Introduction to spectral theory of unbounded operators.

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Definition

Definition

A Banach space $\mathcal{A}$ is a Banach algebra if there exits a multiplication on $\mathcal{A}$ such that $\mathcal{A}$ is an algebra with

1. $\forall x, y \in \mathcal{A} : \| xy \| \leq \| x \| \cdot \| y \|$.
2. It has an identity $e \in \mathcal{A}$ i.e $\forall x \in \mathcal{A}; x = ex = xe$, suppose that $\| e \| = 1$.

A Banach algebra is a Banach $*$algebra, ($*$algebra) if there exits an involution $f : \mathcal{A} \to \mathcal{A}$ $\forall x, y \in \mathcal{A}, \alpha \in \mathbb{C}, f(x+y) = f(x) + f(y), f(xy) = f(y)f(x), f(\alpha x) = \overline{\alpha} f(x)$, and $f^2(x) = x$.

Definition

A $*$algebra is called a $\mathbb{C}^*$-algebra if we have

$$\forall x \in \mathcal{A}, \|f(x)x\| = \|x^*x\| = \|x\|^2. \tag{1}$$

Remark

Equation (1), says $\mathbb{C}^*$-identity, is equivalent to

$$\forall x \in \mathcal{A}; \|x^*\| = \|x\|.$$

Example:

1. For any space X, the bounded linear operators $\mathcal{B}(X)$, form a Banach algebra with identity $\mathbf{1}_X$.
2. For any Hilbert space $\mathcal{H}, \mathcal{B}(\mathcal{H})$ is a $\mathbb{C}^*$-algebra when it is equipped with the adjoint map

$$* : H \in \mathcal{B}(\mathcal{H}) \mapsto H^* \in \mathcal{B}(\mathcal{H})$$

Duality

If X and Y are normed linear spaces and $T : X \to Y$, then we get a natural map $T^* : Y^* \to X^*$ by

$$T^* f(x) = f(Tx), \ \forall f \in Y^*, x \in X.$$

In particular, if $T \in B(X, Y)$, then $T^* \in B(Y^*, X^*)$. In fact,

$$\|T^*\|_{B(Y^*,X^*)} = \|T\|_{B(X,Y)}.$$

To prove this, note that

$$|T^* f(x)| = |f(Tx)| \leq \|f\| \cdot \|T\| \cdot \|x\|.$$

Therefore $\|T^* f\| \leq \|f\| \cdot \|T\|$, so T^* is indeed bounded, with

$$\|T^*\| \leq \|T\|.$$

Duality

Also, given any $y \in Y$, we can find $g \in Y^*$ such that $|g(y)| = \|y\|$, $\|g\| = 1$. Applying this with $y = Tx$ ($x \in X$ arbitrary), gives

$$\|Tx\| = |g(Tx)| = |T^*gx| \leq \|T^*\| \cdot \|g\| \cdot \|x\| = \|T^*\|\|x\|.$$

This shows that

$$\|T\| \leq \|T^*\|.$$

Note that if $T \in B(X, Y)$, $U \in B(Y, Z)$, then

$$(UT)^* = T^*U^*.$$

Let X, Y be Hilbert spaces. Let $T \in \mathcal{B}(X, Y)$ be a bounded linear transformation.

$$\|T\| = \sup\{\|Ah\|_Y : \|h\|_X \leq 1\}.$$

Then the norm of T satisfies:

$$\|T\|^2 = \|T^*\|^2 = \|T^*T\|$$

where T^* denotes the adjoint of T. Indeed Let $h \in X$ such that $\|h\|_X \leq 1$. Then:

$$\begin{aligned} \|Th\|_Y^2 &= \langle Ah, Ah\rangle_Y = \langle T^*Th, h\rangle_X \\ &\leq \|T^*Th\|_X \|h\|_X (Cauchy - Schwarz\ \ Inequality) \\ &\leq \|T^*T\| \|h\|_X^2 \leq \|T^*T\| \leq \|T^*\| \|T\| \end{aligned}$$

it follows that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \, \|X\| .$$

That is,

$$\|T\| \leq \|T^*\| .$$

By substituting T^* for T, and using $T^{**} = T$ from [Double Adjoint is Itself], the reverse inequality is obtained. Hence

$$\|T\|^2 = \|T^*T\| = \|T^*\|^2 .$$

Examples

Example 1: For any compact Hausdorff space S;

$$\mathcal{C}(S) = \{f : S \to \mathbb{C} \mid f \text{ Continous}\},$$

equipped with the norm

$$\| f \|_\infty = \sup_{x \in S} | f(x) |$$

is a commutative Banach algebra with identity $f = 1$, the involution

$$f^*(x) \equiv \overline{f(x)}$$

transforms it on a $\mathbb{C}^*$-algebra.

Examples

Example 2: The analytic functions

$$f : \mathbf{D}^1 = \{z \in \mathbb{C}; \mid z \mid < 1\} \to \mathbb{C}$$

with norm

$$\| f \|_\infty = \sup_{z \in \mathbf{D}} \mid f(z) \mid, \quad \text{the involution} : f(z) \mapsto \overline{f(\overline{z})}$$

form a commutative Banach algebra, but not a $\mathbb{C}^*$-algebra. With $f(z) = e^{iz}$; we have

$$\| f \|_\infty^2 = e^2 \neq \| f^* f \|_\infty = 1.$$

Definition

For a Banach algebra $\mathcal{A}$ with identity $\mathbf{1}_A$ we define

1. The resolvent set
$$\varrho_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \mathbf{1}_A \text{ Has two sided bounded inverse}\}$$
2. The spectrum of $x \in \mathcal{A}$
$$\sigma_{\mathcal{A}}(x) = \mathbb{C} \setminus \varrho_{\mathcal{A}}(x).$$
3. We call the inverse of $x - \lambda \mathbf{1}_{\mathcal{A}}$, the resolvent and denote as
$$R_\lambda(x) = (x - \lambda \mathbf{1}_{\mathcal{A}}) = \frac{1}{x - \lambda \mathbf{1}_{\mathcal{A}}}.$$

First resolvent formula

Lemma

For any $\lambda, \nu \in \varrho(x)$.

$$\begin{aligned} R_\lambda(x) - R_\nu(x) &= (\lambda - \nu) R_\lambda(x) R_\nu(x) \\ &= (\lambda - \nu) R_\nu(x) R_\lambda(x). \end{aligned}$$

Proof: Multiply both sides with $x - \lambda \mathbf{1}_{\mathcal{A}}$ or $x - \nu \mathbf{1}_{\mathcal{A}}$.

Neumann series

Theorem

Let $\mathcal{A}$ be a Banach algebra with identity and $x, y \in \mathcal{A}$ with x invertible and $\|x^{-1}y\| < 1$, then $x - y$ is invertible,

$$(x-y)^{-1} = \sum_{n=0}^{\infty} (x^{-1}y)^n x^{-1},$$

the series being absolutely convergent and

$$\|(x-y)^{-1}\| \leq \|x^{-1}\| / (1 - \|x^{-1}y\|).$$

Proof:

$$\begin{aligned}\|\sum (x^{-1}y)^n x^{-1}\| &\leq \|x^{-1}\| \sum \|x^{-1}y\|^n \\ &\leq \|x^{-1}\|/(1-\|x^{-1}y\|),\end{aligned}$$

so the sum converges absolutely and the norm bound holds. Also

$$\sum_{n=0}^{\infty}(x^{-1}y)^n x^{-1}(x-y) = \sum_{n=0}^{\infty}(x^{-1}y)^n - \sum_{n=0}^{\infty}(x^{-1}y)^{n+1} = \mathbf{1}_X,$$

and similarly for the product in the reverse order.

Remark

If f is an analytic function, i.e. f can be represented by a convergent power series, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we can define $f(T) = \sum_{n=0}^{\infty} a_n T^n$ (which is defined since $B(X)$ is Banach).

Proposition

Let X be a Banach space, $T \in B(X)$ with $\|T\| < 1$. Then $(I - T)^{-1} \in B(X)$ and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ (the Neumann series) in $B(X)$.

proof Let $S_k = \sum_{n=0}^{k} T^n$. Then, for $k < \ell$,

$$\|S_\ell - S_k\| = \left\| \sum_{k<n\leq\ell} T^n \right\| \leq \sum_{k<n\leq\ell} \|T^n\| \leq \sum_{k<n\leq\ell} \|T\|^n \leq \sum_{n=k+1}^{\infty} \|T\|^n \xrightarrow{k\to\infty} 0$$

Hence, $\{S_k\}$ is Cauchy in $B(X)$, so convergent. Let $S = \lim_{k\to\infty} S_k$ in $B(X)$.

$$(I - T)S_k x = \sum_{n=0}^{k} (T^n - T^{n+1})x = x - T^{k+1}x \xrightarrow{k \to \infty} x$$

since $\| T^{k+1}x\| \leq \| T\|^{k+1}\|x\|$. On the other hand $(I - T)S_k x \to (I - T)Sx$ as $k \to \infty$. Hence,

$$S = (I - T)^{-1}.$$

Proposition

Let $T \in B(X)$. Then $\rho(T) \subseteq \mathbb{C}$ is an open set, i.e. $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is closed, and the resolvent function $\varrho(T) \ni \lambda \mapsto R_\lambda(T) \in B(X)$ is a complex analytic map from $\rho(T)$ to $B(X)$ with

$$\|R_\lambda(T)\| \leq \frac{1}{d(\lambda, \sigma(T))},$$

i.e. for all $\lambda_0 \in \rho(T)$, there exists $r > 0$ such that

$$R_\lambda(T) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n T^n$$

for all $\lambda \in B_r(\lambda_0)$.

Proof: Use that $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ if $\|T\| < 1$ and

$$\begin{aligned} T - (\lambda - \mu) I &= (T - \lambda I)(I - \mu R_\lambda(T)) \\ &= (T - \lambda I) S(\mu). \end{aligned}$$

Then $S(\mu)$ is invertible if $|\mu| \|R_\lambda(T)\| < 1$. Hence,

$$\begin{aligned} R_{\lambda - \mu}(T) &= S(\mu)^{-1} R_\lambda(T) \\ &= \sum_{k=0}^{\infty} \mu^k R_\lambda(T)^{k+1}. \end{aligned}$$

Proposition

Let X, Y be Banach spaces. Then the set of invertible operators in $B(X,Y)$ is an open set. If $X \neq 0$ and $Y \neq 0$, then for $S, T \in B(X)$, T invertible and $\|S - T\| < \|T^{-1}\|^{-1}$ implies S is invertible.

proof: Let $R = T - S$. Then $S = T(I - T^{-1}R) = (I - RT^{-1})T$ where $\|T^{-1}R\| < 1$ and $\|RT^{-1}\| < 1$.

Important implication of Neumann series Theorem

1. $\{x \in \mathcal{A} | 0 \in \varrho(x)\}$ is open.
2. $\forall x \in \mathcal{A}$; $\varrho(x)$ is an open subset of $\mathbb{C}$, so $\sigma(x)$ is a closed set.
3. $\forall x \in \mathcal{A}$, the resolvent

$$\lambda \mapsto R_\lambda(x) = (x - \lambda \mathbf{1}_\mathcal{A})^{-1}$$

is an $\mathcal{A}$-valued analytic function. In particular

$$\lim_{\lambda \to \lambda_0} \frac{R_\lambda(x) - R_{\lambda_0}(x)}{\lambda - \lambda_0} = R^2_{\lambda_0}(x).$$

$$\forall f \in \mathcal{A}^* : \varrho(x) \ni \lambda \mapsto f(R_\lambda(x)) \in \mathbb{C} \text{ is analytic.}$$

4. $\forall x \in \mathcal{A}, \sigma(A) \neq \emptyset$ and it is a compact subset of the disc of radius $\| x \|$.

Definition

Let Ω be an open set of $\mathbb{C}$, and $\mathcal{A}$ is a Banach space. Let $f : \Omega \to \mathcal{A}$. We say that f is analytic in Ω if for any $\lambda_0 \in \Omega$

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f'(\lambda_0),$$

exists. It is equivalent to $\varphi \circ f : \mathbb{C} \to \mathbb{C}$ is analytic for any $\varphi \in \mathcal{A}'$.

Suppose that $\sigma(x) = \emptyset$, so $\varrho(x) = \mathbb{C}$, we conclude that $R_\lambda(x)$ is an entire function with value in $\mathcal{A}$. For $\mid \lambda \mid > \| x \|$,

$$R_\lambda(x) = -\sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}.$$

So

$$\| R_\lambda(x) \| \leq \frac{1}{\mid \lambda \mid - \| x \|}.$$

$R_\lambda(x)$ is a bounded and entire function, so by Liouville theorem, we deduce that $R_\lambda(x)$ is constant on $\mathbb{C}$. As

$$\lim_{|\lambda| \to \infty} R_\lambda(x) = 0.$$

We get $R_\lambda(x) = 0, \forall \lambda \in \mathbb{C}$, which is absurd.

Remark

The fact that the spectrum of an element of $\mathcal{A}$ is non empty it is a generalization of the fact that any matrix of $\mathcal{M}_n(\mathbb{C})$ has at least one eigenvalue.

Spectral radius formula

Theorem

$\forall x \in \mathcal{A}$

1. $\lim_{n \to +\infty} \| x^n \|^{1/n}$ *exists and equal* $r(x)$.
2. $$r(x) = \sup\{| \lambda | \ \ | \lambda \in \sigma(x)\}.$$

Remark

An element of an algebra $\mathcal{A}$ is invertible or not is a property which is purely algebraic. So the spectrum and the spectral radius of x depend only on the algebraic structure of $\mathcal{A}$ and not of the metric or the topology, but the limit in the last theorem depends on the properties of the metric of $\mathcal{A}$. It is one of the remarkable aspects of the theorem, which affirms the correspondence of two quantities with different origins.

Remark

The algebra $\mathcal{A}$ could be a subalgebra of another Banach algebra $\mathcal{B}$. So it is possible for an $x \in \mathcal{A}$ to be non invertible in $\mathcal{A}$ and invertible in $\mathcal{B}$. So the spectrum of x depends on the algebra. If we note by $\sigma_{\mathcal{A}}(x)$ (resp. $\sigma_{\mathcal{B}}(x)$) the spectrum of x relatively to $\mathcal{A}$ (resp. $\mathcal{B}$), so $\sigma_{\mathcal{A}}(x) \subset \sigma_{\mathcal{B}}(x)$. The spectral radius is the same in $\mathcal{A}$ and $\mathcal{B}$.

Proof:

1. Set $a_n = Ln\|x^n\|$, then
$$\forall n, m \in \mathbb{N}; a_{n+m} \leq a_n + a_m.$$
Fix $k \in \mathbb{N}$ and write $n = mk + r; 0 \leq r \leq k-1$
$$a_n \leq ma_k + \max_{0\leq r\leq k-1} a_r, \Rightarrow \limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k}$$
$$\Rightarrow \limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_k \frac{a_k}{k} \leq \liminf_{n\to-\infty} \frac{a_n}{n}.$$
2. Let α be the limit of $\| x^n \|^{\frac{1}{n}}$. Let $\lambda \in \sigma(x)$, so $\lambda^n \in \sigma(x^n)$ so
$$| \lambda^n | \leq \| x^n \| .$$
We get that $r(x) = sup_{\lambda\in\sigma(x)} | \lambda | \leq \alpha.$

The opposite inequality is based on the theory of holomorphic functions and entire series. Let $\Omega = D(0, \frac{1}{r(x)})$, if $r(x) = 0, \Omega = \mathbb{C}$. Consider $f : \Omega \to \mathcal{A}$ defined $f(0) = 0$ and

$$f(\lambda) = R_{1/\lambda}(x), \quad \lambda \in \Omega \backslash \{0\}.$$

Using the properties of the resolvent we can write that for $0 < |\lambda| < \frac{1}{\|x\|}$

$$f(\lambda) = -\sum_{n=0}^{+\infty} \lambda^{n+1} x^n.$$

Let R be the radius of convergence of the power series $R \geq d(0, \Omega^c) = \frac{1}{r(x)}$. Using Hadamard formula

$$\frac{1}{R} = \limsup_{n \to +\infty} \| x^n \|^{\frac{1}{n}},$$

So finally

$$\limsup_{n\to+\infty} \| x^n \|^{\frac{1}{n}} \leq r(x).$$

Application: Volterra Integral Kernels

Let $K : [0,1]\times[0,1] \to \mathbb{C}$, continous $V_K : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$

$$f \mapsto \int_0^t K(t,s)f(s)ds.$$

We have $\| V_K \|_\infty \leq \| K \|_\infty \| f \|_\infty$ So $V_K \in \mathcal{B}(\mathcal{C}([0,1]))$.

$$(V_K^n f)(t) = \int_{0\leq s_1 \leq \cdots \leq s_n \leq t} K(t,s_n)K(s_n,s_{n-1})\cdots K(s_2,s_1)f(s_1)ds_1\cdots ds_n.$$

$$\| V_K^n f \|_\infty \leq \| K \|_\infty^n \| f \|_\infty \cdot \sup_{t\in[0,1]} Vol\{(s_1,\cdots,s_n) \mid 0 \leq s_1 \leq \cdots \leq t\}$$

$$\leq \frac{\| K \|_\infty^n}{n!} \cdot \| f \|_\infty .$$

So $r(V_K) = \lim_{n\to\infty} \|V_K^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{\|K\|_\infty}{(n!)^{1/n}} = 0.$ and $\sigma(V_K) = \{0\}$. (Hint: $Ln(n!) \approx nLn(n)$)

Functional calculus of operators

How we can define $f(x)$ for a large class of functions f and (un)bounded linear operator x?

1. Polynomial functional calculus.
2. Analytic functional calculus.
3. Continuous functional calculus.
4. Measurable functional calculus.

Let $\mathcal{A}$ be a Banach algebra with identity and $P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0, a_j \in \mathbb{C}$ a polynomial. If $x \in \mathcal{A}$, then

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{A}.$$

Spectral mapping

Theorem

$$\forall x \in \mathcal{A}: \ \sigma(P(x)) = P(\sigma(x)) = \{P(\lambda) \in \mathbb{C}; \lambda \in \sigma(x)\}.$$

Proof:

Lemma

Let $x_1, \cdots, x_n \in \mathcal{A}$ be mutually committing, then

$$y = x_1 \cdots x_n \text{ invertible} \Leftrightarrow x_1, \cdots, x_n \text{ are each invertible}$$

Proof:

1. $\Rightarrow x_1(x_2 \cdots x_n)y^{-1} = yy^{-1} = \mathbf{1}_A$
$y^{-1}(x_1 \cdot x_2 \cdots x_n) = \mathbf{1}_{\mathcal{A}} = y^{-1}(x_2 \cdots x_n)x_1 =$
$y^{-1}(x_1 \cdots x_n) = y^{-1}y = \mathbf{1}_{\mathcal{A}}$ So x_1 has left and right inverses.
So it is invertible and are the same.
2. $\Leftarrow y^{-1} = x_n^{-1} \cdots x_1^{-1}$.

Proof of the spectral mapping

Proof of the spectral mapping Let $\lambda \in \sigma(P(x)) \Leftrightarrow q(x) = P(x) - \lambda$ is not invertible. $Q(t) = (t - \mu_1) \cdots (t - \mu_n)$. As $x - \mu_i$ and $x - \mu_j$ commute for any i, j, applying the last Lemma we get

$$\lambda \in \sigma(P(x)) \Leftrightarrow \exists j, \mu_j \in \sigma(x) \Leftrightarrow \lambda \in P(\sigma(x)).$$

Spectral mapping-Analytic function

Let $f : \mathbb{C} \to \mathbb{C}$, an entire function $f(t) = \sum_{n=0}^{\infty} a_n t^n$. For

$$x \in \mathcal{A}, f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}. \tag{2}$$

More general function $f : B(0,r) \to \mathbb{C}$ analytic with $r > r(x)$. Using Cauchy integral formula we write

$$f(x) = \frac{1}{2\pi i} \oint_{|\lambda|=r} f(\lambda)(\lambda - x)^{-1} d\lambda$$

Definition

Definition

Let $x \in \mathcal{A}$ and $G \subset \mathbb{C}$ open connected domain such that $\sigma(x) \subset G$. Let $f : G \to \mathbb{C}$ analytic and $\Gamma \subset G \cap \varrho(x)$ a contour. We set

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda)(\lambda - x)^{-1} d\lambda \in \mathcal{A}. \tag{3}$$

Proposition

The equation (3) define an application from the algebra of analytic functions on $G \supset \sigma(x)$ to $\mathcal{A}$. This map is linear and satisfies for $f, g : G \to \mathbb{C}$ analytic on $G \supset \sigma(x)$ and Γ_f, Γ_g admissible contours s.t $\Gamma_f \cap \Gamma_g = \emptyset$.

$$f(x)g(x) = (fg)(x).$$

$$f(x) = \frac{1}{2\pi i}\oint_{\Gamma_f} f(\lambda)(\lambda - x)^{-1} d\lambda, \; g(x) = \frac{1}{2\pi i}\oint_{\Gamma_g} g(\mu)(\mu - x)^{-1} d\mu.$$

$$\begin{aligned} f(x)g(x) &= \frac{1}{(2\pi i)^2}\oint_{\Gamma_f}\oint_{\Gamma_g} f(\lambda)g(\mu)(\lambda-x)^{-1}(\mu-x)^{-1}d\lambda d\mu \\ &= \frac{1}{(2\pi i)^2}\oint_{\Gamma_g}(\oint_{\Gamma_f}\frac{1}{\lambda-\mu}f(\lambda)d\lambda)g(\mu)(\mu-x)^{-1}d\mu \\ &- \frac{1}{(2\pi i)^2}\oint_{\Gamma_f}(\oint_{\Gamma_g}\frac{1}{\lambda-\mu}g(\mu)d\mu)f(\lambda)(\lambda-x)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\oint_{\Gamma_f}(\lambda-x)^{-1}g(\lambda)f(\lambda)d\lambda = (fg)(x). \end{aligned}$$

So we get an algebraic homomorphism

Theorem

$\forall x \in \mathcal{A}$ and analytic $f : G \to \mathbb{C}$ on domain $G \supset \sigma(x)$.

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) | \lambda \in \sigma(x)\}.$$

Proof

If $\mu \notin f(\sigma(x))$, then $G \ni \lambda \to g(\lambda) = (f(\lambda) - \mu)^{-1}$ is analytic. So $g(x)$ is the inverse of $f(x) - \mu$, so $\mu \notin \sigma(f(x))$.
If $\mu \in f(\sigma(x))$; then $\exists \lambda \in \sigma(x); \mu = f(\lambda)$. Then

$$g(z) = \frac{f(z) - f(\lambda)}{z - \lambda},$$

has a false singularity at $z = \lambda$. Hence is analytic on G. So,

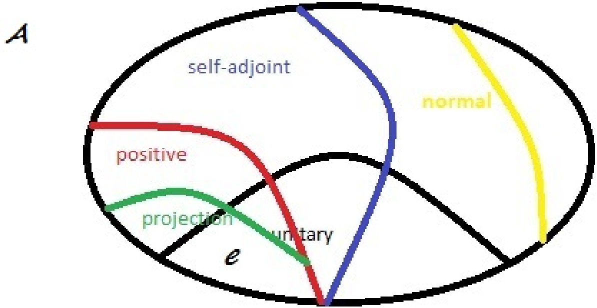
$$f(x) - \mu = (x - \lambda)g(x) = g(x)(x - \lambda).$$

So $f(x) - \mu$ is not invertible since $\lambda \in \sigma(x)$ i.e $\mu \in \sigma(f(x))$.

Some particular elements of a $*$-algebra $\mathcal{A}$

Definition

1. $x \in \mathcal{A}$ is normal iff $x^*x = xx^*$.
2. $x \in \mathcal{A}$ is self adjoint iff $x^* = x$.
3. $x \in \mathcal{A}$ is positif iff $\exists y \in \mathcal{A}; x = yy^*$.
4. $x \in \mathcal{A}$ is projection iff $x^2 = x = x^*$.
5. $x \in \mathcal{A}$ is unitary iff $x^*x = xx^* = \mathbf{1}_{\mathcal{A}}$.


$\mathcal{A}$
self-adjoint
normal
positive
projection
unitary
e

Theorem

If x is normal in $\mathbb{C}^*$-algebra $\mathcal{A}$, then $r(x) = \| x \|$.

Proof:

$$\| x^2 \| = \| xx^* \| = \| x \|^2 .$$

By induction $n \in \mathbb{N}^*$,

$$\| x^{2n} \| = \| x \|^{2n} .$$

So,

$$r(x) = \lim_{n\to\infty} \| x^{2n} \|^{1/2n} = \| x \| .$$

Remark

The norm of a $\mathbb{C}^*$-algebra $\mathcal{A}$ is uniquely determined by the algebraic structure

$$\| x \|^2 = \| xx^* \| = r(xx^*) = \sup\{| \lambda |; \lambda \in \sigma(xx^*)\}.$$

Theorem

Let $x \in \mathcal{A}$.

1. *If x is unitary, then $\sigma(x) \subset \partial \mathbf{D}$.*
2. *If x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.*

Proof:

Let $x \in \mathcal{A}$ be an unitary operator, then

$$\| x \|^2 = \| xx^* \| = \| \mathbf{1}_{\mathcal{A}} \| = 1.$$

As $x^{-1} = x^*$, then $0 \notin \sigma(x)$,

$$x^{-1} - \lambda^{-1} = x^{-1}\lambda^{-1}(\lambda - x) \forall \lambda \neq 0,$$

we conclude that

$$\lambda \in \sigma(x) \Leftrightarrow \lambda^{-1} \in \sigma(x^{-1}).$$

Let $y = e^{ix} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n$. As the involution $*$ is a continuous map on $\mathcal{A}$, then $y^* = e^{-ix}$, and

$$y^* y = y y^* = \mathbf{1}_{\mathcal{A}}.$$

So y is unitary operator and

$$\sigma(y) \subset \partial \mathbf{D}$$

and

$$\sigma(y) = e^{i\sigma(x)} \subset \partial \mathbf{D} \Leftrightarrow \sigma(x) \subset \mathbb{R}.$$

Unbounded operators on Hilbert spaces and their spectral theory: Basics

In the following we consider X, Y two Hilbert spaces and linear operator $A : \mathcal{D}(A) \subset X \to Y$. We suppose that $\mathcal{D}(A)$ is dense in X.

Examples: Maximal multiplication operator associated with measurable $f : M \to \mathbb{C}$ over some measure space (M, μ)

$$\mathcal{D}(M_f) = \{\psi \in L^2(M, \mu) | M_f\psi = f\psi \in L^2(M, \mu)\}.$$

Lemma

Suppose that (M, μ) is σ-finite. Then we have equivalence

1. $M_f \in B(L^2(M, \mu))$
2. $f \in L^\infty(M, \mu)$

Proof: $" \Leftarrow "$, for all $\psi \in \mathcal{D}(M_f)$:

$$\|M_f\psi\| = (\int |f\psi|^2 d\mu)^{\frac{1}{2}} \leq \|f\|_\infty \cdot \|\psi\|.$$

$" \Rightarrow "$ As (M, μ) is σ-finite, $\exists (M_n)_n$:

$$M = \cup_n M_n, \mu(M_n) < \infty.$$

Suppose that

$$\|M_f\| = \sup\{\|M_f\psi\| | \psi \in \mathcal{D}(M_f), \|\psi\| = 1\} < \infty.$$

Consider $\chi_{n,A} = \chi_{\{x \in M_n |, |f(x)| > A\}}, A \in [0, \infty)$.

$$\begin{aligned} A^2 \cdot \mu\{x \in M_n | |f(x)| > A\} &\leq \int |f|^2 |\chi_{n,A} d\mu \\ &\leq \|M_f\|^2 \mu\{x \in M_n | |f(x)| > A\}. \end{aligned}$$

This gives that

$$\mu\{x \in M_n | |f(x)| > A\} = 0, \text{when } A > \|M_f\|, \forall n.$$

$\Rightarrow f \in L^\infty(M, \mu)$. Thus M_f is an unbounded operator with $\mathcal{D}(M_f) \neq L^2(M, \mu)$ in case $f \notin L^\infty(M, \mu)$.

Differential operator on $I = (0, 1)$

$$T_0 : C^1 \to L^2(I),\, T_0\psi = -i\psi'$$

$$f_n(x) = x^n, n \in \mathbb{N},\, T_0 f_n(x) = -inx^{n-1}, \frac{\|T_0 f_n\|}{\|f_n\|} = \frac{n\sqrt{2n+1}}{\sqrt{2(n-1)}}.$$

$$T_{max} : W^{1,2}(I) \to L^2(I),\, T_{max}\psi = -i\psi'$$

Here

$$W^{1,2}(I) = \{\psi : I \to \mathbb{C} | \psi, \psi' \in L^2(I)\}.$$

It is an Hilbert space when equipped by the norm

$$\|\psi\|^2_{W^{1,2}} = \|\psi\|^2_{L^2} + \|\psi'\|^2_{L^2}.$$

Both operators are unbounded. T_{max} is an extension of T_0.

Definition

Let $B : \mathcal{D}(B) \to Y$ and $A : \mathcal{D}(A) :\to Y$. We say that A is an **extension** of B, if $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $Ax = Bx$ for all $x \in \mathcal{B}$, we write

$$B \subset A.$$

Closed and closable operators

Definition

Let $A : \mathcal{D}(A) \to Y$ be a linear operator on Hilbert spaces X, Y with $\mathcal{D}(A)$ is dense in X

1. We call the graph of A the set

$$Grph(A) = \{(x, Ax) \in X \times Y; x \in \mathcal{D}(A)\},$$

   and the graph norm of $x \in \mathcal{D}(A)$ is $\|x\|_A = \|(x, Ax)\|_{X \times Y}$.
2. A is said to be closed if $Graph(A)$ is a closed subset of $X \times Y$, with respect to the topology induced by $\|(x, y)\|^2_{X \times Y} = \|x\|_X + \|y\|_Y$.
3. We call A is closable if it has a closed extension. We denote the smallest closed extension of A by $\overline{A}$.

Remark

$X \times Y$ is an Hilbert space with the scalar product

$$\langle (x,y),(x',y')\rangle_{X\times Y} = \langle x,x'\rangle_X + \langle y,y'\rangle_Y.$$

Lemma

$G \subset X \times Y$ is a graph of an operator $A : \mathcal{D}(A) \to Y$ if and only if G is a subspace with the property:

$$(0, y) \in G \Rightarrow y = 0.$$

Proof: $\Leftarrow$ Let $(x, y), (x, y') \in G$ as G is a subspace we get that $(0, y - y') \in G \Rightarrow y = y'$ so for every $x \in X$, there is at most one $y \in Y$ such that $(x, y) \in G$. So the map $A : \mathcal{D}(A) \to Y$ with

$$\mathcal{D} = \{x \in X | \exists y \in Y : (x, y) \in G\},$$

we set

$$Ax = y.$$

It is a well defined as linear operator with $Graf(A) = G$.

Lemma

Let $(A, \mathcal{D}(A))$ be a linear operator. A is closable if and only if $\overline{Graph(A)}$ is a graph.

Proof: $\Leftarrow$ Let $B : \mathcal{D}(B) \to Y$, with

$$\mathcal{D}(B) = \{x \in X; \exists y \in Y : (x, y) \in \overline{Graph(A)}\},$$

we define

$$Bx = y.$$

It is a linear operator with $Graph(B) = \overline{Graph(A)}$ and $Graph(A) \subset Graph(B)$, and hence $\mathcal{D}(A) \subset \mathcal{D}(B)$.

$\Rightarrow$ Let $B : \mathcal{D}(B) \to Y$ be a closed extension of A. If $(0, y) \in \overline{Graph(A)}$, then $(0, y) \in Graph(B)$; i.e $y = 0$.

Characterization of closed operators

Theorem

For a linear operator $A : \mathcal{D}(A) \to Y$ densely defined on $\mathcal{D}(A) \subset X$ the following properties are equivalent

1. *A is closed.*
2. *$(\mathcal{D}(A), \|\cdot\|_A)$ is complete.*
3. *If $(x_n)_n \subset \mathcal{D}(A)$ with x_n converges to x and Ax_n converges to y then $x \in \mathcal{D}(A)$ and $Ax = y$.*

Proof: (1) $\Rightarrow$ (3) Let $(x_n) \subset \mathcal{D}(A)$ with x_n converges to x and Ax_n converges to y. Then $(x_n, Ax_n) \in Graph(A)$, with

$$\| (x_n, Ax_n) - (x, y) \|_{X \times Y} \to 0.$$

Thus $(x, y) \in \overline{Graph(A)} = Graph(A)$, i.e $x \in \mathcal{D}(A)$ and $Ax = y$.
(3) $\Rightarrow$ (2) Let $(x_n) \subset \mathcal{D}(A)$ be a Cauchy sequences w.r.t. $\| \cdot \|_A$. Then (x_n) is a Cauchy sequence w.r.t. $\| \cdot \|_X$ and (Ax_n) is a Cauchy sequence w.r.t. $\| \cdot \|_Y$. Completeness of X and Y imply :$\exists x \in X, y \in Y$ such that

$$\|x_n - x\|_X \to 0, \|Ax_n - y\|_Y \to 0.$$

Thus $x \in \mathcal{D}(A)$ and $y = Ax$ and

$$\|(x_n, Ax_n) - (x, y)\|_{X \times Y} \to 0.$$

(2) $\Rightarrow$ (1) Let $(x_n, Ax_n) \subset Graph(A)$ converges to (x, y). Then $(x_n) \subset \mathcal{D}(A)$ is a Cauchy sequence w.r.t. $\|\cdot\|_A$, and hence

$$\exists x' \in \mathcal{D}(A) : \|x' - x_n\|_A \to 0, x_n \to x' \quad , Ax_n = Ax'.$$

Uniqueness of the limit in X and Y yields that $x = x'$ and $Ax' = y$.

Examples

Example 1: Dirac Delta function on $X = L^2((-1,1)), \mathcal{D}(A) = C((-1,1))$,

$$(A\psi)(x) = \psi(0).$$

This operator is not closable as there exists $(\psi_n) \subset C((-1,1))$ with $\psi_n(0) = 1$ and $\|\psi_n\| \to 0$ and $A\psi_n = 1 \neq 0$.

Differentiation operators on $I \subset \mathbb{R}$

Example1:

$$T_{max} : W^{1,2}(I) \to L^2(I),\, T_{max}\psi = -i\psi'$$

Here

$$W^{1,2}(I) = \{\psi : I \to \mathbb{C} | \psi, \psi' \in L^2(I)\}.$$

T_{max} is closed since $\|\cdot\|_{T_{max}} = \|\cdot\|_{W^{1,2}}$ and $W^{1,2}(I)$ is a Hilbert space with norm $\|\cdot\|_{W^{1,2}}$.

$$T_0 : C^1 \to L^2(I)$$

T_0 is closable.

Remark

The closure $\overline{A}$ of a closable operator $A : \mathcal{D}(A) \to Y$ is uniquely defined through

$$\mathcal{D}(\overline{A}) = \{x \in X | \exists (x_n) \subset \mathcal{D}(A) : x_n \to x; (Ax_n) \text{ converges}\}$$

$$\overline{A}x = \lim_{n\to\infty} Ax_n.$$

Definition

Let $A : \mathcal{D}(A) \subset X \to Y$ be a **densely** defined linear operator on Hilbert spaces X, Y. The operator $A^* : \mathcal{D}(A^*) \to X$, with

$$\mathcal{D}(A^*) = \{y \in Y | \exists y^* \in X : \langle Ax, y \rangle_Y = \langle x, y^* \rangle_X, \forall x \in \mathcal{D}(A)\}.$$

$$A^* y = y^*,$$

is called the adjoint of A.

Example: Differential operators T_0 and T_{max}

Example: Let $\psi \in \mathcal{C}_c^\infty(I)$ and $\varphi \in W^{1,2}(I)$. Then,

$$\langle T_0\psi, \varphi\rangle = \int_a^b -i\psi(x) \cdot \overline{\varphi(x)} dx = \Big[- i\psi(x) \cdot \overline{\varphi}(x)\Big]_a^b \quad (4)$$

$$+ \int_a^b \psi(x) \cdot \overline{-i\varphi'(x)} dx = \langle \psi, T_{max}\varphi\rangle. \quad (5)$$

Thus

$$T_0^* = T_{max}.$$

It is possible to describe the adjoint using the graph. Let

$$J : X \times Y \to Y \times X$$

$$(x, y) \mapsto J((x, y)) = (-y, x)$$

J is an isometric isomorphism.

Lemma

Let $A : \mathcal{D}(A) \to Y, B : \mathcal{D}(B) \to Y$ be two operators densely defined on X.

1. $Graph(A^*) = (JGraph(A))^{\perp} = J(Graph(A)^{\perp})$.
2. $B \subset A \Rightarrow A^* \subset B^*$.

Proof:

(1) By definition of A^*

$$\begin{aligned} Graph \quad (\quad A^*) &= \{(y,z) \in Y \times X | \langle Ax, y\rangle = \langle x, z\rangle, \forall x \in \mathcal{D}(A)\} \\ &= \{(y,z) \in Y \times X | \langle(-Ax, x), (y,z)\rangle_{Y\times X} = 0, \forall x \in \mathcal{D}(A)\} \\ &= \{(y,z) \in Y \times X | \langle J(v,w), (y,z)\rangle_{Y\times X} = 0, \forall v, w \in Graph(A) \\ &= \Big(J(Graph(A))\Big)^{\perp} = J\Big(Graph(A)\Big)^{\perp}. \end{aligned}$$

(2)

$$\begin{aligned} Graph(B) \subset Graph(A) &\Rightarrow J(Graph(B)) \subset J(Graph(A)) \\ &\Rightarrow (J(Graph(A)))^{\perp} \subset (J(Graph(B)))^{\perp} \\ &\Rightarrow Graph(B^*) \supset Graph(A^*). \end{aligned}$$

Theorem

Let $A : \mathcal{D}(A) \to Y, \mathcal{D}(A) \subset X$ be a densely defined operator on Hilbert spaces X, Y. Then,

1. *A^* is closed.*
2. *If A admits a closure $\overline{A}$, then $\overline{A}^* = A^*$.*
3. *A^* is densely defined if and only if A is closable.*
4. *If A is closable, then it is closure $\overline{A}$ is $(A^*)^*$.*

Proof:

(1) Since $V^{\perp}$ is closed for any V, the graph $Graph(A^*)$ is closed by previous lemma.

(2)

$$
\begin{aligned}
Graph\left(\overline{A}^*\right) &= \left(J(Graph(\overline{A}))\right)^{\perp} = (J(\overline{Graph(A)}))^{\perp} = (\overline{J(Graph(A))})^{\perp} \\
&= (J(Graph(A)))^{\perp} = Graph(A^*).
\end{aligned}
$$

(3) We have

$$\begin{aligned}\overline{Graph(A)} &= (Graph(A)^{\perp})^{\perp} = (J^{-1}(Graph(A^*)))^{\perp}(bytheprecedentlemr\\ &= \{(x,y) \in X \times Y \mid \langle J^{-1}(z, A^*z), (x,y)\rangle_{X\times Y} = 0,\\ \forall &\quad z \in \mathcal{D}(A^*)\}\\ &= \{(x,y) \in X \times Y \mid \langle A^*z, x\rangle_X = \langle z, y\rangle_Y, \forall z \in \mathcal{D}(A^*)\}.\end{aligned}$$

Thus $(0,y) \in \overline{Graph(A)} \Leftrightarrow y \in \mathcal{D}(A^*)^{\perp}$.

$$\overline{Graph(A)} \text{ is a graph } \Leftrightarrow \mathcal{D}(A^*) \text{ is dense.}$$

(4) Using (3), we conclude that A^{**} is well defined and

$$\begin{aligned} Graph(A^{**}) &= (J^{-1}(Graph(A^{*})))^{\perp} = (J^{-1}J(Graph(A)^{\perp}))^{\perp} \\ &= (Graph(A)^{\perp})^{\perp} = \overline{Graph(A)}, i.e\ \overline{A} = A^{**}. \end{aligned}$$

Definition

A densely defined linear operator $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subset X$ on Hilbert space X is called

1. Symmetric iff $A \subset A^*$.
2. Self-adjoint iff $A = A^*$.
3. Essentially self adjoint iff A^* is self adjoint.

Remark

If A is essentially self-adjoint operator then $A \subset A^{**} = A^*$ i.e A is symmetric.

Theorem

1. *Every symmetric operator A is closable with $\overline{A} \subset A^*$*
2. *Equivalent statements*
   1. *A is e.s.a. $(A^{**} = A^*)$.*
   2. *$\overline{A} = A^*$.*
   3. *$\overline{A}$ is self-adjoint, in this case $\overline{A}$ is the unique self adjoint extension of A.*

Proof:

(1) A^* is closed extension of A.

(2) $(1) \Rightarrow (2)$, $\overline{A} = A^{**} = A^*$

$(2) \Rightarrow (3) : \overline{A} = A^{**} = \overline{A}^*$

$(3) \Rightarrow (1) A^* = \overline{A}^* = \overline{A} = A^{**}$ (last therem)

If $\tilde{A}$ is a s.a. extension of A, then $\tilde{A} = \tilde{A}^* \subset A^* = \overline{A} \subset \tilde{A} \Rightarrow \tilde{A} = \overline{A}$.

Example: Maximal multiplication operator, with measurable $f : M \to \mathbb{R}$ over some σ-finite measure space (M, μ)

$$\mathcal{D}(M_f) = \{\psi \in L^2(M, \mu) \mid f\psi L^2(M, \mu)\}$$

$$M_f \psi = f\psi.$$

Let $(x, y) \in Graph(M_f^*), y = M_f^* x$, then for $\psi \in \mathcal{D}(M_f)$.

$$\left| \int \overline{x} f \psi d\mu \right| \leq \| y \| \cdot \| \psi \|,$$

so $\psi \mapsto \int \overline{x} f \psi d\mu = \langle M_f \psi, x \rangle = \langle \psi, y \rangle$ extends uniquely to a bounded functional on $L^2(M, \mu)$, i.e $f\overline{x} \in L^2(M, \mu)$ and $\overline{y} = f\overline{x}$ Therefore $(x, y) \in Graph(M_f^*) \Leftrightarrow y \in \mathcal{D}(M_f)$ and $y = fx$. So

$$M_f = M_f^*.$$

Particular case: $M = \mathbb{R}^d, \mu =$ Lebesgue measure $f(k) = \mid k \mid^2$, define a self-adjoint operator M_f. The Fourier transformation

$$\mathfrak{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

$$(\mathfrak{F}\psi)(x) = \int_{\mathbb{R}^d} e^{-ik\cdot x}\psi(k)\frac{dk}{(2\pi)^{d/2}}.$$

Define a unitary transformation with

$$\mathfrak{F}M_f\psi = -\Delta\mathfrak{F}\psi,$$

and $\mathfrak{F}(\mathcal{D}(M_f)) = \mathcal{D}(L)$, the Laplacian

$$\mathcal{D}(L) = \{\psi \in L^2(\mathbb{R}^d) | \Delta\psi \in L^2(\mathbb{R}^d)\} = \mathfrak{F}\mathcal{D}(M_f).$$

$$L\psi = -\Delta\psi.$$

Remark

Using similar reasoning allows to conclude that all differential operators of the form $Pol(\nabla)$ *are self-adjoint provided* $Pol(ik) \in \mathbb{R}$ *for all* $k \in \mathbb{R}^d$.

We recall that, $T_0\psi = -i\psi', \mathcal{D}(T_0) = \mathcal{C}_c^\infty(I)$, and $\mathcal{D}(T_{max}) = W^{1,2}(I)$. $T_0 \subset T_{max} = T_0^*$. So T_0 is a symmetric operator

$$\mathcal{D}(\overline{T_0}) = \{\psi \in W^{1,2}(I) | \psi(a) = \psi(b) = 0\} = W_0^{1,2}(I).$$

$$\overline{T}_0\psi = -i\psi',$$

it is not essentially self adjoint.

For $\beta \in [0, 2\pi)$ let

$$\mathcal{D}(T_\beta) = \{\psi \in W^{1,2}(I) | \psi(b) = e^{i\beta}\psi(a)\}.$$

$$T_\beta \psi = -i\psi'.$$

Then

1. $T_0 \subset T_\beta \subset T_{max}$
2. $T_\beta \subset T_\beta^*$ i.e T_β symmetric, since $\varphi, \psi \in \mathcal{D}(T_\beta)$

$$_\beta\psi, \varphi\rangle = \int_a^b -i\psi'(x)\overline{\varphi(x)}dx \quad (6)$$

$$= [-i\psi(x)\overline{\varphi(x)}]_a^b + \int_a^b \psi(x)\overline{-i\varphi'(x)}dx \quad (7)$$

$$= \langle\psi, T_\beta\varphi\rangle. \quad (8)$$

$T_\beta^* = T_\beta$, since $\forall \varphi \in \mathcal{D}(T_\beta^*), \psi \in \mathcal{D}(T_\beta)$

$$\int_a^b -i\psi'(x)\overline{\varphi(x)}dx = \langle T_\beta \psi, \varphi \rangle = \langle \psi, T_\beta^* \varphi \rangle \quad (9)$$

$$= \int_a^b \psi(x)\overline{-i\varphi'(x)}dx. \quad (10)$$

So T_0 has infinitely many self adjoint extensions

Theorem

For any densely defined linear operator A on a Hilbert space X.

$$\overline{RanA} \oplus kerA^* = X.$$

Proof: It suffices to prove that $kerA^*$ is the orthogonal complement of $RanA$. Let $u \in RanA$ and $v \in kerA^*$. Then there exists $f \in \mathcal{D}(A)$ such that $u = Af$. We compute

$$\langle u, v \rangle = \langle Af, v \rangle = \langle f, A^* v \rangle = 0.$$

and thus $kerA^* \subset (RanA)^\perp$. Now let $w \in (RanA)^\perp$. For $u = Af \in RanA$, we have

$$0 = \langle u, w \rangle = \langle Af, w \rangle = \langle f, A^* w \rangle, \ \forall f \in \mathcal{D}(A).$$

(Notice that $\langle Af, w \rangle = 0$ implies that $w \in \mathcal{D}(A^*)$.) As $\mathcal{D}(A)$ is dense, it follows that $A^* w = 0$, that is $(RanA)^\perp \subset kerA^*$.

Theorem

Let $A : \mathcal{D} \to X$ be a symmetric operator with the property that $Ran(A) = X$. Then A is selfadjoint.

Proof: As $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, it suffice to show that if $f \in \mathcal{D}(A^*)$ then $f \in \mathcal{D}(A)$.
Let $g = A^*f$. As $rang(A) = X$, there exists $h \in \mathcal{D}(A)$ so that $g = Ah$.

$$\forall v \in \mathcal{D}(A), \ \langle Av, f\rangle = \langle v, A^*f\rangle = \langle v, g\rangle = \langle v, Ah\rangle = \langle Av, h\rangle.$$

If $u \in X$ is arbitrary, there exists $v \in \mathcal{D}(A) = X$ such that $u = Av$. Hence we have

$$\langle u, f\rangle = \langle u, h\rangle \forall u \in X.$$

So $f = h \in \mathcal{D}(A)$.

Theorem

Let T be a symmetric operator, the following assertions are equivalents

1. T *is self-adjoint.*
2. T *is closed and* $\ker(T^* \pm i) = \{0\}$.
3. $Ran(T \pm i) = X$.

(1) $\Rightarrow$ (2): Let T is self-adjoint and $\varphi \in \mathcal{D}(T^*) = \mathcal{D}(T)$ such that $\varphi \in Ker(T^* \pm i)$. So

$$\mp i\langle \varphi, \varphi \rangle = \langle \mp i\varphi, \varphi \rangle = \langle T^*\varphi, \varphi \rangle = \langle T\varphi, \varphi \rangle = \langle \varphi, T^*\varphi \rangle = \pm i\langle \varphi, \varphi \rangle.$$

So $\varphi = 0$.

(2) $\Rightarrow$ (3): Let $y \in Ran(T \pm i)^{\perp}$, then $\langle (T \pm i)x, y \rangle = 0$ for any $x \in \mathcal{D}(T)$. So $y \in \mathcal{D}(T^*)$ and $T^*y = \pm iy$. So $y \in Ker(T^* \mp i) = \{0\}$ so $Ran(T \pm i)$ dense in X. Let's prove that $Ran(T \pm i)$ is closed. Indeed for all $x \in \mathcal{D}(T)$.

$$\| (T \pm i)x \|^2 = \| Tx \|^2 + \| x \|^2,$$

as T is symmetric. This yields that if $x_n \in \mathcal{D}(T)$ a sequence such that $(T \pm i)x_n \to y$, so x_n converges to x. As T is closed we deduce that $x \in \mathcal{D}(T)$ and $(T \pm i)x = y$. So $y \in Ran(T \pm i)$ so $Ran(T \pm i) = X$.

(3) $\Rightarrow$ (1) Let $x \in \mathcal{D}(T^*)$, as $Ran(T \pm i) = X$ there exists $y \in \mathcal{D}(T)$ such that $(T - i)y = (T^* - i)x$. As $T \subset T^*$, we have $x - y \in \mathcal{D}(T^*)$ and $(T^* - i)(x - y) = 0$. So

$$x - y \in ker(T^* - i) = Rang(T + i)^{\perp} = X^{\perp} = \{0\}.$$

So $x = y \in \mathcal{D}(T)$ and $\mathcal{D}(T) = \mathcal{D}(T*)$

Example 1: Let $X = l^2(\mathbb{N})$, let A be the operator with domain

$$\mathcal{D}(A) = \{x = (x_n)_{n\in\mathbb{N}} : x_n \neq 0, \text{for finitely many } n\}$$

and

$$Ax := (\sum_{i=1}^{\infty} x_i, 0, 0, 0, \cdots).$$

Let's determine A^*. Let e_n be the standard unit vector. Pick $y \in \mathcal{D}(A^*)$, then

$$1 \cdot \overline{y}_1 = \langle Ae_n, y\rangle = \langle e_n, A^*y\rangle = 1 \cdot \overline{(A^*y)_n}, \forall n \in \mathbb{N},$$

this yields that $A^*y = 0$, and we obtain $y_1 = 0$. So for any $y \in \mathcal{D}(A^*)$ we have $y_1 = 0$ and $A^*y = 0$.

Now consider the linear operator B given by

$$\mathcal{D}(B) = \{(y_n)_n \in l^2(\mathbb{N}) : y_1 = 0\}, By = 0.$$

Let $y \in \mathcal{D}(B)$,

$$\langle Ax, y\rangle = \langle x, By\rangle \forall x \in \mathcal{D}(A).$$

There for, $y \in \mathcal{D}(A^*)$ and $A^*y = By$. So

$$\mathcal{D}(A^*) = \{(y_n)_n \in l^2(\mathbb{N}) : y_1 = 0\}; A^*y = 0, \forall y \in \mathcal{D}(A^*).$$

Since $\mathcal{D}(A^*)$ is not dense in l^2, the operator A is not closable.

Example 2: Let $X = L^2([0,1])$, $\mathcal{D}(T_0) = \mathcal{C}_c^\infty((0,1))$

$$T_0 f = -f".$$

By integration by part we see that T_0 is symmetric. Let's compute for $f \in \mathcal{D}(T_0)$,

$$\langle T_0 f, 1\rangle = -\int_0^1 f" 1 = [-f'1]_0^1 + \int_0^1 f'1' = 0.$$

So $1 \in \mathcal{D}(T^*)$ and moreover $T^*1 = 0$. So $(1,0) \in Graph(T_0^*)$. For any $x \in [0,1]$, and $f \in \mathcal{D}$ we have

$$\begin{aligned} \mid f(x) \mid &= \mid \int_0^x \int_0^1 f"(s) ds dt \mid \leq \int_0^x \int_0^1 \mid f"(s) \mid ds dt \\ &\leq \int_0^1 \int_0^1 \mid f"(s) \mid ds dt = \int_0^1 \mid f"(s) \mid ds \leq \| T_0 f \| \end{aligned}$$

In particular if $\| T_0 f \| \leq \frac{1}{2}$; then $| f(x) | \leq \frac{1}{2}, \forall x \in [0,1]$, so $| -f(x) | \geq \frac{1}{2}$ and $\| 1 - f \| \geq \frac{1}{2}$. So in all cases we have

$$\| 1 - f \|^2 + \| T_0 f - 0 \|^2 \geq \frac{1}{4}.$$

So $(1,0) \notin \overline{Graph T_0}$ and T is not essentially self-adjoint.
$T_0^* f = -f''$, with
$\mathcal{D}(T_0^*) = H^2([0,1]) = \{f \in \mathcal{C}^1([0,1]) : f'' \in L^2([0,1])\}$

Definition

Let $A : \mathcal{D}(A) :\to X, \mathcal{D}(A) \subset X$ be a closed linear operator in some Hilbert space X. Then

$$\varrho(A) = \{\lambda \in \mathbb{C} | A - \lambda \text{ has a bounded inverse}\}.$$

Is the resolvent set and

$$\sigma(A) = \mathbb{C} \backslash \varrho(A),$$

the spectrum of A and $R_\lambda(A) = (A - \lambda)^{-1}$ is the inverse of $A - \lambda$.

The spectrum of a closed linear operator $A : \mathcal{D}(A) :\to X, \mathcal{D}(A) \subset X$, decomposes into the following components

1. $\sigma_p(A) = \{\lambda \in \mathbb{C} | ker(A_\lambda) \neq \{0\}\}$. It is called the point spectrum or set of eigenvalues of A. Every $x \in ker(A - \lambda) \backslash \{0\}$ is called eigenvectors of A with eigenvalue $\lambda \in \sigma_p(A)$.
2. $\sigma_r(A) = \{\lambda \in \mathbb{C} | ker(A - \lambda) = \{0\}, \overline{Range(A - \lambda)} \neq X\}$. Is called the residual spectrum of A.
3. $\sigma_c(A) = \{\lambda \in \mathbb{C} | ker(A - \lambda) = \{0\}; range(A - \lambda) \neq X; \overline{Range(A - \lambda)} = X\}$. Is called continuous spectrum.

Theorem

For any closed operator $A : \mathcal{D}(A) \subset X \to X$, we have the following disjoint decomposition

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A).$$

Others decomposition of the spectrum exists in case $A = A^*$.

1. Lebesgue decomposition $\sigma_{pp}(A) = \overline{\sigma_p(A)}$ pure point spectrum,
$$\sigma_c(A) = \sigma_{sc}(A) \cup \sigma_{ac}(A).$$
2. $\sigma_{disc}(A) = \{\lambda; \text{isolated eigenvalue of } A \text{ with finit multiplicity}\}$
$$\sigma_{ess}(A) = \sigma(A) \backslash \sigma_{disc}(A).$$

Spectrum of multiplication operators

Example: $M_f : L^2(M,\mu) \to L^2(M,\mu)$. Let $\lambda \in \mathbb{C}$, then

$$\begin{aligned} \lambda - A \quad & is \quad injective \\ & \Leftrightarrow \quad \{\varphi \in L^2(M,\mu), (\lambda - f(x))\varphi(x) = 0 \ a.e. \Rightarrow \varphi(x) = 0 \ a.e.\} \\ & \Leftrightarrow \quad \lambda - f(x) \neq 0 \ a.e. \Leftrightarrow \mu(\{x \in M \mid f(x) = \lambda\}) = 0. \end{aligned}$$

$$\sigma_p(M_f) = \{\lambda \in \mathbb{C} \ \mu(\{x \in M \mid f(x) = \lambda\}) > 0\}.$$

Let $\lambda \in \mathbb{C}\backslash\sigma_p(M_f)$. So, $M_f - \lambda \mathbf{1}_M$ has an inverse:

$$(M_f - \lambda \mathbf{1}_M)\psi = \varphi \Leftrightarrow (f - \lambda \mathbf{1}_M)\psi = \varphi(x) \; a.e. \Leftrightarrow \psi(x) = \frac{1}{f(x) - \lambda}\varphi \; a.e.$$

So

$$(M_f - \lambda \mathbf{1}_M)^{-1} = M_{\frac{1}{f-\lambda}},$$

with domain

$$\mathcal{D} = \{\varphi \in L^2(M,\mu) | M_{\frac{1}{f-\lambda}}\varphi \in L^2(M,\mu)\}.$$

$$M_{\frac{1}{f-\lambda}} \text{ is bounded} \Leftrightarrow \frac{1}{f-\lambda} \in L^\infty.$$

So

$$\varrho(M_f) = \{\lambda \in \mathbb{C} | \exists K > 0 \; s.t. |\lambda - f(x)| \geq K \; a.e.\}.$$

Let $\lambda \in \mathbb{C}\backslash(\sigma_p(A) \cup \varrho(A))$. So $\mu(\{x \in M \mid f(x) = \lambda\}) = 0$, but on the other hand $\mu(\{x \in M \|\, \lambda - f(x) \,|< \varepsilon\}) > 0$, for every $\varepsilon > 0$. Is the range of $(M_f - \lambda \mathbf{1}_M)$ dense or not? Let for $n \in \mathbb{N}$,

$$E_n = \{x \in M \mid f(x) - \lambda \,|\geq \frac{1}{n}\}.$$

For every $\psi \in L^2(M, \mu)$, we have $\chi_{E_n}\psi$ is the image under $(M_f - \lambda \mathbf{1}_M)$; :; of $\frac{1}{f(x)-\lambda}\chi_{E_n}(x)\psi(x) \in L^2(M, \mu)$. We have $\chi_{E_n}\psi$ converges pointwise to ψ, so by dominated convergence theorem, we get convergence in L^2. So, the range of $(M_f - \lambda \mathbf{1}_M)$ is dense. So

$$\sigma_r(M_f) = \emptyset.$$

Lemma

Let $A : \mathcal{D}(A) \to X, \mathcal{D} \subset X$, be a closed linear operator in Hilbert space X. Then

$$\varrho(A^*) = \overline{\varrho(A)} \quad \text{and} \quad \sigma(A^*) = \overline{\sigma(A)}.$$

Proposition

Let $A : \mathcal{D}(A) \to X, \mathcal{D} \subset X$ *be self-adjoint. Then*

1. $\sigma(A) \subset \mathbb{R}$.
2. $\sigma_r(A) = \emptyset$.
3. *If* 0 *is not in the spectrum of* A*, then* $A^{-1} : A(\mathcal{D}(A)) \to X$ *is self-adjoint.*

Proof:

1. $\sigma(A) = \overline{\sigma(A)}$.
2. If $ker(A-\lambda) = \{0\}$, then

$$\overline{(A-\lambda)(\mathcal{D}(A))} = \Big((A-\lambda)(\mathcal{D}(A))\Big)^{\perp\perp} = (ker(A-\lambda))^{\perp} = X,$$

thus $\sigma_r(A) = \emptyset$.

Theorem

Let A be a self-adjoint operator. Then $\lambda \in \sigma(A)$ if and only if there exists a sequence $\{u_n\}_n \subset \mathcal{D}(A)$, such that $\| u_n \|= 1$ and $\| (A - \lambda)u_n \|\to 0$ as $n \to +\infty$.

Proof: Let $\lambda \in \sigma(A)$. Two cases arises:

1. $ker(A-\lambda) \neq \{0\}$ i.e λ is an eigenvalue. Let f be an eigenvector Then let $u_n = f$ for any n with $\| f \| = 1$.
2. $ker(A-\lambda) = \{0\}$. Then $Ran(A-\lambda)$ is dense but not equal to X, so $(A-\lambda)^{-1}$ exist but it is unbounded.

Consequently, If there exists a sequence $\{v_n\}_n \subset \mathcal{D}((A-\lambda)^{-1}), \| v_n \| = 1$ such that

$$\| (A-\lambda)^{-1} v_n \| \to \infty.$$

Let $u_n = [(A-\lambda)^{-1} v_n] \| (A-\lambda)^{-1} v_n \|^{-1}$, then $\{u_n\}_n \subset \mathcal{D}(A), \| u_n \| = 1$, and

$$\| (A-\lambda) u_n \| = \| v_n \| \| (A-\lambda)^{-1} v_n \|^{-1} \to 0.$$

Conversely: Let $\lambda \in \varrho(A)$. Then there exists $M > 0$, such that for any $u \in X$

$$\| R_\lambda(A)u \| \leq M \| u \| .$$

Let $v = R_\lambda(A)u$, for $v \in \mathcal{D}(A)$ so that

$$\| v \| \leq M \| (A - \lambda)v \|,$$

and thus no sequence having the properties described can exist.

Perturbation theory

Definition

$B : \mathcal{D}(B) \to X$ is called A bounded with respect to $A : \mathcal{D} \to X$ densely defined operator if

1. $\mathcal{D}(A) \subset \mathcal{D}(B)$
2. There exists $a, b \in [0, \infty)$; $\forall x \in \mathcal{D}(A)$:

$$\| Bx \| \leq a \| Ax \| + b \| x \| .$$

Kato-Rellich Theorem

Theorem

Let $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subset X$ be a selfadjoint operator on some Hilbert space X and $B : \mathcal{D}(A) \to X$ be symmetric and A-bounded with relative bound < 1. Then

$$A + B : \mathcal{D}(A) \to X \text{ is selfadjoint}.$$

First we note that $A+B:\mathcal{D}(A)\to X$ is symmetric as

$$\forall x,y\in\mathcal{D}(A):\ \langle (A+B)x,y\rangle = \langle Ax,y\rangle + \langle Bx,y\rangle$$
$$= \langle x,Ay\rangle + \langle x,By\rangle = \langle x,(A+B)y\rangle.$$

Let $x\in\mathcal{D}(A)$ and $\eta\in\mathbb{R}\backslash\{0\}$. Then

$$\|(A+i\eta)x\|^2 = \|Ax\|^2 + \eta^2\|x\|^2.$$

Implies that for $x=(A+i\eta)^{-1}y, y\in X$:

$$\|A(A+i\eta)^{-1}y\| < \|y\| \text{ and } \|(A+i\eta)^{-1}y\| \leq \frac{1}{|\eta|}\|y\|$$

$$\Rightarrow \|B(A+i\eta)^{-1}y\| \leq a\|A(A+i\eta)^{-1}y\| + b\|(A+i\eta)^{-1}y\|$$

$$< a\|y\| + \frac{b}{\eta}\|y\|.$$

So by Neumann Theorem $C = 1 + B(A + i\eta)^{-1}$ is invertible and $range(C) = X$. As $(A + i\eta)\mathcal{D}(A) = X$, we have

$$X = C(A + i\eta)(\mathcal{D}(A)) = (A + B + i\eta)(\mathcal{D}(A)).$$

Thus $A + B$ is self-adjoint operator.

Remark

It can be proved that if A is essentially selfadjoint operator and $B : \mathcal{D}(A) \to X$ is symmetric with A-bound less than one , then $A + B : \mathcal{D} \to X$ is e.s.a. and

$$\overline{A+B} = \overline{A} + \overline{B}.$$

Theorem

Let A be a selfadjoint operator, with domain $\mathcal{D}(A)$ and B a compcat operator. Then $A + B$ is a selfadjoint operator on domain $\mathcal{D}(A)$ and

$$\sigma_{ess}(A) = \sigma_{ess}(A+B).$$

Application

If $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ is real valued. Then

$$H = -\Delta + M_V,$$

is selfadjoint on $\mathcal{D}(-\Delta) = W^{2,2}(\mathbb{R}^3)$ and e.s.a. on $\mathcal{C}_c^\infty(\mathbb{R}^3)$.

Lemma

$\forall f \in W^{2,2}(\mathbb{R}^3), \forall a > 0, \exists b \in \mathbb{R}$

$$\|f\|_\infty \leq a\|\Delta f\|_2 + b\|f\|_2.$$

Lower bounded operators and quadratic forms

Definition

1. Let $a(\cdot,\cdot)$ be a sesquilinear form defined on a dense domain $\mathcal{D}(a)$. We say that a is semibounded, if there exists $m \in \mathbb{R}$ such that
$$a(u,u) \geq m \parallel x \parallel^2 \ \ \forall u \in \mathcal{D}(a).$$
If the largest m is positive, we say that that is definite positive.
2. A symmetric operator S is said to be bounded from below if
$$\langle Su, u \rangle \geq m \parallel u \parallel^2, \forall u \in \mathcal{D}(S),$$
with some $m \in \mathbb{R}$.

Remark

The inner product

$$\langle u, v\rangle_a = (1-m)\langle u, v\rangle + a(u, v),$$

satisfies

$$\| u \|_a \geq \| u \|, \ \forall u \in \mathcal{D}(a).$$

Theorem

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let H_1 be a dense subspace of H . Assume that an inner product $\langle \cdot, \cdot \rangle_1$ is defined on H_1 in a such a way that $(H_1, \langle \cdot, \cdot \rangle_1)$ is a Hilbert space and with some $m > 0$ we have

$$m \parallel f \parallel^2 \leq \parallel f \parallel_1^2, \forall f \in H_1.$$

Then there exists a unique self-adjoint operator T on H such that for $\mathcal{D}(T) \subset H_1$ and $\langle Tf, g \rangle = \langle f, g \rangle_1$, for all $f \in \mathcal{D}(T), g \in H_1$, where T is bounded from below with lower bound m. The operator T can be defined by the equalities

$$\mathcal{D}(T) = \{f \in H_1 : \exists \overline{f} \in H, s.t \langle f, g \rangle_1 = \langle \overline{f}, g \rangle \forall g \in H_1\}, \qquad (11)$$

and $Tf = \overline{f}$, where $\mathcal{D}(T)$ is dense in H_1 w.r.t. $\| \cdot \|_1$.

Proof: First we check if such an operator defined by (11) exists since H_1 is dense $\overline{f}$ exists and is uniquely determined. The mapping $f \to \overline{f}$ is linear and so (11) define a linear operator, we denote it by

$$T : (H, \langle \cdot, \cdot \rangle) \to (H, \langle \cdot, \cdot \rangle),\ \mathcal{D}(T) \subset H_1.$$

But we can also define

$$T_0 : (H_1, \langle \cdot, \cdot \rangle_1) \to (H, \langle \cdot, \cdot \rangle)$$

with $\mathcal{D}(T) = \mathcal{D}(T_0)$. Then for all $f \in \mathcal{D}(T)$ we have

$$Tf = T_0 f,$$

and for all $f \in H_1$ we have by (11)

$$\langle f, g \rangle_1 = \langle \overline{f}, g \rangle = \langle T_0 f, g \rangle.$$

Also, for all $f \in \mathcal{D}(T), g \in H_1$ we have

$$\langle T_0 f, g\rangle = \langle f, T_0^* g\rangle_1 \Leftrightarrow \langle f, g\rangle_1 = \langle f, T_0^* g\rangle_1 \Rightarrow T_0^* g = g,$$

for all $g \in H_1$ i.e $\mathcal{D}(T_0^*) = H_1$. Furthermore, define

$$J : (H, \langle\cdot,\cdot\rangle) \to (H_1, \langle\cdot,\cdot\rangle)$$

with $\mathcal{D}(J) = H_1$ and $Jf = f$. Then for all $g \in H_1, f \in \mathcal{D}(T)$ we have

$$\langle f, Jg\rangle_1 = \langle f, g\rangle_1 = \langle f, T_0^* g\rangle_1.$$

Thus $J = T_0^*$.

Assume that J is closed, then $T_0^* = J$ is densely defined. Thus, since $\mathcal{D}(T) = \mathcal{D}(T_0)$, we have that T is densely defined in H_1 w.r.t. $\|\cdot\|_1$ and consequently in H w.r.t. $\|\cdot\|$. By (11) we have for all $f, g \in \mathcal{D}(T)$

$$\begin{aligned} \langle Tf, g\rangle &= \langle f, g\rangle_1 = \overline{\langle g, f\rangle_1} \\ &= \overline{\langle Tg, f\rangle} \; as \; g \in \mathcal{D}(T) \\ &= \langle f, Tg\rangle. \end{aligned}$$

So T is symmetric.

Assume that selfadjoiness of T follows if $Range(T) = H$. Let $f \in \mathcal{H}$ be arbitrary. Then

$$g \mapsto \langle f, g \rangle$$

is a continuous linear functional on H_1 because

$$|\langle f, g \rangle| \leq \|f\| \|g\| \leq m^{-1/2} \|f\| \cdot \|g\|_1.$$

There fore there exists an $\overline{f} \in H_1$ such that

$$\langle f, g \rangle = \langle \overline{f}, g \rangle_1,$$

for all $g \in H_1$ by Riesz Theorem. This means that $\overline{f} \in \mathcal{D}(T)$ and $f = T\overline{f}$. The semi-boundeness follows from

$$\langle Tf, f \rangle = \langle f, f \rangle_1 \geq m \|f\|^2, \forall f \in \mathcal{D}(T).$$

Uniqueness. If S satisfies $\mathcal{D}(S) \subset H_1$ and

$$\langle Sf, g \rangle = \langle f, g \rangle_1 .$$

Then $S = T|_{\mathcal{D}(S)}$, i.e $S \subseteq T \subseteq T^* \subseteq S^*$. If S is self adjoint this implies

$$S = T.$$

Theorem

Assume H is a Hilbert space. $\mathcal{D}$ is a dense subspace of H and $s(\cdot,\cdot)$ is a semi-bounded symmetric sesquilinear form on $\mathcal{D}$ with lower bound m. Let $\|\cdot\|_s$ be compatible with $\|\cdot\|$. Then there exists a unique semi-bounded selfadjoint operator T with lower bound m such that $\mathcal{D}(T) \subseteq H_s$ and $\langle Tf, g\rangle = s(f,g)$ for all $f \in \mathcal{D} \cap \mathcal{D}(T), g \in \mathcal{D}$. We have

$$\mathcal{D}(T) = \{f \in H_s : \exists \overline{f} \in H, s.t. s(f,g) = \langle \overline{f}, g\rangle \forall g \in \mathcal{D}\}. \qquad (12)$$

Where $Tf = \overline{f}$ for $f \in \mathcal{D}(T)$. H_s is the completion of $(\mathcal{D}, \|\cdot\|_s)$.

Replace $(H_1, \langle \cdot, \cdot \rangle)$ by $(H_s, \langle \cdot, \cdot \rangle)$ in the last theorem. Then we obtain exactly one self adjoint operator T_0 such that $\mathcal{D}(T_0) \subseteq H_s$ and

$$\langle T_0 f, g \rangle = \langle f, g \rangle_s = (1-m)\langle f, g \rangle + s(f, g),$$

for all $f \in \mathcal{D}(T_0), g \in H_s$. Also, T_0 is semi-bounded with lower bound 1 because

$$\begin{aligned} \langle f, T_0 f \rangle &= \langle f, f \rangle_s = (1-m)\langle f, f \rangle + s(f, f) \\ &\geq (1-m) \parallel f \parallel^2 + m \parallel f \parallel^2 = \parallel f \parallel^2 . \end{aligned}$$

Define $T = T_0 - (1 - m)$. Then

$$\begin{aligned}
\langle Tf, f\rangle &= \langle (T_0 - (1-m))f, f\rangle \\
&= \langle T_0 f, f\rangle - \langle (1-m)f, f\rangle \\
&\geq \| f \|^2 - \| f \|^2 + m \| f \|_s^2 \\
&= m \| f \|_s^2 .
\end{aligned}$$

1. $\mathcal{D}(T) \subseteq H_s$ follows easily from $\mathcal{D}(T_0) \subseteq H_s$. This because shifting an operator by a constant does not change the domain.

2. $$\begin{aligned} \langle Tf, g \rangle &= \langle [T_0 - (1-m)]f, g \rangle \\ &= \langle T_0 f, g \rangle - \langle (1-m)f, g \rangle \\ &= (1-m)\langle f, g \rangle + s(f,g) - \langle (1-m)f, g \rangle \\ &= s(f,g) \end{aligned}$$

for all $f \in \mathcal{D} \cap \mathcal{D}(T_0), g \in \mathcal{D}$. Uniqueness of T follows from uniqueness of T_0.

Friedrichs Extension Theorem

Theorem

Let S be a semi-bounded symmetric operator with lower bound $m > 0$. Then there exists a semi-bounded self-adjoint extension of S with lower bound m. If we define

$$s(f,g) = \langle Sf, g\rangle, \forall f, g \in \mathcal{D}(S),$$

for H_s, the completion of $(D(S), \|\cdot\|_s)$ then we have the operator T defined by

$$\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s$$

and $Tf = S^*f$ for all $f \in \mathcal{D}(T)$ is a selfadjoint extension of S with lower bound m. The operator T is the only selfadjoint extension of S having the property $\mathcal{D}(T) \subseteq H_s$.

Proof: By the last theorem we know there exists a unique selfadjoint operator T with $\mathcal{D}(T) \subseteq H_s$ and

$$\langle Tf, g\rangle = s(f, g) = \langle Sf, g\rangle, \forall f \in \mathcal{D}(S) \cap \mathcal{D}(T),$$

and m is lower bound for T. We have by (12)

$$\mathcal{D}(T) = \{f \in H_s : \exists \overline{f} \in H, \overline{s}(f, g) = \langle \overline{f}, g\rangle \forall \langle, \quad \forall g \in \mathcal{D}(S)\}.$$

Let $(f_n)_n \in \mathcal{D}(S)$ such that

$$\|f_n - f\| \to 0.$$

Then we obtain

$$\begin{aligned}
\overline{s}(f,g) &= \lim_{n\to\infty} \overline{s}(f_n,g) = \lim_{n\to\infty} (\langle f_n, g\rangle_s - (1-m)\langle f_n, g\rangle) \\
&= \lim_{n\to\infty} ((1-m)\langle f_n, g\rangle + s(f_n,g) - (1-m)\langle f_n, g\rangle) \\
&= \lim_{n\to\infty} s(f_n,g) = \lim_{n\to\infty} \langle Sf_n, g\rangle \\
&= \lim_{n\to\infty} \langle f_n, Sg\rangle = \lim_{n\to\infty} \langle f, Sg\rangle,
\end{aligned}$$

because $\|\cdot\|_s$ is compatible with $\|\cdot\|$. So we can replace $\overline{s}(f,g)$ with $\langle f, Sg\rangle$.

We have to show T is an extension of S:

1. From definition of $\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s$. Also $T = S^* \mid_{\mathcal{D}(T)}$.
2. Since S is symmetric then $S \subseteq S^*$. Also $\mathcal{S} \subseteq H_s$ by construction. Thus

$$\mathcal{D}(S) \subseteq \mathcal{D}(S^*) \cap H_s = \mathcal{D}(T).$$

Furthermore, since S is symmetric then $S = S^* \mid_{\mathcal{D}(S)}$ which means that, by 1, $S = T \mid_{\mathcal{D}(S)}$. Thus we have $S \subseteq T$.

(Uniqueness) Let A be an arbitrary self adjoint extension of S such that $\mathcal{D}(A) \subseteq H_s$. Then since $S \subseteq A$ we have $A \subseteq S^*$ which means $\mathcal{D}(A) \subseteq \mathcal{D}(S^*)$ and $A = S^* \mid_{\mathcal{D}(A)}$. Also,

$$\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s,$$

which means $\mathcal{D}(A) \subset \mathcal{D}(T)$ and so

$$\begin{aligned} A &= S^* \mid_{\mathcal{D}(A)} \\ &= T \mid_{\mathcal{D}(A)} . \end{aligned}$$

Thus $A \subseteq T$ which implies $T = T^* \subseteq A^* = A$ and so $A = T$.

Some useful estimate

Lemma

Let T be a self-adjoint operator and densely defined. For $\lambda \in \mathbb{C}\backslash\mathbb{R}$, the operator R_λ is everywhere defined on X, and the norm is estimated by

$$\|R_\lambda\| \leq \frac{1}{\mid Im\lambda \mid}.$$

Proof: For $\lambda = x + iy$ and $v \in \mathcal{D}(T)$,

$$\begin{aligned} \mid (T - \lambda)v \mid^2 &= \mid (T+x)v \mid^2 + \langle (T-x)v, iyv\rangle + \langle iyv, (T-x)v\rangle + y^2 \mid v \mid^2 \\ &= \mid (T+x)v \mid^2 - iy\langle (T-x)v, v\rangle + iy\langle v, (T-x)v\rangle + y^2 \mid v \mid^2 \\ &= \mid (T-x)v \mid^2 + y^2 \mid v \mid^2 \geq y^2 \mid v \mid^2 . \end{aligned}$$

Thus, for $y \neq 0, (T-\lambda)v \neq 0$. On $(T-\lambda)\mathcal{D}(T)$, there is an inverse R_λ of $T - \lambda$, and for $w = (T-\lambda)v, v \in \mathcal{D}(T)$

$$\mid w \mid = \mid (T-\lambda)v \mid \geq \mid y \mid \cdot \parallel v \parallel = \mid y \mid \parallel R_\lambda (T-\lambda)v \parallel = \mid y \mid \cdot \parallel R_\lambda w \parallel$$

which gives

$$\parallel R_\lambda w \parallel \leq \frac{1}{\mid Im\lambda \mid} \cdot \parallel w \parallel \ (\text{for } (T-\lambda)v, v \in \mathcal{D}(T)).$$

Thus, the operator norm on $(T-\lambda)\mathcal{D}(T)$ satisfies $\| R_\lambda \| \leq \frac{1}{Im\lambda}$ as claimed. It remains to show that $(T-\lambda)\mathcal{D} = X$, the hole space. If

$$\langle (T-\lambda)v, w\rangle = 0, \ \forall v \in \mathcal{D}(T).$$

So $T-\lambda$ can be defined on w as $(T-\lambda)^* w = 0$, this gives $Tw = \overline{\lambda} w$, so $w = 0$. Thus, $(T-\lambda)\mathcal{D}(T)$ is dense in X. As T is closed we get it is equal to X.

Definition

Let $x \in \mathcal{A}$ and λ an isolated point of $\sigma(x)$. Let Γ_{λ_0} be an admissible contour i.e a closed contour around λ_0 such that the closure of the region bounded by Γ_{λ_0} intersects $\sigma(x)$ only at λ_0,

$$P_{\lambda_0} = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} R_\lambda(A) d\lambda,$$

is called Riesz integral for x and λ_0.

Proposition: Let P_{λ_0} be a Riesz integral for x and λ_0.

1. P_{λ_0} is a projection.
2. $Ker(x-\lambda_0) \subset RanP_{\lambda_0}$.
3. If $\mathcal{A}$ is a Hilbert space and x is self adjoint, then P_{λ_0} is orthogonal projection onto $ker(x-\lambda_0)$.

Proof: (1) Let Γ_{λ_0} and $\Gamma_{\lambda_0}^{\tilde{}}$ be two admissible contours for defining P_{λ_0}, we suppose that Γ_{λ_0} is contained in the interior of the region bounded by $\Gamma_{\lambda_0}^{\tilde{}}$.

$$P_{\lambda_0}^2 = \frac{1}{(2\pi i)^2}\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\Gamma_{\lambda_0}^{\tilde{}}} d\mu R_\lambda(x) R_\mu(x) d\mu \qquad (13)$$

$$= \frac{1}{(2\pi i)^2}\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\Gamma_{\lambda_0}^{\tilde{}}} (\mu-\lambda)^{-1}[R_\lambda(x) - R_\mu(x)] d\mu. \qquad (14)$$

Using the residue theorem, we get:

$$\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\Gamma_{\lambda_0}^{\tilde{}}} (\mu-\lambda)^{-1} R_\lambda(x) d\mu = 2\pi i \oint_{\Gamma_{\lambda_0}} R\lambda(x) d\lambda.$$

For the second integral we get that

$\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\Gamma_{\lambda_0}^{\tilde{}}} (\mu-\lambda)^{-1} R_\mu d\mu = \oint_{\Gamma_{\lambda_0}^{\tilde{}}} R_\mu(x) d\mu \oint_{\Gamma_{\lambda_0}} (\mu-\lambda)^{-1} d\lambda = 0.$

(2). Let $f \in ker(x - \lambda_0)$. Then for $\lambda \neq \lambda_0$

$$(x - \lambda_0)^{-1} f = (\lambda_0 - \lambda)^{-1} f.$$

We show that $P_{\lambda_0} f = f$, so $f \in Ran P_{\lambda_0}$. By the definition of P_{λ_0} we find that

$$P_{\lambda_0} f = \frac{1}{2\pi} \oint_{\Gamma_{\lambda_0}} (x - \lambda)^{-1} f d\lambda \quad (15)$$

$$= \frac{1}{2\pi} \oint_{\Gamma_{\lambda_0}} \oint_{\Gamma_{\lambda_0}} (\lambda_0 - \lambda)^{-1} f d\lambda = f \quad (16)$$

(3) Let x be an Hilbert space and suppose that $x = x^*$ (Exercise: show that $P_{\lambda_0} = P^*_{\lambda_0}$). We must show now that $Ran P_{\lambda_0} \subset ker(x - \lambda_0)$. We compute

$$(x - \lambda_0) P_{\lambda_0} = \frac{1}{2\pi} \oint_{\Gamma} (x - \lambda_0)(x - \lambda)^{-1} d\lambda \quad (17)$$

Consider U_{λ_0} denote the interior of Γ_{λ_0}. On $U_{\lambda_0} \backslash \{\lambda_0\}$, the operator $(\lambda - \lambda_0)(x-\lambda)^{-1}$ is analytic, operator and satisfies

$$|\lambda_0 - \lambda| \|(x-\lambda)^{-1}\| \leq |\lambda_0 - \lambda| d(\lambda, \sigma(x))^{-1}. \qquad (19)$$

We can choose Γ_{λ_0}, so that λ_0 is the closest point of $\sigma(x)$ to Γ_{λ_0}. So $|\lambda_0 - \lambda| \|(x-\lambda)^{-1}\| \leq 1$ and this function is uniformly bounded on $U_{\lambda_0} \backslash \{\lambda_0\}$. It follows that $(\lambda_0 - \lambda)(x-\lambda)^{-1}$ extends to analytic function on U_{λ_0} so by Cauchy theorem the integral(19) vanishes. This gives that $RanP_{\lambda_0} \subset Ker(x - \lambda_0)$.

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