Some problems of singular perturbation

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A spectral theorem

- $\mathcal{H}$ a separable Hilbert space.
- $\mathcal{D}$ a dense subspace of $\mathcal{H}$
- q a non-negative quadratic form on $\mathcal{D}$.

Define on $\mathcal{D}$ the scalar product

$$\forall u, v \in \mathcal{D},\ \langle u, v\rangle_q \, = \, q(u, v) + \langle u, v\rangle_{\mathcal{H}}.$$

Assume :

1. $(\mathcal{D}, \|\cdot\|_q)$ is a Hilbert space.
2. The injection from $(\mathcal{D}, \|\cdot\|_q)$ into $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is compact.

Then : there exists a Hilbert basis $(\phi_n)_{n\geqslant 1}$ of $\mathcal{H}$ and a non-decreasing, unbounded, sequence $(\lambda_n)_{n\geqslant 1}$ of $[0, \infty)$ such that, for all $n \geqslant 1,\ \phi_n$ is a solution to the following eigenvalue problem

$$\left\{ \begin{array}{c} \phi_n \in \mathcal{D}, \\ \forall v \in \mathcal{D},\ q(\phi_n, v) \, = \, \lambda_n \langle \phi_n, v\rangle_{\mathcal{H}}. \end{array} \right.$$

Comments

$$\begin{array}{ccc} \mathcal{H} & \to & \ell^2(\mathbb{N},\mathbb{C}) \\ u & \mapsto & ((u_n = \langle \phi_n, u\rangle)_{n\geqslant 1} \end{array}$$

is an isometric isomorphism.
For $s \geqslant 0$ we denote by $\mathcal{D}_s$ the preimage under this isomorphism of

$$\left\{ (u_n)_{n\geqslant 1},\ \sum_{n\geqslant 1} (1+\lambda_n^2)^{\frac{s}{2}} |u_n|^2 < \infty. \right\}.$$

For any $u \in \mathcal{D}_2$ and any $v \in \mathcal{D}_1$ we have

$$q(u,v) \,=\, \sum_{n\geqslant 1} \lambda_n \overline{u_n} v_n \,=\, \langle f, v\rangle_{\mathcal{H}}.$$

By setting $f = Au$, we define a selfadjoint operator with domain $\mathcal{D}_2$.

$$\forall u \in \mathcal{D}_2,\ \langle Au, u\rangle = q(u).$$

Observe that an eigenfunction is necessarily in $\bigcap_{s\geqslant 0} \mathcal{D}_s$.

Examples

1. $[a,b] \subset \mathbb{R}$, $\mathcal{H} = L^2((a,b))$, $\mathcal{D} = H^1((a,b))$

$$q(u) = \int_a^b |u'(x)|^2\, dx + \int_a^b V(x)|u(x)|^2 dx$$

   with a potential V that is continuous and non-negative.
2. The same with $\mathcal{D} = H_0^1((a,b))$.
   In both cases, compact injection from H^1 into L^2.
3. Schrödinger operator on the line with a confining potential
4. Riemannian Laplacian on a compact manifold.
5. Euclidean Laplacian on a compact domain with piecewise smooth boundary.

$$q(u) = \int_\Omega |\nabla u|^2 \,.$$

   $\mathcal{D} = H_0^1$ or H^1 if the latter is defined.
6. Friedrichs extension.

Min-Max principle

For any finite dimensional vector space $F \subset \mathcal{D}$ we define

$$\Lambda(F) \;=\; \max\left\{\frac{q(u)}{\|u\|^2},\; u \in F,\; u \neq 0\right\}$$

We then have

$$\lambda_n = \min\left\{\Lambda(F),\; F \subset \mathcal{D},\; \dim(F) \;=\; n\right\}.$$

Comments

- The proof of the spectral theorem can be derived from the following iterative construction :
  - ϕ_1 realize the minimum of $q(x)/\|x\|^2$ and $V_1 = \mathbb{C} \cdot \phi_1)$
  - ϕ_{n+1} realizes the minimum of $q(x)/\|x\|^2$ on $V_n^\perp$ and $V_{n+1} = V_n + \mathbb{C} \cdot \phi_n$.

  The compactness assumption ensures the existence of ϕ_n at each step.

Regular perturbation

We want to compare a quadratic form q to a reference quadratic form a (q is a perturbation of a).
The perturbation is regular if

- The two quadratic forms a and q are defined on the same domain.
- The difference $q - a$ is small compared to a

$$\exists c < 1,\ \forall u \in \mathcal{D}\ |q(u) - a(u)| \leqslant\ ca(u).$$

The Min-Max characterization of eigenvalues implies

$$\forall n,\ |\lambda_n(a) - \lambda_n(q)| \leqslant\ c\lambda_n(a)$$

Relaxing the control estimate

When q and a are defined on the same domain then we can set $r = q - a$.
The min-max characterization then implies the following estimate :

$$\forall n,\ \lambda_n(q) \leqslant \lambda_n(a) + r_n$$

Where r_n is the largest eigenvalue of the quadratic form $q - a$ restricted to the n-dimensional vector space spanned by the first n eigenvectors of a.
Comparing

$$q(u) = \int_0^1 |u'(x)|^2\, dx + \int_0^1 V(x)|u(x)|^2 dx$$

$$a(u) = \int_0^1 |u'(x)|^2 dx$$

on $H_0^1(]0,1[)$, we find

$$\forall n,\ |\lambda_n(q) - \lambda_n(a)| \leqslant \|V\|_\infty .$$

Singular perturbation

The perturbation is singular when we try to compare quadratic forms that act on different domains.

Examples

- Semiclassical Schrödinger operator at $h = 0$.
- For $\Omega \subset \mathbb{R}^2$ and $\mathcal{D} = H_0^1(\Omega)$ the family of quadratic forms

  $$q_t(u) = t^2 \int_\Omega |\partial_x u|^2 + \int_\Omega |\partial_y u|^2$$

  becomes singular at $t = 0$.
- Changing the boundary condition is a singular perturbation.

Dirichlet-Neumann bracketing

An example : let

$$q(u) = \int_a^b |u'(x)|^2 dx + \int_a^b V(x)|u(x)|^2 dx$$

for a smooth potential V, on the domain $\mathcal{D} = H_0^1(]a,b[)$. For $c \in]a,b[$ define

$$\mathcal{D}_D = \{u \in \mathcal{D},\ u(c) = 0\}$$
$$\mathcal{D}_N = \left\{u \in H^1(]a,b[\backslash\{c\}),\ u(a) = u(b) = 0\right\}.$$

We have $\mathcal{D}_D \subset \mathcal{D} \subset \mathcal{D}_N$ so that

$$\forall n,\ \lambda_n(q_N) \leqslant \lambda_n(q) \leqslant \lambda_n(q_D)$$

Application : Weyl's formula for a semiclassical Schrödinger operator with a confining potential.

Core or not core

Let q be a positive quadratic form on a domain $\mathcal{D}$ so that $(\mathcal{D}, q)$ is a Hilbert space. Let A be the corresponding self-adjoint operator. Let D be a subset of $\mathcal{D}$ that is still dense in $\mathcal{H}$ then

- either D is dense in $\mathcal{D}$. In that case the corresponding operator is the same and D can be used in the min-max characterization to the price of changing the min for an inf.
- Or the closure of D, denoted by $\mathcal{D}_0$ is associated to a different operator A_0. Observe that

$$\forall n,\ \lambda_n(A) \leqslant \lambda_n(A_0).$$

Two applications

- Let V be a confining potential. C_0^∞ is a core for the corresponding Schrödinger operator. Denote by $(\lambda_n(x))_{n\geqslant 1}$ the spectrum of the corresponding Schrödinger operator on the interval $[-x,x]$ with Dirichlet boundary condition. Then

$$\forall n \geqslant 1,\ \lambda_n(x) \underset{x\to+\infty}{\longrightarrow} \lambda_n$$

- Suppose that $\mathcal{D}_0$ has codimension k in $\mathcal{D}$ then

$$\forall n \geqslant 1,\ \lambda_n(A_0) \leqslant \lambda_{n+k}(A).$$

(for any subspace $F \subset \mathcal{D}$ of dimension $n+k$ contains a subspace of $\dim(F\cap\mathcal{D}_0) \geqslant n$).