Pointwise convergence to initial data

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Warning: Many arbitrarily small ε have been identified with 0.

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I will review recent progress for Carleson's question for the Schrödinger equation $i\partial_t u + \Delta u = 0$ with initial data u_0 in the Sobolev space $H^s(\mathbb{R}^n)$. That is, for which s can we be sure that $u(x,t) \to u_0(x)$ as $t \to 0$ for almost every $x \in \mathbb{R}^n$. First I will present examples, due to Bourgain, Lucà and myself, which show that $s \geq \frac{1}{2} - \frac{1}{2(n+1)}$ is necessary. We will see that spread out interference-type behaviour becomes a problem when $n \geq 2$. I will then present maximal estimates, due to Du, Guth, Li and Zhang, which show that $s > \frac{1}{2} - \frac{1}{2(n+1)}$ is sufficient. Loosely speaking, these are a consequence of Strichartz-type estimates that improve for spread out solutions. We will also consider the fractal dimension version of the problem and the analogous questions for other PDE.

Summary

- Part 1: Set-up and introduction to the PDEs.
- Part 2: Convergence for the heat equation.
- Part 3: Decay of the Fourier transform of fractal measures.
- Part 4: Convergence for the wave equation.
- Part 5(a): Counterexample for the Schrödinger equation.
- Part 5(b): Convergence for the Schrödinger equation.

Part 0:
Basic properties of the Fourier transform

For Schwartz functions $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$, we write

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad g^\vee(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\, d\xi,$$

so that

$$f = (\widehat{f}\,)^\vee, \qquad \text{(Inversion formula)}$$

$$\widehat{\Delta f}(\xi) = -|\xi|^2 \widehat{f}(\xi), \qquad \text{where} \quad \Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

$$(\widehat{f}\,\widehat{g})^\vee(x) = f * g(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy,$$

$$\int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, d\xi, \qquad \text{(Plancherel)}$$

and

$$\|f\|_{L^2} := \left(\int_{\mathbb{R}^n} |f(x)|^2\, dx \right)^{1/2} = \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2} = \|\widehat{f}\,\|_{L^2}.$$

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so that

$$f = \big(\widehat{f}\,\big)^{\vee}, \qquad \text{(Inversion formula)}$$

$$\widehat{\Delta f}(\xi) = -|\xi|^2 \widehat{f}(\xi), \qquad \text{we write} \quad \varphi(\Delta) f := \big(\varphi(-|\cdot|^2)\widehat{f}\,\big)^{\vee},$$

$$\big(\widehat{f}\,\widehat{g}\big)^{\vee}(x) = f * g(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy,$$

$$\int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, d\xi, \qquad \text{(Plancherel)}$$

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Part 1:

PDEs to ODEs using the Fourier transform

The heat equation

$$\begin{cases} \partial_t u &= \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) &= u_0 \quad \text{in} \quad \mathbb{R}^n. \end{cases}$$

Taking the Fourier transform of the equation we obtain

$$\begin{cases} \partial_t \widehat{u}(\xi,t) &= -|\xi|^2 \widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) &= \widehat{u}_0(\xi). \end{cases}$$

Solving the ODE (with fixed ξ) this yields

$$\widehat{u}(\xi,t) = e^{-t|\xi|^2}\widehat{u}_0(\xi)\,.$$

Inverting the Fourier transform, we write

$$u(x,t) = e^{t\Delta}u_0(x) := \left(e^{-t|\cdot|^2}\widehat{u}_0\right)^{\vee}.$$

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Taking the Fourier transform of the equation we obtain

$$\left\{ \begin{array}{rcl} \partial_t \widehat{u}(\xi,t) & = & -|\xi|^2 \widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) & = & \widehat{u}_0(\xi). \end{array} \right.$$

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$$u(x,t) = e^{t\Delta} u_0(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi)\, d\xi\,.$$

The Schrödinger equation

$$\begin{cases} \partial_t u = i\Delta u & \text{in} \quad \mathbb{R}^n \times \mathbb{R} \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Taking the Fourier transform of the equation we obtain

$$\begin{cases} \partial_t \widehat{u}(\xi) = -i|\xi|^2 \widehat{u}(\xi, t) \\ \widehat{u}(\xi, 0) = \widehat{u}_0(\xi). \end{cases}$$

Solving the ODE this yields

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The wave equation

$$\begin{cases} \partial_{tt} u &= \Delta u & \text{in} & \mathbb{R}^n \times \mathbb{R} \\ u(\cdot, 0) &= u_0 & \text{in} & \mathbb{R}^n \\ \partial_t u(\cdot, 0) &= u_1 & \text{in} & \mathbb{R}^n. \end{cases}$$

Taking the Fourier transform of the equation we obtain

$$\begin{cases} \partial_{tt} \widehat{u}(\xi, t) &= -|\xi|^2 \widehat{u}(\xi, t) \\ \widehat{u}(\xi, 0) &= \widehat{u}_0(\xi) \\ \partial_t \widehat{u}(\xi, 0) &= \widehat{u}_1(\xi). \end{cases}$$

Solving the ODE this yields

$$\widehat{u}(\xi, t) = \cos(t|\xi|)\widehat{u}_0(\xi) + \tfrac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\,.$$

Inverting the Fourier transform, we write

$$u(\cdot, t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1.$$

The initial data

The Bessel potential G_s is defined via its Fourier transform by

$$\widehat{G_s}(\xi) := (1 + |\cdot|^2)^{-s/2}$$

and satisfies

$$G_s(x) \leq c_{n,s}|x|^{-(n-s)}.$$

We take the initial data u_0 in the Bessel potential space

$$H^s(\mathbb{R}^n) := \{\, G_s * g \, : \, g \in L^2(\mathbb{R}^n) \,\}$$

with norm

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} = \|g\|_{L^2}.$$

Lemma (Pointwise convergence for smooth data)

Let $f \in H^s(\mathbb{R}^n)$ with $s > n/2$. Then

$$\lim_{t\to 0} e^{t\Delta} f(x) = f(x) \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof: Taking $s < n/2 + 2$,

$$\begin{aligned}
(2\pi)^{n/2}|e^{t\Delta}f(x) - f(x)| &= \left| \int \widehat{f}(\xi)|\xi|^s \frac{e^{ix\cdot\xi}(e^{-t|\xi|^2} - 1)}{|\xi|^s} d\xi \right| \\
&\le \Big(\int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \Big)^{1/2} \Big(\int \frac{|e^{-t|\xi|^2} - 1|^2}{|\xi|^{2s}} d\xi \Big)^{1/2} \\
&\le t^{\frac{s-n/2}{2}} \|f\|_{H^s} \Big(\int \frac{|e^{-|y|^2} - 1|^2}{|y|^{2s}} dy \Big)^{1/2} \\
&= t^{\frac{s-n/2}{2}} \|f\|_{H^s} \Big(\int \frac{\min\{|y|^2, 1\}^2}{|y|^{2s}} dy \Big)^{1/2} \\
&\le C_s t^{\frac{s-n/2}{2}} \|f\|_{H^s}.
\end{aligned}$$

A similar calculation works for the Schrödinger equation. $\square$

Hausdorff measure

For $A \subseteq \mathbb{R}^n$ a Borel set,

$$\mathcal{H}^\alpha_\delta(A) := \inf \left\{ \sum_i \delta_i^\alpha \ : \ A \subset \bigcup_i B(x_i, \delta_i), \quad \delta_i < \delta \right\}.$$

Definition

The α-Hausdorff measure of A is

$$\mathcal{H}^\alpha(A) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(A).$$

Hausdorff dimension

Remark

There exists a unique $\alpha_0 \in [0, n]$ *such that*

$$\mathcal{H}^\alpha(A) = \begin{cases} \infty & \textit{if} \quad \alpha < \alpha_0 \\ 0 & \textit{if} \quad \alpha > \alpha_0. \end{cases}$$

Definition

α_0 is the Hausdorff dimension of the set A:

$$\dim(A) := \alpha_0.$$

Definition (Frostman measures)

We say a positive Borel measure μ with $\operatorname{supp}(\mu) \subset B(0,1)$ is *α-dimensional* if

$$c_\alpha(\mu) := \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} \frac{\mu(B(x,r))}{r^\alpha} < \infty.$$

$$\begin{aligned} E_{\alpha'}(\mu) := \iint \frac{d\mu(x)d\mu(y)}{|x-y|^{\alpha'}} &= \int \sum_{j=-1}^{\infty} \int_{|x-y|\sim 2^{-j}} \frac{d\mu(x)}{|x-y|^{\alpha'}} d\mu(y) \\ &\leq \int \sum_{j=-1}^{\infty} c_\alpha(\mu) 2^{-j\alpha} 2^{j\alpha'} d\mu(y) \\ &\lesssim c_\alpha^2(\mu) < \infty \quad \text{if } \alpha > \alpha'. \end{aligned}$$

Lemma (Frostman)

Let $A \subset \mathbb{R}^n$ be a Borel set. The following are equivalent:

- $\mathcal{H}^\alpha(A) = 0$;
- $\mu(A) = 0$ *for all α-dimensional μ.*

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$$\lesssim c_\alpha^2(\mu) < \infty \quad \text{if } \alpha > \alpha'.$$

Lemma (Frostman) $\quad \mu(A) \leq \sum_i \mu(B(x_i,\delta_i)) \leq c_\alpha(\mu) \sum_i \delta_i^\alpha$

Let $A \subset \mathbb{R}^n$ be a Borel set. The following are equivalent:

- $\mathcal{H}^\alpha(A) = 0$;
- $\mu(A) = 0$ *for all α-dimensional μ.*

Control of singularities

Lemma

Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all α-dimensional μ,

$$\|f\|_{L^1(d\mu)} \lesssim c_\alpha(\mu)\|f\|_{H^s}.$$

Thus if $f \in H^s(\mathbb{R}^n)$, then

$$\mu\Big\{x \in \mathbb{R}^n : f(x) = \infty\Big\} = 0 \quad \forall\ \alpha\text{-dimensional } \mu$$

whenever $\alpha > n - 2s$, so that by Frostman's lemma,

$$\mathcal{H}^\alpha\Big\{x \in \mathbb{R}^n : f(x) = \infty\Big\} = 0,$$

whenever $\alpha > n - 2s$, so that

$$\dim\Big\{x \in \mathbb{R}^n : f(x) = \infty\Big\} \le n - 2s.$$

Control of singularities

Lemma

Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all α-dimensional μ,

$$\|f\|_{L^1(d\mu)} \lesssim c_\alpha(\mu)\|f\|_{H^s}.$$

Proof: Writing $f = G_s * g = \big((1 + |\cdot|^2)^{-s/2}\widehat{g}\big)^\vee$, it suffices to prove

$$\|G_s * g\|_{L^1(d\mu)} \lesssim \sqrt{E_{n-2s}(\mu)}\,\|g\|_{L^2(\mathbb{R}^n)}.$$

By Fubini's theorem and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|G_s * g\|_{L^1(d\mu)} &\leq \int\int G_s(x-y)\,d\mu(x)\,|g(y)|\,dy \\ &\leq \|G_s * \mu\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Thus it remains to prove that

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By two applications of Plancherel's theorem,

$$\begin{aligned}
\|G_s * \mu\|^2_{L^2(\mathbb{R}^n)} &= \left\|(1+|\cdot|^2)^{-s/2}\widehat{\mu}\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \int \widehat{\mu}(\xi)\,(1+|\xi|^2)^{-s}\,\overline{\widehat{\mu}(\xi)}\,d\xi \\
&\leq \int \mu * G_{2s}(y)\,d\mu(y) \\
&\leq c_{n,s}\iint \frac{d\mu(x)d\mu(y)}{|x-y|^{n-2s}} = c_{n,s}E_{n-2s}(\mu),
\end{aligned}$$

where we used that $\left((1+|\cdot|^2)^{-s}\right)^{\vee} = G_{2s} \leq c_{n,s}|\cdot|^{-(n-2s)}$. $\square$

Optimality of the control of singularities lemma

If $\dim(A) = \alpha$ with $\alpha < n - 2s$, then we can take a γ such that

$$\alpha < \gamma < n - 2s.$$

Then

$$\mathbf{1}_{B(0,1)}\mathrm{dist}(\cdot, A)^{-\gamma} \in L^2(\mathbb{R}^n)$$

but on the other hand

$$u_0 := G_s * \left[\mathbf{1}_{B(0,1)}\mathrm{dist}(\cdot, A)^{-\gamma}\right] = \infty \quad \text{on } A.$$

So there is initial data $u_0 \in H^s(\mathbb{R}^n)$ which is singular on a set of dimension $\alpha < n - 2s$.

Proposition (Maximal estimates imply convergence)

Let $\alpha > \alpha_0 \geq n - 2s$. Suppose that, for all α-dimensional μ,

$$\Big\| \sup_{0<t<1} |u(\cdot, t)| \Big\|_{L^1(d\mu)} \leq C_\mu \|u_0\|_{H^s}.$$

Then, for all $u_0 \in H^s$,

$$\dim \Big\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} u(t,x) \neq u_0(x) \Big\} \leq \alpha_0.$$

Proof: We are required to prove that for all $\alpha > \alpha_0$,

$$\mathcal{H}^\alpha \Big\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} u(t,x) \neq u_0(x) \Big\} = 0$$

whenever $u_0 \in H^s$. By Frostman's lemma, this follows by showing

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whenever $u_0 \in H^s$. By Frostman's lemma, this follows by showing

$$\mu \Big\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} |u(t,x) - u_0(x)| > \lambda \Big\} = 0, \quad \forall\, \lambda > 0,$$

whenever μ is α-dimensional.

Take $h \in H^{n/2+1}$ such that $\|u_0 - h\|_{H^s} < \varepsilon$, and note that

$$|u(x,t) - u_0(x)| \leq |u(x,t) - u_h(x,t)| + |u_h(x,t) - h(x)| + |h(x) - u_0(x)|,$$

where u_h denotes the solution with initial data h. Then

$$\begin{aligned}
&\mu\{\, x \,:\, \limsup_{t\to 0} |u(x,t) - u_0(x)| > \lambda \,\} \\
\leq\ &\mu\{\, x \,:\, \limsup_{t\to 0} |u(x,t) - u_h(x,t)| > \lambda/3 \,\} \\
+\ &\mu\{\, x \,:\, \limsup_{t\to 0} |u_h(x,t) - h(x)| > \lambda/3 \,\} \\
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&\le \mu\{\, x \,:\, \limsup_{t\to 0} |u_{u_0-h}(x,t)| > \lambda/3 \,\} \\
&+ \mu\{\, x \,:\, \limsup_{t\to 0} |u_h(x,t) - h(x)| > \lambda/3 \,\} \\
&+ \mu\{\, x \,:\, \limsup_{t\to 0} |h(x) - u_0(x)| > \lambda/3 \,\}.
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We use the maximal estimate for the first term and the third term can be bounded by the control of singularities lemma so that

$$\mu\{\, x \,:\, \lim_{t\to 0} |u(x,t) - u_0(x)| > \lambda \,\}\lambda \le C_\mu \, \|u_0 - h\|_{H^s(\mathbb{R}^n)} \le C_\mu \, \varepsilon.$$

Letting ε tend to zero, then λ tend to zero, we are done. $\square$

Part 2:
Convergence for the heat equation

Theorem (Maximal estimate for the heat equation)

Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all α-dimensional μ,

$$\Big\| \sup_{0<t<1} |e^{t\Delta} f| \Big\|_{L^1(d\mu)} \leq C_\mu \|f\|_{H^s}.$$

Proof: By linearising the operator, it will suffice to prove

$$\left| \iint e^{ix\cdot\xi} e^{-t(x)|\xi|^2} \, \widehat{f}(\xi) \, d\xi \, w(x) \, d\mu(x) \right|^2 \lesssim E_{n-2s}(\mu) \, \|f\|_{H^s}^2,$$

whenever $t : \mathbb{R}^n \to (0, \infty)$ and $w : \mathbb{R}^n \to \mathbb{S}^1$ are measurable.
Now, by Fubini and Cauchy-Schwarz, the LHS is bounded by

$$\int |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \int \left| \int e^{ix\cdot\xi} e^{-t(x)|\xi|^2} w(x) \, d\mu(x) \right|^2 \frac{d\xi}{(1 + |\xi|^2)^s}.$$

Squaring out the integral, it will suffice to show that

$$\iiint e^{i(x-y)\cdot\xi} e^{-(t(x)+t(y))|\xi|^2} \frac{d\xi}{(1 + |\xi|^2)^s} w(x) w(y) \, d\mu(x) d\mu(y) \lesssim E_{n-2s}(\mu).$$

Thus, it remains to prove that, for $0 < s < n/2$,

$$\left| \int e^{i(x-y)\cdot\xi} e^{-(t(x)+t(y))|\xi|^2} \frac{d\xi}{(1+|\xi|^2)^s} \right| \lesssim \frac{1}{|x-y|^{n-2s}}$$

uniformly for all choices of $t(x), t(y) > 0$. Recalling that

$$\left(e^{-\lambda|\cdot|^2}\right)^\vee = \frac{1}{\lambda^{n/2}} e^{-|\cdot|^2/\lambda} \quad \text{and} \quad \left((1+|\cdot|^2)^{-s}\right)^\vee =: G_{2s} \leq c_{n,s}|\cdot|^{-(n-2s)},$$

this would follow from

$$\frac{1}{\lambda^{n/2}} e^{-|\cdot|^2/\lambda} * \frac{1}{|\cdot|^{n-2s}} \lesssim \frac{1}{|\cdot|^{n-2s}}.$$

uniformly in λ. By changing variables, this is equivalent to

$$e^{-|\cdot|^2} * \frac{1}{|\cdot|^{n-2s}} \lesssim \frac{1}{|\cdot|^{n-2s}},$$

which can be checked by direct calculation. $\square$

Corollary

Let $f \in H^s$ with $0 < s < n/2$. Then

$$\dim \left\{ x \in \mathbb{R}^n : \quad \lim_{t \to 0} e^{t\Delta} f(x) \neq f(x) < \infty \right\} \leq n - 2s.$$

As we saw before, $f \in H^s$ can be singular on a set of dimension less than $n - 2s$ and so this is optimal.

Part 3:
Decay for the Fourier transform of fractal measures

$\widehat{\delta_{x_1=0}}(\xi_1, \overline{\xi}) = \int_{\mathbb{R}^{n-1}} e^{-i\overline{x}\cdot\overline{\xi}}\, d\overline{x}$ is independent of ξ_1.

Thus, the Fourier transform of certain $(n-1)$-dimensional measures do not decay in every direction.

But perhaps they decay on average......

Let $\beta_n(\alpha)$ denote the supremum of the numbers β for which

$$\frac{1}{|\mathbb{S}_r^{n-1}|} \int_{\mathbb{S}_r^{n-1}} |\widehat{\mu}(\omega)|^2 d\sigma_r(\omega) \lesssim C_\mu \,(1+r)^{-\beta}$$

whenever $r > 0$ and μ is α-dimensional and supported in $B(0,1)$.

Question (Mattila (1987))

What is $\beta_n(\alpha)$?

Equivalently $\beta_n(\alpha)$ is the supremum of the numbers β for which

$$\left\| (g d\sigma_r)^\vee \right\|_{L^1(d\mu)} \lesssim C_\mu \, r^{\frac{n-1}{2}} \,(1+r)^{-\beta/2} \|g\|_{L^2(\mathbb{S}_r^{n-1})}.$$

Best known results

$$\beta_2(\alpha) = \begin{cases} \alpha, & \alpha \in (0, 1/2], \\ 1/2, & \alpha \in [1/2, 1], \\ \alpha/2, & \alpha \in [1, 2], \end{cases} \quad \begin{array}{l} \text{Mattila (1987)} \\ \\ \text{Wolff (1999).} \end{array}$$

$$\beta_3(\alpha) \geq \begin{cases} \alpha, & \alpha \in (0, 1], \\ 1, & \alpha \in [1, \frac{3}{2}], \\ \alpha - \frac{\alpha}{3}, & \alpha \in [\frac{3}{2}, 3], \end{cases} \quad \begin{array}{l} \text{Mattila (1987)} \\ \\ \text{Du-Guth-Ou-Wang-Wilson-Zhang.} \end{array}$$

$$\beta_{n\geq 4}(\alpha) \geq \begin{cases} \alpha, & \alpha \in (0, \frac{n-1}{2}], \\ \frac{n-1}{2}, & \alpha \in [\frac{n-1}{2}, \frac{n}{2}], \\ \alpha - \frac{\alpha}{n}, & \alpha \in [\frac{n}{2}, n], \end{cases} \quad \begin{array}{l} \text{Mattila (1987)} \\ \\ \text{Du-Zhang.} \end{array}$$

The initial data

The Riesz potential I_s is defined via its Fourier transform by

$$\widehat{I_s}(\xi) := |\cdot|^{-s}$$

and satisfies

$$I_s(x) \leq c_{n,s}|x|^{-(n-s)}.$$

We now take the initial data in the Riesz potential space

$$\dot{H}^s(\mathbb{R}^n) := \{\, I_s * g \;:\; g \in L^2(\mathbb{R}^n) \,\}$$

with norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} = \|g\|_{L^2}.$$

Lemma (Bridging lemma)

Let $f \in \dot{H}^s(\mathbb{R}^n)$ with $0 < s < n/2$ and $\beta_n(\alpha) > n - 2s$. Then

$$\dim \left\{ x \in \mathbb{R}^n \ : \ \lim_{t \to 0} e^{it(-\Delta)^{m/2}} f(x) \neq f(x) \right\} \leq \alpha.$$

Proof: It will suffice to prove, for all α-dimensional μ,

$$\left\| \sup_{0<t<1} |e^{it(-\Delta)^{m/2}} f| \right\|_{L^1(d\mu)} \lesssim C_\mu \|f\|_{\dot{H}^s(\mathbb{R}^n)} .$$

Writing in polar coordinates,

$$\begin{aligned} |e^{it(-\Delta)^{m/2}} f(x)| &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-it|\xi|^m} e^{ix\cdot\xi} \, \widehat{f}(\xi) \, d\xi \right| \\ &= \left| \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-itr^m} \int_{\mathbb{S}_r^{n-1}} e^{ix\cdot\omega} \, \widehat{f}(\omega) \, d\sigma_r(\omega) \, dr \right| \\ &\leq \frac{1}{(2\pi)^{n/2}} \int_0^\infty \left| \int_{\mathbb{S}_r^{n-1}} e^{ix\cdot\omega} \, \widehat{f}(\omega) \, d\sigma_r(\omega) \right| dr. \end{aligned}$$

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|e^{it(-\Delta)^{m/2}} f(x)| &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-it|\xi|^m} e^{ix\cdot\xi} \, \widehat{f}(\xi) \, d\xi \right| \\
&= \left| \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-itr^m} \int_{\mathbb{S}_r^{n-1}} e^{ix\cdot\omega} \, \widehat{f}(\omega) \, d\sigma_r(\omega) \, dr \right| \\
&\leq \frac{1}{(2\pi)^{n/2}} \int_0^\infty \left| \left(\widehat{f} d\sigma_r \right)^\vee (x) \right| \, dr.
\end{aligned}$$

$$|e^{it(-\Delta)^{m/2}}f(x)| \lesssim \int_0^\infty \left|\left(\widehat{f}d\sigma_r\right)^\vee(x)\right| dr,$$

so that, by Fubini,

$$\Big\| \sup_{0<t<1} |e^{it(-\Delta)^{m/2}}f| \Big\|_{L^1(d\mu)} \lesssim \int_0^\infty \big\|\left(\widehat{f}d\sigma_r\right)^\vee\big\|_{L^1(d\mu)} dr.$$

By the dual version of the Mattila inequality,

$$\big\|\left(\widehat{f}d\sigma_r\right)^\vee\big\|_{L^1(d\mu)} \leq C_\mu\, r^{\frac{n-1}{2}}(1+r)^{-\beta/2}\|\widehat{f}\|_{L^2(\mathbb{S}_r^{n-1})}.$$

for all $\beta < \beta_n(\alpha)$, so that

$$\Big\| \sup_{0<t<1} |e^{it(-\Delta)^{m/2}}f| \Big\|_{L^1(d\mu)} \leq C_\mu \int_0^\infty r^{\frac{n-1}{2}}(1+r)^{-\beta/2}\|\widehat{f}\|_{L^2(\mathbb{S}_r^{n-1})} dr.$$

Finally, by Cauchy-Schwarz,

$$\leq C_\mu \left(\int_0^\infty (1+r)^{-\beta} r^{n-1-2s} dr\right)^{1/2} \left(\int_0^\infty \|\widehat{f}\|^2_{L^2(\mathbb{S}_r^{n-1})} r^{2s} dr\right)^{1/2}$$

$$\leq C_\mu \|f\|_{\dot{H}^s(\mathbb{R}^n)},$$

where for the final inequality we must take $\beta > n - 2s$. □

Part 4:

Convergence for the wave equation

Recall that, with initial data $u(\cdot,0)=u_0$ and $\partial_t u(\cdot,0)=u_1$, the solution to the wave equation satisfies

$$\begin{aligned}
\widehat{u}(\xi,t) &= \qquad \cos(t|\xi|)\widehat{u_0}(\xi) \qquad + \quad \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\\
&= \frac{1}{2}(e^{it|\xi|}+e^{-it|\xi|})\widehat{u_0}(\xi) \ + \frac{1}{2}\frac{(e^{it|\xi|}-e^{-it|\xi|})}{i|\xi|}\widehat{u_1}(\xi)\\
&= e^{it|\xi|}\frac{1}{2}\Big(\widehat{u_0}(\xi)+\frac{\widehat{u_1}(\xi)}{i|\xi|}\Big)+e^{-it|\xi|}\frac{1}{2}\Big(\widehat{u_0}(\xi)-\frac{\widehat{u_1}(\xi)}{i|\xi|}\Big)\\
&=: e^{it|\xi|}\widehat{f_+}(\xi) \qquad\qquad\qquad + e^{-it|\xi|}\widehat{f_-}(\xi).
\end{aligned}$$

With this notation, we can write

$$u(\cdot,t)=e^{it(-\Delta)^{1/2}}f_+ + e^{-it(-\Delta)^{1/2}}f_-.$$

If the initial data is in $\dot{H}^s\times\dot{H}^{s-1}$, both f_+ and f_- belong to $\dot{H}^s$.

Thus convergence of $e^{it(-\Delta)^{1/2}}f$ to f for all $f\in\dot{H}^s$ implies convergence of $u(\cdot,t)$ to u_0 for all $(u_0,u_1)\in\dot{H}^s\times\dot{H}^{s-1}$.

Now $\beta(\alpha) \geq \frac{n-1}{n}\alpha$, so if $\alpha > \frac{n}{n-1}(n-2s)$ then $\beta(\alpha) > n-2s$. Thus, by the bridging lemma,

Corollary

Let u be a solution to the Schrödinger equation with initial data in H^s or to the wave equation with initial data in $\dot{H}^s \times \dot{H}^{s-1}$. Then

$$\dim \left\{ x \in \mathbb{R}^n \ : \ \lim_{t \to 0} u(x,t) \neq u_0(x) \right\} \leq \frac{n}{n-1}(n-2s).$$

In particular,

Corollary

Let u be a solution to the Schrödinger equation with initial data in H^1 or to the wave equation with initial data in $\dot{H}^1 \times L^2$. Then

$$\dim \left\{ x \in \mathbb{R}^n \ : \ \lim_{t \to 0} u(x,t) \neq u_0(x) \right\} < n-1.$$

Part 5:
The Schrödinger equation

Lebesgue a.e. convergence for Schrödinger

In 1979, Carleson asked for which s is it true that

$$\lim_{t\to 0} e^{it\Delta}f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \quad \forall\, f \in H^s(\mathbb{R}^n)?$$

Improvements made by:

Carleson (1979), Dahlberg-Kenig (1982),
Carbery/Cowling (1985/83), Sjölin/Vega (1987/88),
Bourgain (1991/92), Moyua-Vargas-Vega (1996/99),
Tao-Vargas-Vega (1998), Tao-Vargas (2000), Tao (2003),
Lee (2006), Bourgain (2013), Lucà-R. (2015), Bourgain (2016),
Du-Guth-Li (2017), Du-Guth-Li-Zhang (2018), Du-Zhang (2018).

Lebesgue a.e. convergence for Schrödinger

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Best known sufficient condition for convergence:

- $s \geq 1/4$ with $n = 1$ (Carleson);
- $s > 1/3$ with $n = 2$ (Du-Guth-Li);
- $s > \frac{1}{2} - \frac{1}{2(n+1)}$ with $n \geq 3$ (Du-Zhang).

Best known necessary condition for convergence:

- $s \geq 1/4$ with $n = 1$ (Dahlberg-Kenig);
- $s \geq \frac{1}{2} - \frac{1}{2(n+1)}$ with $n \geq 2$ (Bourgain).

Part 5(a):

$s \geq \frac{1}{2} - \frac{1}{2(n+1)}$ is necessary for Lebesgue a.e. convergence

Proof

Lemma (Nikišin-Stein maximal principle)

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \quad a.e.\ x \in \mathbb{R}^n,$$

for all $f \in H^s(\mathbb{R}^n)$ if and only if there is a constant C such that

$$\Big\| \sup_{0<t<1} |e^{it\Delta} f| \Big\|_{L^2(B(0,1))} \le C \|f\|_{H^s(\mathbb{R}^n)}$$

for all $f \in H^s(\mathbb{R}^n)$.

So it suffices to show that, if

$$\Big\| \sup_{0<t<1} |e^{it\Delta} f| \Big\|_{L^2(B(0,1))} \lesssim R^s \|f\|_2,$$

whenever $\operatorname{supp} \widehat{f} \subset \{\xi : |\xi| \le R\}$, then $s \ge \frac{1}{2} - \frac{1}{2(n+1)} = \frac{n}{2(n+1)}$.

The concentrated example

Consider initial data f defined by

$$\widehat{f}(\xi) = \mathbf{1}_{|\xi|\leq \frac{1}{10}R^{1/2}} \quad \text{so that} \quad \|f\|_2 \leq R^{n/4}.$$

Then, if $(x,t) \in X \times T$, where

$$X := B(0, R^{-1/2}) \quad \text{and} \quad T := (0, R^{-1}],$$

there is no cancellation in the integral:

$$|e^{it\Delta}f(x)| = \left| \frac{1}{(2\pi)^{n/2}} \int_{|\xi|\leq \frac{1}{10}R^{1/2}} e^{ix\cdot\xi} e^{-it|\xi|^2} d\xi \right| \geq cR^{n/2}.$$

The travelling concentrated example

Instead Dahlberg-Kenig took

$$f_{dk}(x) = e^{i\frac{1}{2}x\cdot\theta} f(x),$$

where $\theta \in \mathbb{R}^n$, so that

$$|e^{it\Delta} f_{dk}(x)| = |e^{it\Delta} f(x - t\theta)| \geq cR^{n/4}$$

whenever

$$x \in X + t\theta \quad \text{and} \quad t \in T = (0, R^{-1}).$$

This yields

$$\sup_{0<t<1} |e^{it\Delta} f_{dk}(x)| \geq cR^{n/2}$$

whenever

$$x \in \bigcup_{t\in T} X + t\theta.$$

When $n = 1$, we can take $\theta = R$, so that

$$(0,1) \subset \bigcup_{t\in T} X + t\theta.$$

Conclusion that $s \geq 1/4$ is necessary when $n = 1$

Plugging into the maximal estimate,

$$\Big\| \sup_{0<t<1} |e^{it\Delta} f_{dk}| \Big\|_{L^2(0,1)} \leq CR^s \|f_{dk}\|_2,$$

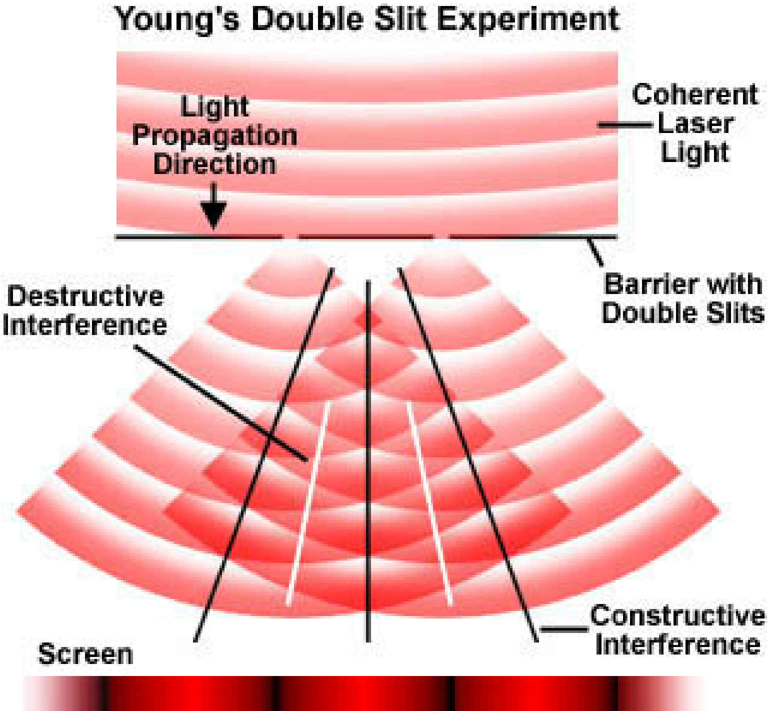
and recalling that when $x \in (0, 1)$,

$$\sup_{0<t<1} |e^{it\Delta} f_{dk}(x)| \geq cR^{1/2} \quad \text{and} \quad \|f_{dk}\|_2 \leq R^{1/4},$$

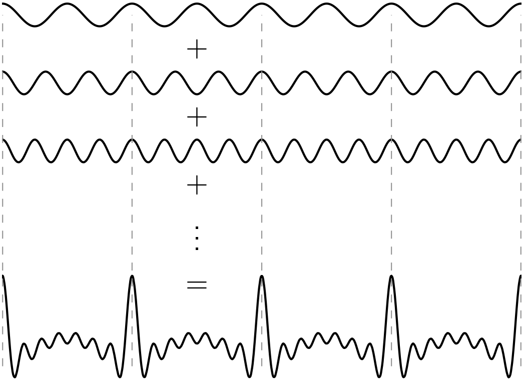
we obtain

$$cR^{1/4} \leq CR^s.$$

Letting $R \to \infty$, we see that $s \geq 1/4$. $\square$


Young's Double Slit Experiment
Light Propagation Direction
Coherent Laser Light
Barrier with Double Slits
Destructive Interference
Constructive Interference
Screen

Constructive interference with different frequencies



The Barceló-Bennett-Carbery-Ruiz-Vilela example

Consider the frequencies

$$\Omega := \{\xi \in 2\pi R^{1-\kappa}\mathbb{Z}^n \,:\, |\xi| \leq R\} + B(0, \tfrac{1}{10}),$$

for $0 < \kappa < 1$,

and initial data defined by

$$\widehat{f_{bbcrv}} = \mathbf{1}_\Omega,$$

so that

$$\|f_{bbcrv}\|_2 = \sqrt{|\Omega|} \leq R^{\frac{n\kappa}{2}}.$$

This was originally considered in the context of Mattila's question regarding decay of the Fourier transform of measures.

Periodic constructive interference

The interference pattern reappears periodically for a short time:

$$|e^{it\Delta} f_{bbcrv}(x)| \geq c|\Omega|,$$

whenever $(x, t) \in X \times T$,

where

$$X := \left\{x \in R^{\kappa-1}\mathbb{Z}^n \,:\, |x| \leq 1\right\} + B(0, R^{-1}),$$

and

$$T := \left\{t \in \frac{1}{2\pi} R^{2(\kappa-1)}\mathbb{Z} \,:\, 0 < t < R^{-1}\right\}.$$

Periodic constructive interference

In order to avoid cancellation in the integral

$$|e^{it\Delta} f_{bbcrv}(x)| = \left| \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{ix\cdot\xi} e^{-it|\xi|^2} d\xi \right| \geq c|\Omega|,$$

this time X is in some sense the dual-set of Ω:

$$x \cdot \xi \in \left(R^{\kappa-1}\mathbb{Z}^n\right) \cdot \left(2\pi R^{1-\kappa}\mathbb{Z}^n\right) = 2\pi\mathbb{Z}.$$

and T is some sense the dual-set of $\Omega \cdot \Omega$:

$$t\xi \cdot \xi \in \left(\frac{1}{2\pi} R^{2(\kappa-1)}\mathbb{Z}\right) \left(2\pi R^{1-\kappa}\mathbb{Z}^n\right) \cdot \left(2\pi R^{1-\kappa}\mathbb{Z}^n\right) = 2\pi\mathbb{Z}.$$

Periodic constructive interference

Thus

$$|e^{it\Delta} f_{bbcrv}(x)| \geq c|\Omega|$$

whenever $(x, t) \in X \times T$.

But the interference always reappears in the same places so

$$\sup_{0<t<1} |e^{it\Delta} f_{bbcrv}(x)| \geq c|\Omega|$$

only for $x \in X$.

The travelling interference example

Instead we take

$$f_\theta(x) = e^{i\frac{1}{2}x\cdot\theta} f(x),$$

where $\theta \in \mathbb{R}^n$, so that

$$|e^{it\Delta} f_\theta(x)| = |e^{it\Delta} f(x - t\theta)|,$$

which yields

$$\sup_{0<t<1} |e^{it\Delta} f_\theta(x)| \geq c|\Omega|$$

whenever

$$x \in \bigcup_{t\in T} X + t\theta.$$

If $n = 1$ and $\kappa < 1/3$, we can take $\theta = R^\kappa$ so that

$$(0,1) \subset \bigcup_{t\in T} X + t\theta.$$

Lemma (Lucà-R.)

If $0 < \kappa < \frac{1}{n+2}$*, then there exists* $\theta \in \mathbb{R}^n$ *such that*

$$B(0, 1/2) \subset \bigcup_{t \in T} X + t\theta.$$

This is optimal in the sense that it is not true for $\kappa > \frac{1}{n+2}$.

After scaling and quotienting out $\mathbb{Z}^n$, this follows from quantitive ergodic theory on the torus $\mathbb{T}^n$.

Lemma (Lucà-R.)

Let $0 < \delta < 1$*. Then, there exists* $\theta \in \mathbb{S}^{n-1}$ *such that for all* $y \in \mathbb{T}^n$ *there is a* $t \in R^\delta \mathbb{Z} \cap (0, R)$ *such that*

$$\|y - t\theta\| \leq R^{-\frac{1-\delta}{n}}.$$

Conclusion that $s \geq \frac{n}{2(n+2)}$ is necessary

Plugging into the maximal estimate,

$$\left\| \sup_{0<t<1} |e^{it\Delta} f_\theta| \right\|_{L^2(B(0,1))} \leq CR^s \|f_\theta\|_2,$$

and recalling that when $x \in B(0, 1/2)$,

$$\sup_{0<t<1} |e^{it\Delta} f_\theta(x)| \geq c|\Omega| \quad \text{and} \quad \|f_\theta\|_2 = \sqrt{|\Omega|},$$

we obtain

$$c\sqrt{|\Omega|} \leq CR^s.$$

Then as $|\Omega| \geq R^{n\kappa}$, this yields

$$cR^{\frac{n\kappa}{2}} \leq CR^s.$$

Letting $\kappa \to \frac{1}{n+2}$ and $R \to \infty$, we see that $s \geq \frac{n}{2(n+2)}$. $\square$

Combining the examples

Writing $x = (x_1, \overline{x}) \in \mathbb{R}^n$, we consider

$$f(x) = f_{dk}(x_1) f_\theta(\overline{x})$$

with $\kappa < \frac{1}{2(n+1)}$ and $\theta \in \mathbb{R}^{n-1}$.

Note that

$$e^{it\Delta} f(x) = e^{it\Delta} f_{dk}(x_1) e^{it\Delta} f_\theta(\overline{x}).$$

In order to make the first factor large, we must take t near to x_1/R.

Thus we do not have as many good times as before.

However, we have taken fewer waves than before (smaller κ).

By the ergodic lemma we can still find a $\theta \in \mathbb{R}^{n-1}$ and enough good t's (near to x_1/R), such that the integral of $e^{it\Delta} f_\theta(\overline{x})$ has no cancellation for all $\overline{x} \in B(0, 1/2)$.

Conclusion that $s \geq \frac{n}{2(n+1)}$ is necessary

Plugging into the maximal estimate,

$$\left\| \sup_{0<t<1} |e^{it\Delta} f_{dk}\, e^{it\Delta} f_\theta| \right\|_{L^2\big((0,1)\times B(0,1)\big)} \leq CR^s \|f_{dk}\|_2 \|f_\theta\|_2,$$

and recalling that when $(x_1, \overline{x}) \in (0,1) \times B(0,1/2)$,

$$\sup_{0<t<1} |e^{it\Delta} f_{dk}\, e^{it\Delta} f_\theta| \geq cR^{1/2}|\Omega| \quad \text{and} \quad \|f_{dk}\|_2 \|f_\theta\|_2 \leq R^{1/4}\sqrt{|\Omega|}$$

we obtain

$$cR^{1/4}\sqrt{|\Omega|} \leq CR^s.$$

Then as $|\Omega| \geq cR^{(n-1)\kappa}$, we see that

$$s \geq 1/4 + \frac{(n-1)\kappa}{2}.$$

Finally we let $\kappa \to \frac{1}{2(n+1)}$, so that $s \geq \frac{n+1}{4(n+1)} + \frac{n-1}{4(n+1)} = \frac{n}{2(n+1)}$. $\square$

Part 5(b):

$s > \frac{n}{2(n+1)}$ is sufficient for Lebegue a.e. convergence.

Proof

By summing a geometric series, it suffices to show

$$\int_{B(0,1)} \sup_{0<t<1} |e^{it\Delta} f(x)|^2 dx \lesssim R^{\frac{n}{n+1}} \|f\|_2^2 .$$

whenever $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n \,:\, R \leq |\xi| \leq 2R\}$.

By scaling, this can be rewritten

$$\int_{B(0,R)} \sup_{0<t<R^2} |e^{it\Delta} f(x)|^2 dx \lesssim R^{\frac{n}{n+1}} \|f\|_2^2 .$$

whenever $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n \,:\, 1 \leq |\xi| \leq 2\}$.

By Lee's temporal localisation lemma, this would follow from

$$\int_{B(0,R)} \sup_{0<t<R} |e^{it\Delta} f(x)|^2 dx \lesssim R^{\frac{n}{n+1}} \|f\|_2^2 .$$

whenever $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n \,:\, 1 \leq |\xi| \leq 2\}$.

Covering $B(0,R) \times [0,R]$ by disjoint cubes $Q \times I$ of sidelength 1,

$$\int_{B(0,R)} \sup_{0<t<R} |e^{it\Delta} f(x)|^2 dx \lesssim \sum_{Q} \int_Q \int_0^R |e^{it\Delta} f(x)|^2 dt dx$$
$$\lesssim \sum_{Q,I} \Big(\int_{Q\times I} |e^{it\Delta} f(x)|^{p_n} dx dt \Big)^{\frac{2}{p_n}},$$

where $p_n = \frac{2(n+1)}{n-1}$.

Summing over all the cubes ($\lesssim R^{n+1}$) for which

$$\int_{Q\times I} |e^{it\Delta} f(x)|^{p_n} dx dt \lesssim R^{-(n+1)} \|f\|_2^{p_n},$$

we get a good enough bound.

On the other hand, as $|e^{it\Delta} f(x)| \le \|\widehat{f}\|_1 \le \|f\|_2$, we have

$$\int_{Q\times I} |e^{it\Delta} f(x)|^{p_n} dx dt \lesssim \|f\|_2^{p_n}.$$

The pigeonhole principle

We can divide the remaining cubes $Q \times I$ into $(n+1)\log R$ classes $\mathcal{Q}_j$ for which

$$2^{-j-1}\|f\|_2^{p_n} < \int_{Q\times I} |e^{it\Delta}f(x)|^{p_n}dxdt \leq 2^{-j}\|f\|_2^{p_n}.$$

Now, leaving only a single $Q \times I$ for each Q, we have

$$\int_{B(0,R)} \sup_{0<t<R} |e^{it\Delta}f(x)|^2 dx \lesssim \sum_j \sum_{Q\times I\in\mathcal{Q}_j} \int_{Q\times I} |e^{it\Delta}f(x)|^2 dxdt$$

so we can find a single j for which

$$\int_{B(0,R)} \sup_{0<t<R} |e^{it\Delta}f(x)|^2 dx \lesssim \log R \sum_{Q\times I\in\mathcal{Q}_j} \int_{Q\times I} |e^{it\Delta}f(x)|^2 dxdt.$$

Theorem (Spread-improving Strichartz estimate)

Let $p_n = \frac{2(n+1)}{n-1}$. *Then*

$$\left(\sum_{Q\times I \in \mathcal{Q}_j} \int_{Q\times I} |e^{it\Delta} f(x)|^{p_n} dxdt \right)^{1/p_n} \lesssim (\#\mathcal{Q}_j)^{-\frac{1}{n+1}} R^{\frac{n}{2(n+1)}} \|f\|_2$$

Using this, the proof is completed by

$$\begin{aligned} &\left(\sum_{Q\times I \in \mathcal{Q}_j} \int_{Q\times I} |e^{it\Delta} f(x)|^{2} dxdt \right)^{1/2} \\ &\leq \left| \bigcup_{Q\times I\in\mathcal{Q}_j} Q \times I \right|^{\frac{1}{n+1}} \left(\sum_{Q\times I \in \mathcal{Q}_j} \int_{Q\times I} |e^{it\Delta} f(x)|^{p_n} dxdt \right)^{1/p_n} \\ &\lesssim (\#\mathcal{Q}_j)^{\frac{1}{n+1}} (\#\mathcal{Q}_j)^{-\frac{1}{n+1}} R^{\frac{n}{2(n+1)}} \|f\|_2 \\ &\lesssim R^{\frac{n}{2(n+1)}} \|f\|_2. \qquad \square \end{aligned}$$

An ingredient for spread-improving Strichartz: Decoupling

Theorem (Bourgain-Demeter)

Let $q_d = \frac{2(d+2)}{d}$ *and write* $f = \sum_\tau f_\tau$*, where* $\widehat{f_\tau}$ *are supported on pieces of diameter* $R^{-1/2}$*. Then*

$$\left(\int_{B(0,R)} |e^{it\Delta} f(x)|^{q_d} \, dx \right)^{\frac{1}{q_d}} \lesssim \left(\sum_\tau \left(\int_{B(0,R)} |e^{it\Delta} f_\tau(x)|^{q_d} \, dx \right)^{\frac{2}{q_d}} \right)^{\frac{1}{2}}.$$

This is used in $d = n - 1$ dimensions after a dimension reduction.

Part 5(c):

Refined convergence for the Schrödinger equation

Maximal estimate for the Schrödinger equation

Theorem

Let $n/4 \leq s < n/2$ and $\alpha > n-2s$. Then, for all α-dimensional μ,

$$\Big\| \sup_{0<t<1} |e^{it\Delta} f| \Big\|_{L^1(d\mu)} \leq C_\mu \|f\|_{\dot{H}^s}.$$

Proof: By the same proof as for the heat equation, one finally arrives to the inequality

$$\left| e^{-i|\cdot|^2} * \frac{1}{|\cdot|^{n-2s}} \right| \leq C_{n-2s} \frac{1}{|\cdot|^{n-2s}},$$

This can also be shown to be true by more difficult direct calculation as long as $n/4 \leq s < n/2$. □

Corollary

Let $f \in H^s$ with $n/4 \leq s < n/2$. Then

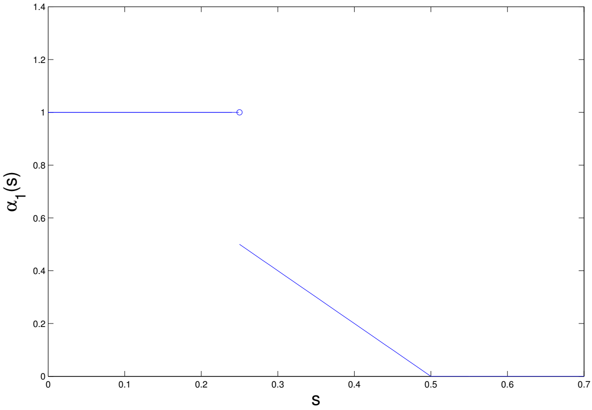
$$\dim \left\{ x \in \mathbb{R}^n \quad \lim_{t \to 0} e^{it\Delta} f(x) \neq f(x) \right\} \leq n - 2s.$$

Again this is sharp in the range $s \geq n/4$.

We cannot go below this regularity in one dimension due to the necessary condition of Dahlberg-Kenig.

In the next section we will see that neither can we go below this regularity in higher dimensions using a fractal version of the Lucà-R.-necessary condition.

$$\alpha_n(s) := \sup_{f\in H^s(\mathbb{R}^n)} \dim \left\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} e^{it\Delta} f(x) \neq f(x) \right\}$$



$$\alpha_n(s) = n - 2s, \qquad n/4 \leq s \leq n/2.$$

What about lower regularity ($s < n/4$) in higher dimensions?

Best known bounds in higher dimensions

$$\alpha_n(s) \leq \begin{cases} n, & s \in [0, \frac{n}{2(n+1)}) \\ n+1-\frac{2(n+1)s}{n}, & s \in [\frac{n}{2(n+1)}, \frac{n}{4}) \\ n-2s, & s \in [\frac{n}{4}, \frac{n}{2}] \end{cases}$$

(Du-Guth-Li, Du-Zhang)
(Barceló-Bennett-Carbery-R.)

$$\alpha_n(s) \geq \begin{cases} n, & s \in [0, \frac{n}{2(n+1)}) \\ n+\frac{n}{n-1}-\frac{2(n+1)s}{n-1}, & s \in [\frac{n}{2(n+1)}, \frac{n+1}{8}) \\ n+1-\frac{2(n+2)s}{n}, & s \in [\frac{n+1}{8} \frac{n}{4}) \\ n-2s, & s \in [\frac{n}{4}, \frac{n}{2}] \end{cases}$$

(Dahlberg-Kenig, Bourgain)
(Lucà-R.)
(Lucà-R.)
(Žubrinić)

$\alpha_n(s) \geq n + 1 - \frac{2(n+2)s}{n}$ when $\frac{n}{2(n+2)} \leq s \leq \frac{n}{4}$

This follows from:

Theorem (Lucà-R.)

Let $n/2 \leq \alpha \leq n$ and suppose that

$$\dim \left\{ x \in \mathbb{R}^n \quad \lim_{t \to 0} e^{it\Delta} f(x) \neq f(x) \right\} < \alpha$$

whenever $f \in H^s(\mathbb{R}^n)$. Then

$$s \geq \frac{n}{2(n+2)} \Big(n - \alpha + 1 \Big).$$

Proof

The Nikišin-Stein maximal principle does not hold in this context, and so we first give a direct proof of the Lebesgue measure result.

We consider a sum of the previous initial data

$$f := \sum_{j>1} f_{\theta_j}, \qquad \theta_j \in \mathbb{S}^{n-1},$$

where we take $R = 2^j$ and normalise in a different way, so that

$$f_{\theta_j}(x) := e^{i\frac{1}{2}\theta_j \cdot x} f_j(x), \qquad \widehat{f_j} = 2^{-j(n\kappa-\varepsilon)}\chi_{\Omega_j},$$

$$\Omega_j := \left\{\xi \in 2\pi 2^{j(1-\kappa)}\mathbb{Z}^n \ : \ |\xi| \leq 2^j\right\} + B(0, \tfrac{1}{10}).$$

Note that $|\Omega_j| \simeq 2^{jn\kappa}$, so that $\|f_j\|_{H^s} \simeq 2^{-j\frac{n\kappa}{2}+j\varepsilon+js}$.

Then if $s < \frac{n\kappa}{2} - \varepsilon$ we can sum so that $f \in H^s$.

To generalise to the fractal case we will take $\frac{1}{n+2} \leq \kappa < \frac{n-\alpha+1}{n+2}$.

By the previous calculations, for all $x \in E_j := \cup_{t \in T_j} X_j + t\theta_j$, where

$$X_j := \left\{x \in 2^{j(\kappa-1)}\mathbb{Z}^n \,:\, |x| \leq 2\right\} + B(0, 2^{-j}),$$

$$T_j := \left\{t \in \frac{1}{2\pi} 2^{2j(\kappa-1)}\mathbb{Z} \,:\, 0 < t < 2^{-j}\right\},$$

there is a $t_j(x) \in T_j$ such that $|e^{it_j(x)\Delta} f_{\theta_j}(x)| \gtrsim 2^{j\varepsilon}$.

One can also show (essentially) that $|e^{it_j(x)\Delta} \sum_{k \neq j} f_{\theta_k}(x)| \leq C$.

If $\kappa < \frac{1}{n+2}$, then $B(0, 1/2) \subset \bigcap_{j>1} E_j$, and we are done.

If $\kappa \geq \frac{1}{n+2}$, we consider the limit set

$$\limsup_{j \to \infty} E_j := \bigcap_{j>1} \bigcup_{k>j} E_k$$

and prove that this is α-dimensional.

For this we use that the limit is 'α-Hausdorff dense'.

Falconer's density theorem

Consider the Hausdorff content $\mathcal{H}^\alpha_\infty$ defined by

$$\mathcal{H}^\alpha_\infty(E) := \inf\left\{\sum_i \delta_i^\alpha \;:\; E \subset \bigcup_i B(x_i, \delta_i)\right\}.$$

Theorem (Falconer (1985))

Suppose that, for all balls $B_r \subset B(0,1)$ of radius r,

$$\liminf_{j\to\infty} \mathcal{H}^\alpha_\infty(E_j \cap B(x, r)) \geq cr^\alpha. \tag{\dagger}$$

Then $\dim\left(\limsup_{j\to\infty} E_j\right) \geq \alpha$.

The proof is completed by checking the density condition (†) with $E_j = \bigcup_{t\in T_j} X_j + t\theta_j$ using a variant of the ergodic lemma. □