A UNIFIED APPROACH TO THREE THEMES IN HARMONIC ANALYSIS

VICTOR LIE

1. Formulation of the problem - informal.

This course aims to present a new and unified approach to three main distinct themes in Harmonic Analysis:

- (I) The Linear Hilbert Transform and Maximal Operator along variable curves (see (2) and (3));
- (II) Carleson Type operators in the presence of curvature (see (4) and (5));
- (III) The bilinear Hilbert transform and maximal operator along variable curves (see (6) and (7)).

Specifically, we plan to investigate the following

Main Problem.(Informal) *For each point $x \in \mathbb{R}$ we associate a curve $\Gamma_x = (t,\, \gamma(x,t))$ in the plane, where here $t \in \mathbb{R}$ and*

(1)
$$\gamma_x(\cdot) := \gamma(x,\cdot) : \mathbb{R} \to \mathbb{R}\,,$$

is a real function obeying some "suitable" smoothness and non-zero curvature conditions in the t-parameter. Define now the variable family of curves in the plane $\Gamma \equiv \{\Gamma_x\}_{\{x\in\mathbb{R}\}}$.

Task: Under minimal conditions on the curve Γ (e.g. minimal regularity in the x-variable), study the L^p-boundedness, $1 \leq p \leq \infty$, of the following operators:

- **the linear Hilbert transform along Γ** *defined as*

(2)
$$H_\Gamma \,:\, S(\mathbb{R}^2) \longmapsto S'(\mathbb{R}^2)\,,$$
$$H_\Gamma(f)(x,y) := \text{p.v.} \int_{\mathbb{R}} f(x-t,\, y+\gamma(x,t))\, \frac{dt}{t}\;;$$

Key words and phrases. Wave-packet analysis, Hilbert transform and Maximal operator along vector fields in the presence of curvature, Bilinear Hilbert transform and Maximal operator along curves, Zygmund's differentiation conjecture, Carleson's Theorem.

The author was supported by the National Science Foundation under Grant No. DMS-1500958.

- **the (sub)linear maximal operator along** Γ *defined as*

(3)
$$M_\Gamma\,:\,S(\mathbb{R}^2)\longmapsto L^\infty(\mathbb{R}^2)\,,$$
$$M_\Gamma(f)(x,y):=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-t,\,y+\gamma(x,t))|\,dt\,;$$

- **the** γ **- Carleson operator** *given by*

(4)
$$C_\gamma\,:\,S(\mathbb{R})\longmapsto L^\infty(\mathbb{R})\,,$$
$$C_\gamma f(x):=p.v.\int_{\mathbb{R}}f(x-t)\,e^{i\,\gamma(x,t)}\,\frac{dt}{t}\,;$$

- **the** γ **- Hardy-Littlewood operator** *given by*

(5)
$$M_\gamma\,:\,S(\mathbb{R})\longmapsto L^\infty(\mathbb{R})\,,$$
$$M_\gamma f(x):=\sup_{a>0}\left|\frac{1}{2a}\int_{-a}^{a}f(x-t)\,e^{i\,\gamma(x,t)}\,dt\right|\,;$$

- **the bilinear Hilbert transform along** Γ *defined as*

(6)
$$H_\Gamma^{\mathcal{B}}\,:\,S(\mathbb{R})\times S(\mathbb{R})\longmapsto S'(\mathbb{R})\,,$$
$$H_\Gamma^{\mathcal{B}}(f,g)(x):=\text{p.v.}\int_{\mathbb{R}}f(x-t)\,g(x+\gamma(x,t))\,\frac{dt}{t}\,;$$

- **the (sub)bilinear maximal operator along** Γ *defined as*

(7)
$$M_\Gamma^{\mathcal{B}}\,:\,S(\mathbb{R})\times S(\mathbb{R})\longmapsto L^\infty(\mathbb{R})\,,$$
$$M_\Gamma^{\mathcal{B}}(f,g)(x):=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-t)\,g(x+\gamma(x,t))|\,dt\,;$$

2. Historical context; Motivation.

The above three themes have a very deep and inter-related history[1]:

- (I) originates in the area of PDE through the study of the constant coefficient parabolic operators initiated by F. Jones ([18]), E. Fabes ([12]) and E. Fabes and M. Riviere ([11]). Departing from this, E. Stein and S. Wainger and a bit later Nagel, Riviere, and Wainger initiated a systematic study of this theme that gradually relied on Van der Corput estimates, orthogonality methods, Fourier methods, Littlewood-Paley square function techniques etc. ([38], [30], [32], [31], [35], [36], [39]). Further important developments appeared after mid eighties - see e.g. [7], [33], [4], [2] and

[1]The bibliography below is far from being exhaustive.

more recently [6] and [22]. In a different context, theme (I) can be related - as a suitable model case - with the celebrated Zygmund's differentiation conjecture along vector fields (see e.g. [1], [16], [10]).

- (II) can be traced back to the seminal 1910's conjecture of Luzin ([29]) regarding the almost everywhere convergence of the Fourier series associated to functions in $L^2(\Pi)$. The positive resolution of this conjecture was delivered by L. Carleson half a century later in [5] by developing fundamental new tools that are now part of what we call the area of time-frequency analysis. His result is essentially equivalent with the (weak) L^2 boundedness of the operator introduced in (4) in the case $\gamma(x,t) = a(x)\,t$ with $a(\cdot)$ being an arbitrary measurable function. Several extensions of Carleson's result appeared later with two of the best known of them provided by R. Hunt ([17], L^p case for $1 < p < \infty$) and Sjölin ([34], the higher dimensional analogue). Closely related with the development of (I) and (II) is the above mentioned work of E. Stein and S. Wainger on the singular oscillatory integral expressions/operators in [38] which later led Stein to consider maximal singular oscillatory integrals on Heisenberg groups and eventually formulate the so called Polynomial Carleson Operator Conjecture ([37], [40]) whose main (one dimensional) case was solved by the author in [26]. This latter result is essentially equivalent with the (weak) L^2 boundedness of the operator in (4) in the case $\gamma(x,t) = \sum_{j=1}^{d} a_j(x)\,t^j$ where here $d \in \mathbb{N}$ fixed and $a_j(\cdot)$ are arbitrary measurable functions.

- (III) has - for suitable curves $\gamma(x,t) = \gamma(t)$ - early correspondences in ergodic theory specifically with the long-studied problem of understanding the L^p-norm convergence of (non-)conventional bilinear averages. Within Harmonic Analysis, the plenary manifestation of this theme derives from the study of Cauchy transform along Lipschitz curves initiated by A. Calderon ([3], [9]). Indeed, in this setting Calderon was naturally led to conjecture the boundedness[2] of the Bilinear Hilbert Transform which is the object defined in (6) for the *zero-curvature case* $\gamma(x,t) = t$. This conjecture of Calderon was solved by M. Lacey and C. Thiele in the celebrated works [19] and [20]. The analogue result for (7) was proved by M. Lacey in [21].

For the *nonzero-curvature case*, e.g. $\gamma(x,t) = \gamma(t) = \sum_{j=2}^{d} a_j\,t^j$, with $d > 1$ and $\alpha_j \in \mathbb{R}$ [3], the first explicit example was studied in [24], in the special case of pure monomials. There, X. Li proved the $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ boundedness of $H_\Gamma^{\mathcal{B}}$ relying on the so-called σ-uniformity concept introduced in [8] and inspired by Gowers's work in [13]. In [27], [28] the author proved the maximal possible range up to the end-points for $H_\Gamma^{\mathcal{B}}$ with γ belonging to a suitable classes of curves $\mathcal{NF}$ that includes in particular any Laurent polynomial with no term of degree ± 1. The proof of these last results combines elements of time-frequency analysis (Gabor frames) with orthogonality methods.

[2]For Hölder exponents within the "Banach triangle".

[3]Essential here is that the term $a\,t$ is excluded from the definition of $\gamma(t)$.

3. Main results - informal

Main Theorem. *Let $\Gamma \equiv \{\Gamma_x\}_{x\in\mathbb{R}}$ be a family of (twisted) variable curves defined by $\Gamma_x = (t, \gamma(x,t))$ with $\gamma:\ \mathbb{R}^2 \to \mathbb{R}$ measurable.*

Assume now that

$$\gamma \in \mathbf{M}_x\mathbf{NF}_t\,,$$

that is, at an informal level one has[4]

(8)

- $\gamma_t(\cdot) := \gamma(\cdot,\, t)$ *is* $x-$measurable *for every* $t \in \mathbb{R}\setminus\{0\}$*;*
- $\gamma_x(\cdot) := \gamma(x,\, \cdot)$ *is finitely piecewise* $t-$smooth *within the class* $C^{2+}(\mathbb{R}\setminus\{0\})$ *for almost everywhere* $x \in \mathbb{R}$*;*
- γ *is "non-flat" (that is it obeys a non-zero curvature condition) near origin and infinity; [In particular, outside of a controlled region,* γ *can be decomposed in a finite number of pieces on which it has* $x-$*uniform non-vanishing curvature in the* $t-$*variable.]*
- γ *obeys a suitable non-degeneracy condition.*

Then for any $1 < p < \infty$ *we have that*

(1) H_Γ *and* M_Γ *are bounded operators from* $L^p(\mathbb{R}^2)$ *to* $L^p(\mathbb{R}^2)$*;*
(2) C_γ *and* M_γ *are bounded operators from* $L^p(\mathbb{R})$ *to* $L^p(\mathbb{R})$*.*

Observation 1. *Following some of the key ideas introduced in our study - under suitable conditions imposed on* γ *- one can also obtain the* L^p *boundedness of the Bilinear Hilbert transform and Maximal operator corresponding to* (6) *and* (7)*. Thus, we are able to provide a unitary method for all the operators defined in the statement of our Main Problem, and in particular to identify and highlight as natural a common approach to both the singular and the maximal operators within each class of objects belonging to the itemization (I) - (III) at the beginning of our introduction. Moreover, as a consequence of these methods, we are able to immediately encompass and generalize many of the previous results - to only mention here few -* [40], [14], [15], [23], [24], [25], [27], [28] *and part of* [16].

We end this section by quoting two consequences of our Main Theorem:

Corollary 1. *Let* $d \in \mathbb{N}$ *and* [5]

$$\gamma(x,t) := \sum_{j=1}^{d} a_j(x)\, t^{\alpha_j}\,, \tag{9}$$

where here $\{\alpha_j\}_{j=1}^d \subset \mathbb{R}\setminus\{0,\,1\}$ *and* $\{a_j\}_{j=1}^d$ *measurable functions.*

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[4] Below, the class C^{2+} simply means the standard $C^{2+\delta}$ where here δ can be any number strictly greater than zero.

[5] Throughout the paper, for convenience, we allow a notational abuse and introduce the following convention: given $\alpha,\, t \in \mathbb{R}$ we let t^α stand for either $|t|^\alpha$ or $\mathrm{sgn}\,(t)\,|t|^\alpha$.

Then, one has that

$$\gamma \in \mathbf{M}_x\mathbf{NF}_t\,. \tag{10}$$

Corollary 2. *Let* $1 < p < \infty$, $d \in \mathbb{N}$ *and* $\{\alpha_j\}_{j=1}^d \subset \mathbb{R} \setminus \{1\}$. *Then the following operators are bounded operators from* $L^p(\mathbb{R})$ *to* $L^p(\mathbb{R})$*:*

- *the* generalized Polynomial Carleson-type operator *given by*

$$C_{\vec{\alpha},d}f(x) := \sup_{\{a_j\}_{j=1}^d \subset \mathbb{R}} \left| p.v. \int_{\mathbb{R}} f(x-t)\, e^{i\sum_{j=1}^d a_j\, t^{\alpha_j}}\, \frac{dt}{t} \right|, \qquad f \in \mathcal{S}(\mathbb{R})\,; \tag{11}$$

- *the* generalized Polynomial Hardy-Littlewood-type operator *given by*

$$M_{\vec{\alpha},d}f(x) := \sup_{\substack{\{a_j\}_{j=1}^d \subset \mathbb{R} \\ a>0}} \left| \frac{1}{2a} \int_{-a}^{a} f(x-t)\, e^{i\sum_{j=1}^d a_j\, t^{\alpha_j}}\, dt \right|, \qquad f \in \mathcal{S}(\mathbb{R})\,; \tag{12}$$

4. Aim of our course

Our course will revolve around two main components:

- we will start by presenting the historical evolution of our three themes and discuss the inter-connections among them as described in Section 2;
- we will outline some of the main ideas in proving our main theorem and provide a unitary treatment of the above three themes keeping though our main focus on the first two and only very briefly - if the time allows - touch the third theme concerning the (sub)bilinear operators given by (6) and (7).

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Department of Mathematics, Purdue, IN 46907 USA
E-mail address: vlie@math.purdue.edu

Institute of Mathematics of the Romanian Academy, Bucharest, RO 70700, P.O. Box 1-764, Romania.