Spectral theory on combinatorial and quantum graphs

Topic 4 Introduction to quantum graphs.

Evans Harrell
Georgia Tech
www.math.gatech.edu/~harrell

القيروان

Atlanta

November, 2016

Illustrative examples

1. An interval, $V = 0$.
2. The regular Y-graph, $V = 0$.

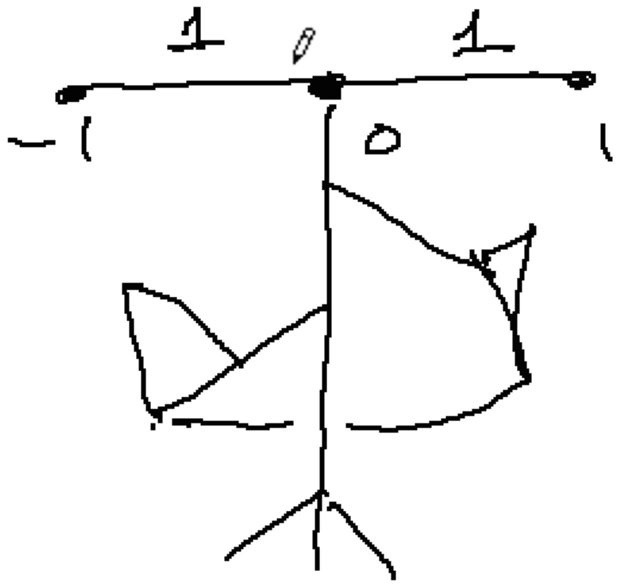
   eigenvalues determined by $\lambda = k^2$,

   $\sum \tan(k L_j) = 0$.

   - if two lengths are same?

     The eigenfunction can = 0 on large parts of the graph!

$k = \sqrt{\lambda}$



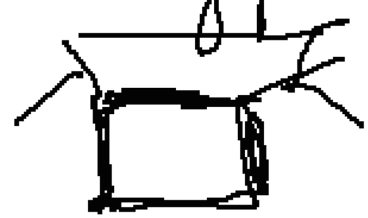
$-\psi'' = k^2 \psi$

$\sin(\pi k x)$

$= 0$ at vertex

extend by 0 on the 3rd edge

An efn can be compactly supported



$\sin(4\pi x)$

Illustrative examples

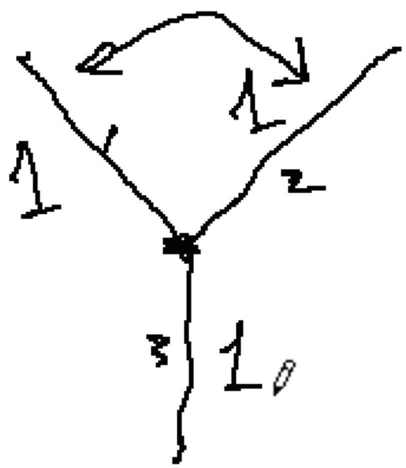
1. An interval, $V = 0$.
2. The regular Y-graph, $V = 0$.

   eigenvalues determined by $\lambda = k^2$,

   $\sum \tan(k L_j) = 0$.

   - if two lengths are same?

     The eigenfunction can = 0 on large parts of the graph!



Permute 2 branches (edge)

Given an efn. Ψ

Then $\Psi(P(G))$ also efn.

$\Psi \pm \Psi_P$ also efn.

$\Rightarrow$ $k=1$ $\quad \sin(k\pi x_e)$

on $e_{1,2}$

0 on e_3

A basis for this eigenspace with $\lambda = 1^2\pi^2$

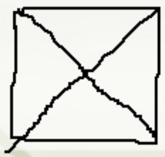
Illustrative examples

1. An interval, $V = 0$.
2. The regular Y-graph, $V = 0$.

   eigenvalues determined by $\lambda = k^2$,

   $\sum \tan(k L_j) = 0$.

   - if two lengths are same?
3. K_4, $V=0$, all lengths are same (Exercise)



What happens when you…

- Add or increase an edge? (Say, when V=0)?
- Identify two vertices? √
- Impose a Dirichlet condition on a vertex?

Interlacing thms

$$\sum_e \int_e |f'^2| + V|f|^2 + \text{cont.}$$

Change graph in various ways

Identify them.

$\lambda_k \leq \tilde{\lambda}_k \leq \lambda_{k+1}$

$\lambda_k \uparrow$? $\lambda_k \downarrow$?

λ_k min max

$$= \inf_{\substack{\text{subst } M \\ \dim k}} \sup \langle f, Hf \rangle \qquad \|f\| = 1 \text{ satisfy conds}$$

2 sets of test fns in min max

$$\lambda_n = \inf_{\dim S = n} \sup_{\substack{\|f\|=1 \\ f \in S}} \langle Hf, f \rangle$$

Suppose I wish to test for $\lambda_{k+1}(G)$ ← has no id of vert.

$\phi_1 \cdots \phi_{k+1}$ norm. e'vecs

I can use them to test for λ_k^* if I can arrange the extra cont.

At least 1 of the $\phi_1 \cdots \phi_{k+1}$ differs at $\phi_1(v_1) \neq \phi_l(v_2)$

Otherwise there are suitable test fns for λ_k^{α} & done

$\phi_1 - \alpha_1 \phi_l$

$\phi_2 - \alpha_2 \phi_l$

$\vdots$

choose α's so that New continuity condit is satisfied.

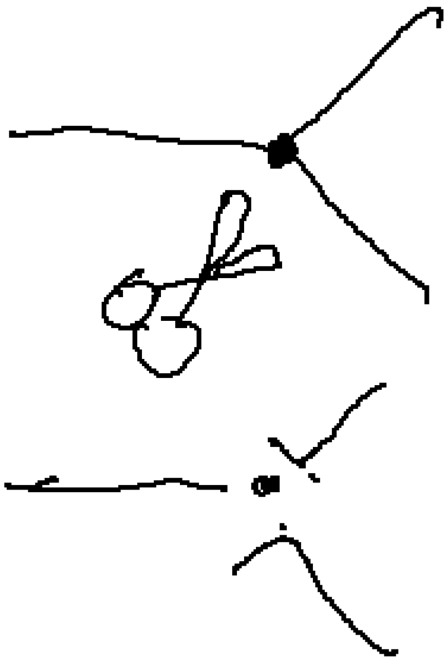
k test fns, with energies $\leq \lambda_{k+1} \Rightarrow \lambda_k^* \leq \lambda_{k+1}$

What happens when you…

- Add or increase an edge? (Say, when V=0)?
- If you increase the search space, quantities defined by an infimum, like variational eigenvalues, can only go down.

What happens when you…

- Add or increase an edge? (Say, when V=0)?
- Impose a Dirichlet condition on a vertex? (And what is a Dirichlet condition in the weak sense?)
  - Like pinning down a vertex
- Likewise for Neumann?
  - Like cutting the edges loose



K   cont + no new rest in Q form.

$$\sum_e \int_e |f'|^2$$

K → N

No longer require continuity at the now "free" ends

minimax, min over larger set

adding N $\lambda_k \downarrow$

Replacing the K conditions by N is like cutting the edges away from the vertex. Inserting an N condition moves eigenvalues down.

Dir in weak sense

begin with fns $f \in C^\infty$
support disjoint from
the end of edge.
complete space in H^1 norm

H^1_0 (subset of H^1) smaller space than H^1

e'values $\lambda_k \uparrow$ $\lambda_k^N \le \lambda_k \le \lambda_k^D$

Replacing the K conditions by D is like pinning functions down. Inserting a D condition reduces the test function space and moves eigenvalues up.

Weyl asymptotic expression

- How are eigenvalues λ_j asymptotically distributed as $j \to \infty$?
- On an isolated interval, both D and N conditions lead to
  - $\lambda_j = (j\pi/L)^2$, except that in one case $j \geq 0$ and in the other $j \geq 1$.
  - $((j\pm1)\pi/L)^2 = (j\pi/L)^2 (1 + O(1/j))$
- If you prefer to ask how many eigenvalues are $\leq k^2$, $N(k) = (L/\pi)\, k + O(1)$, and this will be true even if we have a union of independent intervals.

How do we calculate the eigenvalues of quantum graphs?

- We know how to connect edges along vertices and how to solve an ode on an edge, but we need to put this information together.
- We borrow ideas from scattering theory, and construct a “secular determinant”

Connections at vertices

- Let us consider one vertex at a time, and orient edges outward. We can write the conditions of continuity and the Kirchhoff condition as follows. Let **f** be the vector of values of a function at 0 along edge **e** = 1,2,… d_v, and let **f**′ be the analogous vector of derivatives.

Connections at vertices

- We can capture the continuity and Kirchhoff conditions as

$$A\mathbf{f} + B\mathbf{f}' = \mathbf{0},$$

where

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

Connections at vertices

A basis of "scattering states" at a vertex v is of the form

$e^{-ikx_e} + \sigma_{ee} e^{ikx_e}$ on one edge e, and

$\sigma_{ee'} e^{ikx_{e'}}$ on the other edges e'.

A calculation shows that $A\mathbf{f} + B\mathbf{f}' = \mathbf{0}$ implies that, as a matrix,

$$\sigma(k) = -(A + ikB)^{-1}(A - ikB).$$

A technical lemma

- Noticing that $AB^* = 0$, calculations show that for real $k \neq 0$,
  - $(A \pm i\,k\,B)\,(A^* \mp i\,k\,B^*) = AA^* + k^2\,BB^*$
- A related operator is
  - $\sigma(\kappa) := -(A+ikB)^{-1}\,(A - ikB)$,

    which is unitary (for each k):

$$\begin{aligned}
\sigma(k) &= -(A+ikB)^{-1}(A-ikB)(A^*+ikB^*)(A^*+ikB^*)^{-1} \\
&= -(A+ikB)^{-1}(A+ikB)(A^*-ikB^*)(A^*+ikB^*)^{-1} \\
&= -(A^*-ikB^*)(A^*+ikB^*)^{-1} \\
&= (\sigma(k)*)^{-1}
\end{aligned}$$

The edge scattering matrix on $\vec{\mathcal{E}}$.

$$\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$$

(It is unitary and depends on k.)

The "bond" scattering matrix

- This is the solution operator of the ODE on the directed edges, which connects initial conditions at an edge in the basis $\exp(\pm\, i\, k\, x_e)$ to the values at the other end of the edge (reverse orientation!) in the basis $\exp(\pm\, i\, k\, x_{-e})$

The "bond" scattering matrix

$$\begin{pmatrix} \cdots & 0 & e^{ikL_e} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & e^{-ikL_e} & 0 & \cdots \end{pmatrix}$$

- These are the entries connecting e and – e, and the same thing happens at other such pairs. Again, it's a unitary operator, called exp(i k **L**).

How do we calculate the eigenvalues of quantum graphs?

- There is a consistent solution on the entire graph iff there is a nonzero vector γ in the directed edge space such that:

  $$\sigma \exp(i\, k\, \mathbf{L})\, \gamma = \gamma.$$

- Thus the eigenvalues $\lambda = k$ are the solutions of the *secular equation*:

  $$\det(I - \sigma \exp(i\, k\, \mathbf{L})) = 0.$$

Control of eigenfunctions

- The “landscape”method: Find a positive function that dominates the eigenfunctions. (Filoche et al., Steinerberger; current research by EH with Maltsev.)
- If $-u'' + V u \leq 0$, then u has no local maximum. (Maximum principle for QG’s.)

Proof of maximum principle

$V(x) \geq 0$

$-u'' + V(x)u \leq 0$ u no local max.

on edges

$u'(x_m) = 0$

$u''(x) \leq 0$ contra.

Meanwhile also might max at a vertex. $u_e'(v^+) \leq 0$

If at any e $u_e'(v) < 0$

$\exists e'$ $u_{e'}'(v) > 0$ contra.

$u_e' = 0$ at v bec K.

Control of eigenfunctions

- Suppose that $H \Upsilon \geq 1$ (including K conditions) on some connected part of the QG. Then if ψ is an eigenfunction,

$$|\psi|(x) \leq C\, \Upsilon(x) + \text{(boundary values)}$$

Suppose $\Upsilon > 0$

$H\Upsilon \geq 1$

$\swarrow \lambda \|\Psi\|_{\mathcal{I}}$

$$H(\pm\Psi - C\Upsilon) = \pm\lambda\Psi - \lambda\|\Psi\|_{\infty(\mathcal{I})} 1$$

$$\leq 0$$

max princ applies

$\pm\Psi - C\Upsilon$ no local max

$$|\Psi| \leq C\Upsilon$$

Challenges for the future

- Spectral conditions to determine a *quantum* graph uniquely, both the graph structure and the potential. Can the graph structure be seen independently of the potential?
- What "universal" constraints characterize the possible spectra?
- Where do the eigenfunctions concentrate? Are there explicit bounds that reflect this?
  - "Landscape functions"
- Other properties of eigenfunctions.

Some references for quantum graphs and their spectra

- G. Berkolaiko and P. Kuchment, Introduction to Quantum Graphs.
- G. Berkolaiko, an Elementary Introduction to Quantum Graphs (recently on the arxiv).