Eigenvalue spacings for 1D Schrödinger operators

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The setting

We consider a semiclassical Schrödinger operator P_h on the half-line $[0, \infty)$, and Dirichlet (or Neumann) boundary condition at 0 :

$$P_h u = -h^2 u'' + V(x) u.$$

- The potential V is smooth on $(0, +\infty)$, continuous on $[0, \infty)$ and $V(0) = 0$.
- The operator P_h is self-adjoint and we study its spectrum in an energy window $K = [a, b] \subset [0, \infty)$.

Assumptions

We make the following assumptions on V and $[a, b]$.

1. $\exists \delta > 0,\ \liminf V \ > \ b + \delta$ for $x \to +\infty$.
2. $\exists \eta > 0,\ \forall x > 0,\ V(x) \in [a, b] \implies V'(x) \geqslant \eta.$

These assumptions have the following consequences :

- $(H1)$ implies that the spectrum in $[a, b]$ is discrete.
- $(H2)$ implies that there is only one well (no tunnel effect here).

Remark : the spectrum is simple (because the problem reduces to a second-order differential equation).

Results - 1 : non-critical energies

The following results can be extracted from the literature that starts with Titchmarsch, Olver, Hörmander, Maslov .... and develops into semiclassical analysis (see e.g. the textbooks by Helffer-Robert, Dimassi-Sjöstrand, Zworski ....).

Theorem

Under the standing assumptions, if moreover $a > 0$ then there exists $c > 0$ and h_0 such that, for $h \leqslant h_0$ and any two distinct eigenvalues E_h and E'_h of P_h,
$$|E_h - E'_h| \geqslant ch.$$

Results - 2 : Bottom of the well

Theorem

Assume there exists $\gamma \geqslant 1$ and $W \in C^\infty([0,\infty))$ such that $V(x) = x^\gamma W(x)$ for $x \geqslant 0$, and $\liminf V > 0$. For any $M > 0$, there exists $c > 0$ and h_0 such that, for $h \leqslant h_0$ and any two distincts eigenvalues E_h and E'_h of P_h, if $E_h < Mh^{\frac{2\gamma}{\gamma+2}}$ then

$$|E_h - E'_h| \geqslant ch^{\frac{2\gamma}{\gamma+2}}.$$

Actually for both results, much more is known. Asymptotic expansions for E_h are known (see Bohr-Sommerfeld rules, Airy, harmonic approximation)

Approximate solutions

There are well-known techniques to study the solutions to a second order differential equation of the form

$$-h^2 u'' + Wu = 0.$$

We distinguish between

- The classically allowed region where $W(x) > 0$.
- The classically forbidden region where $W(x) < 0$.
- Points where W vanishes are called turning points.

Away of the turning points, approximate solutions can be constructed using WKB, Liouville-Green expansions i.e. by making the Ansatz

$$u_h(x) = \exp(\frac{i}{h} S(x)) \sum_{k \geqslant 0} h^k b_k(x).$$

WKB expansion 1/2

Compute

$$\exp(-\tfrac{i}{h}S(x))(P_h - E)(\exp(\tfrac{i}{h}S(x))b) =$$

$$\begin{aligned} &\left((S')^2 + V - E\right) b \\ &-h \cdot (2iS'b' + iS''b) \\ &-h^2 \cdot (b'') \end{aligned}$$

By solving successively

1. The eiconal equation : $(S')^2 + V - E = 0$,
2. the transport equation : $2S'b_0 + S''b_0 = 0$,
3. the higher order transport equations :
   $2S'b_{k+1} + S''b_{k+1} = ib''_k$,

we find approximate solutions in the classically allowed and in the classically forbidden region.

WKB expansions 2/2

Fix I a compact interval in the classically allowed region, we find a real phase S, and sequence $(a_k^{\pm})_{k\geqslant 1}$. We set

$$\phi_{\pm}(x) \,=\, (E - V(x))^{-\frac{1}{4}}\, e^{\pm\frac{i}{h}S(x)} \left(1 + \sum_{k\geqslant 1} h^k a_k^{\pm}(x)\right).$$

in the sense of asymptotic expansions and we obtain

$$(P_h - E)\phi_{\pm} \,=\, O(h^{\infty})\|\phi_{\pm}\|_{L^2(I)}.$$

The latter means that

$$\forall N, \exists C_N,\ \|(P_h - E)\phi_{+}\|_{L^2(I)} \leqslant C_N \cdot h^N \|\phi_{\pm}\|_{L^2(I)}.$$

A sequence (E_h, ϕ_h) satisfying this estimate will be called a $O(h^N)$ quasisolution.

Near the turning point

In the classically forbidden region, the same method applies except that the phase now is purely imaginary so that $\phi_\pm$ exhibit exponentially growing/decaying behaviour.
There are several ways to deal with the turning point.

- The Cherry-Langer transform whose idea is to seek a change of coordinates and a change of function so that the equation becomes equivalent to its first order approximation. The latter is then solved using Airy functions. Keeping tracks of the error terms gives an asymptotic expansion near the turning point.
- Another method is to seek the solution using the Maslov Ansatz :

$$u_h(x) = \frac{1}{\sqrt{2\pi h}} \int \exp(\frac{i}{h}(x\xi - T(\xi))) \sum_{k\geqslant 0} h^k b_k(\xi)\, d\xi.$$

  Such an Ansatz is motivated by a phase space analysis : in the ξ variable, the $e^{\frac{i}{h}F(\xi)} \sum_{k\geqslant 0} h^k b_k$ is the same WKB ansatz.

The Maslov Ansatz

Using the Maslov Ansatz leads to the same steps :

- an eiconal equation for F,
- a homogenous transport equation for b_0
- inhomogenous transport equations giving b_{k+1} knowing b_k.

Semiclassical analysis provides a nice geometric interpretation that in particular explains this symmetry between the WKB and Maslov Ansatz : the computation takes places on the lagrangian submanifold $\xi^2 + V(x) = E$ that can be parametrized by x and/or ξ.

Going back and forth between the x and ξ representation is done by using the h-Fourier transform and the stationary phase expansion. This is one of the foundations of semiclassical analysis.

Approximate eigenvalues 1/2

With our assumptions, when V is smooth on $[0, \infty)$, the Maslov Ansatz provides us with a $O(h^\infty)$ quasisolution on $[0, \infty)$.
When the Maslov Ansatz satisfies the boundary condition at 0 we obtain an approximate eigenfunction (a quasimode) and thus an estimate on the eigenvalues.
Normalize the phase T so that $T = 0$ at the turning point. The stationary phase expansion yields that

$$u_h(0) = C \left(\exp(\frac{i}{h} S(E) + \frac{\pi}{4})\big(1 + \sum_{k \geqslant 0} h^k a_k^+\big) \right.$$

$$\left. + \exp(-\frac{i}{h} S(E) - \frac{\pi}{4})\big(1 + \sum_{k \geqslant 0} h^k a_k^-\big) \right)$$

Approximate eigenvalues 2/2

We can write

$$u_h(0) = C\cos(\frac{i}{h}S_h(E) + \frac{\pi}{4})$$

so that that whenever

$$S_h(E_h) = \frac{\pi}{2} + kh\pi,\ (k \in \mathbb{Z})$$

we have an approximate eigenvalue.
The semiclassical action S_h has a asymptotic expansion and S_0 is linked with the classical action $\int \mathbb{1}\{|\xi|^2 + V(x) \leqslant E\} dx d\xi$.
We thus obtain eigenvalues that satisfy the order h spacing.

From approximate to exact

In order to prove that all eigenvalues are obtained by the preceding method we have to prove that any true eigenfunction is close to the Maslov Ansatz. This can be done using the following steps.

- Adapt the method of variation of constants replacing solutions by $O(h^\infty)$ quasisolutions. Use it to prove that in the classically allowed region, any true solution has a WKB expansion.
- Prove a non-concentration (or control estimate) saying that any true solution has mass in the classically allowed region.
- Combine these two facts to prove that there exists $c > 0$ so that $hu_h'(0)/\|u_h\| \geqslant c$.
- Compute the semiclassical wronskian of the Maslov Ansatz and of the true solution in the classically forbidden region and at 0 to conclude.

Scaling in the well

Let (u_h, E_h) be a solution to the equation

$$-h^2 u_h'' + (V(x) - E)u_h = 0.$$

We set $v_h(y) = u_h(h^\alpha y)$ for $\alpha = \frac{2}{\gamma+2}$ we find that

$$-v_h'' + (y^\gamma W(h^\alpha y) - e_h) v_h = 0$$

where we have set $e_h = h^{-\frac{2\gamma}{\gamma+2}} E_\gamma$.
Heuristics : the equation on the half-line

$$-G'' + (y^\gamma W(0) - e)G = 0.$$

has a discrete spectrum $0 < e_0 < e_1 \dots$ and the small eigenvalues in the spectrum of P_h are given, at leading order by $h^{\frac{2\gamma}{\gamma+2}} e_j$.

More details

We compare the two following operators on the half-line :

$$Q_h v = -v'' + (y^\gamma W(h^\alpha y)v$$
$$A_h v = -v'' + (y^\gamma W(0))v.$$

Let F_A be the vector space that is spanned by the n- first eigenvalues of A_h.
Any function in F is exponentially decaying for large y so that there exists M so that

$$\forall v \in F,\ |\langle Qv, v\rangle - \langle Av, v\rangle| \leqslant Mh^\alpha \|v\|^2$$

This gives

$$\lambda_N(A) \geqslant \lambda_N(Q) + Mh^\alpha.$$

To prove the converse, we must get the exponential decay of eigenfunctions of Q_h with bounded energy. This can be done by convexity type estimate.

Perspectives

1. The intermediate regime $E_h \to 0$, $h^{-\frac{2\gamma}{\gamma+2}} E_h \to \infty$.
2. Potentials with singularities.