On the absolute continuous spectrum of discrete operators

CIMPA school: Théorie spectrale des graphes et des variétés

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1. General theory of self-adjoint operators
2. Explicit examples of spectra coming from the analysis on infinite graphs
3. Continuous functional calculus
4. The essential spectrum
5. Borelian functional calculus
6. Absolute continuous spectrum
7. Limiting absorption principle and Mourre theory

Notation: Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces, we denote by $\mathcal{L}(X, Y)$ the set of continuous linear maps acting from X to Y. Endowed with the norm

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x\in X, \|x\|_X=1} \|Tx\|_Y,$$

we have that $\mathcal{L}(X, Y)$ is a Banach space.

$\mathbb{N}$ is the set of non-negative integers (be careful $0 \in \mathbb{N}$) and $\mathbb{Z}$ is the set of integers.

We focus on the study of complex Hilbert spaces.

Definition

Given a complex vector space X, a scalar product is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$:

1) $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \overline{\lambda} \langle y, z \rangle$,
2) $\langle z, x + \lambda y \rangle = \langle z, x \rangle + \lambda \langle z, y \rangle$,
3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4) $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

A vector space X endowed with a scalar product is a pre-Hilbert space.

Note that the third line gives $\langle x, x \rangle \geq 0$.

Remark

Here we take the convention to be anti-linear with respect to the first variable. It is a choice.

Proposition

Let $(X, \langle\cdot,\cdot\rangle)$ *be a pre-Hilbert space. We set* $\|x\| := \sqrt{\langle x, x\rangle}$. *We have that* $\|\cdot\|$ *is a norm, i.e., for all* $x, y \in X$ *and* $\lambda \in \mathbb{C}$

1. $\|x\| = 0$ *if and only if* $x = 0$,
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$,
3. $\|x + y\| \leq \|x\| + \|y\|$.

If $(X, \|\cdot\|)$ *is complete, we say that* X *is a Hilbert space.*

Proposition

We say that $(e_n)_{n\in\mathbb{N}}$ *is a Hilbert basis for a Hilbert space* $(\mathcal{H}, \|\cdot\|)$, *if*

1. $\langle e_n, e_m\rangle = \delta_{n,m}$ *for all* $n, m \in N$. *In particular,* $\|e_n\| = 1$ *for all* $n \in \mathbb{N}$,
2. $\sum_{n\in\mathbb{N}} \mathbb{C}e_n = \mathcal{H}$.

Remark

Sometimes it is useful to take $\mathbb{Z}$ *instead of* $\mathbb{N}$ *in this definition.*

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Definition

A metric space (X, d) is separable if and only if there is a countable set $F \subset X$ such that F is dense in X.

Proposition

Given a Hilbert space $(\mathcal{H}, \|\cdot\|)$. The following statements are equivalent:

1. *$\mathcal{H}$ is separable,*
2. *$\mathcal{H}$ has a Hilbert basis.*

Proof:

$2 \implies 1$: Given $(e_n)_n$ a Hilbert basis, take $F := \cup_n \mathbb{Q} e_n$.
$1 \implies 2$: We have $F = \cup_n f_n$ with $f_n \in \mathcal{H}$. Use Gram-Schmidt on $(f_n)_n$. □

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From now on, all the Hilbert space are complex and separable.

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Remark

From now on, all the Hilbert space are complex and separable.

Two main examples:

1) Set $\mathcal{H} := \ell^2(\mathbb{N};\mathbb{C}) := \{f : \mathbb{N} \to \mathbb{C}, \text{ such that } \sum_n |f_n|^2 < \infty\}$ endowed with

$$\langle f, g\rangle := \sum_{n\in\mathbb{N}} \overline{f_n} g_n,$$

for $f, g \in \ell^2(\mathbb{N};\mathbb{C})$.

For all $n \in N$, set $e_n : \mathbb{N} \to C$ given by $e_n(m) := \delta_{n,m}$. We have that $(e_n)_{n\in N}$ is a Hilbert basis.

2) Set $\mathcal{H} := L^2([-\pi,\pi];\mathbb{C})$, endowed with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)} g(x)\, dx,$$

with $f, g \in \mathcal{H}$.

For all $n \in Z$, set $e_n(x) := e^{inx}$. We have that $(e_n)_{n\in\mathbb{Z}}$ is a Hilbert basis.

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with $f, g \in \mathcal{H}$.

For all $n \in Z$, set $e_n(x) := e^{\mathrm{i}nx}$. We have that $(e_n)_{n\in\mathbb{Z}}$ is a Hilbert basis.

We turn to the polarisation properties.

Proposition

Let $\mathcal{X}$ be $\mathbb{C}$-vector space. We take $\mathcal{Q} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ to be a sesquilinear form which is linear on the right and anti-linear on the left, i.e.,

1) $\mathcal{Q}(x, y + \lambda z) = \mathcal{Q}(x, y) + \lambda \mathcal{Q}(y, z)$,
2) $\mathcal{Q}(x + \lambda y, z) = \mathcal{Q}(x, z) + \overline{\lambda} \mathcal{Q}(y, z)$,

for all $x, y, z \in \mathcal{X}$ et $\lambda \in \mathbb{C}$. Set $\mathcal{Q}(x) := \mathcal{Q}(x, x)$ (because this is not necessarily real!). We have the following identity of polarisation:

$$\mathcal{Q}(x, y) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \mathcal{Q}(\mathrm{i}^k x + y).$$

Proof:

Develop the right hand side. □

Remark

In particular we get:

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \|\mathrm{i}^k x + y\|^2.$$

In other words, given a norm that comes from a scalar product, we can recover the scalar product.

Remark

When the vector space is real a bilinear form $\mathcal{Q}$ satisfies:

$$Q(x, y) = \frac{1}{4}\left(\mathcal{Q}(x+y) - \mathcal{Q}(x-y)\right),$$

for all $x, y \in \mathcal{X}$.

Corollary

Given $\mathcal{H}$ a Hilbert space and S, T two bounded operators. If

$$\langle x, Sx\rangle = \langle x, Tx\rangle, \text{ pour tout } x \in \mathcal{X}$$

then $S = T$.

Proof:

Set $\mathcal{Q}_1(x,y) := \langle x, Sy\rangle$ and $\mathcal{Q}_2(x,y) := \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$. There are quadratic forms.

By hypothesis we have $\mathcal{Q}_1(x) = \mathcal{Q}_2(x)$ for all $x \in \mathcal{H}$. In particular we have:

$$\langle x, Sy\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \mathcal{Q}_1(\mathrm{i}^k x + y) = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \mathcal{Q}_2(\mathrm{i}^k x + y) = \langle x, Ty\rangle.$$

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Proposition (Riesz's isomorphism)

Set $\phi \in \mathcal{H}'$, where $\mathcal{H}'$ is the set of anti-linear continuous forms defined on $\mathcal{H}$. Then there exists a unique $x_\phi \in \mathcal{H}$ such that

$$\phi(x) = \langle x, x_\phi \rangle,$$

for all $x \in \mathcal{H}$. Moreover $\|x_\phi\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}'}$.

Remark

Here we have chosen the space of anti-linear forms instead of the space of linear forms. It seems a bit peculiar but this provides that

$$\Phi : \begin{cases} \mathcal{H}' & \to \mathcal{H} \\ \phi & \mapsto x_\phi \end{cases}$$

is a (linear) isometry of Hilbert spaces.

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Proposition

Set $T \in \mathcal{L}(\mathcal{H})$. There is a unique $S \in \mathcal{L}(\mathcal{H})$ so that

$$\langle x, Ty \rangle = \langle Sx, y \rangle,$$

for all $x, y \in \mathcal{H}$. We denote it by $T^* := S$. Moreover, we have:

$$\|T\| = \|T^*\|.$$

Remark

We have $T^{**} = T$.

Proposition

Given $T \in \mathcal{L}(X)$, we have:

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

Definition

Let $T \in \mathcal{L}(\mathcal{H})$,

1) T *is normal if* $T^*T = TT^*$.
2) T *is self-adjoint if* $T = T^*$.
3) T *is unitary if* $T^*T = TT^* = \mathrm{Id}$.

Remark

Set $T \in \mathcal{L}(\mathcal{H})$. *Let* $U \in \mathcal{L}(\mathcal{H})$, *be unitary. Then,* $\sigma(T) = \sigma(UTU^*)$.

Exercise

Set $T \in \mathcal{L}(\mathcal{H})$. *We have that* T *is unitary if and only* T *is surjective and is an isometry, i.e.,* $\|Tx\| = \|x\|$, *for all* $x \in \mathcal{H}$.

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Definition

Let $T \in \mathcal{L}(\mathcal{H})$.

1) *The resolvent set of T is:*

$$\rho(T) := \{\lambda \in \mathbb{C},\ \lambda \mathrm{Id} - T \text{ is invertible}\}.$$

2) *If $\lambda \in \rho(T)$, we define the resolvent $R_\lambda(T)$ (or simply R_λ) of T at λ by*

$$R_\lambda(T) := (\lambda \mathrm{Id} - T)^{-1}.$$

3) *The spectrum of T is*

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

4) *We say that $\lambda \in \mathbb{C}$ is an eigenvalue of T if $\lambda \mathrm{Id} - T$ is not injectif, i.e.,* $\ker(\lambda \mathrm{Id} - T) \neq \{0\}$. *The point spectrum is given by:*

$$\sigma_{\mathrm{p}}(T) := \overline{\{\lambda \in \mathbb{C},\ \ker(\lambda \mathrm{Id} - T) \neq \{0\}\}}.$$

Remark

If $\lambda \in \rho(T)$, $R_\lambda(T) \in \mathcal{L}(\mathcal{H})$ (Banach's Theorem).

Remark

We have:

1) *When $\mathcal{H}$ is of finite dimension and $T \in \mathcal{L}(\mathcal{H})$, the rank theorem states that T is surjective if and only if T is injective if and only if it is bijective. In particular*

$$\sigma_p(T) = \sigma(T), \text{ when } \dim \mathcal{X} < \infty$$

   The situation is very different in infinite dimension.

2) *The point spectrum is usually different from the set of eigenvalues.*

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$.

1) *If $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$. In particular $\sigma(T) \subset \overline{D(0, \|T\|)}$. Moreover,*

$$\|(\lambda - T)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|}.$$

2) *$\rho(T)$ is open and non-empty in $\mathbb{C}$.*
3) *$\sigma(T)$ is compact and non-empty in $\mathbb{C}$.*
4) *$\sigma_{\mathrm{p}}(T) \subset \sigma(T)$.*

Definition

Given $T \in \mathcal{L}(\mathcal{H})$. We call spectral radius:

$$\mathrm{rad}(T) := \inf\{r, \sigma(T) \subset \overline{B}(0, r)\}.$$

Proposition

Let $H \in \mathcal{L}(\mathcal{H})$, we have

$$\mathrm{rad}(H) = \lim_{n\to\infty} \|H^n\|^{1/n}.$$

Moreover, if H is self-adjoint, then $\mathrm{rad}(H) = \|H\|$. In particular $\|H\|$ or $\|H\|$ belongs to $\sigma(H)$.

Remark

For

$$H := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we have $\mathrm{rad}(H) < \|H\|$.

Proposition (Identities of the resolvent)

Let S, T be bounded operators in $\mathcal{H}$.

1) *Suppose that $\lambda \in \rho(S) \cap \rho(T)$. We have:*

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S)R_\lambda(S).$$

2) *Suppose that $\lambda, \mu \in \rho(T)$, then*

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)R_\lambda(T)R_\mu(T) = (\mu - \lambda)R_\mu(T)R_\lambda(T).$$

   In particular R_λ and R_μ commute.

3) *The map $R_\cdot(T) := \lambda \mapsto R_\lambda(T)$ acting from $\rho(T)$ into $\mathcal{GL}(\mathcal{H})$ is analytic with derivative:*

$$\frac{dR_\lambda}{d\lambda} = -R_\lambda^2.$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then

1) $\sigma(T) \subset \mathbb{R}$.
2) *For $z \in \mathbb{C} \setminus \mathbb{R}$, we have $z \notin \sigma(T)$ and*

$$\|(z\mathrm{Id} - T)^{-1}\| \leq \frac{1}{\Im(z)}.$$

3) *Let λ_1 and λ_2 two distincts eigenvalues of T, Then* $\ker(\lambda_1 \mathrm{Id} - T) \perp \ker(\lambda_2 \mathrm{Id} - T)$.
4) *T has at most a countable number of eigenvalues.*

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle x, Ay\rangle| = \sup_{\|x\|=1} |\langle x, Ax\rangle|.$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Let

$$m := \inf\{\langle x, Tx\rangle, \, x \in \mathcal{H} \text{ with } \|x\| = 1\}$$
$$M := \sup\{\langle x, Tx\rangle, \, x \in \mathcal{H} \text{ with } \|x\| = 1\}.$$

Then $\sigma(T) \subset [m, M]$. Moreover, m and M belong to $\sigma(T)$.

We can also compute the spectrum with the help of approximate eigenvalues:

Proposition

Let $H \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then $\lambda \in \sigma(H)$ if and only if

$$\exists f_n \in \mathcal{H},\ \|f_n\| = 1 \ \text{and}\ \|(H - \lambda) f_n\| \to 0.$$

Some examples :

Proposition

Let $\mathcal{H} := \ell^2(\mathbb{N};\mathbb{C})$. *Let* $F \in \ell^\infty(\mathbb{N};\mathbb{C})$. *We denote by* $F(Q)$ *the operator of multiplication by* F, *i.e., for all* $f \in \mathcal{H}$,

$$(F(Q)f)(n) := F(n)f(n), \quad \text{for all } n \in \mathbb{N}.$$

1. $F(Q)$ *is bounded.*
2. $F(Q)$ *is normal.*
3. $F(Q)$ *is self-adjoint if and only if* $F(n) \in \mathbb{R}$, *for all* $n \in \mathbb{N}$.
4. $F(Q)$ *is unitary if and only if* $|F(n)| = 1$, *for all* $n \in \mathbb{N}$.
5. $\cup_{n\in\mathbb{N}}\{F(n)\}$ *is the set of eigenvalues of* $F(Q)$.
6. $\sigma(F(Q)) = \overline{\cup_{n\in\mathbb{N}}\{F(n)\}}$.
7. $F(Q)$ *is compact if and only if* $\lim_{n\to\infty} F(n) = 0$.
8. $F(Q)$ *is of finite rank if and only if* F *has finite support.*

Exercise

Give F *such that* $\sigma(F(Q)) = [0,2]$.

Proposition

Let $\mathcal{H} := L^2([0,1];\mathbb{C})$. *Let* $F \in \mathcal{C}^0([0,1];\mathbb{C})$. *We denote by* $F(Q)$ *the operator of multiplication by* F, *i.e., for all* $f \in \mathcal{H}$,

$$(F(Q)f)(x) := F(x)f(x), \quad \text{for all } x \in [0,1].$$

1. $F(Q)$ *is bounded.*
2. $F(Q)$ *is normal.*
3. $F(Q)$ *is self-adjoint if and only if* $F(x) \in \mathbb{R}$, *for all* $x \in [0,1]$.
4. $F(Q)$ *is unitary if and only if* $|F(x)| = 1$, *for all* $x \in [0,1]$.
5. $\{\lambda, \mathrm{Leb}(F^{-1}(\lambda)) > 0\}$ *is the set of eigenvalues of* $F(Q)$. *The eigenvalues are of infinite multiplicity.*
6. $\sigma(F(Q)) = F([0,1])$.
7. $F(Q)$ *is compact if and only if* $F \equiv 0$

Exercise

State this result for $F \in L^\infty([0,1],\mathbb{C})$.

Let $\mathcal{H} := \ell^2(\mathbb{Z};\mathbb{C})$. We define the adjacency matrix by:

$$(\mathcal{A}f)(n) := f(n-1) + f(n+1), \quad \text{for } f \in \mathcal{H}.$$

It is a self-adjoint operator. Indeed we have for all $g, f \in \mathcal{H}$:

$$\langle f, \mathcal{A}g\rangle = \sum_{n\in\mathbb{Z}} \overline{f(n)}\,(g(n+1)+g(n-1)) = \sum_{n\in\mathbb{Z}} \overline{f(n+1)+f(n-1)}g(n) = \langle \mathcal{A}f, g\rangle$$

The Fourier transform $\mathscr{F} : \ell^2(\mathbb{Z}) \to L^2([-\pi,\pi])$ is defined by

$$(\mathscr{F}f)(x) := \frac{1}{\sqrt{2\pi}} \sum_n f(n)e^{-ixn}, \text{ for all } f \in \ell^2(\mathbb{Z}) \text{ and } x \in [-\pi,\pi].$$

It is unitary and its inverse is given by:

$$(\mathscr{F}^{-1}f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)e^{ikx}\,dx, \text{ for all } f \in L^2([\pi,\pi]) \text{ and } k \in \mathbb{Z}.$$

We take advantage of the Fourier Transform to study $\mathcal{A}$ and set:

$$\tilde{\mathcal{A}} := \mathscr{F}\mathcal{A}\mathscr{F}^{-1}.$$

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We take advantage of the Fourier Transform to study $\mathcal{A}$ and set:

$$\tilde{A} := \mathscr{F}\mathcal{A}\mathscr{F}^{-1}.$$

Let $f \in L^2([-\pi, \pi])$. We have:

$$(\tilde{A}f)(x) = \mathscr{F}(\mathcal{A}\mathscr{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn} (\mathcal{A}\mathscr{F}^{-1}f)(n)$$
$$= \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn} \left((\mathscr{F}^{-1}f)(n+1) + (\mathscr{F}^{-1}f)(n-1) \right)$$
$$= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} \left(e^{\mathrm{i}(n+1)t} f(t) + e^{\mathrm{i}(n-1)t} f(t) \right) \, dt$$
$$= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} e^{\mathrm{i}nt} 2\cos(t) f(t) \, dt = 2\cos(t) f(t).$$

Therefore

$$\tilde{A} := \mathscr{F}\mathcal{A}\mathscr{F}^{-1} = 2\cos(Q).$$

In particular:

$$\sigma(\mathcal{A}) = [-2, 2]$$

and $\mathcal{A}$ has no eigenvalue.

Exercise

Compute the spectrum of $\mathcal{A}$ using the approximate eigenvalues approach.

Let $f \in L^2([-\pi, \pi])$. We have:

$$
\begin{aligned}
(\tilde{\mathcal{A}} f)(x) &= \mathscr{F}(\mathcal{A}\mathscr{F}^{-1} f)(x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn} (\mathcal{A}\mathscr{F}^{-1} f)(n) \\
&= \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn} \Big((\mathscr{F}^{-1} f)(n+1) + (\mathscr{F}^{-1} f)(n-1) \Big) \\
&= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} \Big(e^{\mathrm{i}(n+1)t} f(t) + e^{\mathrm{i}(n-1)t} f(t) \Big)\, dt \\
&= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} e^{\mathrm{i}nt} 2\cos(t) f(t)\, dt = 2\cos(t) f(t).
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Let $f \in L^2([-\pi, \pi])$. We have:

$$(\tilde{\mathcal{A}}f)(x) = \mathscr{F}(\mathcal{A}\mathscr{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn}(\mathcal{A}\mathscr{F}^{-1}f)(n)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_n e^{-\mathrm{i}xn} \left((\mathscr{F}^{-1}f)(n+1) + (\mathscr{F}^{-1}f)(n-1) \right)$$

$$= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} \left(e^{\mathrm{i}(n+1)t} f(t) + e^{\mathrm{i}(n-1)t} f(t) \right) \, dt$$

$$= \frac{1}{2\pi} \sum_n e^{-\mathrm{i}xn} \int_{-\pi}^{\pi} e^{\mathrm{i}nt} 2\cos(t) f(t) \, dt = 2\cos(t) f(t).$$

Therefore

$$\tilde{\mathcal{A}} := \mathscr{F}\mathcal{A}\mathscr{F}^{-1} = 2\cos(Q).$$

In particular:

$$\sigma(\mathcal{A}) = [-2, 2]$$

and $\mathcal{A}$ has no eigenvalue.

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Compute the spectrum of $\mathcal{A}$ using the approximate eigenvalues approach.

Let $\mathcal{H} := \ell^2(\mathbb{N}; \mathbb{C})$. For $f \in \ell^2(\mathbb{N})$, we define the adjacency matrix by:

$$(\mathcal{A}f)(n) := \begin{cases} f(n-1) + f(n+1), & \text{if } n \geq 1, \\ f(1), & \text{if } n = 0. \end{cases}$$

The Fourier transform $\mathscr{F} : \ell^2(\mathbb{N}) \to L^2_{\mathrm{odd}}([-\pi, \pi])$ is defined by

$$(\mathscr{F} f)(x) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}} f(n+1)\sin((n+1)x), \text{ for all } f \in \ell^2(\mathbb{N}) \text{ and } x \in [-\pi, \pi].$$

It is unitary.

We take advantage of this Fourier transform and obtain similarly

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Let $\mathcal{H} := \ell^2(\mathbb{Z};\mathbb{C}^2)$, endowed with the scalar product

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \langle f(n), g(n)\rangle_{\mathbb{C}^2} = \sum_{n\in\mathbb{Z}} \overline{f_1(n)}g_2(n) + \overline{f_2(n)}g_2(n).$$

where $f, g \in \mathcal{H}$, $f(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix}$, and $g(n) = \begin{pmatrix} g_1(n) \\ g_2(n) \end{pmatrix}$.

Set $m \geq 0$. The Dirac discrete operator, acting on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, is defined by

$$D_m := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix},$$

where $d := \mathrm{Id} - \tau$ and τ is the right shift, defined by

$$\tau f(n) = f(n+1), \text{ for all } f \in \ell^2(\mathbb{Z}, \mathbb{C}).$$

Note that $\tau^* f(n) = f(n-1)$, for all $f \in \ell^2(\mathbb{Z}, \mathbb{C})$.

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The operator D_m is self-adjoint and we have:

$$D_m^2 = \begin{pmatrix} \Delta + m^2 & 0 \\ 0 & \Delta + m^2 \end{pmatrix},$$

where $\Delta = 2 - \mathcal{A}_{\mathbb{Z}}$. Recall that $\sigma(\Delta) = 2 - \sigma(\mathcal{A}_{\mathbb{Z}}) = [0, 4]$.

Since we have a direct sum, we have:

$$\sigma(D_m^2) = [m^2, m^2 + 4].$$

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To remove the square above D_m, we define the symmetry S on $\ell^2(\mathbb{Z}, \mathbb{C})$ by

$$Sf(n) = f(-n)$$

and the unitary operator on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$

$$U := \begin{pmatrix} 0 & \mathrm{i}S \\ -\mathrm{i}S & 0 \end{pmatrix}.$$

Clearly $U = U^* = U^{-1}$. We have that

$$UD_mU = -D_m.$$

In particular, we have

$$\sigma(D_m) = \sigma(-D_m) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right].$$

Exercise

Show that D_m is unitarily equivalent to $\begin{pmatrix} \sqrt{m^2+2-2\cos(Q)} & 0 \\ 0 & -\sqrt{m^2+2-2\cos(Q)} \end{pmatrix}$, *which acts in $L^2([\pi,\pi], \mathbb{C}^2)$. Compute the spectrum in an alternative way.*

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Let $\mathscr{V}$ be a finite or countable set and let $\mathscr{E} := \mathscr{V} \times \mathscr{V} \to \{0, 1\}$ such that

$$\mathscr{E}(x, y) = \mathscr{E}(y, x), \quad \text{for all } x, y \in \mathscr{V}.$$

We say that $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ is an non-oriented graph with *edges* $\mathscr{E}$ and *vertices* $\mathscr{V}$.

We say that $x, y \in \mathscr{V}$ are *neighbours* if $\mathscr{E}(x, y) = 1$. We write: $x \sim y$ and $\mathscr{N}(x) := \{y \in \mathcal{V}, x \sim y\}$.

The *degree* of $x \in \mathscr{V}$ is given by:

$$\deg_G(x) := |\{y \in \mathscr{E} \mid x \sim y\}|.$$

Hypotheses: $\deg_G(x) < \infty$ and $\mathscr{E}(x, x) = 0$ for all $x \in \mathscr{V}$.

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Set $\mathcal{H} := \ell^2(\mathscr{V};\mathbb{C})$, endowed with $\langle f, g\rangle = \sum_{x\in\mathscr{V}} \overline{f(x)}g(x)$.

The *Laplacian* is given by:

$$\Delta f(x) = \sum_{y\sim x}(f(x) - f(y)), \quad \text{for all } f \in \mathcal{C}_c(\mathscr{V}).$$

The *adjacency matrix* is given by

$$\mathcal{A}f(x) = \sum_{y\sim x} f(y), \quad \text{for all } f \in \mathcal{C}_c(\mathscr{V}).$$

Note that $\Delta = \deg_{\mathcal{G}}(Q) - \mathcal{A}$. They are both symmetric on $\mathcal{C}_c(\mathcal{V})$.

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Proposition

We have:

1. $$\Delta \text{ bounded} \iff \mathcal{A} \text{ bounded} \iff \deg(\cdot) \text{ bounded.}$$

   In particular, is $\deg(\cdot)$ *is bounded then* Δ *and* $\mathcal{A}$ *are self-adjoint.*

2. $$0 \leq \langle f, \Delta f\rangle \leq 2\langle f, \deg(Q) f\rangle, \text{ for all } f \in \mathcal{C}_c(\mathscr{V}).$$

   In particular, $\sigma(\Delta) \subset [0, 2\sup_{x\in\mathcal{V}} \deg(x)]$.

Proof:

We start with the second point.

$$\langle f, \Delta f\rangle = \frac{1}{2}\sum_{x\in\mathscr{V}}\sum_{y\in\mathcal{V}} \mathscr{E}(x,y)|f(x) - f(y)|^2$$
$$\leq \sum_{x\in\mathscr{V}}\sum_{y\sim x}(|f(x)|^2 + |f(y)|^2) = 2\langle f, \deg(Q) f\rangle,$$

for $f \in \mathcal{C}_c(\mathscr{V})$.

We turn to the first point. For Δ, using 2) and that $\langle \delta_x, \Delta\delta_x\rangle = \deg(x)$ we have the equivalence between Δ and $\deg$.

We focus on $\mathcal{A}$.

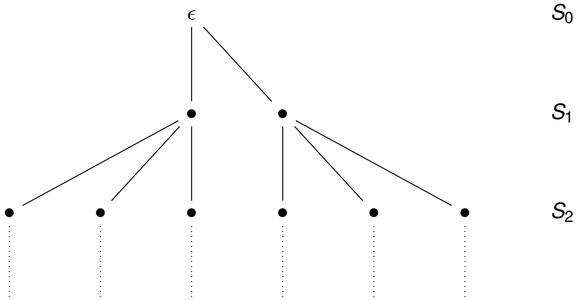
$$|\langle f, \mathcal{A}f\rangle| = \left|\sum_{x\in\mathcal{V}} \overline{f(x)} \sum_{y\sim x} f(y)\right| \leq \frac{1}{2}\sum_{x}\sum_{y\sim x}\left(|f(x)|^2 + |f(y)|^2\right) = \langle f, \deg(Q)f\rangle.$$

and on the other side, since $\mathcal{E}(x,y) \in \{0,1\}$, we have:

$$\|\mathcal{A}f\|^2 = \sum_x \left|\sum_{y\sim x}\mathcal{E}(x,y)f(y)\right|^2 \geq \sum_x\sum_{y\sim x}\mathcal{E}(x,y)\,|f(y)|^2 = \sum_x\sum_{y\sim x}\mathcal{E}(x,y)\,|f(x)|^2$$
$$= \langle f, \deg(Q)f\rangle.$$

which ends the proof. □

Consider a tree $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, a connected graph with no cycle. Due to its structure, one can take any point of V to be a root. We denote it by ϵ.



We define inductively the *spheres* S_n by $S_{-1} = \emptyset$, $S_0 := \{\epsilon\}$, and $S_{n+1} := \mathcal{N}(S_n) \setminus S_{n-1}$. Given $n \in \mathbb{N}$, $x \in S_n$, and $y \in \mathcal{N}(x)$, one sees that $y \in S_{n-1} \cup S_{n+1}$.

We write $x \sim> y$ and say that x is a *son* of y, if $y \in S_{n-1}$, while we write $x <\sim y$ and say that x is a *father* of y, if $y \in S_{n+1}$.

Notice that ϵ has no father.

Given $x \neq \epsilon$, note that there is a unique $y \in V$ with $x \sim> y$, i.e., everyone apart from ϵ has one and only one father. We denote the father of x by $\overleftarrow{x}$.

Given $x \in S_n$, we set $\ell(x) := n$, the *length* of x. The *offspring* of an element x is given by

$$\text{off}(x) := |\{y \in \mathscr{N}(x), y \sim> x\}|,$$

i.e., it is the number of sons of x. When $\ell(x) \geq 1$, note that $\text{off}(x) = \deg(x) - 1$.

We consider the tree $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ with uniform offspring sequence $(b_n)_{n\in\mathbb{N}}$, i.e., every $x \in S_n$ has b_n sons. We define:

$$(Uf)(x) := \mathbf{1}_{\{\cup_{n\geq 1} S_n\}}(x) \frac{1}{\sqrt{b_{\ell(\overleftarrow{x})}}} f(\overleftarrow{x}), \text{ for } f \in \ell^2(\mathcal{V}).$$

Easily, one get $\|Uf\| = \|f\|$, for all $f \in \ell^2(\mathcal{V})$. The adjoint U^* of U is given by

$$(U^* f)(x) := \frac{1}{\sqrt{b_{\ell(x)}}} \sum_{y \sim > x} f(y), \text{ for } f \in \ell^2(\mathcal{V}).$$

Note that one has:

$$(\mathcal{A}_G f)(x) = \sqrt{b_{\ell(\overleftarrow{x})}}\, (Uf)(x) + \sqrt{b_{\ell(x)}}\, (U^* f)(x), \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

Supposing now that $b_n \geq 1$ for all $n \in \mathbb{N}$, we construct invariant subspaces for $\mathcal{A}_G$.

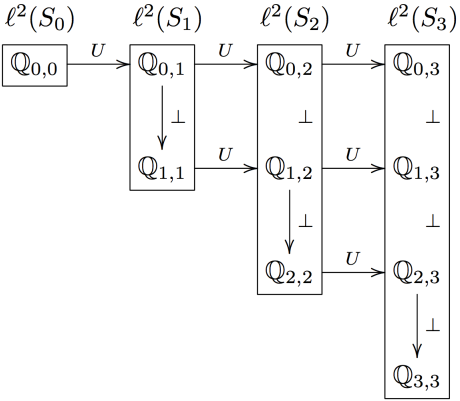
We start by noticing that $\dim \ell^2(S_n) = \prod_{i=0,\dots,n-1} b_n$, for $n \geq 1$ and $\dim \ell^2(S_0) = 1$. Therefore, as U is an isometry, $U\ell^2(S_n) = \ell^2(S_{n+1})$ if and only if $b_n = 1$.

Set $\mathbb{Q}_{0,0} := \ell^2(S_0)$ and $\mathbb{Q}_{0,k} := U^k \mathbb{Q}_{0,0}$, for all $k \in \mathbb{N}$. Note that $\dim \mathbb{Q}_{0,k} = \dim \ell^2(S_0) = 1$, for all $k \in \mathbb{N}$. Moreover, given $f \in \ell^2(S_k)$, one has $f \in \mathbb{Q}_{0,k}$ if and only if f is constant on S_k.

We define recursively $\mathbb{Q}_{n,n+k}$ for $k, n \in \mathbb{N}$. Given $n \in \mathbb{N}$, suppose that $\mathbb{Q}_{n,n+k}$ is constructed for all $k \in \mathbb{N}$, and set

- $\mathbb{Q}_{n+1,n+1}$ as the orthogonal complement of $\bigoplus_{i=0,\dots,n} \mathbb{Q}_{i,n+1}$ in $\ell^2(S_{n+1})$,
- $\mathbb{Q}_{n+1,n+k+1} := U^k \mathbb{Q}_{n+1,n+1}$, for all $k \in \mathbb{N} \setminus \{0\}$.

We sum-up the construction in the following diagram:



We point out that $\dim \mathbb{Q}_{n+1,n+1} = \dim \mathbb{Q}_{n+1,n+k+1}$, for all $k \in \mathbb{N}$ and stress that it is 0 if and only if $b_n = 1$. Notice that $U^* \mathbb{Q}_{n,n} = 0$, for all $n \in \mathbb{N}$.

Set finally $\mathbb{M}_n := \bigoplus_{k\in\mathbb{N}} \mathbb{Q}_{n,n+k}$ and note that $\ell^2(G) = \bigoplus_{n\in\mathbb{N}} \mathbb{M}_n$. Moreover, one has that canonically $\mathbb{M}_n \simeq \ell^2(\mathbb{N}; \mathbb{Q}_{n,n}) \simeq \ell^2(\mathbb{N}) \otimes \mathbb{Q}_{n,n}$. In this representation, the restriction $\mathcal{A}_n$ of $\mathcal{A}$ to the space $\mathbb{M}_n$ is given by the following tensor product of Jacobi matrices:

$$\mathcal{A}_n \simeq \begin{pmatrix} 0 & \sqrt{b_n} & 0 & 0 & \cdots \\ \sqrt{b_n} & 0 & \sqrt{b_{n+1}} & 0 & \ddots \\ 0 & \sqrt{b_{n+1}} & 0 & \sqrt{b_{n+2}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \otimes \mathbf{1}_{\mathbb{Q}_{n,n}}.$$

Now $\mathcal{A}$ is given as the direct sum $\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ in $\oplus_{n\in\mathbb{N}} \mathbb{M}_n$.

In particular, for a binary tree, i.e, $b_n = 2$ for all $n \in \mathbb{N}$,

$$\mathcal{A}_n \simeq \sqrt{2} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \ddots \\ 0 & 1 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \otimes \mathbf{1}_{\mathbb{Q}_{n,n}}.$$

Hence, $\mathcal{A}$ is the infinite direct sum of copies of $\sqrt{2}\,\mathcal{A}_{\mathbb{N}}$.

We obtain that

$$\sigma(\mathcal{A}) = [-2\sqrt{2}, 2\sqrt{2}].$$

We define the class of antitrees. The *sphere* of radius $n \in \mathbb{N}$ around a vertex $v \in \mathcal{V}$ is the set $S_n(v) := \{w \in \mathcal{V} \mid d_G(v, w) = n\}$. A graph is an *antitree*, if there exists a vertex $v \in \mathcal{V}$ such that for all other vertices $w \in \mathcal{V} \setminus \{v\}$

$$\mathscr{N}(w) = S_{n-1}(v) \cup S_{n+1}(v),$$

where $n = d_G(v, w) \geq 1$. The distinguished vertex v is the *root* of the antitree. Antitrees are bipartite and enjoy *radial symmetry*.

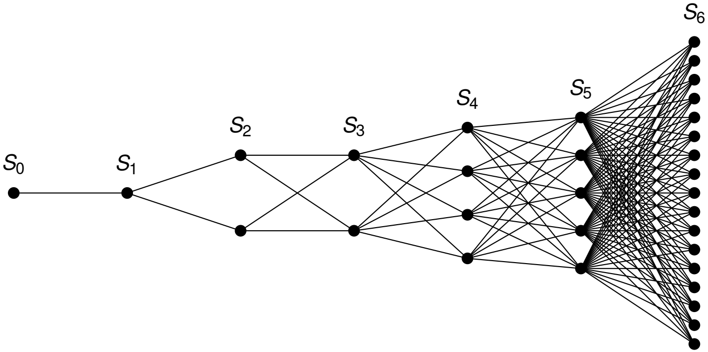


Figure: An antitree with spheres $S_0, \ldots, S_6$.

We denote the root by v, the spheres by $S_n := S_n(v)$, and their sizes by $s_n := |S_n|$. Further, $|x| := d(v, x)$ is the distance of $x \in V$ from the root.

The operator $P : \ell^2(\mathcal{V}) \to \ell^2(\mathcal{V})$, given by

$$Pf(x) := \frac{1}{s_{|x|}} \sum_{y \in S_{|x|}} f(y), \text{ for all } f \in \ell^2(\mathcal{V}) \text{ and } x \in \mathcal{V},$$

averages a function over the spheres. Thereby, $P = P^2 = P^*$ is the orthogonal projection onto the space of radially symmetric functions in $\ell^2(\mathcal{V})$. A function $f : \mathcal{V} \to \mathbb{C}$ is radially symmetric, if it is constant on spheres, i.e., for all nodes $x, y \in \mathcal{V}$ with $|x| = |y|$, we have $f(x) = f(y)$.

For all radially symmetric f, we define $\tilde{f} : \mathbb{N} \to \mathbb{C}$, $\tilde{f}(|x|) := f(x)$, for all $x \in \mathcal{V}$. Note that

$$P\ell^2(V) = \{f \colon \mathcal{V} \to \mathbb{C}, f \text{ radially symmetric}, \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n\in\mathbb{N}}),$$

where $(s_n)_{n\in\mathbb{N}}$ is now a sequence of weights.

The key observation is that

$$\mathcal{A} = P\mathcal{A}P \text{ and } \widetilde{\mathcal{A}Pf}(|x|) = s_{|x|-1}\widetilde{Pf}(|x|-1) + s_{|x|+1}\widetilde{Pf}(|x|+1),$$

for all $f \in \mathcal{C}_c(V)$, with the convention $s_{-1} = 0$.

Using the unitary transformation

$$U : \ell^2(\mathbb{N}, (s_n)_{n\in\mathbb{N}}) \to \ell^2(\mathbb{N}), \quad U\tilde{f}(n) = \sqrt{s_n}\tilde{f}(n),$$

we see that $\mathcal{A}$ is unitarily equivalent to the direct sum of 0 on $(P\ell^2(V))^\perp$ and a Jacobi matrix acting on $\ell^2(\mathbb{N})$ with 0 on the diagonal and the sequence $(\sqrt{s_n}\sqrt{s_{n+1}})_{n\in\mathbb{N}}$ on the off-diagonal.

$$\mathcal{A} \simeq 0 \oplus \begin{pmatrix} 0 & \sqrt{s_0}\sqrt{s_1} & 0 & 0 & \cdots \\ \sqrt{s_0}\sqrt{s_1} & 0 & \sqrt{s_1}\sqrt{s_2} & 0 & \ddots \\ 0 & \sqrt{s_1}\sqrt{s_2} & 0 & \sqrt{s_2}\sqrt{s_3} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In particular, if $s_n = 2$ for all $n \in \mathbb{N}$, $\sigma(\mathcal{A}) = [-2, 2]$ and 0 is the only eigenvalue. It is of infinite multiplicity.

Definition

Let H be a bounded self-adjoint operator. We set:

$$\mathcal{H}_{\rm p} := \mathcal{H}_{\rm p}(H) := \overline{\{f \in \ker(\lambda - H),\ \lambda \in \sigma_{\rm p}(H)\}}$$

the spectral subspace associated to $\sigma_{\rm p}(H)$. We set also:

$$\mathcal{H}_{\rm c} := \mathcal{H}_{\rm c}(H) := \mathcal{H}_{\rm p}^{\perp}$$

the spectral subspace associated to continuous spectrum of H.

Theorem (RAGE)

Let H be self-adjoint in $\mathcal{H}$ and K be a compact operator in $\mathcal{H}$. Let $\phi_0 \in \mathcal{H}_c(H)$. We have:

$$\frac{1}{T}\int_0^T \|Ke^{itH}\phi_0\|^2\, dt \to 0, \quad \text{as } T \to \infty,$$

where $e^{itH}\phi$ is the unique solution to the Schrödinger equation:

$$\begin{cases} \mathrm{i}(\partial_t\phi)(t) & = (H\phi)(t) \\ \phi(0) & = \phi_0. \end{cases}$$

Remark

In the previous examples, by taking $K = 1_X(Q)$, where X is a finite set, we see that the if the initial condition is taken in the spectral subspaces associated to the continuous spectrum of H then it escapes, in average, every compact set.

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We refer to C. Rojas-Molina's course for a proof and a different presentation. We also mention that she uses this theorem to prove the spectrum is purely point almost surely in the setting of random Schrödinger operators acting on $\mathbb{Z}^d$.

Theorem (RAGE)

Let H be self-adjoint in $\mathcal{H}$ and K be a compact operator in $\mathcal{H}$. Let $\phi_0 \in \mathcal{H}_c(H)$. We have:

$$\frac{1}{T}\int_0^T \|Ke^{itH}\phi_0\|^2\, dt \to 0, \quad \text{as } T \to \infty,$$

where $e^{itH}\phi$ is the unique solution to the Schrödinger equation:

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The aim is to localise more precisely in the spectrum a vector. For instance, one would like to know around which energy a ϕ is taken in $\mathcal{H}_c$. We shall build the continuous functional calculus.

We take for instance $\mathcal{A}_\mathbb{Z}$. We have that

$$\mathcal{A}_\mathbb{Z} = \mathscr{F} 2\cos(Q) \mathscr{F}^{-1},$$

where $\mathscr{F}$ was a unitary transform.

Given $f \in \mathcal{C}(\sigma(\mathcal{A}_\mathbb{Z}))$, we can define the

$$f(\mathcal{A}_\mathbb{Z}) := \mathscr{F} f(2\cos(Q)) \mathscr{F}^{-1}$$

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For a general self-adjoint operator H, it is complicate to find such a unitary transformation so we will build directly $f(H)$ by first considering polynomials and then by proceeding by density.

We aim at defining the continuous functional calculus for bounded self-adjoint operator. We start with polynomials. We define the operator $P(T) \in \mathcal{L}(\mathcal{H})$ by:

$$P(T) := \sum_{k=0}^{n} a_k T^k, \text{ when } P(X) := \sum_{k=0}^{n} a_k X^k, \text{ with } n \in \mathbb{N} \text{ and } a_i \in \mathbb{C}.$$

Note that, given $P, Q \in \mathbb{C}[X]$ and $\lambda, \mu \in \mathbb{C}$, we have:

$$(\lambda P + \mu Q)(T) = \lambda(P(T)) + \mu(Q(T))$$
$$(PQ)(T) = P(T)Q(T) = Q(T)P(T).$$

Proposition (Spectral mapping)

Given $T \in \mathcal{L}(\mathcal{H})$ and $P \in \mathbb{C}[X]$, we have:

$$P(\sigma(T)) = \sigma(P(T))$$

Proof:

We proceed by contraposition. Let $\lambda \in \mathbb{C}$. We have λ root of $P(\lambda) - P$. There exists $Q \in \mathbb{C}[X]$ such that $P(\lambda) - P(X) = (\lambda - X)Q(X)$, then

$$P(\lambda)\mathrm{Id} - P(T) = (\lambda\mathrm{Id} - T)Q(T) = Q(T)(\lambda\mathrm{Id} - T).$$

If $P(\lambda) \notin \sigma(P(T))$, we set $S := (P(\lambda)\mathrm{Id} - P(T))^{-1}$. We get:

$$(\lambda\mathrm{Id} - T)Q(T)S = \mathrm{Id} = SQ(T)(\lambda\mathrm{Id} - T).$$

This implies that $\lambda\mathrm{Id} - T$ is invertible with inverse $Q(T)S = SQ(T)$. In particular $\lambda \notin \sigma(T)$.

We turn to the equality. It is enough to deal with $\deg P = n \geq 1$. Let $\mu \in \sigma(P(T))$ and $\lambda_1, \ldots, \lambda_n$ roots of $P - \mu$. We have:

$$P(X) - \mu = c(X - \lambda_1)\ldots(X - \lambda_n),$$

for some $c \neq 0$. This gives:

$$P(T) - \mu\mathrm{Id} = c(T - \lambda_1\mathrm{Id})\ldots(T - \lambda_n\mathrm{Id}).$$

Since $\mu \in \sigma(P(T))$, $P(T) - \mu\mathrm{Id}$ is not invertible, there exist $i_0 \in \{1, \ldots n\}$ such that $(T - \lambda_{i_0})$ is not invertible, then $\lambda_{i_0} \in \sigma(T)$. Moreover, $P(\lambda_{i_0}) = \mu$. □

Let $P \in \mathbb{C}[X]$ be given by $P = \sum_{k=0}^{n} a_k X^k$, we set:

$$\overline{P} := \sum_{k=0}^{n} \overline{a_k} X^k \quad \text{and} \quad |P|^2 := P\overline{P}.$$

We estimate the norm of $P(T) := \sum_{k=0}^{n} a_k T^k$.

Proposition

Soit $P \in \mathbb{C}[X]$. Alors $P(T)^* = \overline{P}(T)$ et

$$\|P(T)\| = \max_{t \in \sigma(T)} |P(t)|.$$

Note that we have a max because $\sigma(T)$ is compact and P is continuous.

Proof:

The fact that $P(T)^* = \overline{P}(T)$ follows from $T^* = T$. As seen above

$$\|P(T)\|^2 = \|P(T)P(T)^*\| = \|P(T)\overline{P}(T)\| = \||P|^2(T)\|.$$

Note then that $|P|^2(T)$ is self-adjoint because

$$\langle x, |P|^2(T)y\rangle = \langle x, P(T)\overline{P}(T)y\rangle = \langle \overline{P}(T)P(T)x, y\rangle = \langle |P|^2(T)x, y\rangle,$$

for all $x, y \in \mathcal{H}$. Moreover $|P|^2(T) \geq 0$ because

$$\langle x, |P|^2(T)x\rangle = \langle \overline{P}(T)x, \overline{P}(T)x\rangle \geq 0,$$

for all $x \in \mathcal{H}$. By the spectral radius and by spectral transfert, we see that

$$\|P(T)\|^2 = \||P|^2(T)\| = \max \sigma(|P|^2(T)) = \max_{t\in\sigma(T)} |P|^2(t) = \left(\max_{t\in\sigma(T)} |P(t)|\right)^2.$$

which gives the result. □

We recall the theorem of Stone-Weierstrass.

Theorem (Stone-Weierstrass)

Let K a Hausdorff compact space. Let $\mathcal{A}$ be a sub-algebra of $\mathcal{C}(K;\mathbb{C})$, endowed with the uniform norm, with the following properties:

1. *If $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A}$.*
2. *$\mathcal{A}$ separates points, i.e., for all $x \neq y$ in K, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.*
3. *the identity belongs to $\mathcal{A}$.*

Then $\overline{\mathcal{A}} = \mathcal{C}(K;\mathbb{C})$.

We deduce the main theorem.

Theorem (Continuous functional calculus)

Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. There exists a unique continuous morphism $\Phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ (of $*$-algebra) satisfying:

1) $\Phi(P) = P(T)$, *for all* $P \in \mathbb{C}[X]$,
2) $\Phi(f + \lambda g) = \Phi(f) + \lambda\Phi(g)$,
3) $\Phi(fg) = \Phi(f)\Phi(g)$,
4) $\Phi(\bar{f}) = (\Phi(f))^*$,

for all $f, g \in \mathcal{C}(\sigma(T))$ *and* $\lambda \in \mathbb{C}$. *Moreover,* Φ *is an isometry, i.e.,*

$$\|\Phi(f)\| = \max_{t \in \sigma(T)} |f(t)|, \text{ for all } f \in \mathcal{C}(\sigma(T)).$$

Remark

We denote $\Phi(f)$ *by* $f(T)$.

Proof:

We set

$$\Phi_0 : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \Phi_0(f) := f(T)$$

where

$$\mathcal{A} := \{P|_{\sigma(T)}, \text{ with } P \in \mathbb{C}[X]\},$$

endowed with the sup norm.

First note that if P and Q are two polynomials with the same restriction to $\sigma(T)$. Then,

$$\|P(t) - Q(T)\| = \|(P - Q)(T)\| = \max_{t\in\sigma(T)} |(P - Q)(t)| = 0.$$

This means that $P(T) = Q(T)$. Therefore Φ_0 is well-defined.

Notice that Φ_0 is an isometry. By Stone-Weierstrass'Theorem we see that $\mathcal{A}$ is dense in $\mathcal{C}(\sigma(T))$, for the sup norm. By density, there exists a unique linear map

$$\Phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi|_{\mathcal{A}} = \Phi_0$ and such that $\|\Phi\|_{\mathcal{L}(\mathcal{C}(\sigma(T)),\mathcal{H})} = \|\Phi_0\|_{\mathcal{L}(\mathcal{A},\mathcal{H})}$. Moreover, since Φ_0 satisfy 2, 3 et 4 and that is an isometry, by density Φ also satisfies the points. □

Remark

We stress that if $\lambda \in \rho(T)$, we obtain:

$$\|(\lambda \mathrm{Id} - T)^{-1}\| = \frac{1}{d(\lambda, \sigma(T))}.$$

This equality does not hold true in general for bounded operators.

Proposition (Spectral mapping)

Given $T \in \mathcal{L}(\mathcal{H})$ self-adjoint and $f \in \mathcal{C}(\sigma(T); \mathbb{C})$. Then,

$$\sigma(f(T)) = f(\sigma(T)).$$

Proof:

Let $\lambda \notin f(\sigma(T))$. We set $g(t) := (\lambda - f(t))^{-1}$. We have $g \in \mathcal{C}(\sigma(T))$. By functional calculus,

$$g(T)(\lambda\mathrm{Id} - f(T)) = (\lambda\mathrm{Id} - f(T))g(T) = \mathrm{Id}.$$

Then, $\lambda \notin \sigma(f(T))$, i.e, $\sigma(f(T)) \subset f(\sigma(T))$.

Set now $\lambda \in f(\sigma(T))$. For all $n \in \mathbb{N}$, we choose $g_n \in \mathcal{C}_c(\mathbb{R}; [0, 1])$ being 1 in λ and 0 away from $[\lambda - 1/n, \lambda + 1/n]$. By functional calculus,

$$\|(\lambda\mathrm{Id} - f(T))g_n(T)\| = \max_{t\in[\lambda-1/n,\lambda+1/n]\cap\sigma(T)} |(\lambda - f(t))g_n(t)| \to 0,$$

when $n \to \infty$.

Note also that $\|g_n(T)\| = 1$. Then, there exists a sequence x_n with norm 1 such that $\|g_n(T)x_n\| \geq 1/2$. We set

$$y_n := \frac{g_n(T)x_n}{\|g_n(T)x_n\|}.$$

We have $\|y_n\| = 1$ and

$$\|(\lambda\mathrm{Id} - f(T))y_n\| \leq 2\|(\lambda\mathrm{Id} - f(T))g_n(T)\| \cdot \|x_n\| \to 0.$$

In particular $\lambda \in \sigma(f(T))$. □

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint, $f \in \mathcal{C}(\sigma(T))$ and $g \in \mathcal{C}(f\sigma(T))$. Then,

$$g(f(T)) = (g \circ f)(T).$$

Recall that $f(\sigma(T)) = \sigma(f(T))$. Then $g(f(T))$ has a meaning by applying the functional calculus for $f(T)$.

Proof:

Set

$$\mathcal{A} := \{g \in \mathcal{C}(f\sigma(T)), g(f(T)) = (g \circ f)T\}.$$

Clearly $\mathcal{A}$ is an algebra and $\mathcal{A}$ contains the function 1. Moreover, the function g defined by $g(x) = x$ is in $\mathcal{A}$, because $g(f(T)) = f(T)$ and $g \circ f = f$. Besides, the functions separates points. Take now $g \in \mathcal{A}$. We have:

$$\overline{g}(f(T)) = (g(f(T)))^* = ((g \circ f)(T))^* = \overline{g} \circ f(T),$$

the $\mathcal{A}$ is stable by conjugaison. By Stone-Weirstrass, we get: $\overline{\mathcal{A}} = \mathcal{C}(f\sigma(T))$.
It remains to show that $\mathcal{A}$ is closed. Let $g_n \in \mathcal{C}(f\sigma(T))$ that tends to $g \in (f\sigma(T))$ for the sup norm. By functional calculus for $f(T)$, we see that $\|g(f(T)) - g_n(f(T))\| \to 0$, when $n \to \infty$. Then, by functional calculus for T, as $g_n \circ f$ tends uniformly to $g \circ f$, we have that $\|(g \circ f)(T)) - (g_n \circ f)(T)\| \to 0$, when $n \to \infty$. Then $g \in \mathcal{A}$ and $\mathcal{A}$ is closed. □

Exercise

Let H be a self-adjoint operator.

1. *Prove that*
$$\sigma(H) = \{\lambda \in \mathbb{R}, \varphi(H) \neq 0, \text{ for all } \varphi \in \mathcal{C}(\mathbb{R};\mathbb{C}) \text{ with } \varphi(\lambda) \neq 0\}$$
2. *Prove that*
$$e^{\mathrm{i}tH} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}tH)^n}{n!},$$
where the left hand side is given by functional calculus.
3. *Prove that $e^{\mathrm{i}tH}$ is unitary.*

Exercise

Let $H \in \mathcal{L}(\mathcal{H})$ such that $\langle f, Hf \rangle \geq 0$, for all $f \in \mathcal{H}$.

1. *Prove that H is self-adjoint. (Hint: Use the polarisation identity)*
2. *Prove that $\sigma(H) \subset [0, \infty[$.*
3. *Prove that there is (a unique) T self-adjoint with $\sigma(T) \subset [0, \infty[$, such that $T^2 = H$. It is the square root of H.*

We give now more or less explicit ways to deal with the functional calculus of H self-adjoint.

The Fourier approach Let $f \in L^1(\mathbb{R}; \mathbb{C})$. Set

$$\hat{f}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f(t) e^{-\mathrm{i}t\xi} \, dt,$$

Assume that $\hat{f} \in L^1(\mathbb{R}; \mathbb{C})$. Then we have:

$$f(H) = \int_{\mathbb{R}} \hat{f}(\xi) e^{\mathrm{i}\xi H} \, d\xi,$$

where the integral exists in $\mathcal{L}(\mathcal{H})$.

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Where do we use that $\hat{f} \in L^1(\mathbb{R}; \mathbb{C})$? Prove the equality.

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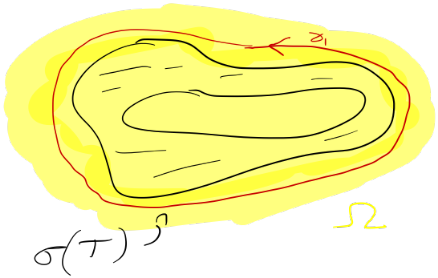
The Holomorphic approach Let f be holomorphic in an open neighbourhood Ω of $\sigma(H)$, where is H is bounded

$$f(H) = \int_\Gamma f(z)(H-z)^{-1}\, dz,$$

where the integral exists in $\mathcal{L}(\mathcal{H})$ and Γ is a contour with indice 1 that circumvents $\sigma(T)$.



Helffer-Sjöstrand's formula

For $\rho \in \mathbb{R}$, let $\mathcal{S}^\rho$ be the class of function $\varphi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ such that

$$\forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t\in\mathbb{R}} \langle t\rangle^{-\rho+k} |\partial_t^k \varphi(t)| < \infty. \tag{1}$$

We also write $\varphi^{(k)}$ for $\partial_t^k \varphi$. Equiped with the semi-norms defined by (1), $\mathcal{S}^\rho$ is a Fréchet space. Leibniz' formula implies the continuous embedding:

$$\mathcal{S}^\rho \cdot \mathcal{S}^{\rho'} \subset \mathcal{S}^{\rho+\rho'}.$$

Lemma

Let $\varphi \in \mathcal{S}^\rho$ with $\rho \in \mathbb{R}$. For all $l \in \mathbb{N}$, there is a smooth function $\varphi^\mathbb{C} : \mathbb{C} \to \mathbb{C}$, call an almost analytic extension of φ, such that:

$$\varphi^\mathbb{C}|_\mathbb{R} = \varphi, \qquad \left|\frac{\partial \varphi^\mathbb{C}}{\partial \overline{z}}(z)\right| \leq c_1 \langle \Re(z)\rangle^{\rho-1-l} |\mathrm{Im}(z)|^l$$

$$\mathrm{supp}\varphi^\mathbb{C} \subset \{x + \mathrm{i}y \mid |y| \leq c_2\langle x\rangle\},$$

$$\varphi^\mathbb{C}(x+\mathrm{i}y) = 0, \ \text{if } x \notin \mathrm{supp}\varphi.$$

for constants c_1, c_2 depending on the semi-norms (1) of φ in $\mathcal{S}^\rho$.

Let $\rho < 0$ and $\varphi \in \mathcal{S}^\rho$. The bounded operator $\varphi(A)$ can be recover by Helffer-Sjöstrand's formula:

$$\varphi(A) = \frac{\mathrm{i}}{2\pi} \int_\mathbb{C} \frac{\partial \varphi^\mathbb{C}}{\partial \overline{z}} (z - A)^{-1} dz \wedge d\overline{z},$$

where the integral exists in the norm topology.

Exercise

Using $\|(z-A)^{-1}\| \leq 1/|\mathrm{Im}(z)|$, show that the integral converges in norm.

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Definition

Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. We set

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{R}, \lambda \text{ is an isolated eigenvalue of finite multiplicity}\},$$
$$\sigma_{\mathrm{ess}}(T) := \sigma(T) \setminus \sigma_{\mathrm{d}}(T).$$

These spectra are called discret and essential, respectively.

Proposition

Let T be self-adjoint in $\mathcal{H}$ of infinite dimension, then $\sigma_{\mathrm{ess}}(T) \neq \emptyset$.

Proof:

Suppose that the spectrum is purely discret. Since it is contained in a compact there is a sub-sequence of eigenvalues that converges to a point of the spectrum. The later is not isolated. Contradiction. □

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We link this notion to the functional calculus.

Proposition

Let T be a self-adjoint operator acting in $\mathcal{H}$ and $\lambda \in \sigma(T)$ isolated.

1) $\lambda \in \sigma_{\mathrm{p}}(T)$.
2) *Given $\varphi \in \mathcal{C}(\sigma(T))$ defined by* 1 *on* λ *and* 0 *elsewhere, we have that $\varphi(T)$ is an orthogonal projection with range* $\ker(\lambda \mathrm{Id} - T)$.

Proof:

First since $\varphi(\lambda) = 1$, $\varphi(T)$ is a projection. Indeed,

$$\|\varphi^2(T) - \varphi(T)\| = \sup_{t\in\sigma(T)} |\varphi^2(t) - \varphi(t)| = |\varphi^2(\lambda) - \varphi(\lambda)| = 0.$$

Moreover, the projection is orthogonal because φ is with real values and therefore $\varphi(T)^* = \overline{\varphi}(T) = \varphi(T)$. Then we show that $\mathrm{Im}\varphi(T) \subset \ker(\lambda\mathrm{Id} - T)$. We have:

$$\|(\lambda\mathrm{Id} - T)\varphi(T)\| = \sup_{t\in\sigma(T)} |(\lambda - t)\varphi(t)| = 0.$$

Take now $x \in \ker(\lambda\mathrm{Id} - T)$. We have:

$$\begin{aligned}(\mathrm{Id} - \varphi(T))x &= \Phi\big(\underbrace{(1 - \varphi(\cdot))(\lambda - \cdot)^{-1}}_{\in\mathcal{C}(\sigma(T))}(\lambda - \cdot)\big)x \\ &= \Phi\big((1 - \varphi(\cdot))(\lambda - \cdot)^{-1}\big)(\lambda\mathrm{Id} - T)x = 0\end{aligned}$$

Then $\mathrm{Im}\varphi(T) = \ker(\lambda\mathrm{Id} - T)$. Finally since $\varphi(T) \neq 0$ by functional calculus and then $\lambda \in \sigma_{\mathrm{p}}(T)$. □

Proposition

Let T be self-adjoint in $\mathcal{H}$ and $\lambda \in \sigma(T)$. Then,

1) $\lambda \in \sigma_{\mathrm{d}}(T)$, *if and only if there exists* $\varepsilon > 0$ *and* $\varphi \in \mathcal{C}(\sigma(T); \mathbb{R})$ *such that* $\mathrm{supp}(\varphi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$ *with* $\varphi(\lambda) = 1$ *and such that* $\varphi(T)$ *is compact.*
2) $\lambda \in \sigma_{\mathrm{ess}}(T)$, *if and only if for all* $\varepsilon > 0$ *and for all* $\varphi \in \mathcal{C}(\sigma(T); \mathbb{R})$ *such that* $\mathrm{supp}(\varphi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$ *with* $\varphi(\lambda) = 1$*, we have that* $\varphi(T)$ *is non-compact.*

Note that in both cases that, since $\lambda \in \sigma(T)$ and that $\varphi(\lambda) = 1$, functional calculus ensures that $\varphi(T) \neq 0$.

Proof:

Note that 1) and 2) are equivalent (by taking the negation).

We suppose that there exist $\varepsilon > 0$ and $\varphi \in \mathcal{C}(\sigma(T);\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subset [\lambda-\varepsilon, \lambda+\varepsilon]$ with $\varphi(\lambda) = 1$ and such that $\varphi(T)$ is compact.

Suppose that λ is not isolated. There exist a sequence $\lambda_n \in \sigma(T)$ (note that λ could belong to an interval) such that $\lambda_n \to \lambda$. By spectral mapping, the spectrum of $\varphi(T)$ is contained in $\varphi(\lambda_n)$ and $1 = \varphi(\lambda)$. By continuity we have $\varphi(\lambda_n) \to 1$. This is a contradiction with the fact that $\varphi(T)$ is compact (because 0 is the only possible accumulation point). Contradiction.

We have that λ is isolated. Let $\varphi_0 \in \mathcal{C}(\sigma(T))$ with $\varphi_0(\lambda) = 1$ and 0 elsewhere. We have:

$$\|\varphi_0(H) - \varphi_0(H)\varphi(H)\| = \max_{t\in\sigma(T)} |\varphi_0(t) - \varphi_0(t)\varphi(t)| = |\varphi_0(\lambda) - \varphi_0(\lambda)\varphi(\lambda)| = 0.$$

Then $\varphi_0(T) = \varphi_0(T)\varphi(T)$ is compact, because it is a product of a compact operator and a bounded operator. By the previous proposition $\varphi_0(T)$ is a orthogonal projection with image $\ker(\lambda\mathrm{Id} - T)$. Since it is compact we deduce that it is finite (Riesz theorem). In particular $\lambda \in \sigma_{\mathrm{d}}(T)$. □

Theorem (Weyl)

Let T and V be two self-adjoint operators on $\mathcal{H}$. If $V \in \mathcal{K}(\mathcal{H})$, i.e., compact, then

$$\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T + V).$$

Proof:

We set

$$\mathcal{A} := \{\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T+V)), \varphi(T) - \varphi(T+V) \in \mathcal{K}(\mathcal{H})\}$$

First $\mathcal{A}$ is an algebra. 1 is in $\mathcal{A}$ because $\mathrm{Id} - \mathrm{Id} = 0$ is compact. Then by taking $\varphi(t) = t$, we see that $\varphi(T) - \varphi(T+V) = -V \in \mathcal{K}(\mathcal{H})$. This function separates points. Suppose now that $\varphi \in \mathcal{A}$, we have:

$$\overline{\varphi}(T) - \overline{\varphi}(T+V) = (\varphi(T))^* - (\varphi(T+V))^* = (\varphi(T) - \varphi(T+V))^* \in \mathcal{K}(\mathcal{H}).$$

Because the adjoint of a compact operator is compact. By Stone-Weirstrass we deduce that $\overline{\mathcal{A}} = \mathcal{C}(\sigma(T) \cup \sigma(T+V))$. It remains to show that $\mathcal{A}$ is closed. Let $\varphi_n \in \mathcal{A}$ that tends to $\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T+V))$ for the uniform norm. We have $\|\varphi_n(T) - \varphi(T)\| \to 0$ and $\|\varphi_n(T+V) - \varphi(T+V)\| \to 0$ when $n \to \infty$. In particular,

$$\varphi_n(T) - \varphi_n(T+V) \to \varphi(T) - \varphi(T+V),$$

in norm then $\varphi(T) - \varphi(T+V) \in \mathcal{K}(\mathcal{H})$, because $\mathcal{K}(\mathcal{H})$ is closed.

Finally since $\varphi(T) - \varphi(T+V)$ is compact for all $\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T+V))$ the previous proposition gives $\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(T+V)$. □

We turn to a characterisation of the essential spectrum.

Proposition (Weyl's criterion)

Let T be self-adjoint on $\mathcal{H}$. Then $\lambda \in \sigma_{\mathrm{ess}}(T)$ if and only if there exist $f_n \in \mathcal{H}$ such that :

$$\|f_n\| = 1, \quad f_n \rightharpoonup 0 \quad et \quad \|(\lambda \mathrm{Id} - T)f_n\| \to 0,$$

when $n \to \infty$ and where $\rightharpoonup$ denotes the weak convergence.

Proof:

Let $\lambda \in \sigma_{\mathrm{ess}}(T)$. Suppose first that λ is isolated. We have that λ is an eigenvalue of infinite multiplicity. Take $(f_n)_n$ to be an orthonormal basis of $\ker(\lambda \mathrm{Id} - T)$.

Suppose now that λ is not isolated. There exist $\lambda_n \in \sigma(T)$, two by two distinct, such that $\lambda_n \to \lambda$, when $n \to \infty$. Up to a sub-sequence or considering $-T$, we can suppose that λ_n is strictly increasing. We then construct $\varphi_n \in \mathcal{C}(\sigma(T); [0,1])$ such that $\varphi_n(\lambda_n) = 1$ and such that $\mathrm{supp}\varphi_n \subset [(2\lambda_n + \lambda_{n-1})/3, (2\lambda_n + \lambda_{n+1})/3]$. In particular, φ_n has support two by two disjoint and $\|\varphi_n(T)\| = 1$. Take now $x_n \in \mathcal{H}$ such that $\|\varphi_n(T)x_n\| \geq 1/2$. We have

$$f_n := \frac{\varphi_n(T)x_n}{\|\varphi_n(T)x_n\|}$$

which is of norm 1. We see that f_n tends weakly to 0 because for $n \neq m$

$$\langle f_n, f_m \rangle = \langle \frac{x_n}{\|\varphi_n(T)x_n\|}, \frac{\overbrace{\varphi_n(T)\varphi_m(T)}^{=0} x_m}{\|\varphi_m(T)x_m\|} \rangle = 0,$$

due to the support of φ_n and by functional calculus. Finally we have:

$$\|(\lambda \mathrm{Id} - T)f_n\| \leq 2\|(\lambda \mathrm{Id} - T)g_n(T)\| \cdot \|x_n\| \to 0,$$

by functional calculus. □

In all the examples that we have considered earlier we have, by denoting by H the operator considered and by taking $V(Q)$ with $\lim_{x\to\infty} V(x) = 0$, we have that

$$V(Q) \text{ is a compact operator}$$

and therefore

$$\sigma_{\mathrm{ess}}(H) = \sigma(H) = \sigma_{\mathrm{ess}}(H + V).$$

Exercise

Let $\mathcal{H} := \ell^2(\mathbb{Z};\mathbb{C})$. *Let* $V : \mathbb{Z} \to \mathbb{R}$ *such that* $c^{\pm} := \lim_{n\to\pm\infty} V(n)$ *exists and is finite. Using that* $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$, *prove that*

$$\begin{aligned}\sigma_{\mathrm{ess}}(\mathcal{A}_{\mathbb{Z}} + V(Q)) &= [-2 + c^{-}, 2 + c^{-}] \cup [-2 + c^{+}, 2 + c^{+}] \\ &= [-2, 2] + \{c_{-}, c_{+}\}.\end{aligned}$$

Exercise

Same exercice but use the Weyl's criterion.

Theorem

Let $\mathcal{G} := (\mathcal{E}, \mathcal{V})$ *be a binary tree. Let* $\mathcal{H} := \ell^2(\mathcal{V}; \mathbb{C})$. *Let* $\hat{\mathcal{V}} := \mathcal{V} \cup \partial\mathcal{V}$ *be the hyperbolic compactification of* $\mathcal{V}$. *Suppose that* $V : \mathcal{V} \to \mathbb{R}$ *is bounded and extends continuously to* $\hat{\mathcal{V}}$.

Then we have:

$$\sigma_{\mathrm{ess}}(\mathcal{A} + V(Q)) = \left[-2\sqrt{2}, 2\sqrt{2}\right] + V(\partial\mathcal{V}).$$

The aim now is to define the spectral measure of an operator. We would like to be able to define $1_X(H)$, where X is a Borelian set.

In a second step we will relate some properties of the measure to the dynamical behaviour of the Schrödinger equation.

Let $H \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Let $f \in \mathcal{H} \setminus \{0\}$. By functional calculus, we have that

$$\Phi : \mathcal{C}(\sigma(T); C) \to C, \quad \text{given by} \quad \Phi(\varphi) := \langle f, \varphi(H) f \rangle$$

is continuous and positive (if $\varphi \geq 0$ then $\Phi(\varphi) \geq 0$).

Therefore by Riesz-Markov's Theorem there is a unique measure m_f such that

$$\langle f, \varphi(H) f \rangle = \int_{\sigma(H)} \varphi(t)\, dm_f(t).$$

Definition

The measure m_f is called the spectral measure of H associated to f.

Remark

If $\|f\| = 1$, note that m_f is a probability measure.

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Definition

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Remark

If $\|f\| = 1$, note that m_f is a probability measure.

Given $\varphi \in \mathcal{B}(\sigma(H)) = \mathcal{B}(\sigma(H); \mathbb{C})$, i.e, a borelian bounded function, we set:

$$\langle f, \varphi(H) f \rangle := \int_{\sigma(H)} \varphi(t)\, dm_f(t).$$

We now explain why $\varphi(H)$ is a well-defined bounded operator (why does $\varphi(H)$ is linear? Does it depend on the choice of f?).

Given $\varphi \in \mathcal{C}(\sigma(H))$. For $f \in \mathcal{H}$, we set

$$B_\varphi(f, f) := \langle f, \varphi(H) f \rangle = \int_{\sigma(H)} \varphi(t)\, dm_f(t)$$

and stress that m_f is a bounded measure. Indeed,

$$m_f(\sigma(H)) = \int_{\sigma(H)} 1\, dm_f(t) = \langle f, 1(H) f \rangle = \|f\|^2,$$

because $1(H) = \mathrm{Id}$. (recall the starting point with polynomials).

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because $1(H) = \mathrm{Id}$. (recall the starting point with polynomials).

We also set

$$B_\varphi(f,g) := \langle f, \varphi(H)g\rangle$$

Recallying the polarisation formula

$$B(f,g) = \frac{1}{4}\sum_{k=0}^{3} i^k B(i^k f + g, i^k f + g).$$

We see that there is a complex measure $m_{f,g}$ such that:

$$B_\varphi(f,g) = \int_{\sigma(H)} \varphi(t)\, dm_{f,g}(t), \text{ where } m_{f,g} := \frac{1}{4}\sum_{k=0}^{3} i^k m_{i^k f+g}.$$

Notice that:

$$m_{\lambda f+g,h} = \overline{\lambda} m_{f,h} + m_{g,h} \quad \text{and} \quad m_{h,\lambda f+g} = \lambda m_{h,f} + m_{h,g}.$$

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We now take $\varphi \in \mathcal{B}(\sigma(H))$. We extend the definition of B_φ in the following way:

$$B_\varphi(f,g) := \int_{\sigma(H)} \varphi(t)\, dm_{f,g}(t).$$

By the property of the measure we see that:

B_φ is a sesquilinear form.

We now prove that it is continuous. First we note that:

$$|B_\varphi(f,f)| \le \|\varphi\|_\infty \int_{\sigma(H)} 1\, dm_f(t) = \|\varphi\|_\infty \langle f, 1(H) f\rangle = \|\varphi\|_\infty \|f\|^2.$$

We aim at showing:

$$|B_\varphi(f,g)| \le \|\varphi\|_\infty \|f\| \cdot \|g\|, \quad \text{for all } f, g \in \mathcal{H}.$$

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Set:

$$\alpha := \sup_{\|f\|=1} |B_\varphi(f,f)|.$$

It is enough to show that $|B_\varphi(f,g)| \leq \alpha$ for all f et g such that $\|f\| = \|g\| = 1$. If $B_\varphi(f,g) = 0$ there is nothing to do. We set

$$\lambda := \frac{\overline{B_\varphi(f,g)}}{|B_\varphi(f,g)|}.$$

Note that $|\lambda| = 1$. By polarisation, we have:

$$|B_\varphi(f,g)| = B_\varphi(f,\lambda g) = \Re B_\varphi(f,\lambda g) = \Re\left(\frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \underbrace{B_\varphi(i^k f + \lambda g, i^k f + \lambda g)}_{\in\mathbb{R}}\right)$$

$$= \frac{1}{4}\left(B_\varphi(f+\lambda g, f+\lambda g) - B_\varphi(-f+\lambda g, -f+\lambda g)\rangle\right)$$

$$\leq \frac{\alpha}{4}\left(\|f+\lambda g\|^2 + \|-f+\lambda g\|^2\right) \leq \alpha,$$

where we used in the last line that $\|x\| = \|y\| = |\lambda| = 1$.

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where we used in the last line that $\|x\| = \|y\| = |\lambda| = 1$.

We turn to the existence of $\varphi(H)$.

Note that $f \mapsto B_\varphi(f,g)$ is a continuous anti-linear from. Therefore there exists $T(g)$ such that

$$B_\varphi(f,g) = \langle f, T(g)\rangle, \quad \text{for all } f \in \mathscr{H}.$$

It is easy to see that $T(g_1 + \lambda g_2) = T(g_1) + \lambda T(g_2)$.

Moreover, by Riesz'isomorphism, we get:

$$\|Tg\| = \|f \mapsto B_\varphi(f,g)\| \leq \|\varphi\|_\infty \|g\|.$$

Therefore T is a linear bounded operator. We denote it by $\varphi(H)$. □

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The hardest part is done, with few more efforts one can show:

Theorem

Let H be self-adjoint operator acting on $\mathcal{H}$. There is a unique map $\hat{\Phi} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ such that:

1. $\hat{\Phi}(\varphi + \lambda\psi) = \hat{\Phi}(\varphi) + \lambda\hat{\Phi}(\psi)$,
2. $(\hat{\Phi}(\varphi))^* = \hat{\Phi}(\overline{\varphi})$,
3. $\hat{\Phi}(\varphi \times \psi) = \hat{\Phi}(\varphi)\hat{\Phi}(\psi)$,
4. $\hat{\Phi}(x) = H$,
5. *If $\phi_n(x) \to \phi(x)$ for all $x \in \sigma(H)$ and if $\sup_n \|\phi_n\|_\infty < \infty$ then for all $f \in \mathcal{H}$, $\Phi(\phi_n)f \to \Phi(\phi)f$, as $n \to \infty$.*

Moreover we have:

1. $\|\hat{\Phi}(H)\| \leq \|\varphi\|_\infty$
2. *If $Hf = \lambda f$, then $\Phi(\varphi)f = \varphi(\lambda)f$,*
3. *If $\varphi \geq 0$ then $\sigma(\Phi(\varphi)) \geq 0$.*

Remark

As before we denote $\Phi(\varphi)$ by $\varphi(H)$.

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As before we denote $\Phi(\varphi)$ by $\varphi(H)$.

Remark

Given a borel set $\mathcal{I} \subset \sigma(H)$, *we have that* $E_{\mathcal{I}}(H) := 1_{\mathcal{I}}(H)$ *is an orthogonal projector. Moreover,*

$$\langle f, E_{\mathcal{I}}(H)f\rangle = \int_{\mathcal{I}} dm_f(t) = m_f(\mathcal{I}).$$

and

$$\langle f, E_{\mathcal{I}}(H)g\rangle = \int_{\mathcal{I}} dm_{f,g}(t) = m_{f,g}(\mathcal{I}).$$

Therefore $\mathcal{I} \to E_{\mathcal{I}}(H)$ *is a measure with projector values in* $\mathcal{L}(\mathcal{H})$.

Using for instance the Bochner integral, we can prove that for $\varphi \in \mathcal{B}(\sigma(H))$

$$\varphi(H) = \int_{\sigma(H)} \varphi(t) dE_t(H).$$

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Exercise

Let H be a self-adjoint operator. Recalling that

$$\sigma(H) = \{\lambda \in \mathbb{R}, \varphi(H) \neq 0, \text{ for all } \varphi \in \mathcal{C}(\mathbb{R}; \mathbb{C}) \text{ with } \varphi(\lambda) \neq 0\},$$

Prove that

$$\sigma(H) = \{\lambda \in \mathbb{R}, E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H) \neq 0, \text{ for all } \varepsilon > 0\}.$$

There is a link between the spectrum and nature of the spectral measure.

Exercise

$E\{\lambda\}(H) \neq 0$ if and only if λ is an eigenvalue of H. Moreover $E_{\{\lambda\}}(H)$ is an orthogonal projector with image $\ker(\lambda - H)$.

Definition

Let μ be a borel sigma-finite measure on $\mathbb{R}$.

1. *We say that $x \in \mathbb{R}$ is an atom for μ if $\mu(\{x\}) > 0$.*
2. *We say that μ is continuous if μ has no atom.*
3. *We say that μ is supported by borel set Σ if $\mu(\mathbb{R} \setminus \Sigma) = 0$.*
4. *We say that μ is absolutely continuous with respect to the Lebesgue measure if $\mu(\mathcal{I}) = 0$ when $\mathrm{Leb}(\mathcal{I}) = 0$. We denote it by $\mu \ll \mathrm{Leb}$.*
5. *We say that μ is singular with respect to the measure ν when there exists a borel set Σ such that $\mu(\mathbb{R} \setminus \Sigma) = 0$ and $\nu(\Sigma) = 0$. We denote it by $\mu \perp \nu$.*

Theorem (Radon-Nykodim)

Let μ be a borel sigma-finite measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then there exists $f \in L^1_{\mathrm{loc}}(\mathbb{R}, dx)$ such that

$$\mu(A) = \int_A f(x)\, dx,$$

for all A borel sets.

We now turn to the decomposition of the spectral measure.

Theorem (Lebesgue decomposition)

Given μ be a borel sigma-finite measure on $\mathbb{R}$. There are measures μ^{p} and μ^{c} which are purely atomic and continuous, respectively, such that:

$$\mu = \mu^{\mathrm{p}} + \mu^{\mathrm{c}}.$$

We have $\mu^{\mathrm{p}} \perp \mu^{\mathrm{c}}$.

Moreover, there are measures μ^{ac} and μ^{sc}, which are continuous with respect to the Lebesgue measure and singular with respect to it, respectively, such that:

$$\mu^{\mathrm{c}} = \mu^{\mathrm{ac}} + \mu^{\mathrm{sc}}.$$

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We have $\mu^{\mathrm{ac}} \perp \mu^{\mathrm{sc}}$.

Given $f \in \mathcal{H}$ and H self-adjoint, we have:

$$\begin{aligned}
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Using the separability of the space and cyclic vectors, we infer:

Theorem

Let H be self-adjoint in $\mathcal{H}$, there are closed (Hilbert) subspaces $\mathcal{H}^{\mathrm{p}}$, $\mathcal{H}^{\mathrm{ac}}$, and $\mathcal{H}^{\mathrm{sc}}$ such that

$$\mathcal{H} = \mathcal{H}^{\mathrm{p}} \oplus \underbrace{\mathcal{H}^{\mathrm{ac}} \oplus \mathcal{H}^{\mathrm{sc}}}_{\mathcal{H}^{\mathrm{c}}}$$

and, denoting by m_f the spectral measure of H associated to f,

1. *if $f \in \mathcal{H}^{\mathrm{p}}$ then m_f is atomic,*
2. *if $f \in \mathcal{H}^{\mathrm{ac}}$ then m_f is absolutely continuous with respect to the Lebesgue measure,*
3. *if $f \in \mathcal{H}^{\mathrm{sc}}$ then m_f is singularly continuous with respect to the Lebesgue measure.*

We denote by P^{p}, P^{ac}, and P^{sc} the respective projection.

Moreover, $\varphi(H)\mathcal{H}^X \subset \mathcal{H}^X$, for $X \in \{\mathrm{p}, \mathrm{ac}, \mathrm{sc}\}$ and $\varphi \in \mathcal{B}(\mathbb{R})$.

Remark

Note that

$$P^{\mathrm{p}} = E_{\sigma_{\mathrm{p}}}(H).$$

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We now decompose the spectrum. Set $X \in \{\mathrm{p}, \mathrm{ac}, \mathrm{sc}\}$ and let

$$\sigma^X(H) := \sigma^X(H|_{\mathcal{H}^X}).$$

We have:

$$\sigma(H) = \sigma^{\mathrm{p}}(H) \cup \sigma^{\mathrm{ac}}(H) \cup \sigma^{\mathrm{sc}}(H).$$

Be careful: We do not have in general that the different spectra are two by two disjoint. We could have mixed spectrum. For instance, by taking a direct sum, it is easy to construct an example such that

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Proposition

Given $f \in \mathcal{H}^{\mathrm{ac}}$. Let K be a compact operator. Then

$$Ke^{-\mathrm{i}tH} f \to 0, \qquad \text{as } t \to \infty.$$

Remark

Recall that given $f \in \mathcal{H}^{\mathrm{sc}} \subset \mathcal{H}^{\mathrm{c}}$ and K a compact operator, the RAGE's theorem ensures a priori solely:

$$\frac{1}{T}\int_0^T \|Ke^{-\mathrm{i}tH} f\|^2 \, dt \to 0, \qquad \text{as } T \to \infty.$$

Remark

Take $K = 1_X(Q)$, where X is a finite set in the examples of graphs, by denoting by H the studied operator, we see that for $f \in \mathcal{H}^{\mathrm{ac}}$ we have

$$1_X(Q)e^{-\mathrm{i}tH} f \to 0, \qquad \text{as } t \to \infty.$$

The particle escapes to infinity.

Proposition

Given $f \in \mathcal{H}^{\mathrm{ac}}$. Let K be a compact operator. Then

$$Ke^{-\mathrm{i}tH} f \to 0, \qquad \text{as } t \to \infty.$$

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Proof:

Let $f \in \mathcal{H}^{\mathrm{ac}}$ and let $g \in \mathcal{H}$. We denote by

$$m_{g,f}(\mathcal{I}) := \langle g, E_{\mathcal{I}}(H)f \rangle.$$

This measure is purely absolutely continuous with respect to the Lebesgue measure because, for $\mathcal{I}$ such that $\mathrm{Leb}(\mathcal{I}) = 0$, we have:

$$|m_{g,f}(\mathcal{I})| = |\langle g, E_{\mathcal{I}}(H)f \rangle| \leq \|g\|^2 \cdot \|E_{\mathcal{I}}(H)f\|^2 = 0.$$

By the Riemann-Lebesgue's Theorem, we have that

$$t \mapsto \widehat{m_{g,f}}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}xt}\, dm_{g,f}(x) \in \mathcal{C}_0(\mathbb{R}),$$

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Therefore for $\{g_j\}_{j=1,\dots,N} \subset \mathcal{H}$, we get: $\langle \sum_j g_j, e^{-\mathrm{i}tH} f \rangle \to 0$, as $t \to 0$. By density of the finite rank operator in the set of compact operator, for $K \in \mathcal{K}(\mathcal{H})$, we obtain:

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We now decompose the spectrum of $\mathcal{A}_{\mathbb{Z}}$ which acts on $\mathcal{H} := \ell^2(\mathbb{Z})$.

First note that the different spectra are stable by unitary equivalence. We recall that $\mathcal{A}_{\mathbb{Z}}$ is unitarily equivalent to

$$\varphi(Q) \text{ in } L^2(-\pi, \pi),$$

where $\varphi(x) := 2\cos(x)$. Note that $\varphi(Q)L^2(0,\pi) \subset L^2(0,\pi)$ and $\varphi(Q)L^2(-\pi,0) \subset L^2(-\pi,0)$.

Take f in $L^2(0,\pi)$. Set $\mathcal{I} \subset (0,\pi)$ such that $\mathrm{Leb}(\mathcal{I}) = 0$.

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Do the same with f in $L^2(-\pi, 0)$. Therefore, that the spectrum of $\mathcal{A}_{\mathbb{Z}}$ is purely absolutely continuous with respect to the Lebesgue measure.

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Exercise

Let $\varphi \in \mathcal{C}^1([-\pi, \pi]; \mathbb{R})$ *such that* $\varphi'(x) = 0$ *if and only if* $x \in [-1, 1]$. *Let* $H := \varphi(Q)$ *in* $L^2([-\pi, \pi])$. *Show that:*

$$\sigma^{\mathrm{p}}(H) = \{\varphi(0)\}, \quad \sigma^{\mathrm{ac}}(H) = \varphi([-\pi, \pi]), \text{ and } \sigma^{\mathrm{sc}}(H) = \emptyset.$$

Set $f : [0,1] \to [0,1]$ given by:

$$f(x) := \begin{cases} 3x, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ 0, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 3x - 2, & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

For $n \in \mathbb{N}^*$, set $E_{n+1} := f^{-1}(E_n)$, where $E_0 := [0, 1]$. This gives

$$E_1 = [0, 1/3] \cup [2/3, 1],$$
$$E_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9]$$

and so on. We have

$$C := \cap_{n \in \mathbb{N}} E_n.$$

This is the triadic Cantor set. Note that C is compact, $C \neq \emptyset$, and $\mathrm{Leb}(C) = 0$.

Let $\alpha \in \mathcal{C}([0,1])$ be constructed as follows.

$$\alpha(x) := \begin{cases} \frac{1}{2}, & \text{for } x \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ \frac{1}{4}, & \text{for } x \in \left(\frac{1}{9}, \frac{2}{9}\right), \\ \frac{3}{4}, & \text{for } x \in \left(\frac{7}{9}, \frac{8}{9}\right), \\ etc... \end{cases}$$

and extended by continuity on $[0,1]$.

The function α is strictly increasing and its derivative is 0 almost everywhere. The Cantor measure is defined by prescribing

$$\mu_C(a,b) := \alpha(b) - \alpha(a).$$

and extending it to the Borel sets. We have that $\mu_C(C) = 1$ and that $\mathrm{Leb}(C) = 0$. Note also that $\mu_C(x) = 0$, for all $x \in C$. Therefore μ_C is singular continuous with respect to the Lebesgue measure.

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Using the spectral theorem, we establish Stone's formula:

$$\frac{1}{2}\langle f, (E_{[a,b]}(H) + E_{(a,b)}(H))f\rangle = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_a^b \mathrm{Im}(\langle f, (H-\lambda-\mathrm{i}\varepsilon)^{-1} f\rangle)\, dx.$$

where $f \in \mathcal{H}$.

Proposition

Let H be self-adjoint in Hilbert space $\mathcal{H}$. Set $a < b$. Suppose that there is $f \in \mathcal{H}$ such that

$$c(f) := \sup_{\varepsilon\in(0,1)} \sup_{\lambda\in(a,b)} |\mathrm{Im}(\langle f, (H-\lambda-\mathrm{i}\varepsilon)^{-1} f\rangle)| < \infty$$

Then $E_{(a,b)}(H)f \in \mathcal{H}^{\mathrm{ac}}$.

Assume that $\{f, c(f) < \infty\}$ is dense in $\mathcal{H}$, then:

$$\sigma(H)|_{(a,b)} = \sigma^{\mathrm{ac}}(H)|_{(a,b)}, \quad \sigma^{\mathrm{p}}(H)|_{(a,b)} = \sigma^{\mathrm{sc}}(H)|_{(a,b)} = \emptyset.$$

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$$c(f) := \sup_{\varepsilon\in(0,1)} \sup_{\lambda\in(a,b)} |\mathrm{Im}(\langle f, (H-\lambda-\mathrm{i}\varepsilon)^{-1} f\rangle)| < \infty$$

Then $E_{(a,b)}(H)f \in \mathcal{H}^{\mathrm{ac}}$.

Assume that $\{f, c(f) < \infty\}$ is dense in $\mathcal{H}$, then:

$$\sigma(H)|_{(a,b)} = \sigma^{\mathrm{ac}}(H)|_{(a,b)}, \quad \sigma^{\mathrm{p}}(H)|_{(a,b)} = \sigma^{\mathrm{sc}}(H)|_{(a,b)} = \emptyset.$$

Using the spectral theorem, we establish Stone's formula:

$$\frac{1}{2}\langle f, (E_{[a,b]}(H) + E_{(a,b)}(H))f\rangle = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_a^b \mathrm{Im}(\langle f, (H-\lambda-\mathrm{i}\varepsilon)^{-1} f\rangle)\, dx.$$

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Proof:

Set $f \in \mathcal{H}$. By Stone's formula and the fact that given a set J, $\|E_J(H)f\| \leq \|E_{\overline{J}}(H)f\|$, we have for $c < d$

$$0 \leq \langle f, E_{(c,d)}(H))f\rangle \leq \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_c^d \mathrm{Im}(\langle f, (H - \lambda - \mathrm{i}\varepsilon)^{-1} f\rangle)\, dx.$$

Set $S := \cup_{i=1}^N (a_i, b_i)$ is open in (a, b), where the intervals are taken two by two disjoint. Suppose first that $N < \infty$. We have:

$$\|E_S(H)f\|^2 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_S \mathrm{Im}(\langle f, (H - \lambda - \mathrm{i}\varepsilon)^{-1} f\rangle)\, dx.$$
$$\leq C \sum_i \int_{a_i}^{b_i} dx = C \cdot \mathrm{Leb}(S).$$

Suppose then that $N = \infty$. For $m \in \mathbb{N}$, set $S_m := \cup_{i=1}^m (a_i, b_i)$.

$$\|E_S(\varphi(Q))f\|^2 = \lim_{m \to \infty} \|E_{S_m}(\varphi(Q))f\|^2 \leq C \lim_{m \to \infty} \mathrm{Leb}(S_m) = C \cdot \mathrm{Leb}(S).$$

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Proposition (Putnam)

Let H be a a bounded self-adjoint operator acting in a Hilbert space $\mathcal{H}$. Suppose that there is a bounded self-adjoint operator A, such that:

$$[H, \mathrm{i}A] = C^* C,$$

where C is a bounded and injective operator. Then,

$$\sup_{\varepsilon>0} \sup_{\lambda\in\mathbb{R}} \left| \left\langle f, \mathrm{Im}(H-\lambda-\mathrm{i}\varepsilon))^{-1} f \right\rangle \right| \leq 4\|A\| \cdot \|(C^*)^{-1} f\|^2,$$

for all $f \in \mathcal{D}((C^*)^{-1})$. In particular, the spectrum of H is purely absolutely continuous.

Remark

Note that $(C^*)^{-1}$ is an unbounded operator with dense domain, since C is injective.

Proof:

Set $R(z) := (z - H)^{-1}$. Then

$$\begin{aligned}\|CR(\lambda \pm \mathrm{i}\varepsilon)\|^2 &= \|R(\lambda \mp \mathrm{i}\varepsilon)C^* CR(\lambda \pm \mathrm{i}\varepsilon)\| \\ &= \|R(\lambda \mp \mathrm{i}\varepsilon)[H, \mathrm{i}A]R(\lambda \pm \mathrm{i}\varepsilon)\| \\ &= \|R(\lambda \mp \mathrm{i}\varepsilon)[H - \lambda \mp \mathrm{i}\varepsilon, \mathrm{i}A]R(\lambda \pm \mathrm{i}\varepsilon)\| \\ &\leq \|AR(\lambda \pm \mathrm{i}\varepsilon)\| + \|R(\lambda \mp \mathrm{i}\varepsilon)A\| + 2\varepsilon\|R(\lambda \mp \mathrm{i}\varepsilon)AR(\lambda \pm \mathrm{i}\varepsilon)\| \leq 4\|A\|/\varepsilon.\end{aligned}$$

Therefore, we obtain

$$2\|C\mathrm{Im}R(\lambda \pm \mathrm{i}\varepsilon)C^*\| = \|2\mathrm{i}\varepsilon CR(\lambda + \mathrm{i}\varepsilon)R(\lambda - \mathrm{i}\varepsilon)C^*\| \leq 8\|A\|.$$

Therefore,

$$\sup_{\varepsilon>0} \sup_{\lambda\in\mathbb{R}} \left|\left\langle f, \Im(H - \lambda - \mathrm{i}\varepsilon))^{-1} f\right\rangle\right| \leq 4\|A\| \cdot \|(C^*)^{-1} f\|^2.$$

Stone's formula ensures that the measure given by $\|E_{(\cdot)}(H)f\|^2$ is purely-absolutely continuous for all $f \in \mathcal{D}((C^*)^{-1})$. Since the domain is dense in $\mathscr{H}$ and that $\mathcal{H}^{\rm ac}$ is closed, we obtain the result. □

Here we have proved a stronger result than the absence of singularly continuous spectrum

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For the *a.c.* spectrum it would suffice to have on the right hand side a constant that depends on f. Here we have an explicit dependency of f that is uniform in a certain sense.

The bound that we obtain is in fact equivalent to the global propagation estimate:

$$\int_{\mathbb{R}} \|C^* e^{-\mathrm{i}tH} f\|^2 dt \leq c\|f\|^2,$$

for some $c > 0$ and all $f \in \mathcal{H}$.

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We now aim at perturbation theory.

Theorem

Let H be a self-adjoint operator. There exists a compact and self-adjoint operator K such that

$$\sigma^{\mathrm{pp}}(H+K)\cap\sigma^{\mathrm{ess}}(H)=\sigma^{\mathrm{ess}}(H).$$

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Theorem (Kato-Rosenblum)

Let H be a self-adjoint operator. Let T be self-adjoint and trace class, i.e., T compact such that $\sum_i |\lambda_i(T)| < \infty$. Then, $\mathcal{H}^{\rm ac}(H)$ is unitarily equivalent to $\mathcal{H}^{\rm ac}(H+T)$. In particular,

$$\sigma^{\rm ac}(H) = \sigma^{\rm ac}(H+T).$$

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Even if $\mathcal{H}^{\rm ac}(H) = \mathcal{H}$ the theorem does not guarantee that $\mathcal{H}^{\rm ac}(H+T) = \mathcal{H}$. We could have that $\mathcal{H}^{\rm sc}(H+T) \neq 0$.

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We now prove the remark. Given a self-adjoint operator H and $f \in \mathcal{H}$. Set $m_f(\cdot) := \langle f, E_{(\cdot)}(H)f \rangle$. We define the *Borel transform* of m_f by setting:

$$F_{m_f}(x) := \int_{\mathbb{R}} \frac{dm_f(\xi)}{\xi - x}.$$

The de la Vallée-Poussin's result links the boundary value of F_{m_f} with the Lebesgue decomposition of m_f.

Theorem (Vallée-Poussin)

Let

$$A_{m_f} := \{x, \lim_{\varepsilon \to 0^+} F_{m_f}(x + \mathrm{i}\varepsilon) = \infty\}$$

and

$$B_{m_f} := \{x, \lim_{\varepsilon \to 0^+} F_{m_f}(x + \mathrm{i}\varepsilon) \text{ is finite and } \mathrm{Im} F_{m_f}(x + \mathrm{i}0^+) > 0\}.$$

Then, $m_f(\mathbb{R} \setminus (A_{m_f} \cup B_{m_f})) = 0$, $m_f^{\mathrm{ac}}(\mathbb{R} \setminus B_{m_f}) = 0$, $m_f^{\mathrm{s}}(\mathbb{R} \setminus A_{m_f}) = 0$.

Let $L^2([0,1], \mathrm{Leb}|_{[0,1]} + m_C)$. We see that

$$\sigma(Q) = [0,1], \sigma^{\mathrm{ac}}(Q) = [0,1], \text{ and } \sigma^{\mathrm{sc}}(Q) = C.$$

For $\lambda \in \mathbb{R}$, we set

$$H_\lambda := Q + \lambda P_{\{1\}},$$

where

$$P_{\{1\}} := 1\langle 1, \cdot \rangle.$$

We have that for $\lambda \in \mathbb{R} \setminus \{0\}$,

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Proof:

A direct computation gives:

$$\frac{1}{\pi}\mathrm{Im}F_m(x+\mathrm{i}0^+) = \begin{cases} 1, & x \in (0,1), \\ 1/2, & x \in \{0,1\}, \\ 0, & x \notin [0,1]. \end{cases}$$

and for $x \in (0,1)$:

$$\mathrm{Re}F_m(x+\mathrm{i}0^+) = \ln\left(\frac{x}{1-x}\right),$$

for $x \in (0,1)$. Since for any measure μ we have

$$\mathrm{Im}F_\mu(x_0+\mathrm{i}\varepsilon) \geq \mu\left(\{y, |x-y| \leq \varepsilon\}\right),$$

we infer:

$$F_{\mu_C}(x+\mathrm{i}0^+) = \begin{cases} +\infty, & x \in C, \\ 0, & x \notin C, \quad \text{since the measure is not supported here} \end{cases}$$

Recall that

$$F_{\mu_\lambda}(z) = \langle 1, (H_\lambda - z)^{-1} 1 \rangle = \int (x-z)^{-1} d\mu_\lambda(x),$$

i.e., μ_λ is the spectral measure associated to H_λ and to the vector 1.

We now turn to the study of μ_λ and focus on $F_\lambda(z)$, for all $z \in \mathbb{C} \setminus \mathbb{R}$. The resolvent identity gives

$$(H_\lambda - z)^{-1} = (H_0 - z)^{-1} - \lambda (H_\lambda - z)^{-1} P_1 (H_0 - z)^{-1}.$$

This gives:

$$F_{\mu_\lambda}(z) = F_{\mu_0}(z) - \lambda F_{\mu_\lambda}(z) F_{\mu_0}(z).$$

Therefore

$$F_{\mu_\lambda}(z) = \frac{F_{\mu_0}(z)}{1 + \lambda F_{\mu_0}(z)}.$$

This yields

$$\mathrm{Im}(F_{\mu_\lambda}(z)) = \frac{\mathrm{Im}(F_{\mu_0}(z))}{\left(1 + \lambda \mathrm{Re}(F_{\mu_0}(z))\right)^2 + \lambda^2 \mathrm{Im}(F_{\mu_0}(z))^2}.$$

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The singular part of the spectrum of H_λ is supported by:

$$A_\lambda := \{x, \lim_{\varepsilon \to 0^+} F_{\mu_\lambda}(x + \mathrm{i}\varepsilon) = \infty\}.$$

Given $\lambda \neq 0$, we see that $[0, 1] \cap A_\lambda = \emptyset$. Therefore there is no singular spectrum for H_λ. The spectrum of H_λ is purely absolutely continuous. $\square$

It is very complicated to apply the Putnam theorem in practice because of the boundedness of A.

We sacrifice the boundedness of A in the Putnam theorem and try to exploit the positivity of a commutator.

We start with $\varphi(Q) := 2\cos(Q)$ on $\mathcal{H} := L^2(-\pi, \pi)$. For $f \in \mathcal{C}_c^\infty((-\pi, \pi))$ we set:

$$A_0 f := \frac{1}{2}\left(\mathrm{i}\partial_x \varphi'(Q) + \varphi'(Q)\mathrm{i}\partial_x\right).$$

This operator is essentially self-adjoint and we denote by A_0 its closure.

For $f \in \mathcal{C}_c^\infty((-\pi,\pi))$, we have:

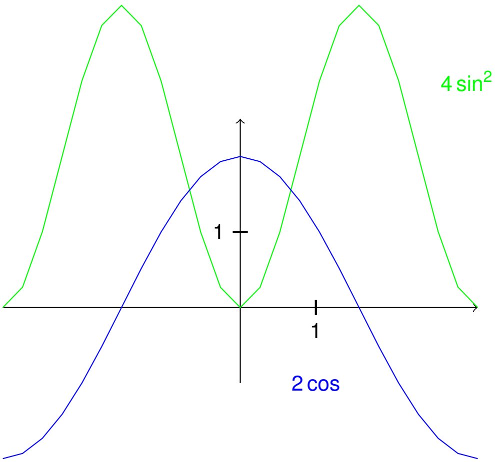
$$\begin{aligned}
2[\varphi(Q), \mathrm{i}A_0]f &= -[\varphi(Q), \partial_x\varphi'(Q) + \varphi'(Q)\partial_x]f \\
&= (\partial_x\varphi'(Q) + \varphi'(Q)\partial_x)\varphi(Q)f - \varphi(Q)(\partial_x\varphi'(Q) + \varphi'(Q)\partial_x)f \\
&= \varphi''(Q)\varphi(Q)f + (\varphi'(Q))^2 f + \varphi'(Q)\varphi(Q)f' + (\varphi'(Q))^2 f + \varphi'(Q)\varphi(Q)f' \\
&\quad - \left(\varphi''(Q)\varphi(Q)f + \varphi'(Q)\varphi(Q)f' + \varphi'(Q)\varphi(Q)f'\right) \\
&= 2\varphi^2(Q)f.
\end{aligned}$$

In other words, using the density of $\mathcal{C}_c^\infty$ in $\mathcal{H}$, we infer:

$$[\varphi(Q), \mathrm{i}A_0] = (\varphi'(Q))^2.$$

This gives:

$$[\varphi(Q), \mathrm{i}A_0] = 4\sin^2(Q) = (2 - 2\cos(Q))(2 + 2\cos(Q)).$$



Remark

Note that $4\sin^2(x) = 0$ *if and only if* $\cos'(x) = 0$.

The operator $4\sin^2(Q)$ is injective and non-negative. Taking apart that A_0 is unbounded, we are in the setting of Putnam's theory. We hope to deduce that $2\cos(Q)$ is purely ac by this method.

Take $\mathcal{I}$ be a closed subset included in the interior of $[-2,2]=\sigma(\varphi(Q))$. We have:

$$E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), \mathrm{i}A_0]E_{\mathcal{I}}(\varphi(Q)) = E_{\mathcal{I}}(\varphi(Q))(2-\varphi(Q))(2+\varphi(Q))E_{\mathcal{I}}(\varphi(Q))$$

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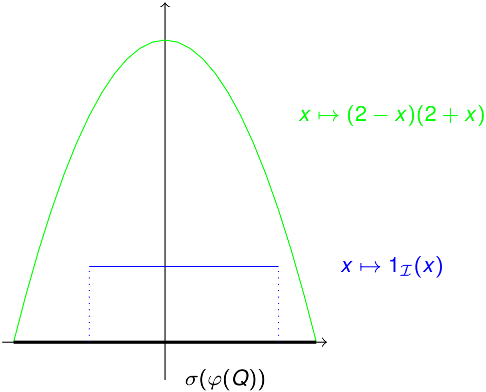
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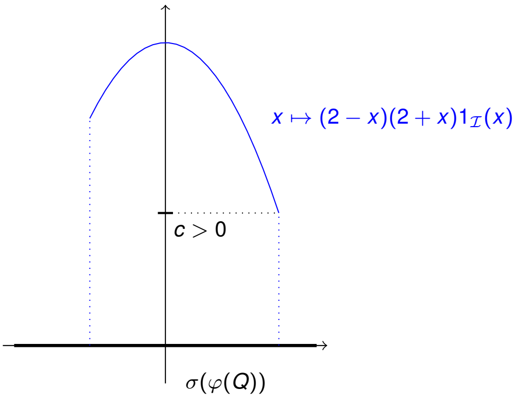
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There is $c > 0$, for all $f \in \mathcal{H}$,

$$\begin{aligned}\langle f, E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), \mathrm{i}A_0]E_{\mathcal{I}}(\varphi(Q))f\rangle &= \langle E_{\mathcal{I}}(\varphi(Q))(2-\varphi(Q))(2+\varphi(Q))E_{\mathcal{I}}(\varphi(Q))f\rangle \\ &= \int_{\sigma(\varphi(Q))} 1_{\mathcal{I}}(x)(2-x)(2+x)1_{\mathcal{I}}(x)dm_f(\varphi(Q))(x) \\ &\geq c\int_{\sigma(\varphi(Q))} 1_{\mathcal{I}}(x)dm_f(\varphi(Q))(x) \\ &= c\langle E_{\mathcal{I}}(\varphi(Q))f, E_{\mathcal{I}}(\varphi(Q))f\rangle.\end{aligned}$$

In other words we have that there is $c > 0$ such that

$$E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), \mathrm{i}A_0]E_{\mathcal{I}}(\varphi(Q)) \geq cE_{\mathcal{I}}(\varphi(Q)),$$

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We now go back to $\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C})$ and will go into perturbation theory. Recall that the Fourier transform $\mathscr{F} : \ell^2(\mathbb{Z}) \to L^2([-\pi, \pi])$ is defined by

$$(\mathscr{F} f)(x) := \frac{1}{\sqrt{2\pi}} \sum_n f(n) e^{-ixn}, \text{ for all } f \in \ell^2(\mathbb{Z}) \text{ and } x \in [-\pi, \pi].$$

The adjacency matrix is given by:

$$(\mathcal{A}_{\mathbb{Z}} f)(n) := f(n-1) + f(n+1), \quad \text{for } f \in \mathcal{H}.$$

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Moreover, for $f \in \mathcal{C}_c(\mathbb{Z})$, the set of function with compact support, we have:

$$Af := \mathscr{F}^{-1} A_0 \mathscr{F} f = \mathrm{i} \left(\frac{1}{2}(U^* + U) + Q(U^* - U) \right) f,$$

where

$$Uf(n) := f(n-1) \quad \text{and } (U^* f)(n) = f(n+1).$$

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Thanks to the previous calculus, we have:

$$[\mathcal{A}_{\mathbb{Z}}, \mathrm{i}A] = (2 - \mathcal{A}_{\mathbb{Z}})(2 + \mathcal{A}_{\mathbb{Z}})$$

and, given $\mathcal{I}$ closed included in the interior of $[-2, 2]$, the spectrum of $\mathcal{A}_{\mathbb{Z}}$, there is a positive constant $c > 0$:

$$E_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}})[\mathcal{A}_{\mathbb{Z}}, \mathrm{i}A]E_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}}) \geq cE_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}}),$$

in the form sense, i.e. when applied to $f \in \mathcal{H}$ on both side.

We now add a perturbation. Let $V : \mathbb{Z} \to \mathbb{R}$ be such that

$$\lim_{n\to\pm\infty} V(n) = 0 \quad \text{and} \quad \lim_{n\to\pm\infty} n(V(n) - V(n+1)) = 0.$$

In particular, we have :

$$V(Q) \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad Q(V(Q) - V(Q+1)) \in \mathcal{K}(\mathcal{H}).$$

Take $f \in \mathcal{C}_c$. We have:

$$\begin{aligned}[U^*, V(Q)]f(n) &= (U^* V(Q)f)(n) - (V(Q)U^* f)(n) \\ &= (V(Q)f)(n+1) - V(n)f(n+1) = (V(n+1) - V(n))f(n+1) \\ &= ((V(Q+1) - V(Q))U^* f)(n).\end{aligned}$$

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$$[U^*, V] = (V(Q+1) - V(Q))U^* \quad \text{and} \quad [U, V] = (V(Q-1) - V(Q))U.$$

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2[V(Q), \mathrm{i}A]f &= 2\left[V(Q), \mathrm{i}\cdot\mathrm{i}\left(\frac{1}{2}(U^* + U) + Q(U^* - U)\right)\right] \\
&= [(U^* + U) + Q(U^* - U), V(Q)]\, f \\
&= [U^*, V]f + [U, V]f + Q[U^*, V]f - Q[U, V]f, \text{ since } [Q, V(Q)] = 0 \\
&= \underbrace{(V(Q+1) - V(Q))}_{\text{compact}} U^* f + \underbrace{(V(Q-1) - V(Q))}_{\text{compact}} Uf \\
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We plug this information into the previous estimate. We set $H := \mathcal{A}_{\mathbb{Z}} + V(Q)$

$$
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[H, \mathrm{i}A] &= [\mathcal{A}_{\mathbb{Z}}, \mathrm{i}A] + [V(Q), \mathrm{i}A] = (2 - \mathcal{A}_{\mathbb{Z}})(2 + \mathcal{A}_{\mathbb{Z}}) + \text{ compact} \\
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Recall that, by the Weyl's Theorem, $\sigma_{\mathrm{ess}}(H) = [-2, 2]$, therefore by taking $\mathcal{I}$ being closed in the interior of the essential spectrum of H we get, there are $c := \inf_{x \in \mathcal{I}}(2 - x)(2 + x) > 0$ and a compact operator K such that

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E_{\mathcal{I}}(H)[H, \mathrm{i}A]E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H) + \underbrace{E_{\mathcal{I}}(H)KE_{\mathcal{I}}(H)}_{\text{compact}},
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General Theory:
Given a bounded operator H acting in a complex Hilbert space $\mathscr{H}$ and $k \in \mathbb{N}$, one says that $H \in \mathcal{C}^k(A)$ if $t \mapsto e^{-\mathrm{i}tA} H e^{\mathrm{i}tA} f$ is $\mathcal{C}^k$ for all $f \in \mathcal{H}$.

Proposition

Let H be a bounded operator and A be a self-adjoint operator The following assertions are equivalent:

1. $H \in \mathcal{C}^1(A)$.
2. *There is a constant* $c > 0$ *such that*

$$|\langle Hf, Af\rangle - \langle Af, Hf\rangle| \leq c\|f\|^2, \tag{2}$$

for all $f \in \mathcal{D}(A)$.

Note that, by density of $\mathcal{D}(A)$, (2) defines a bounded operator that we denote by $[H, A]_\circ$, or simply $[H, A]$ when no confusion can arise.

We can prove that the derivative of $t \mapsto e^{-\mathrm{i}tA} H e^{\mathrm{i}tA} f$ for $t = 0$ is equal to $[H, A]_\circ$.

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Proposition ("Virial Theorem")

Let $H \in \mathcal{C}^1(A)$ with H bounded and self-adjoint and A self-adjoint.

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$$E_{\mathcal{I}}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H) + K,$$

   where $K \in \mathcal{K}(\mathcal{H})$, then H has a finite number of eigenvalue in $\mathcal{I}$, counted with multiplicity.

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Proof:

Let f be an eigenfunction of H associated to $\lambda \in \mathcal{I}$. We have

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We change slightly the approach. Set for $\tau \neq 0$,

$$A_\tau := \frac{1}{\mathrm{i}\tau}(e^{\mathrm{i}A\tau} - \mathrm{Id})$$

Note that for $g \in \mathcal{D}(A)$,

$$\lim_{\tau\to 0} A_\tau g = Ag.$$

Moreover, we have for all $g \in \mathcal{H}$

$$[A, H]_\circ g = \lim_{\tau\to 0} \frac{1}{\mathrm{i}\tau}\left(e^{\mathrm{i}\tau A} H e^{-\mathrm{i}\tau A} - H\right) g = \lim_{\tau\to 0} \frac{1}{\mathrm{i}\tau}[e^{\mathrm{i}\tau A}, H] e^{-\mathrm{i}\tau A} g = \lim_{\tau\to 0} [A_\tau, H] g.$$

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\end{aligned}
$$

We turn to the point 2. We apply the strict Mourre estimate to f, where $Hf = \lambda f$ and $\lambda \in \mathcal{I}$. Note first that

$$f = E_{\{\lambda\}}(H)f = E_{\mathcal{I}}(H)f.$$

Therefore, we get

$$\begin{aligned}\|f\|^2 = \|E_{\mathcal{I}}(H)f\|^2 &\leq \frac{1}{c}\langle f, E_{\mathcal{I}}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H)f\rangle \\ &= \frac{1}{c}\langle E_{\mathcal{I}}(H)f, [H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H)f\rangle \\ &= \frac{1}{c}\langle f, [H, \mathrm{i}A]_\circ f\rangle = 0.\end{aligned}$$

Therefore H has no eigenvalue in $\mathcal{I}$.

We turn to the point 2. We apply the strict Mourre estimate to f, where $Hf = \lambda f$ and $\lambda \in \mathcal{I}$. Note first that

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We now prove the point 1. Suppose that, for all $n \in \mathbb{N}$, there are $\lambda_n \in \mathcal{I}$ and $f_n \in \mathcal{H}$ such that $\|f_n\| = 1$ and $Hf_n = \lambda f_n$.

We apply the Mourre estimate to f_n. We get:

$$\begin{aligned} 0 = \langle f_n, E_{\mathcal{I}}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H) f_n\rangle &\geq c\langle E_{\mathcal{I}}(H) f_n, E_{\mathcal{I}}(H) f_n\rangle + \langle E_{\mathcal{I}}(H) f_n, KE_{\mathcal{I}}(H) f_n\rangle \\ &\geq c \underbrace{\langle f_n, f_n\rangle}_{=1} + \langle f_n, Kf_n\rangle \\ &\geq c - \|Kf_n\|^2 \end{aligned}$$

Or $f_n \rightharpoonup 0$ and K compact, therefore $\|Kf_n\| \to 0$. With n large enough we obtain a contradiction with the fact that $c > 0$. □

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Proposition

Let $H \in \mathcal{C}^1(A)$ with H bounded and self-adjoint and A self-adjoint. Assume that the following Mourre estimate holds true

$$E_{\mathcal{I}}(H)[H, \mathrm{i}A]_{\circ} E_{\mathcal{I}}(H) \geq c E_{\mathcal{I}}(H) + K,$$

where $K \in \mathcal{K}(\mathcal{H})$.

If H has no eigenvalue in $\mathcal{I}$, then for all λ in the interior of $\mathcal{I}$ there is $\mathcal{J} := [\lambda - \varepsilon, \lambda + \varepsilon]$, with $\varepsilon > 0$ small enough, such that

$$E_{\mathcal{J}}(H)[H, \mathrm{i}A]_{\circ} E_{\mathcal{J}}(H) \geq \frac{c}{2} E_{\mathcal{J}}(H),$$

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Proof:

Set $\mathcal{I}_n := [\lambda - 1/n, \lambda + 1/n]$. Since there is no eigenvalue in $\mathcal{I}$, we have that for all $f \in \mathcal{H}$ that

$$\|E_{\mathcal{I}_n}(H)f\|^2 = \int_{\mathcal{I}_n} dm_f(x) \to 0, \quad \text{as } n \to \infty,$$

by dominated convergence.

Since K is compact, we have that $\|KE_{\mathcal{I}_n}(H)\| \to 0$, as $n \to \infty$. Therefore, for n large enough, we obtain that $\|KE_{\mathcal{I}_n}(H)\| \leq c\|E_{\mathcal{I}_n}(H)\|/2$. Therefore we obtain:

$$E_{\mathcal{I}_n}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}_n}(H) \geq \frac{c}{2} E_{\mathcal{I}_n}(H).$$

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Assume that $H \in \mathcal{C}^1(A)$ and

$$E_{\mathcal{I}}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H) \geq c E_{\mathcal{I}}(H).$$

We will deduce some dynamical properties.

Given $f \in \mathcal{H}$ and $f_t := e^{-\mathrm{i}tH} f$ its evolution at time $t \in \mathbb{R}$ under the dynamic generated by the Hamiltonian H, one looks at the Heisenberg picture:

$$\mathscr{H}_f(t) := \langle f_t, A f_t \rangle. \tag{3}$$

As A is an unbounded self-adjoint operator, we take $f := \varphi(H)g$, with $g \in \mathcal{D}(A)$ and $\varphi \in \mathcal{C}_c^\infty(\mathcal{I})$. We can prove that $\mathcal{H}_f$ is well-defined as $e^{-\mathrm{i}tH}\varphi(H)$ stabilises the domain of A. This implies also that $\mathscr{H}_f \in \mathcal{C}^1(\mathbb{R})$.

Remark

Note that $E_{\mathcal{I}}(H)f = E_{\mathcal{I}}(H)\varphi(H)g = \varphi(H)g = f$.

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Since $H \in \mathcal{C}^1(A)$, the commutator $[H, \mathrm{i}A]_\circ$ is a bounded operator. We denote by C its norm.

$$\mathcal{H}_f'(t) = \langle f_t, [H, \mathrm{i}A]_\circ f_t \rangle = \langle f_t, E_\mathcal{I}(H)[H, \mathrm{i}A]_\circ E_\mathcal{I}(H) f_t \rangle.$$

We now use the Mourre estimate above $\mathcal{I}$ and since $e^{\mathrm{i}tH}$ is unitary, one gets:

$$c\|f\|^2 \leq \mathcal{H}_f'(t) \leq C\|f\|^2.$$

Now integrate the previous inequality and obtain

$$ct\|f\|^2 \leq \mathcal{H}_f(t) - \mathcal{H}_f(0) \leq Ct\|f\|^2, \qquad \text{for } t \geq 0$$

The transport of the particle is therefore ballistic with respect to A, we have some transport in the direction given by A. Purely absolutely continuous spectrum is therefore expected.

Theorem

Suppose that H is a bounded and self-adjoint operator and that A is self-adjoint. Assume that $H \in \mathcal{C}^2(A)$ *and that*

$$E_{\mathcal{I}}(H)[H, \mathrm{i}A]_\circ E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H),$$

holds true for some non-empty and closed interval $\mathcal{I}$*. Then:*

1. *The spectrum of H restricted to* $\mathcal{I}$ *is purely absolutely continuous.*
2. *Given* $\mathcal{J}$ *a closed interval included in the interior of* $\mathcal{I}$*, for all* $s > 1/2$ *there is a constant* $c > 0$*, such that the following limiting absorption principle holds true:*

$$\sup_{\lambda\in\mathcal{J}} \sup_{\varepsilon>0} |\langle f, (H - \lambda - \mathrm{i}\varepsilon)^{-1} f\rangle| \leq c\|\langle A\rangle^s f\|^2,$$

where $\langle x\rangle := \sqrt{1 + x^2}$.

3. *There is* $c > 0$ *such that for all* $f \in \mathcal{H}$*,*

$$\int_R \|\langle A\rangle^{-s} e^{-\mathrm{i}tH} E_{\mathcal{J}}(H) f\|^2\, dt \leq c\|f\|^2.$$

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Theorem

Suppose that $H := \mathcal{A}_{\mathbb{Z}} + V(Q)$, *where*

$$\lim_{n\to\pm\infty} V(n) = 0, \lim_{n\to\pm\infty} n(V(n) - V(n+1)) = 0, \text{ and } \sup_n n^2|V(n) - V(n+1)| < \infty$$

Then:

1. *The essential spectrum of* H *is* $\sigma_{\rm ess}(H) = [-2, 2]$.
2. *The eigenvalues of* H *that do not belong to* $\{-2, 2\}$ *are of finite multiplicity and can only accumulate to* $\{-2, 2\}$.
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With more technology, we can prove that

1. Under the hypothesis that there is $\varepsilon > 0$ such that

$$\lim_{n\to\pm\infty} V(n) = 0, \lim_{n\to\pm\infty} n^{1+\varepsilon}(V(n) - V(n+1)) = 0,$$

   the conclusions of the Theorem remain true.

2. Under the hypothesis that $n \mapsto V(n+k) - V(n) \in \ell^1(\mathbb{Z})$ holds true for some $k \in \mathbb{Z}$, we have that

$$\sigma^{\mathrm{sc}}(H) = \emptyset$$

   and that there is no eigenvalue in $(-2, 2)$.

I thank you very to have followed my course.

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W.O. Amrein, A. Boutet de Monvel and V. Georgescu: *C_0-groups, commutator methods and spectral theory of N-body hamiltonians.*, Birkhäuser 1996.

A. Boutet de Monvel and J. Sahbani: *On the spectral properties of discrete Schrödinger operators: the multi-dimensional case*, Rev. Math. Phys. 11 (1999), no. 9, 1061–1078.

H. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry,* Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.

V. Georgescu and C. Gérard: *On the Virial Theorem in Quantum Mechanics*, Commun. Math. Phys. **208**, 275–281, (1999).

C. Gérard: *A proof of the abstract limiting absorption principle by energy estimates*, journal of functional analysis.

S. Golénia and T. Haugomat: *On the a.c. spectrum of 1D discrete Dirac operator*, Methods Funct. Anal. Topology 20 (2014), no. 3, 252–273.

S. Golénia: *Commutator, spectral analysis and applications*, Habilitation à diriger les recherches, https://www.math.u-bordeaux.fr/ sgolenia/Fichiers/HDR.pdf

S. Golénia and T. Jecko: *A new look at Mourre's commutator theory*, Complex Anal. Oper. Theory 1 (2007), no. 3, 399–422.

S. Golénia and C. Schumacher : *The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs*, J. Math. Phys. 52 (2011), no. 6, 063512, 17 pages.

S. Golénia and C. Schumacher : *Comment: "The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs" [J. Math. Phys. (52), 063512 (2011)*, J. Math. Phys. 54 (2013), no. 6, 064101, 4 pages.

E. Mourre: *Absence of singular continuous spectrum for certain self-adjoint operators.* Commun. in Math. Phys. **78**, 391–408, 1981.

P. Perry, I.M. Sigal, and B. Simon: *Absence of singular continuous spectrum in N-body quantum systems* Bull. Amer. Math. Soc. 3 (1980), no. 3, 1019–1023.

C.R. Putnam: *Commutation properties of Hilbert space operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete. **36**. Berlin-Heidelberg-New York: Springer-Verlag. XII, 167 p. 1967.