b_j t^{a_j}} \frac{dt}{t} \right| < K(a_1, \ldots, a_n),$$

  with K independent of $\{b_j\}_{j=1}^n$. This result is based on Van der Corput estimates.
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- This result was extended along the '70 decade in several stages to more general functions $\gamma(t)$ obeying suitable smoothness and non-vanishing curvature conditions (Stein, Wainger, Nagel, Riviere).
- A main breakthrough was the proof of the $L^p(\mathbb{R}^d)$ inequalities $(1 < p \neq 2 < \infty)$ for H_Γ and later for the associated maximal operator M_Γ.

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- The zero curvature case.
- This topic originates in Lebesgue's theory of integration; Lebesgue showed that for any (locally) integrable function over the real line and for almost every point, the value of the integrable function is the limit of infinitesimal averages taken about the point.
- Natural question: what about similar differentiability results in higher dimensions, say for functions on $\mathbb{R}^2$?
- This problem is much more subtle: reason - the existence of "pathological" objects such as Besicovitch sets.
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- In light of these challenging aspects of higher-dimensional differentiation problem, an alternative line of inquiry is offered by studying the problem of differentiation for averages along (variable) one-dimensional sets (curves) in $\mathbb{R}^2$.
- The most representative example in this context is given by Zygmund's conjecture, which, informally, asks about differentiability of averages along families of lines whose directions are described by a Lipschitz vector field.

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Conjecture

(Zygmund) *If* $u : \mathbb{R}^2 \to \mathbb{R}$ *is a Lipschitz vector field then the maximal operator*

$$M_{u,\epsilon_0} f(x,y) := \sup_{0<\epsilon<\epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x-t, y-u(x,y)t)| \, dt \, ,$$

is bounded on $L^p(\mathbb{R}^2)$ *for any* $1 < p < \infty$ *provided* ϵ_0 *is small enough depending on* $\|u\|_{Lip}$.

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(Stein) *If $u : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz vector field the Hilbert transform*

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- Bourgain (1989) proved that for every real analytic function u there exists $\epsilon_0 > 0$ such that M_{u,ϵ_0} is bounded on L^2 (L^p).
- The analogue for H_{u,ϵ_0} was proved by Stein and Street (2012).
- In between, L^p bounds were shown for both M_{u,ϵ_0} and H_{u,ϵ_0} under the assumption of extra-curvature/smoothness (Christ, Nagel, Stein and Wainger)
- A breakthrough in terms of the methods/regularity assumptions is due to Lacey and Li (2006). They showed - up to a conditional bound on a Kakeya type maximal operator - that if u is $C^{1+\epsilon}$ then H_u is bounded on L^2.
- If $u = u(x)$ (single variable) and measurable then H_u is L^p bounded for $p > \frac{3}{2}$ (Bateman and Thiele 2013); a similar result if u is constant along a Lispchitz curve was proved by Guo (2017).

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- If $\gamma(x, y, t) = x\,\tilde{\gamma}(t)$ with $\tilde{\gamma}$ suitable smooth & convex then H_Γ is $L^p(\mathbb{R}^2)$-bdd. for $p > 1$ (Carbery, Wainger, Wright 1995);
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  - Using the annulus estimate above and a square function argument, Di Plinio, Guo, Thiele and Zorin-Kranich, (2017) completed the global result for H_Γ.

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Historical background and motivation (II): II.1. Maximal singular oscillatory integral operators

- (1913) Luzin conjectures that if f is square integrable then its Fourier series converges to f almost everywhere.
- (1922) Kolmogorov constructs an example of an L^1 function whose Fourier series diverges a.e. suggesting that Luzin's conjecture may be false.
- (1966) L. Carleson provides the positive answer to this conjecture, setting the foundation of time-frequency analysis.
- Carleson's proof relied on/equivalent with the L^2 boundedness of the Carleson operator.
- R. Hunt (1969) further proved that $C: L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for $1 < p < \infty$ while Sjölin (1971) extend this result to higher dimensions.

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Motivated by the study of singular integral on the Heisenberg group as well as on the previously discussed work with Wainger on the Hilbert transform along curves, E. Stein proposed the following generalization of Carleson's result:

Conjecture

(Stein, 1995) *Let $\frac{1}{2}_{d,n}$ be the class of all real-coefficient polynomials in d variables with no constant term and of degree less than or equal to n, and let K be a suitable CZ kernel on R^d. Then the Polynomial Carleson operator defined as*

$$C_{d,n}f(x) := \sup_{Q \in \frac{1}{2}_{d,n}} \left| \int_{\mathbb{R}^d} e^{i\,Q(y)}\, K(y)\, f(x-y)\, dy \right| ,$$

obeys, for any $1 < p < \infty$, the bound

$$\|C_{d,n}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} .$$

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(Stein, 1995) *Let* $\frac{1}{2}_{d,n}$ *be the class of all real-coefficient polynomials in* d *variables with no constant term and of degree less than or equal to* n*, and let* K *be a suitable CZ kernel on* R^d*. Then the Polynomial Carleson operator defined as*

$$C_{d,n}f(x) := \sup_{Q\in\frac{1}{2}_{d,n}} \left| \int_{\mathbb{R}^d} e^{i\,Q(y)}\, K(y)\, f(x-y)\, dy \right| ,$$

obeys, for any $1 < p < \infty$*, the bound*

$$\|C_{d,n}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} .$$

Historical background and motivation (II): II.1. Maximal singular oscillatory integral operators

- In 2001, relying on Van der Corput estimates and TT^* method, Stein and Wainger verified the above conjecture for the case when the supremum in the above expression ranges only through polynomials having **no linear** term. Notice thus that this result does not extend Carleson's Theorem.
- In 2009 (L., PhD thesis) we solved this conjecture in the case $d = 1$, $n = 2$, $p = 2$.
- In 2011 we completely solved the one dimensional case.
- In November 2017, based on our methods, Zorin-Kranich proved the higher dimensional case for $p \geq 2$ and more general CZ kernels (not necessarily translation invariant).
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Historical background and motivation (II): II.2. Connection with the Hilbert transform along curves

- Take $\gamma(x,t) := \sum_{j=1}^{n} a_j(x)\, t^j$ with $a_j(\cdot)$ being measurable functions. Take as usual $\Gamma_x = (t, -\gamma(x,t))$ on $\mathbb{R}^2$ and

$$H_\Gamma f(x,y) := p.v. \int_{\mathbb{R}} f(x-t,\, y+\gamma(x,t))\, \frac{dt}{t}\,.$$

- Now the L^2-boundedness of H_Γ is equivalent via Parseval

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} f(x-t,\eta) \frac{e^{i\,\eta\,\gamma(x,t)}}{t}\, dt \right|^2 dx\, d\eta \lesssim \|f(x,\eta)\|^2_{L^2(\mathbb{R}^2)}\,.$$

- Conclude that the L^2-boundedness of H_Γ implies (and is in fact equivalent) with the L^2 boundedness of the (Polynomial) Carleson operator in $1\,D$.

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Historical background and motivation: III.1. Bilinear Hilbert transform and maximal operators along curves

- The original formulation of this third theme, as with those of the previous two, was cast in terms of a single variable dependence, *i.e.* for curves $\gamma(x,t) \equiv \gamma(t)$.
- General Problem Let $\Gamma := (t, -\gamma(t))$ be a plane curve with γ a suitable (piecewise) smooth real function. Goal: Understand the conditions on the curve Γ under which one has that
  - the bilinear Hilbert transform along the curve Γ

$$H_\Gamma^{\mathcal{B}}(f,g)(x) := \text{p.v.} \int_{\mathbb{R}} f(x-t)\, g(x+\gamma(t)) \frac{dt}{t} \,,$$

  - the (sub)bilinear maximal operator along the curve Γ

$$M_\Gamma^{\mathcal{B}}(f,g)(x) := \sup_{\epsilon>0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x-t)\, g(x+\gamma(t))|\, dt \,,$$

each map $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedly for some $p,\ q,\ r \geq 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

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- The original formulation of this third theme, as with those of the previous two, was cast in terms of a single variable dependence, *i.e.* for curves $\gamma(x,t) \equiv \gamma(t)$.
- The zero-curvature/flat case: $\gamma(t) = a\,t$ with $a \in \mathbb{R} \setminus \{-1, 0\}$.
- This theme arose in the study of the Cauchy transform along Lipschitz curves. Indeed, this study led Calderón to conjecture the $L^p \times L^q \to L^r$ boundedness of the *Bilinear Hilbert transform* (BHT) $H^{\beta}_{\Gamma_a}$ with $\gamma(t) = a\,t$ and $a \in \mathbb{R} \setminus \{-1, 0\}$ for Hölder exponents $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $p, q, r \geq 1$.

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Bilinear Hilbert transform - flat case

- **Definition.** For $\alpha \in \mathbb{R}$ and $f, g \in S(\mathbb{R})$ set the BHT

$$H_\alpha(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)\, g(x-\alpha\, t)\, \frac{dt}{t}\ .$$

- **Origin.** The BHT arose from the study of the Cauchy integral (Hilbert transform) on Lipschitz curves - research initiated by A. Calderon.
- Let $\gamma(x) = x + i\,A(x)$ curve in C with $A' = a \in L^\infty(\mathbb{R})$. The Hilbert transform on γ is given by

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- Use Taylor series

$$\frac{1}{x-y+i(A(x)-A(y))}=\frac{1}{x-y}\sum_{k=0}^{\infty}(-i)^k\left(\frac{A(x)-A(y)}{x-y}\right)^k$$

- Naturally led to the study

$$C_k f(x)=p.v.\int_{\mathbb{R}}\frac{(A(x)-A(y))^k}{(x-y)^{k+1}}f(y)\,dy\,,\quad k\in\mathbb{N}$$

- C_0 Hilbert transform; C_1 - Calderon's first commutator:

$$C_1 f(x)=\int\int_0^1 a(x+\alpha(y-x))\,\frac{1}{x-y}\,f(y)\,d\alpha\,dy$$

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- **Thoerem** (Lacey-Thiele (1997,1999)). For $\alpha \notin \{0,1\}$ the BHT obeys

$$\|H_\alpha(f,g)\|_r \leq C\|f\|_p \, \|g\|_q \,,$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\frac{2}{3} < r < \infty$ and $1 < p,\, q \leq \infty$.

- **Facts.** The BHT has the following symmetries

$$H_\alpha(T_y f, T_y g) = T_y H_\alpha(f,g)\,,\ H_\alpha(D_\lambda f, D_\lambda g) = D_\lambda H_\alpha(f,g)$$

$$H_\alpha(M_{\alpha a} f, M_{-a} g) = M_{(\alpha-1)a} H_\alpha(f,g)$$

- **Consequences.** The extra modulation symmetry suggests the use of the wave-packet theory.

$$H_\alpha(f,g)(x) = \int\int sgn(\xi + \alpha\eta)\, \hat{f}(\xi)\, \hat{g}(\eta)\, e^{i\,(\eta+\xi)\,x}\, d\xi\, d\eta\,.$$

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Bilinear Hilbert transform - nonflat case

- By analogy with the study of the boundedness properties of the Hilbert transform along curves (initiated by Jones/Fabes and Riviere) one can ask the following
- **Problem.** For what class of curves $\Gamma = (t, \gamma(t)) \subset \mathbb{R}^2$ can one provide bounds for the BHT along Γ defined by

$$H_\Gamma(f,g)(x) := \mathrm{p.v.} \int_{\mathbb{R}} f(x-t)g(x-\gamma(t))\frac{dt}{t} \; ?$$

- **Theorem** (X. Li, 2008). If $\Gamma = (t, t^d)$ with $d \in \mathbb{N}$, $d \geq 2$ then

$$H_\Gamma : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \mapsto L^1(\mathbb{R}) \, , .$$

- His proof uses a "half" discretization of the symbol of H_Γ and further relies essentially on the $\sigma-$uniformity concept inspired by the work of Gowers.

Bilinear Hilbert transform - nonflat case

- **Theorem** (L.,2011,2015). If γ is a smooth "non-flat" curve near zero and infinity then $H_\Gamma : L^p(\mathbb{R}) \times L^q(\mathbb{R}) \mapsto L^(r\mathbb{R})$. with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.
- **Observation.** The class of curves contains
- the real polynomial with no constant and no linear term;
- the class of real analytic function near 0 (and ∞) such that $\gamma(0)=\gamma'(0)=0$ $(\gamma(\infty)=\gamma'(\infty)=0)$;
- finite lin. combin. of $|t|^\alpha(\log|t|)^\beta$ with $\alpha,\,\beta\in\mathbb{R}$, $\alpha\notin\{0,\,1\}$;

Bilinear Hilbert transform - nonflat case

- **About the proof**.
- Does not involve the notion of $\sigma-$uniformity used by Li in the monomial case;
- This discretization realizes the fragile equilibrium between the two possible extremes:
- cut too rough the multiplier $\Rightarrow$ can not take advantage of the cancelation offered by the phase
- cut too fine $\Rightarrow$ delicate number theoretical problems involving Van der Corput lemma and Weyl type sums

Recall our goal

Main Problem

(General Formulation) Let $\Gamma_{(x,y)} = (t,\, \gamma(x,y,t))$ *be a variable curve in the plane, where here* $t \in \mathbb{R}$ *and* $(x,y) \in \mathbb{R}^2$ *while*

$$\gamma_{(x,y)}(\cdot) := \gamma(x,y,\cdot) : \mathbb{R} \to \mathbb{R}$$

is a "suitable" real function. Under what conditions on the curve $\Gamma_{(x,y)}$ *- [main target: minimal regularity in x and y] - do we have that the three groups of operators satisfy the natural boundedness range?*

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- In our present study we will focus on the **twisted mutivariable-case** $\gamma(x, y, t) = \gamma(x, t)$ with
- minimal regularity in the **variable** x: for every $t \in \mathbb{R}$ the function $\gamma(\cdot, t)$ is only measurable;
- "non-flatness" and low degree smoothness (minimal - C^2 (piecewise)) in the **variable** t for a.e. $x \in \mathbb{R}$.
- **Motivation/interest**
- develop an extensive study that provides a unitary and sharp method of treating *simultaneously* both the Hilbert transform and the maximal operator along curves - as opposed to the disparate previous ad-hoc techniques
- implement an approach that introduces time-frequency analysis/wave-packet analysis
- gather under the same umbrella several themes in harmonic analysis: maximal singular integral operators in the spirit of Stein-Wainger, Carleson type-operators, Hilbert transform/maximal operators along curves

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- minimal regularity in the **variable** x: for every $t\in\mathbb{R}$ the function $\gamma(\cdot,t)$ is only measurable;
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Theorem (L.,2019)

Let $\gamma(x,t):\ \mathbb{R}^2 \to \mathbb{R}$ *be such*

- $\gamma(\cdot,t)$ *is* <u>*measurable*</u> *for every* $t \in \mathbb{R}$;
- $\gamma(x,\cdot)$ *is* <u>*"non-flat"*</u> *in the variable* t *for a.e.* $x \in \mathbb{R}$ *and piecewise* C^2*-smooth;*
- γ *satisfy suitable* <u>*nondegeneracy*</u> *condition.*

Then, for any $1 < p < \infty$, *one has*

$$\|H_\Gamma f\|_p,\ \|M_\Gamma f\|_p,\ \|C_\gamma f\|_p \lesssim_p \|f\|_p\,.$$

Observation

Examples of $\gamma(x,t)$ for which our results hold:

- $\gamma(x,t) = a(x)\,\gamma(t)$ *with* $a(\cdot)$ *real measurable and* $\gamma(\cdot) \in C^2(\mathbb{R}\setminus\{0\})$ *"non-doubling and uniformly locally convex away from the origin";*
  (e.g. $\gamma(t) = \sum_{j=1}^{d} c_j\, t^{\alpha_j}$ *with* $d \in \mathbb{N}$, $c_j \in \mathbb{R}$ *and* $\alpha_j \in \mathbb{R}\setminus\{-1,1\}$ *and even linear combinations of terms of the form* $|t|^{\alpha}\,|\log|t||^{\beta}$ *with* $\alpha, \beta \in \mathbb{R}$ *and* $\alpha \notin \{-1,0,1\}$*).*
- $\gamma(x,t) = \sum_{j=2}^{d} a_j(x)\, t^j$ *where here* $d \in \mathbb{N}$ *with* $d \geq 2$ *and* $\{a_j\}_j$ *real measurable functions.*
- *More generally,* $\gamma(x,t) = \sum_{j=1}^{d} a_j(x)\, t^{\alpha_j}$ *with* $\alpha_j \in \mathbb{R}\setminus\{-1,1\}$ *and* $\{a_j\}_j$ *as before.*
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Corollary (L.,2019)

Let $\vec{\alpha} := (\alpha_j)_{j=1}^n$ *with* $\{\alpha_j\}_{j=1}^n \subset \mathbb{R} \setminus \{-1, 1\}$ *(distinct) and set*

$$\frac{1}{2}_{\vec{\alpha}} := \{\sum_{j=1}^{n} b_j\, y^{\alpha_j} \,|\, \{b_j\}_{j=1}^n \subset \mathbb{R}\}\,.$$

Then the Generalized Polynomial type Carleson operator defined as

$$C_{\vec{\alpha}} f(x) := \sup_{Q \in \frac{1}{2}_{\vec{\alpha}}} \left| \int_{\mathbb{R}} e^{i\,Q(y)}\, f(x-y)\, \frac{dy}{y} \right|\,,$$

obeys for $1 < p < \infty$ *the bound*

$$\|C_{\vec{\alpha}} f\|_{L^p(\mathbb{R})} \lesssim_{\vec{\alpha},p} \|f\|_{L^p(\mathbb{R})}\,.$$

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Multiplier analysis

- If regarded from the Fourier side, H_Γ takes the form

$$H_\Gamma f(x,y)=\int_{\mathbb{R}^2} e^{i\xi x+i\eta y}\widehat{f}(\xi,\eta)m(\xi,\eta,x)d(\xi,\eta),$$

where the multiplier is given by

$$m(x,\xi,\eta)=p.v.\int_{\mathbb{R}} e^{-i\xi t+i\eta\gamma_x(t)}\frac{dt}{t}\,.$$

- If ρ smooth, compactly supported function, we decompose

$$\frac{1}{t}=\sum_{j\in\mathbb{Z}}2^j\rho(2^j t)\,.$$

- We end the first stage decomposition of m, by writing

$$m=\sum_{j\in\mathbb{Z}}m_j,$$

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- Since we are dealing with a highly oscillatory integrand, it is natural to expect an analysis of the phase according to the principle of non/ stationary phase (PDE - "resonance method").
- If we set the phase function

$$\varphi_{\gamma,x,\xi,\eta}(t) := -\frac{\xi}{2^j}\,t + \eta\,\gamma_x(\frac{t}{2^j})\,,$$

$$\frac{d}{dt}\varphi_{\gamma,x,\xi,\eta}(t) := -\frac{\xi}{2^j} + \eta\,2^{-j}\,\gamma_x'(\frac{t}{2^j})\,.$$

- At the heuristic level, based on the properties of γ

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$$1=\sum_{m,n\in\mathbb{Z}}\phi(\frac{\xi\,2^{-j}}{2^m})\,\phi(\frac{\eta\,2^{-j}\,\gamma_x'(2^{-j})}{2^n})\,,$$

with $\phi\in C_0^\infty(\mathbb{R})$, $\mathrm{supp}\,\{\frac{1}{2}\leq|\xi|\leq 2\}$ and $\sum_{k\in\mathbb{Z}}\phi(\xi/2^k)=1$.

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- We split our multiplier's analysis in three regions:
  - (**I**) the low frequency case - no oscillation present:

$$m_j^L = \sum_{(m,n)\in(\mathbb{Z}_-)^2} m_{j,m,n}\,;$$

  - (**II**) the high frequency far from diagonal case - no stationary points present:

$$m_j^{HF\Delta} = \sum_{(m,n)\in\mathbb{Z}^2\setminus((\mathbb{Z}_-)^2\cup\Delta)} m_{j,m,n}\,;$$

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- **(I)** The low frequency case can be treated **globally** in the j^{th} parameter. Is the **only** case in which one sees the distinction between the maximal operator and the Hilbert transform.
- One gets $|M^L_\Gamma(f)| \lesssim_\gamma M_1 M_2 f$ and respectively $|H^L_\Gamma(f)| \lesssim_\gamma \left(\sum_{k\in\mathbb{Z}} |M_1(f *_y \check{\phi}_k)|^2\right)^{\frac{1}{2}}$.
- **(II)** The high frequency far from diagonal case appeals to the fact we have no stationary points at the phase of the corresponding multiplier and hence one can first integrate by parts and then apply a square function argument combined with vector-valued Calderon-Zygmund theory.
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- **(II)** The high frequency far from diagonal case appeals to the fact we have no stationary points at the phase of the corresponding multiplier and hence one can first integrate by parts and then apply a square function argument combined with vector-valued Calderon-Zygmund theory.
- **(III)** The high frequency diagonal case is of course the main term and its treatment the most difficult part of our proof.

Proof's philosophy - a sketch

- Set now

$$H_{j,m}(f)(x,y) := \int_{\mathbb{R}^2} \hat{f}(\xi,\eta)\, m_{j,m,m}(x,\xi,\,\eta)\, e^{i\xi x}\, e^{i\eta y}\, d\xi d\eta\,.$$

- Then, the main terms for our operators are in this instance

$$H^{main} f := \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{N}} H_{j,m} f$$

$$M^{main} f := \sup_{j\in\mathbb{Z}} |\sum_{m\in\mathbb{N}} H_{j,m} f| \leq \sum_{m\in\mathbb{N}} \sup_{j\in\mathbb{Z}} |H_{j,m} f|\,.$$

- An important observation is the following

$$\|H^{main} f\|_p,\ \|M^{main} f\|_p \lesssim_p \sum_{m\in\mathbb{N}} \|(\sum_{j\in\mathbb{Z}} |H_{j,m}(f)|^2)^{\frac{1}{2}}\|_p\,.$$

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Proof's philosophy - a model
The main L^2-estimate

Theorem

With the above notations, $\exists\, c_\gamma > 0$ *such that:*

$$\left\| \Big(\sum_{j\in\mathbb{Z}} |H_{j,m}(f)|^2\Big)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \lesssim_\gamma 2^{-m\,c_\gamma}\ \|f\|_{L^2(\mathbb{R}^2)}\ .$$

This decay result is sharp.

The proof of this result is based on time-frequency analysis and involves among others Gabor frame decompositions, TT^* method, (non-)stationary phase principle.

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For any $1 < p < \infty$ *the following holds:*

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Open problems

- Is there any interesting interpretation that one can provide for our results in terms of parabolic differential operators...variable coefficients?
- Extend these results such that the curvature in t is not required; this will treat in a unitary fashion both Stein-Wainger type results and Polynomial Carleson operators in the L^2-case.
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THANK YOU!

The H-L maximal function and TT^*-method.

- Let us give a **direct** proof of the L^2 bounds

$$\|Mf\|_{L^2(\mathbb{R}^d)} \lesssim_d \|f\|_{L^2(\mathbb{R}^d)} .$$

- By restricting f to *positive* step functions we have that

$$r \to M_r f(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f \quad \text{continuous} .$$

- Notice that by continuity $Mf = \sup_{r\in\mathcal{Q}} M_r f$; enough to restrict the sup to a **finite** collection $\mathcal{R}$ of r's and prove that our strong L^2 bounds are **independent** of $\mathcal{R}$ and f.
- Let D denote the best constant (we know is finite) of

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The main problem

A fundamental and difficult question in the theory of trigonometric series is what happens between the **two extreme** situations:

- $p = 1$ **divergence** of the Fourier series for functions in L^1 (Kolmogorov);
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Question (A-Qualitative)

What is the **largest** *rearrangement invariant Banach space of functions $Y \subseteq L^1(\mathbb{T})$ for which the partial Fourier sums $S_n(f)(x) := \sum_{k=-n}^{n} \hat{f}(k)\, e^{i\,k\,x}$ converge to $f(x)$ almost everywhere $x \in \mathbb{T}$ for any $f \in Y$?*

Definition

We say that a r.i. (quasi-) Banach space Y **is a** $\mathcal{C}-$ **space** iff $\exists\ C_0 = C_0(Y) > 0$ such that $\|Cf\|_{1,\infty} \leq C_0\,\|f\|_Y \quad \forall\ f \in Y$.

Question (A-Quantitative)

Give a satisfactory description of the **Lorentz spaces** *or (***r.i. (quasi-)Banach spaces** *Y ($Y \subseteq L^1(\mathbb{T})$) that are also $\mathcal{C}-$spaces. If it exists, describe the* **maximal Lorentz** $\mathcal{C}-$**space** Y_0.

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Positive results

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non decreasing convex function with $\phi(0) = 0$ and $\phi(\infty) = \infty$.

Denote with $\phi(L) := \{f \in L(\mathbb{T}) \mid \int_{\mathbb{T}} \phi(|f(x)|)\, dx < \infty\}$.

For the following functions ϕ, $\phi(L)$ is a **Lorentz** $C-$space:

- (Sjölin, 1969) $\phi(x) = x \log^2(10 + x)$.
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Also in terms of **r.i. quasi-Banach** $C-$**spaces:**

- (F. Soria, 1985,1989) $\|Cf\|_{1,\infty} \lesssim \|f\|_B$.
- (Arias de Reyna, 2002) $\|Cf\|_{1,\infty} \lesssim \|f\|_{QA}$.
  $L(\log L)^2 \subsetneq L \log L \log\log L \subsetneq B,\ L \log L \log\log\log L \subsetneq QA \subsetneq L \log L$.

All results are based on extrapolation theory.

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Denote with $\phi(L) := \{f \in L(\mathbb{T}) \mid \int_{\mathbb{T}} \phi(|f(x)|)\, dx < \infty\}$.
For the following functions ϕ, $\phi(L)$ is a **Lorentz** $C-$space:

- (Sjölin, 1969) $\phi(x) = x \log^2(10 + x)$.
- (Sjölin, 1969) $\phi(x) = x \log(10 + x) \log\log(10 + x)$.
- (Antonov, 1996) $\phi(x) = x \log(10 + x) \log\log\log(10 + x)$.

Also in terms of **r.i. quasi-Banach** $C-$**spaces:**

- (F. Soria, 1985,1989) $\|Cf\|_{1,\infty} \lesssim \|f\|_B$.
- (Arias de Reyna, 2002) $\|Cf\|_{1,\infty} \lesssim \|f\|_{QA}$.
  $L(\log L)^2 \subsetneq L\log L \log\log L \subsetneq B$, $L\log L \log\log\log L \subsetneq QA \subsetneq L\log L$.

All results are based on extrapolation theory.

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All results are based on extrapolation theory.

Negative results

If ϕ as below, then $\phi(L)$ is **not** a Lorentz $C-$space:

- (Kolomogorov, 1922) $\phi(u) = u$.
- (Korner, 1981) $\phi(u) = o(u \log\log u)$ as $u \mapsto \infty$.
- (Konyagin, 2000) $\phi(u) = o(u \sqrt{\frac{\log u}{\log\log u}})$ as $u \mapsto \infty$.

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